Saving for Justice

by

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1. Introduction

One of the central issues of welfare macroeconomics is, How much ought a nation to save? The classic answer of Ramsey [11], that savings is optimal if it maximizes the integral of utility over time subject to the national income constraint, has guided research in the area for half a century. Recently this dominion of utilitarianism has come under scrutiny. Mirrlees [7], for instance, has shown how Ramsey's rule breaks down in the presence of technical change and population growth. Moreover, if the scope of the question is not merely efficiency but also justice, the whole framework of utility maximization may be indefensible. Thus, Rawls in his theory of justice remarks:

> ...it seems evident, for example, that the classical principle of utility leads in the wrong direction for questions of justice between generations...maximizing total utility may lead to an excessive rate of accumulation (at least in the near future)...the utilitarian doctrine may direct us to demand heavy sacrifices of the poorer generations for the sake of greater advantages for the later ones that are far better off. But this calculus of advantages which balances the losses of some against benefits to others, appears even less justified in the case of generations than among contemporaries. Even if we cannot define a precise just savings principle, we should be able to aviod this sort of extreme. [12, pp. 286-7].

While Rawls does not explicitly define a just savings principle, a number of recent economic studies have explored the implications of Rawls overall position. Ordover [7] and Ordover and Phelps [8] consider optimal taxation of wages, wealth, and interest in a steady-state economy which satisfies Rawl's difference principle. The major drawback of these papers is their assumption of a steady-state, which leaves aside the important issue of how the steady state should be reached; and if there are multiple steady states, which one should be reached.

This latter set of issues, of direct relevance to the theory of saving, has also received attention [1, 3, 13, 14, 2, 4]. The literature so far has

applied Rawls' difference principle for contemporaries to the intergenerational context. Thus, Arrow has shown [1] that there is no net capital accumulation in a Rawlsian savings program so formulated, as long as intergenerational altruism is sufficiently limited (for results in similar vein, see Dasgupta [3]). Riley [13] and Riley and Phelps [14] have shown that more extensive altruism or overlapping generations are sufficient to ensure a positive net capital accumulation. Calvo [2] has extended the difference principle model to the case of an uncertain technology. Grout [4] has discussed the model from the point of view of game theory. This literature however faces the problem that Rawls explicitly rejects the maximin welfare function as applying to the question of justice between generations:

> It is now clear why the difference principle does not apply to the savings problem. There is no way for later generations to improve the situation of the least fortunate first generation. The principle is inapplicable and it would seem to imply, if anything, that there be no saving at all. Thus, the problem of saving must be treated in another fashion. [12, p. 291]

This paper, then, attempts to construct a just savings principle on a theoretical level that corresponds to Rawls' own suggestions "at a more primitive level" [10, p. 286]. In the next section the savings problem is modeled as a optimal control problem. The major differences between the Rawlsian solution and utilitarian solutions turn out to correspond to differences in the objective function and the space of admissible controls. The third section discusses the Ramsey case of no technological progress and no population growth. Here are proved both the existence of a just savings path and a characterization of the essentially unique just path. The fourth section extends these results to the case of population growth and technological change; the results here can be contrasted to those of Mirrlees. The fifth section compares numerically Rawsian and utilitarian paths. Unanswered questions and further issues are raised in the conclusion.

2. Model and Interpretations

We first present the stationary state model. Let $k(t)$ be the capital stock at time t.

Output $y(t)$ is a function of the capital stock,

(1) $y(t) = f(k(t))$.

We assume that the production function has $f(k) > 0$ for $k > 0$ and $f' > 0$. Denote by $u(t)$ the average saving rate at time t. Obviously,

(2) $0 \leq u(t) \leq 1$

The rate of investment, $\dot{k}(t)$, is given by

(3) $\dot{k}(t) = u(t) \, y(t)$

There is no depreciation of capital. Initial capital stock at time 0 is given by

(4) $k(0) = k_o > 0$.

The target capital stock at time T is given by the interval

(5) $k(T) \; \epsilon \; [k*, \; k^{max}]$.

Here k^{max} is the capital accumulated if $u(t) = 1$ throughout the time period $[0, \; T]$; $k*$, the minimum allowable accumulated capital. $k*$ is assumed to be greater than k_o. Taking $k(t)$ as the state variable and $u(t)$ as the control variable, the objective of the problem is to minimize the maximum control,

(6) $\min \max_t u(t)$

subject to (1) - (5).

Some discussion of this framework (in particular (5) and (6)) is in order, since it is not obvious that the control problem answers to Rawlsian principles. Now Rawls says:

> It is also characteristic of the contract doctrine to define
> a just state of society at which the entire course of accumulation
> aims. This feature derives from the fact that an ideal conception
> of a just basic structure is embedded in the principles chosen in
> the original position. In this respect, justice and fairness con-
> trasts with utilitarian views. The just savings principle can be

> regarded as an understanding between generations to carry their
> fair share of the burden of realizing and preserving a just
> society. The end of the savings process is set up in advance,
> although only the general outlines can be discerned.
>
> ...Of course, the just savings principle applies to what
> a society is to save as a matter of justice. If men wish to
> save for various grand projects, that is another matter. [12,
> pp. 288-9].

On this interpretation, then, k* is the capital stock required for a "just

state of society," while the interval in (5) allows for saving "for various

grand projects."

The objective function (6) derives first of all from the idea that

savings, since it is a burden, ought to be somehow minimized. The special

form of (6) can be grounded on further Rawlsian principles, in particular,

the general conception of justice.

> All social values...are to be distributed equally unless an
> unequal distribution of any, or all, of these values is to every-
> one's advantage. [12, p. 62].

This would seen to apply also to a disvalue like saving. But the conception

only applies if there is a relevant feature for which equality is even possi-

ble. It is obvious that utility ordinarily is not such a feature, since the

richer later generations will be better off. [See, however, [13] and [14]

for cases where utility can be equal.] Again, it would be possible to have

k̇ the same for all generations, but this would seem to impose a much greater

burden on the poorer generations. Here, Rawls says:

> · Thus the persons in the original position are to ask them-
> selves how much they would be willing to save at each stage of
> advance on the assumption that all other generations are to save
> at the same rates. That is, they are to consider their willing-
> ness to save at any given phase of civilization with the under-
> standing that the rates they propose are to regulate the whole
> span of accumulation... In effect, then, they must choose a just
> savings principle that assigns an appropriate rate of accumula-
> tion to each level of advance. Presumably this rate changes
> depending upon the state of society. When people are poor and
> saving is difficult, a lower rate of saving should be required;
> whereas in a wealthier society greater savings may reasonably be
> expected since the real burden is less. [12, p. 287].

Thus, if $\dot{k}$ is the rate of accumulation, then (6) says that the maximum "real burden," $\dot{k}(t)/y(t)$, is to be minimized.

Finally, as far as the finite terminal time T is concerned, Rawls remarks:

> But in any event we are not bound to go on maximizing indefinitely. Indeed, it is for this reason that the savings principle is agreed to after the principles of justice for institutions, even though this principle constrains the difference principle. These principles tell us what to strive for. The savings principle represents an interpretation, arrived at in the original position, of the previously accepted natural duty to uphold and further just institutions. In this case the ethical problem is that of agreeing on a path over time which treats all generations justly during the whole course of society's history. [12, p. 289].

Indeed, from the standpoint of the priority of justice, the just state ought not to be postponed indefinitely. As far as society after time T is concerned, Rawls offers the possible interpretation "of the ideal society as one whose economy is in a steady state of growth (possible zero); and which is at the same time just." [12, p. 286]. Still, one might not be willing to commit to a fixed time interval of length T, even in the original position. Thus, if T itself belongs to an interval $[T_0, T_1]$ of lengths, one satisfies a final Rawlsian dictum:

> Now I believe that it is not possible, at present anyway, to define precise limits on what the rate of savings should be. How the burden of capital accumulation and of raising the standard of civilization and culture is to be shared between generations seems to admit of no definite answer. It does not follow, however, that certain bounds which impose significant ethical constraints cannot be formulated. [12, p. 286].

Our justification for the control problem (1)-(6) comes down to this. As far as we can see, it is consistent with what Rawls says about just saving. Even if Rawls' entire theory were mistaken, the control problem would still account for just savings in other theories, for example, intuitionist. In either event, the framework is a definite alternative to all forms of utilitarianism.

It will prove useful to extend the model to account for population growth and technological change. Specifically, let population grow at a constant rate $\underline{n}$ and let there be Harrod-neutral technological progress at constant rate $\underline{a}$. Now interpret $k(t)$ ϵ $y(t)$ as the capital-labor ratio and output-labor ratio in efficiency labor units, respectively. Amending equation (3) to (3'), to describe the law of motion of $k(t)$:

(3') $\dot{k} = u\ f(k) - (n + a)k$

completes the extension.

Finally, we shall need utilitarian versions of the control problem:

(6') $\max \int_0^\infty L(t) \cdot U(c(t))dt$

subject to (3) or (3') and (4)

where $L(t)$ is population at time t, $c(t)$ is per capita consumption, and U is the instantaneous utlity function. The endpoint restriction on capital is lifted. In the Ramsey case, $L(t) = 1$ and $n = a = 0$. In the Mirrlees case, $L(t) = e^{nt}$ and $n + a \geq 0$. In case the integral (6') is not defined, the overtaking principle replaces it. Thus, utility stream U_1 is better than U_2 if for all t large enough,

$\int_0^t L(t)\ U_1\ dt > \int_0^t L(t)\ U_2\ dt.$

3. Just Saving in a Stationary Economy

This section analyzes the control problem

$$\min_{t} \max \; u(t)$$

(I)   subject to

(2) $\dot{k} = u\, f(k)$

(3) $0 \leq u(t) \leq 1$

(4) $k(0) = k_o$ fixed

(5) $k(T) \in [k*,\, k^{max}]$

Our principal result is the following.

Theorem 1.   There exists a solution to control problem (I).   The solution is essentially unique, and is characterized as being a constant control leading from k_o to $k*$ in time T.

Proof. Let $k(T) = \bar{k} \in [k*,\, k^{max}]$.

Integrating (2), one has

(7) $$\int_{k_o}^{\bar{k}} \frac{dk}{f(k)} = \int_{0}^{T} u(t)\, dt,$$

but the integral on the left-hand side is simply a positive constant depending on $\bar{k}$, denoted $c(\bar{k})$.

Consider the problem

$$\min \; \| u \|$$

subject to (7)

where the norm is taken in the sense of L_∞ space.   We show the solution to this control problem also solves control problem (I).

By the minimum norm duality theorem [6, Theorem 2, Corollary 1, pp. 123-4], there exists a control of minimum norm, $u*(t)$, such that

(8) $\| u* \| = \max\, c(\bar{k})b$

S.T. $\| b \| \leq 1$

the norm on the right-hand side being taken in L_1. Thus, the minimum norm

has the value

$$\frac{c(\overline{k})}{T} \text{ , since } 1 \geq \|\| b \|\| = \int_0^T | b | \, dt.$$

Moreover, the control of minimum norm is aligned with the constant function

$\frac{1}{T}$ on the interval $[0, T]$; which means that

(9) $u*(t) = K[\text{sgn } \frac{1}{T}] = K$, for some constant K.

(8) and (9) together imply that the optimal control $u*(t) = \frac{c(\overline{k})}{T}$, a constant.

The solution is essentially unique, in the sense that any optimal

solution differs from u* only on a set of measure zero. [Lebesgue measure

on $[0, T]$ is meant throughout.] The minimum norm control obviously satisfies

$0 \leq u*(t)$, so it remains to show that $u*(t) \leq 1$; then u* is optimal for the

target $\overline{k}$.

To see this, note that $c'(\overline{k}) = \frac{1}{f(\overline{k})} > 0$, so $c(\overline{k})$ is strictly increasing.

By the definition of $k^{\max}$, $c(k^{\max}) = T$. This $c(\overline{k}) < c(k^{\max}) = T$, and so

$u* = \frac{c(\overline{k})}{T} < 1.$

Finally, by the montomicity of $c(\overline{k})$, the optimal control for problem (I)

is given by

(10) $u*(t) = \frac{1}{T} \int_{k_o}^{k*} \frac{dk}{f(k)}$.

Notice that the just savings rate decreases as the time span T increases,

while the just savings rate increases as k* increases. Notice also that

computing the just savings path will in practice be a much simpler matter than

computing any utilitarian path, since only the evaluation of the integral

in (10) is involved.

4. Just Saving in a Growing Economy

In this section we extend the analysis of the previous section to economies that are capable of growth. The control problem now becomes

$$\min_{t} \max u(t)$$

(II) subject to

(2') $\dot{k} = u\, f(k) - (n + a)k$

(3) $0 \le u(t) \le 1$

(4) $k(0) = k_o$ fixed

(5) $k(t) \in [k*, k^{max}]$

Our principal result of the last section immediately generalizes as:

<u>Theorem 2</u>. There exists a solution to control problem (II). The solution is essentially unique, and is characterized as being a constant control leading from k_o to k* in time T.

<u>Proof</u>. By a well-known result in control theory [5, Theorem 4, Corollary 2, p. 262], there exists an optimal control.

Let U be the set of all controls, the application of which makes k(T) reach the target interval. A control $u \in U$ is inadmissible if there exists another control $\bar{u} \in U$, such that

$\bar{u}(t) \le u(t)$ all t

$\bar{u}(t) < u(t)$ on a set t of positive measure.

If there exists no such control, then u is admissible. Let $U_C \subset U$ be the set of constant controls, and $\varphi(t, u) = k(t)$ be the solution of (2') for $u \in U_C$. We show that

(12) $\dfrac{\partial \varphi}{\partial u} > 0$

when the initial value (4) is fixed. Applying a well-known result in differential equations [10, Theorem 17, p. 197], $\dfrac{\partial \varphi}{\partial u}$ satisfies the linear differential equation

(13) $\dfrac{d}{dt} \dfrac{\partial \varphi}{\partial u} = (uf' - (n + a)) \dfrac{\partial \varphi}{\partial u} + f$

with initial value $\dfrac{\partial \varphi}{\partial u} (t_o, u) = 0$, where f' and f are evaluated at $\varphi(t, u)$.
Solving (13), one has

(14) $\dfrac{\partial \varphi}{\partial u} (t, u) = \displaystyle\int_{t_o}^{t} f \exp \left[- \int_{t_o}^{t} uf' - (n + a) dt \right] dt$

the right hand side of which is clearly positive. In particular,
$\dfrac{dk(T)}{du} = \dfrac{\partial \varphi(T, u)}{\partial u} > 0$; thus the constant control in U_C that leads to k* is

minimal with respect to U_C. Denoting the above control by u*, we claim that

u* is admissible for all U. For, suppose that u* is not admissible. Then

there exists $\bar{u} \, \epsilon \, U$, $\bar{u}(t) \leq$ u* everywhere and $\bar{u}(t) < u(t)$ on a non-negligible

set S. Let (t_o, t_1) be the first open interval found in S. On (t_o, t_1) it

is clear that $\dot{k}(\bar{u}) < \dot{k}$(u*) and since $\dot{k}(\bar{u})$ is never greater than $\dot{k}$(u*)

$$\int_{0}^{T} \dot{k}(\bar{u}) = \int_{(t_o, t_1)} \dot{k}(\bar{u}) + \int_{[0, T] - S} \dot{k}(\bar{u}) + \int_{S - (t_o, t_1)} \dot{k}(\bar{u})$$

$$< \int_{(t_o, t_1)} \dot{k}(u*) + \int_{[0, T] - (t_o, t_1)} \dot{k}(u*) = k* - k_o$$

so that $\bar{u} \notin U$, a contradiction. It is further evident by the preceding

inequality that a constant control which is admissible is minimax.

Remark. Theorem 2 obviously contains Theorem 1 as a special case, with

n + a = 0. However, Theorem 1 is easier to prove and perhaps gains in insight.

Further, unlike the case in Theorem 1, one can give no explicit rule for calcu-

lating the minimax policy here, since the accumulation equation (2') may be

computationally intractable.

5. Simple Examples

It is worthwhile seeing what minimax savings paths look like in fairly

orthodox settins, if only to compare them to utilitarian alternatives. We

consider two examples, one involving a stationary economy, the other, an

economy with growing population and technical progress.

Example 1. Let the production function be $f(k) = k^b$, $0 < b < 1$. Since

the economy is stationary, (10) applies and the minimax savings rate is given

by

$$u^*(t) = \frac{1}{T} \int_{k_o}^{k^*} k^{-b} \; dk = \frac{k^{*1-b} - k_o^{1-b}}{T(1-b)}$$

Substituting in (2) and solving, the time path of capital is given by

$$(15) \quad k(t) = (k_o^{1-b} + (1-b)u^*t)^{1/(1-b)}$$

Table 1 sets out the savings paths (15) for b ranging from .25 to .75, when

$k_o = 4$, $k^* = 8$, and $T = 100$. The paths are very much alike, despite the

great differences in the technology. In table 2 is shown the dependence of

the minimax savings rate on various values of (k^*, T) for $k_o = 4$ and $b = .25$.

The savings rate is never above 7.5%, nor does it change dramatically in

response to changes in the target stock or accumulation period. For higher

b, minimax savings rates follow the general pattern of table 2, but somewhat

lower. A reasonable rate, averaging all these cases, would appear to be

about 3%.

Compare these results to a utilitarian path with the same parameter

values, the objective now being to maximize (6') with instantaneous utility

function

$$(16) \quad u(c) = -c^{-1}.$$

Consumption c is given by the equation

(17) $c(t) = f(k(t)) - \dot{k}(t)$.

By Ramsey's rule, there exists a constant B such that optimal $\dot{k}$ satisfies

(18) $\dot{k} = \dfrac{B - u}{u'} = c$

the last equality following from (16) and the fact that the bliss point B is zero. Substituting (17) into (18), one has that

(19) $\dot{k} = f(k(t))/2$.

In other words, Ramsey's rule here requires a constant savings rate of 50% forever. Obviously, a large capital is bound to accumulate. For instance, k(100) ranges from 138 for b = .25 up to 37483 for b = .75. Utilitarianism thus places a much heavier savings burden on the present generation, for the benefit of later generations that inherit the accumulated capital.

Example 2. (Steady-State Case). Again the production function is $f(k) = k^b$, $0 < b < 1$, where k is the capital-labor ratio. Such a relation is implied by the Cobb-Douglas production function with constant returns to scale and capital share b. In this case, (2') can be solved explicitly, the minimax control being given by

(20) $u^* = (n + a)(k^{*1-b} - k_o^{1-b}e^{-(n+a)T})$.

The time path of the capital-labor ratio is given by

(21) $k(t) = (k_o^{1-b}e^{-(n+a)t} + u^*(n+a))^{1/(1-b)}$

Table 3 shows the dependence of the minimax saving rate on the population growth rate plus the technical progress rate, and on the period of accumulation, based on (20). Calculations are carried out for the parameter values $k_o = 4$, $k^* = 8$, and b = .25. Both increased population growth and a longer period of accumulation result in a higher minimax saving rate. Note also that for large

Table 1. Time paths of k(t), minimax saving

	t = 0	t = 25	t = 50	t = 75	t = 100
b = .25, u* = .026	4	4.93	5.91	6.93	8
b = .5, u* = .017	4	4.87	5.83	6.87	8
b = .75, u* = .011	4	4.81	5.74	6.80	8

Table 2. Dependence of u* on k* and T

$(b = .25, \ k_o = 4)$

T \ k*	6	8	10
50	.027	.051	.075
100	.013	.026	.037
200	.007	.013	.019

T, savings rates here are substantially higher than in Table 2. Measurement in terms of efficiency-units of labor is partly responsible for these results. A fair average of these cases would appear to be 10%.

The example has been framed in such a way that a direct comparision with Mirrlees' results [7, pp. 109-112] is possible. The utilitarian asymptotic saving rate is given by $(a + n)b/(2a)$. These values are reflected in the last column of table 4. The initial savings rates are obtained from Mirrlees' figure 4, after translating the capital-labor ratio into the capital-output ratio. Calculations are based on zero population growth, technical progress at 2%, and $k_o = 4$. For all values of the capital share $\underline{b}$, utilitarian saving rates are higher than minimax saving rates. This feature is explained by the utilitarian target capital-labor ratios, which range from 11.5 (for $b = .25$) to 123596 (for $b = .75$). The utilitarian paths simply accumulate more capital. The minimax saving rate, like the utilitarian rates, is quite sensitive to the value of $\underline{b}$; however, whereas the latter increases as $\underline{b}$ increases, the former decreases as $\underline{b}$ increases.

Several contrasts can be drawn between the two examples. First, whereas population growth and technological change always decrease the utilitarian savings rate, they increase the minimax savings rate for $T \geq 100$. The minimax savings rate is uniformly lower than the utilitarian savings rate. To some extent, of course, this is due to the choice of contemporaneous utility function. Taking $u(c) = -c^{-n}$ with $\underline{n}$ much larger than 1 lowers the utilitarian saving rate. While the minimax path reaches its target in finite time, the utilitarian path reaches its target only asymptotically. Overall, savings rates currently observed in the developed would appear more justifiable on minimax than on utilitarian grounds.

Table 3. Minimax Saving Rates

$(k_o = 4, k^* = 8, b = .25)$

	$T = 50$	$T = 100$	$T = 200$
$n + a = .01$	.030	.037	.044
$n + a = .02$	.074	.087	.094
$n + a = .03$	.124	.138	.142

Table 4. Minimax and Utilitarian
Saving Rates

$(a = .02, n = 0, k_o = 4, T = 100)$

	Minimax	Utilitarian	
		Initial	Asymptotic
$b = .25$	.087	.25	.125
$b = .50$	.051	.42	.25
$b = .75$	.030	.48	.375

6. Conclusion

This paper has argued that minimax savings rates are a reasonable interpretation of Rawls' theory of just saving. In the economic models studied, the minimax criterion requires an equal savings rate on the part of every generation, leading to the minimum accumulation of capital consistent with social justice. The main numerical result is that, if the capital-labor ratio must be raised 100% in the next century to achieve justice, and if the Cobb-Douglas production function with capital share 1/4 adequately forecasts production, then the savings rate should be between 3 and 10%, depending on the rate of population increase and technical progress. Minimax paths are seen to be significantly different from typical utilitarian alternatives.

One interesting open problem is whether there exist economic models for which the minimax path does not lead to a constant savings rate. Certainly, second-order models would make this possible, and perhaps certain time-dependent production functions as well.

The greatest drawback to the present theory is the determination of the just capital stock and period of accumulation. In Rawls' theory, these must be decided in the original position. It seems inevitable that their determination involve the larger question of the social minimum and social stability. For too high a savings rate, even if equal for all generations, surely intensifies social tension.

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