ELASTIC PROBLEM OF A LONG CIRCULAR CYLINDER WITH A SHRINK-FITTED COLLAR

by

Roger Dean Low

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Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State College

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NOTATION

The following notation is used

(r,θ,z)	cylindrical coordinates
σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}	normal stresses in the r,θ, and z directions respectively
$\sigma_{r\theta}$, σ_{rz}, $\sigma_{\theta z}$	shear stresses
$\Theta\ (=\sigma_{rr}+\sigma_{\theta\theta}+\sigma_{zz})$	stress invariant
e_{rr}, $e_{\theta\theta}$, e_{zz}	normal strains in the r,θ, and z directions respectively
$e_{r\theta}$, e_{rz}, $e_{\theta z}$	shear strains
u_r, u_θ, u_z	displacements in the r,θ, and z directions respectively
E	Young's modulus
σ	Poisson's ratio
Δ	Laplace's operator in cylindrical coordinates independent of θ, i.e., $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$
m,n	positive integers

INTRODUCTION

The present investigation is concerned with the formulation and solution of a mixed boundary value problem in the theory of elasticity in cylindrical coordinates. The particular problem is that of determining the stresses and displacements in the interior and on the boundary of an infinite circular cylinder when a finite band of displacement is prescribed on a portion of its lateral surface; the remainder of the surface is stress free. The finite band of displacement could be produced by shrink-fitting a rigid collar onto the cylinder, and because of this the solution of the problem may be of some interest in the realm of machine design.

Harding and Sneddon (1) and Chong (2) have treated the problems of the indentation of a semi-infinite medium by an axially-symmetric rigid punch in both rectangular and cylindrical coordinates. The present problem may be considered as an analogue of these in a medium unbounded in only one dimension rather than two.

Okubo (3) has treated the problem of a finite circular cylinder with a press-fitted collar taking into account the stresses in the collar as well as those in the cylinder. His approach was to assume stress functions for the cylinder and collar in the form of infinite series, and then impose the boundary conditions to evaluate the coefficients. He is thus led to an infinite system of linear equations for the determination of these coefficients.

The method of solution of the proposed problem is to make use of a Fourier sine transform which reduces the problem to the solution of a pair of dual integral equations. A formal solution of these equations is obtained by a method due to Tranter (4), which reduces the problem still further to the solution of an infinite system of linear equations. In spite of the fact that the present method and Okubo's method both require the solution of an infinite system it is felt that the former is somewhat more general in that it is not necessary to assume a particular form for the solution.

The final portion of this thesis is concerned with a numerical example in which an attempt is made to determine the stresses and displacements on the axis of the cylinder and the normal stress on its lateral surface. The results indicate that a more extensive numerical treatment would be required if one were interested in the precise nature of the stress distributions, especially on the boundary of the cylinder. The results on the axis, however, give a good qualitative description of behavior of the stresses and displacements in the cylinder.

REVIEW OF GENERAL EQUATIONS

In a homogeneous, isotropic, elastic medium subjected to surface tractions and/or surface displacements, there exists at each point of the medium a system of stresses and strains related by the generalized Hooke's law. In the sequel it will be seen that cylindrical coordinates (r,θ,z) are best suited for the formulation and solution of the proposed problem. In addition it will be evident that all pertinent quantities are independent of θ. Hence the shear stresses $\sigma_{r\theta}$, $\sigma_{\theta z}$, the shear strains $e_{r\theta}$, $e_{\theta z}$, and the displacement component u_θ are identically zero.

Of the three equilibrium equations which the stresses must satisfy, one is satisfied identically due to the axial symmetry, and the other two are, in the absence of body forces,

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 , \tag{1}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 . \tag{2}$$

The generalized Hooke's law relating the components of stress and strain is

$$\begin{aligned} Ee_{rr} &= \sigma_{rr} - \sigma(\sigma_{\theta\theta} + \sigma_{zz}) , \\ Ee_{\theta\theta} &= \sigma_{\theta\theta} - \sigma(\sigma_{rr} + \sigma_{zz}) , \\ Ee_{zz} &= \sigma_{zz} - \sigma(\sigma_{rr} + \sigma_{\theta\theta}) , \\ Ee_{rz} &= (1 + \sigma)\, \sigma_{rz} . \end{aligned} \tag{3}$$

The strain-displacement relationships are

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{u_r}{r}, \quad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$2e_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}. \tag{4}$$

Finally, the strain components are not independent but are related by the six compatability equations, two of which are satisfied identically in the case of axial symmetry, and the remaining four become, when written in terms of the stresses,

$$\Delta\sigma_{rr} - \frac{2(\sigma_{rr} - \sigma_{\theta\theta})}{r^2} + \frac{1}{1 + \sigma}\frac{\partial^2\Theta}{\partial r^2} = 0,$$

$$\Delta\sigma_{\theta\theta} + \frac{2(\sigma_{rr} - \sigma_{\theta\theta})}{r^2} + \frac{1}{1 + \sigma}\frac{1}{r}\frac{\partial\Theta}{\partial r} = 0,$$

$$\Delta\sigma_{zz} + \frac{1}{1 + \sigma}\frac{\partial^2\Theta}{\partial z^2} = 0, \tag{5}$$

$$\Delta\sigma_{rz} - \frac{\sigma_{rz}}{r^2} + \frac{1}{1 + \sigma}\frac{\partial^2\Theta}{\partial r\partial z} = 0.$$

STATEMENT OF THE PROBLEM

Consider an infinite, homogeneous, isotropic, elastic solid, circular cylinder of radius a. A coaxial, finite, rigid collar is to be shrink-fitted on the cylinder, producing a known contact displacement in the radial direction over the length of the collar. It is then required to find the stresses and displacements throughout the cylinder.

Let the cylinder be referred to a cylindrical coordinate system (r, θ, z), with z measured along the axis of the cylinder. Then, by the obvious axial symmetry, all pertinent quantities are independent of θ. Further, if the origin of coordinates be taken at the midpoint of the collar, attention can be restricted to the region $0 \leq z < \infty$ and $0 \leq r \leq a$.

A stress function $\psi(r,z)$ suitable for axially symmetric elastic problems in which the medium is homogeneous and isotropic, and in which there are no body forces has been given by Love (5, p. 276). His results are:

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\sigma\Delta\psi - \frac{\partial^2\psi}{\partial r^2}\right),$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\sigma\Delta\psi - \frac{1}{r}\frac{\partial\psi}{\partial r}\right), \qquad (6)$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left((2-\sigma)\Delta\psi - \frac{\partial^2\psi}{\partial z^2}\right),$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left((1-\sigma)\Delta\psi - \frac{\partial^2\psi}{\partial z^2}\right).$$

Equilibrium equation (1) is satisfied identically by the stress components given in terms of ψ by equations (6). The second equilibrium equation and the four compatability equations are also satisfied if the stress function is biharmonic, i.e., if $\psi(r,z)$ satisfies the equation

$$\Delta\Delta\psi = \Delta^2\psi = 0 \; . \tag{7}$$

The displacement components are then given by

$$u_r = -\frac{1+\sigma}{E}\frac{\partial^2\psi}{\partial r\partial z} \, ,$$

$$u_z = \frac{1+\sigma}{E}\left(2(1-\sigma)\Delta\psi - \frac{\partial^2\psi}{\partial z^2}\right) \; . \tag{8}$$

BOUNDARY CONDITIONS

Let the region of contact between the shrink-fitted collar and the cylinder be of length 2h, and let the displacement produced by the collar be given by $(-g(z))$, where $g(z)$ is symmetric about $z = 0$. If it is assumed further that the contact surfaces are frictionless, then the boundary conditions are

$$\sigma_{rz}|_{r=a} = 0 \ , \quad 0 < z < \infty , \tag{9}$$

$$u_r|_{r=a} = - g(z) \ , \ 0 < z < h \ , \tag{10}$$

$$\sigma_{rr}|_{r=a} = 0 \ , \ z > h \ , \tag{11}$$

and

all stress and displacement components vanish at infinity. (12)

The minus sign in boundary condition (10) occurs since it is convenient to regard $g(z)$ as positive in the direction of increasing r.

Since the stress and displacement components are given in terms of derivatives of the stress function ψ, the boundary conditions (9) - (12) are essentially conditions on ψ; hence the problem has been reduced to the solution of the mixed boundary value problem consisting of equation (7) and boundary conditions (9) - (12).

REDUCTION TO EQUIVALENT DUAL INTEGRAL EQUATIONS

It will be shown that the solution of the mixed boundary value problem can be reduced to the solution of a pair of dual integral equations. For this purpose it is convenient to introduce the Fourier transform of ψ with respect to z. Inspection of equations (6) and (8) shows that σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, and u_r would be even functions of z if $\psi(r,z)$ were an odd function of z. Also, the even or odd character of σ_{rz} and u_z will be the same as that of $\psi(r,z)$. From the obvious symmetry with respect to the plane z = 0, the components of normal stress and radial displacement must be even functions of z; whereas the components of shear stress and axial displacement must be odd functions of z. Hence $\psi(r,z)$ must be an odd function of z. It is therefore appropriate to consider the Fourier sine transform of ψ given by

$$\bar{\psi}(r,p) = \frac{2}{\pi} \int_0^\infty \psi(r,z) \sin pz \, dz \; . \tag{13}$$

Then, once $\bar{\psi}(r,p)$ has been obtained, $\psi(r,z)$ is given by Fourier's inversion theorem as

$$\psi(r,z) = \int_0^\infty \bar{\psi}(r,p) \sin pz \, dp \; . \tag{14}$$

The procedure to be followed is to transform equation (7) through the use of equation (13), which will lead to a differential equation for $\bar{\psi}$ as a function of r with the transform variable p appearing as a parameter. Once this equation has been solved, subject to boundary conditions equivalent to (9)-(12), inversion by equation (14) will give the solution for the stress function $\psi(r,z)$. Equations (6) and (8) will then give the stress and displacement components.

The transformation of equation (7) can be formally obtained in two steps. Let $f(r,z) = \Delta\psi$ so that $\Delta f = \Delta^2\psi$. Then, by equation (13)

$$\frac{2}{\pi}\int_0^\infty \Delta^2\psi \sin pz \, dz = \frac{2}{\pi}\int_0^\infty \Delta f \sin pz \, dz \; .$$

Now

$$\int_0^\infty \Delta f \sin pz \, dz = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\int_0^\infty f \sin pz \, dz + \int_0^\infty \frac{\partial^2 f}{\partial z^2} \sin pz \, dz$$

$$= \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\int_0^\infty f \sin pz \, dz + \left[\frac{\partial f}{\partial z} \sin pz\right]_0^\infty$$

$$- \left[pf \cos pz\right]_0^\infty - p^2\int_0^\infty f \sin pz \, dz$$

$$= \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - p^2\right)\int_0^\infty f \sin pz \, dz \; ,$$

if it is assumed that the bracketed expressions vanish. Repetition of the above process, after $f(r,z)$ has been replaced by $\Delta\psi$, gives

$$\frac{2}{\pi}\int_0^\infty \Delta^2\psi \sin pz\, dz = \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-p^2\right)^2 \frac{2}{\pi}\int_0^\infty \psi(r,z)\sin pz\, dz$$

$$= \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-p^2\right)^2 \bar{\psi}(r,p) ,$$

in view of equation (13). Hence the differential equation for $\bar{\psi}$ corresponding to the transform of equation (7) is

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-p^2\right)^2 \bar{\psi}(r,p) = 0 . \tag{15}$$

In the process of obtaining equation (15), it has been assumed that the following bracketed expressions, obtained by partial integration, vanish:

$$\left[\frac{\partial}{\partial z}(\Delta\psi)\sin pz\right]_0^\infty = 0 , \quad \left[\Delta\psi \cos pz\right]_0^\infty = 0 ,$$

$$\left[\frac{\partial\psi}{\partial z}\sin pz\right]_0^\infty = 0 , \quad \left[\psi \cos pz\right]_0^\infty = 0 . \tag{16}$$

In view of boundary condition (12), it will be assumed that $\psi(r,z)$ and its derivatives up to and including the third order vanish at infinity; hence the brackets in equations (16) vanish at the upper limit. As to the lower limit, those brackets containing the factor sin pz will vanish provided

$$\frac{\partial}{\partial z}(\Delta\psi) \quad \text{and} \quad \frac{\partial\psi}{\partial z}$$

are bounded for $z = 0$, a condition which must hold if the stresses and displacements are to be finite on the plane $z = 0$. The two brackets containing the factor cos pz will vanish at $z = 0$ since $\psi(r,z)$ is an odd function of z.

To obtain the solution of equation (15), it is convenient to proceed as follows. Let

$$H(r,p) = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - p^2\right) \bar{\psi}(r,p),$$

then

$$\frac{d^2H}{dr^2} + \frac{1}{r}\frac{dH}{dr} - p^2H = 0 . \qquad (17)$$

Equation (17) is Bessel's equation for the modified functions of order zero; hence its solution is

$$H(r,p) = c_1I_0(pr) + c_2K_0(pr) , \qquad (18)$$

where c_1 and c_2 are, in general, functions of the parameter p. Now equation (15) becomes

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - p^2\right) \bar{\psi}(r,p) = c_1I_0(pr)+c_2K_0(pr). \qquad (19)$$

The general solution of equation (19) is of the form

$$\bar{\psi}(r,p) = A(r,p)\, I_0(pr) + B(r,p)\, K_0(pr) , \qquad (20)$$

where the functions $A(r,p)$ and $B(r,p)$ are to be determined by the method of variation of parameters. This method imposes on the two functions, $A(r,p)$ and $B(r,p)$, the conditions

$$I_0'(pr)\,\frac{\partial A}{\partial r} + K_0'(pr)\,\frac{\partial B}{\partial r} = \frac{1}{p}\left[c_1 I_0(pr) + c_2 K_0(pr)\right],$$

$$I_0(pr)\,\frac{\partial A}{\partial r} + K_0(pr)\,\frac{\partial B}{\partial r} = 0\,, \qquad (21)$$

where the primes on $I_0(pr)$ and $K_0(pr)$ denote differentiation with respect to their argument (pr). The determinant of the coefficients in the system (21) is (6, p. 80)

$$\begin{vmatrix} I_0'(pr) & K_0'(pr) \\ I_0(pr) & K_0(pr) \end{vmatrix} = I_0'(pr)K_0(pr) - I_0(pr)K_0'(pr)$$

$$= -W\left\{I_0(pr),\, K_0(pr)\right\}$$

$$= \frac{1}{pr}\,, \qquad (22)$$

where $W\left\{I_0(pr),\, K_0(pr)\right\}$ is the Wronskian of the fundamental solutions of the homogeneous equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\,\frac{du}{dr} - p^2u = 0\,.$$

The solution of the system (21) is easily seen to be

$$\frac{\partial A}{\partial r} = r\left[c_1 I_0(pr)K_0(pr) + c_2 K_0^{\,2}(pr)\right], \qquad (23)$$

$$\frac{\partial B}{\partial r} = -\,r\left[c_1 I_0^{\,2}(pr) + c_2 I_0(pr)K_0(pr)\right]. \qquad (24)$$

From equations (23) and (24) it follows that the functions

$A(r,p)$ and $B(r,p)$ are given by

$$A(r,p) = c_1(p)\int rI_o(pr)K_o(pr)\ dr + c_2(p)\int rK_o^2(pr)\,dr + c_3(p), \quad (25)$$

$$B(r,p) = -\ c_1(p)\int rI_o^2(pr)\ dr - c_2(p)\int rI_o(pr)K_o(pr)\,dr + c_4(p)\ , \quad (26)$$

where $c_3(p)$ and $c_4(p)$ are arbitrary functions of p. Thus the solution of equation (19) is obtained once the integrals in equations (25) and (26) are known. It may be verified by differentiation and the recurrence relationships for the modified Bessel functions that

$$\int rI_o^2(pr)\ dr = \frac{1}{2}\,r^2\left[I_o^2(pr) - I_1^2(pr)\right]\ ,$$

$$\int rK_o^2(pr)\ dr = \frac{1}{2}\,r^2\left[K_o^2(pr) - K_1^2(pr)\right]\ , \quad (27)$$

$$\int rI_o(pr)K_o(pr)\ dr = \frac{1}{2}\,r^2\left[I_o(pr)K_o(pr) + I_1(pr)K_1(pr)\right]\ ,$$

where $I_1(pr)$ and $K_1(pr)$ are the modified Bessel functions of order one.

From equations (25), (26) and (27) it follows upon substitution into equation (20) that the general solution of equation (19) is of the form

$$\bar{\Psi}(r,p) = A(p)\ I_o(pr) + B(p)prI_1(pr) + C(p)K_o(pr) + D(p)prK_1(pr)\ , \quad (28)$$

where $A(p),\ldots, D(p)$ are arbitrary functions of p.

Since the stresses and displacements must be finite at

$r = 0$, inspection of equation (28) shows that $C(p)$ and $D(p)$ must be zero as the Bessel functions $K_0(pr)$ and $K_1(pr)$ are infinite at $r = 0$. Thus the transform of the stress function becomes.

$$\bar{\psi}(r,p) = A(p)I_0(pr) + B(p)prI_1(pr) , \qquad (29)$$

and inversion gives

$$\psi(r,z) = \int_0^\infty \left[A(p)I_0(pr) + B(p)prI_1(pr) \right] \sin pz dp . \qquad (30)$$

The stresses and displacements are found by transforming equations (6) and (8), substituting for $\bar{\psi}$ from equation (29), and then inverting by equation (14). This is formally equivalent to differentiating $\psi(r,z)$ as given by equation (30).

To transform the first three of equations (6) and the first of equations (8), it is convenient to define the Fourier cosine transform of a function $g(r,z)$ as

$$\hat{g}(r,p) = \frac{2}{\pi}\int_0^\infty g(r,z) \cos pz \, dz ,$$

in contrast to its Fourier sine transform defined earlier as

$$\bar{g}(r,p) = \frac{2}{\pi}\int_0^\infty g(r,z) \sin pz \, dz .$$

Thus

$$\widehat{\frac{\partial g}{\partial z}} = \frac{2}{\pi}\int_0^\infty \frac{\partial g}{\partial z} \cos pz \, dz$$

$$= \frac{2}{\pi}\left[g \cos pz \right]_0^\infty + p \frac{2}{\pi}\int_0^\infty g(r,z) \sin pz \, dz$$

$$= p\bar{g}(r,p) \, , \tag{31}$$

where it has been assumed that g(r,z) is such that the bracketed expression vanishes. It will be seen that this is a valid assumption in view of the argument following equations (16). Thus the Fourier cosine transform of σ_{rr} becomes

$$\frac{2}{\pi}\int_0^\infty \sigma_{rr} \cos pz \, dz = \frac{2}{\pi}\int_0^\infty \frac{\partial}{\partial z}\left(\sigma\Delta \, \Phi - \frac{\partial^2 \Phi}{\partial r^2}\right) \cos pz \, dz$$

$$= p \, \overline{\left(\sigma\Delta \, \Phi - \frac{\partial^2 \Phi}{\partial r^2}\right)}$$

$$= p\left[\sigma\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - p^2\right)\bar{\Phi} - \frac{d^2\bar{\Phi}}{dr^2}\right]$$

$$= \sigma \, p\left(\frac{1}{r}\bar{\Phi}' - p^2\bar{\Phi}\right) - (1-\sigma) \, p\bar{\Phi}'' \, ,$$

where the primes on $\bar{\Phi}$ denote differentiation with respect to r. Hence, upon inversion,

$$\sigma_{rr} = \int_0^\infty \left[\sigma \, p\left(\frac{1}{r}\bar{\Phi}' - p^2\bar{\Phi}\right) - (1-\sigma) \, p\bar{\Phi}''\right]\cos pz \, dp. \tag{32}$$

In a similar fashion the other stresses and displacements are given by

$$\sigma_{\theta\theta} = \int_0^\infty \left[\sigma \, p\left(\bar{\Phi}'' - p^2\bar{\Phi}\right) - (1-\sigma) \, \bar{\Phi}'\right] \cos pz \, dp, \tag{33}$$

$$\sigma_{zz} = \int_0^\infty \left[(2-\sigma)p\left(\bar{\Phi}'' + \frac{1}{r}\bar{\Phi}'\right) - (1-\sigma)p^3\bar{\Phi}\right]\cos pz \, dp, \tag{34}$$

$$\sigma_{rz} = \int_0^\infty \left[(1-\sigma)\left(\bar{\Phi}''' + \frac{1}{r}\bar{\Phi}'' - \frac{1}{r^2}\bar{\Phi}'\right) + \sigma p^2\bar{\Phi}'\right] \sin pz \, dp, \tag{35}$$

$$u_r = -\frac{1+\sigma}{E}\int_0^\infty p\bar{\psi}' \cos pz \, dp \,, \tag{36}$$

$$u_z = \frac{1+\sigma}{E}\int_0^\infty \left[2(1-\sigma)\left(\bar{\psi}'' + \frac{1}{r}\bar{\psi}'\right) - (1-2\sigma)p^2\bar{\psi}\right] \sin pz \, dp \,. \tag{37}$$

If the expression for $\bar{\psi}(r,p)$, as given by equation (29), is substituted into equations (32) - (37) there results the following expressions for the stresses and displacements:

$$\sigma_{rr} = \int_0^\infty \left[A(p)\frac{1}{pr}\left\{I_1(pr) - I_0(pr)\right\} - B(p)\left\{(1-2\sigma)I_0(pr) + prI_1(pr)\right\}\right] p^3 \cos pz \, dp \,, \tag{38}$$

$$\sigma_{\theta\theta} = \int_0^\infty \left[-A(p)\frac{1}{pr} I_1(pr) - (1-2\sigma)B(p)I_0(pr)\right] p^3 \cos pz \, dp, \tag{39}$$

$$\sigma_{zz} = \int_0^\infty \left[A(p)I_0(pr) + B(p)\left\{2(2-\sigma)I_0(pr) + prI_1(pr)\right\}\right] p^3 \cos pz \, dp \,, \tag{40}$$

$$\sigma_{rz} = \int_0^\infty \left[A(p)I_1(pr) + B(p)\left\{2(1-\sigma)I_1(pr) + prI_0(pr)\right\}\right] p^3 \sin pz \, dp \,, \tag{41}$$

$$u_r = -\frac{1+\sigma}{E}\int_0^\infty \left[A(p)I_1(pr) + B(p)prI_0(pr)\right] p^2 \cos pz \, dp \,, \tag{42}$$

$$u_z = \frac{1+\sigma}{E}\int_0^\infty \left[A(p)I_0(pr) + B(p)\left\{4(1-\sigma)I_0(pr) + prI_1(pr)\right\}\right] p^2 \sin pz \, dp \,. \tag{43}$$

In the above integral expressions for the stresses and displacements, the functions A(p) and B(p) are still unknown. The boundary conditions are imposed to evaluate them. From equations (9) and (41), it follows that

$$\int_0^\infty \Big[A(p)I_1(pa) + B(p)\big\{2(1-\sigma)I_1(pa) + + paI_0(pa)\big\}\Big] p^3 \cos pz \, dp \equiv 0$$

for all values of z. Hence the integrand must vanish and

$$A(p) = - B(p)\left[2(1-\sigma) + pa\,\frac{I_0(pa)}{I_1(pa)}\right]. \tag{44}$$

Next, equations (10) and (42) imply that

$$-\frac{1+\sigma}{E}\int_0^\infty \Big[A(p)I_1(pa) + + B(p)paI_0(pa)\Big] p^2 \cos pz \, dp = -g(z),$$

which, upon substitution from equation (44), becomes

$$\int_0^\infty I_1(pa)p^2B(p) \cos pz \, dp = -\frac{E}{2(1-\sigma^2)}\, g(z),\ 0 < z < h\ . \tag{45}$$

Finally, equations (11) and (38) require that

$$\int_0^\infty \Big[A(p)\Big\{\frac{1}{pa}\, I_1(pa) - I_0(pa)\Big\} - B(p)\big\{(1-2\sigma)I_0(pa) + + paI_1(pa)\big\}\Big] p^3 \cos pz \, dp = 0,\ z > h,$$

which, upon substitution from equation (44), becomes

$$\int_0^\infty I_1(pa)\left[-2(1-\sigma) + p^2a^2\left(\frac{I_0^2(pa)}{I_1^2(pa)} - 1\right)\right] p^2B(p)\cos pz\, dp = 0, \quad z > h. \quad (46)$$

Equations (45) and (46) are a pair of dual integral equations for the determination of B(p). However, the solution of these equations is more readily obtained if the trigonometric function cos pz is written in terms of a Bessel function of the first kind of order $-\frac{1}{2}$. The desired relationship is

$$\cos pz = \sqrt{\frac{\pi pz}{2}}\; J_{-1/2}(pz)\,. \quad (47)$$

Hence the equations (45) and (46) become, upon substitution from equation (47),

$$\int_0^\infty I_1(pa)B(p)p^{5/2}J_{-1/2}(pz)\,dp = -\frac{E}{\sqrt{2\pi}(1-\sigma^2)}\frac{g(z)}{\sqrt{z}}, \quad 0 < z < h, \quad (48)$$

$$\int_0^\infty I_1(pa)\left[2(1-\sigma) - p^2a^2\left(\frac{I_0^2(pa)}{I_1^2(pa)} - 1\right)\right] B(p)p^{5/2}J_{-1/2}(pz)\,dp = 0, \quad z > h. \quad (49)$$

The form of these equations can be simplified if one defines

$$I_1(pa)B(p)p^{5/2} = \frac{f(p)}{2(1-\sigma) - p^2a^2\left(\dfrac{I_0^{\,2}(pa)}{I_1^{\,2}(pa)} - 1\right)} , \tag{50}$$

$$G(p) = \frac{1}{-2(1-\sigma) + p^2a^2\left(\dfrac{I_0^{\,2}(pa)}{I_1^{\,2}(pa)} - 1\right)} , \tag{51}$$

$$\beta = \frac{E}{\sqrt{2\pi}\,(1 - \sigma^2)} . \tag{52}$$

With definitions (50) - (52), equations (48) and (49) become

$$\int_0^\infty G(p)f(p)J_{-1/2}(pz)\,dp = \beta\,\frac{g(z)}{\sqrt{z}} , \quad 0 < z < h , \tag{53}$$

$$\int_0^\infty f(p)J_{-1/2}(pz)\,dp = 0 , \quad z > h . \tag{54}$$

Once f(p) has been found from the dual integral equations (53) and (54), then B(p) is given by

$$B(p) = -\frac{f(p)G(p)}{p^{5/2}I_1(pa)} . \tag{55}$$

FORMAL SOLUTION OF THE DUAL INTEGRAL EQUATIONS

The method to be used for the solution of the dual integral equations (53) and (54) was first given by Tranter. As Tranter's method is quite recent, it seems appropriate to give the details of his method as applied to the present problem. Briefly, the method is to assume a solution for f(p) in the form of a Neumann series of Bessel functions with undetermined coefficients which satisfies equation (54), and then to choose these coefficients in such a way that equation (53) is also satisfied. This procedure leads to an infinite system of linear equations for the determination of the coefficients.

More precisely, the details of Tranter's method are as follows. Consider the integral.

$$I(m,z) = \int_0^\infty p^{1/2} J_{2m}(ph) J_{-1/2}(pz)\, dp \ , \tag{56}$$

which has the value (7, p. 35)

$$I(m,z) = \begin{cases} \dfrac{1}{h}\sqrt{\dfrac{2}{\pi z}}\ {}_2F_1\left(m+\frac{1}{2},\ -m+\frac{1}{2};\ \frac{1}{2};\ \frac{z^2}{h^2}\right), & 0 < z < h \ , \\ 0 \ , \quad z > h \ , \end{cases} \tag{57}$$

where ${}_2F_1$ is Gauss' hypergeometric function with the indicated arguments. In view of equations (56) and (57) it follows that the second of the dual integral equations is satisfied if f(p) has the form

$$f(p) = \sum_{m=1}^{\infty} a_m p^{1/2} J_{2m}(ph) , \tag{58}$$

where the a_m are the undetermined coefficients mentioned previously. It remains to determine these coefficients so that the first of the dual integral equations is also satisfied. For this purpose a preliminary result is required. Application of Hankel's inversion theorem (8, p. 52) to equation (56) gives

$$p^{-1/2} J_{2m}(ph) = \int_0^{\infty} zI(m,z)J_{-1/2}(pz)\, dz ,$$

and substitution for $I(m,z)$ as given by equation (57) yields

$$p^{-1/2} J_{2m}(ph) = \frac{1}{h}\sqrt{\frac{2}{\pi}}\int_0^h z^{1/2}\, {}_2F_1\left(m+\tfrac{1}{2},\ -m+\tfrac{1}{2};\ \tfrac{1}{2};\frac{z^2}{h^2}\right)J_{-1/2}(pz)\, dz . \tag{59}$$

The hypergeometric function appearing in the integrand of equation (59) can be written in the form

$${}_2F_1\left(m+\tfrac{1}{2},\ -m+\tfrac{1}{2};\tfrac{1}{2};\frac{z^2}{h^2}\right) = \left(1-\frac{z^2}{h^2}\right)^{-1/2} {}_2F_1\left(-m,m;\tfrac{1}{2};\ \frac{z^2}{h^2}\right),$$

when use is made of the well known transformation (7, p. 8)

$${}_2F_1(a,b;c;t) = (1-t)^{c-a-b}\ {}_2F_1(c-a,c-b;c;t).$$

Now the Jacobi polynomials, $\mathcal{F}_m(\alpha,\beta,t)$, are defined in terms of hypergeometric functions as (7, p. 83)

$$\mathcal{F}_m(\alpha,\beta,t) = {}_2F_1(-m, m+\alpha;\ \beta;\ t)\ ;$$

hence equation (59) can be written in the form

$$p^{-1/2}\, J_{2m}(ph) =$$

$$= \frac{1}{h}\sqrt{\frac{2}{\pi}} \int_0^h z^{1/2}(1-z^2)^{-1/2}\, \mathcal{F}_m\left(0,\ \tfrac{1}{2}, \tfrac{z^2}{h^2}\right) J_{-1/2}(pz)\, dz\ , \qquad (60)$$

which is the desired preliminary result.

Substitution of equation (58) into equation (53) gives

$$\int_0^\infty G(p)\left(\sum_{m=1}^\infty a_m p^{1/2}\, J_{2m}(ph)\right) J_{-1/2}(pz)\, dp = \beta \frac{g(z)}{\sqrt{z}},\ 0 < z < h\ ,$$

and if the series is assumed to be uniformly convergent, the order of summation and integration may be interchanged to give

$$\sum_{m=1}^\infty a_m \int_0^\infty p^{1/2}\, G(p) J_{2m}(ph) J_{-1/2}(pz)\, dp = \beta \frac{g(z)}{\sqrt{z}},\ 0 < z < h. \qquad (61)$$

When equation (61) is multiplied by

$$z^{1/2}\left(1 - \frac{z^2}{h^2}\right)^{-1/2} \mathcal{F}_n\left(0,\ \tfrac{1}{2}, \tfrac{z^2}{h^2}\right),\quad (n = 1, 2, 3, \ldots),$$

and integrated with respect to z from 0 to h, assuming that the order of integration may be interchanged, it becomes

$$\sum_{m=1}^{\infty} a_m \int_0^{\infty} p^{1/2} G(p) J_{2m}(ph)\, dp \int_0^h z^{1/2}\left(1-\frac{z^2}{h^2}\right)^{-1/2} \mathcal{F}_n\left(0,\tfrac{1}{2},\frac{z^2}{h^2}\right) J_{-1/2}(pz)\, dz$$

$$= \beta \int_0^h g(z)\left(1-\frac{z^2}{h^2}\right)^{-1/2} \mathcal{F}_n\left(0,\tfrac{1}{2},\frac{z^2}{h^2}\right) dz\ . \qquad (62)$$

In view of the preliminary result, equation (62) can be written in the form

$$\sum_{m=1}^{\infty} a_m \int_0^{\infty} G(p) J_{2m}(ph) J_{2n}(ph)\, dp =$$

$$= \frac{\beta}{h}\sqrt{\frac{2}{\pi}} \int_0^h g(z)\left(1-\frac{z^2}{h^2}\right)^{-1/2} \mathcal{F}_n\left(0,\tfrac{1}{2},\frac{z^2}{h^2}\right) dz\ . \qquad (63)$$

Equation (63) can be written in a more convenient form if use is made of the integral (7, p. 35)

$$\int_0^{\infty} p^{-1} J_{2m}(ph) J_{2n}(ph)\, dp = \frac{\delta_{mn}}{4n}\ . \qquad (64)$$

Hence if one adds and subtracts the integral

$$a_m \int_0^{\infty} p^{-1} J_{2m}(ph) J_{2n}(ph)\, dp$$

in equation (63), one obtains

$$\frac{a_n}{4n} + \sum_{m=1}^{\infty} a_m \int_0^{\infty} \left(G(p) - p^{-1}\right) J_{2m}(ph) J_{2n}(ph)\, dp =$$

$$= \frac{\beta}{h}\sqrt{\frac{2}{\pi}} \int_0^h g(z)\left(1 - \frac{z^2}{h^2}\right)^{-1/2} \mathcal{F}_n\left(0,\tfrac{1}{2},\frac{z^2}{h^2}\right) dz \; . \tag{65}$$

The infinite system (65) may be written in the form

$$\frac{a_n}{4n} + \sum_{m=1}^{\infty} L_{m,n} a_m = R(n) \; , \; (n = 1,2,3, \ldots) \; , \tag{66}$$

if one defines

$$L_{m,n} = \int_0^{\infty}\left(G(p) - p^{-1}\right) J_{2m}(ph) J_{2n}(ph)\, dp \tag{67}$$

and

$$R(n) = \frac{\beta}{h}\sqrt{\frac{2}{\pi}} \int_0^h g(z)\left(1 - \frac{z^2}{h^2}\right)^{-1/2} \mathcal{F}_n\left(0,\tfrac{1}{2},\frac{z^2}{h^2}\right) dz \; . \tag{68}$$

The reason for employing equation (64) to write equation (63) in the form (66) is that for large p the asymptotic behavior of G(p) is of the form

$$G(p) \sim p^{-1} \; ;$$

hence the numerical evaluation of the integrals $L_{m,n}$ will be more feasible. It follows from the symmetry of the problem with respect to the plane z = 0, that g(z) must be an even function of z. Thus one may write

$$g(z) \doteq \sum_{k=0} b_k \mathcal{F}_k\left(0,\tfrac{1}{2},\frac{z^2}{h^2}\right) \; , \tag{69}$$

where the b_k can be determined from the orthogonality properties of the Jacobi polynomials, which are (7, p. 83)

$$\int_0^1 \left(1-t^2\right)^{-1/2} \mathcal{J}_k(0,\tfrac{1}{2},t^2)\,\mathcal{J}_n(0,\tfrac{1}{2},t^2)\,dt = \begin{cases} 0\ , & k\neq n\ , \\ \frac{\pi}{4}\ , & k=n\neq 0\ , \\ \frac{\pi}{2}\ , & k=n=0\ . \end{cases} \tag{70}$$

Hence, substitution of equation (69) into equation (68), gives

$$R(n) = \frac{\beta}{h}\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty} b_k \int_0^h \left(1-\frac{z^2}{h^2}\right)^{-1/2} \mathcal{J}_k\left(0,\tfrac{1}{2},\frac{z^2}{h^2}\right)\mathcal{J}_n\left(0,\tfrac{1}{2},\frac{z^2}{h^2}\right) dz\ . \tag{71}$$

If the substitution $t = {}^z\!/h$ is made in equation (71), it becomes

$$R(n) = \beta\sqrt{\frac{2}{\pi}}\ \sum_{k=0}^{\infty} b_k \int_0^1 \left(1-t^2\right)^{-1/2} \mathcal{J}_k(0,\tfrac{1}{2},t^2)\mathcal{J}_n(0,\tfrac{1}{2},t^2)\,dt\ , \tag{72}$$

and since $n=1,2,3,\ldots$, it follows from equations (70) and (72) that

$$R(n) = \frac{1}{4}\,\beta\sqrt{2\pi}\ \ b_n\ , (n=1,2,3,\ldots). \tag{73}$$

NUMERICAL EXAMPLE

As an example of the previous results, a particular case is considered in which $a = h = 1$, $\sigma = 0.25$, and $g(z) = \varepsilon(1 - z^4)$. The values of the stresses and displacements are computed on the axis of the cylinder and an attempt to determine the normal stress on the lateral surface is made.

The first step is to determine the $L_{m,n}$ appearing as coefficients of the unknowns a_m in equations (66). Since the $L_{m,n}$ are given as infinite integrals in equation (67), and because $G(p)$ behaves asymptotically like p^{-1}; it is assumed that the major contributions of these integrals could be obtained by integrating from 0 to 50. The evaluation of the integrals was done on the I.B.M. 650 at Iowa State College.

From the chosen form of $g(z)$, it readily follows that one may write

$$g(z) = \varepsilon(1 - z^4)$$

$$= \frac{1}{8}\,\varepsilon(5\mathcal{F}_0 + 4\mathcal{F}_1 - \mathcal{F}_2) \,.$$

Hence the b_k appearing in equation (69) are

$$b_0 = {}^{5\varepsilon}/_8 \,, \quad b_1 = {}^{\varepsilon}/_2 \,, \quad b_2 = {}^{-\varepsilon}/_8 \,,$$

and from equation (73) it follows that

$$R(1) = \varepsilon\beta \sqrt{2\pi}/8 \ , \quad R(2) = -\ \varepsilon\beta \sqrt{2\pi}/32 \ ,$$

while all other R's are zero. Thus the first two equations of the infinite system are non-homogeneous while the remainder of them are homogeneous.

The next step is to determine how many equations of the infinite system (66) are required. Twenty equations are considered adequate for reasons to be mentioned later. The next step is to solve the 20 x 20 system for the a_m, and this was also done on the I.B.M. 650. As an internal check on the machine calculations, the equations were solved in groups of 5 x 5, 10 x 10, 15 x 15, and 20 x 20 respectively, and those values of a_m which could be compared agreed very well.

To justify the above assumption that twenty equations are sufficient, the values of the a_m obtained were substituted into equation (58) and the partial sums were evaluated for those values of p at which $J_{2m}(p)$ attains its maxima and minima. It was found that the first five terms accounted for approximately 85 per cent of the sum of all twenty terms. For this reason it seems that twenty of the a_m should give a good approximation to the actual solution of the infinite system.

The integral expressions for the stresses and displacements are given by equations (38) - (43). However, to facilitate this numerical evaluation, it is convenient to have integral expressions for the combinations $\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}$, $\sigma_{rr} + \sigma_{\theta\theta}$ and $\sigma_{\theta\theta}$. The results are, for r = 0

$$\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = \frac{5}{2} \int_0^\infty B(p)p^3 \cos pz \, dp \,, \tag{74}$$

$$\sigma_{rr} + \sigma_{\theta\theta} = \int_0^\infty \left(\frac{1}{2} + p \frac{I_0(p)}{I_1(p)}\right) B(p)p^3 \cos pz \, dp \,, \tag{75}$$

$$\sigma_{\theta\theta} = \frac{1}{2} \int_0^\infty \left(\frac{1}{2} + p \frac{I_0(p)}{I_1(p)}\right) B(p)p^3 \cos pz \, dp \,, \tag{76}$$

$$\sigma_{rz} \equiv 0 \,, \tag{77}$$

$$Eu_z = \frac{5}{4} \int_0^\infty \left(\frac{3}{2} - p \frac{I_0(p)}{I_1(p)}\right) B(p)p^2 \sin pz \, dp \,, \tag{78}$$

$$u_r \equiv 0 \,. \tag{79}$$

For r = 1, the normal stress σ_{rr} is given by

$$\sigma_{rr} = \int_0^\infty \left[p \frac{I_0(p)}{I_1(p)} - \left(p + \frac{3}{2p}\right) \frac{I_1(p)}{I_0(p)}\right] I_0(p)B(p)p^3 \cos pz \, dp. \tag{80}$$

It must be remembered that the function B(p) appearing in equations (74) - (80) is now known and is given by equation (55).

For r = 0, the function B(p) decreases rapidly with increasing p; thus the integrations can be carried out by integrating with a unit interval from 0 to 20. For r = 1, the factor $I_0(p)$, being a rapidly increasing function, cancels the decreasing nature of B(p) and makes it necessary to perform the integration with a much larger upper limit. It is necessary, due to the lack of more extensive tables, to stop the integration at 100 and the result in this case indicates clearly that this is not sufficient to determine the precise nature of the normal stress distribution on the lateral surface.

Each of the integrals in equations (74) - (80) are of the form

$$\int_0^b F(p) \begin{matrix} \sin \\ \cos \end{matrix} pz \, dp ,$$

and a method due to Filon (9), which avoids the numerical difficulty involved in the evaluation of integrals with rapidly oscillating integrands, is used to perform the integrations. The results are indicated in the following figures.

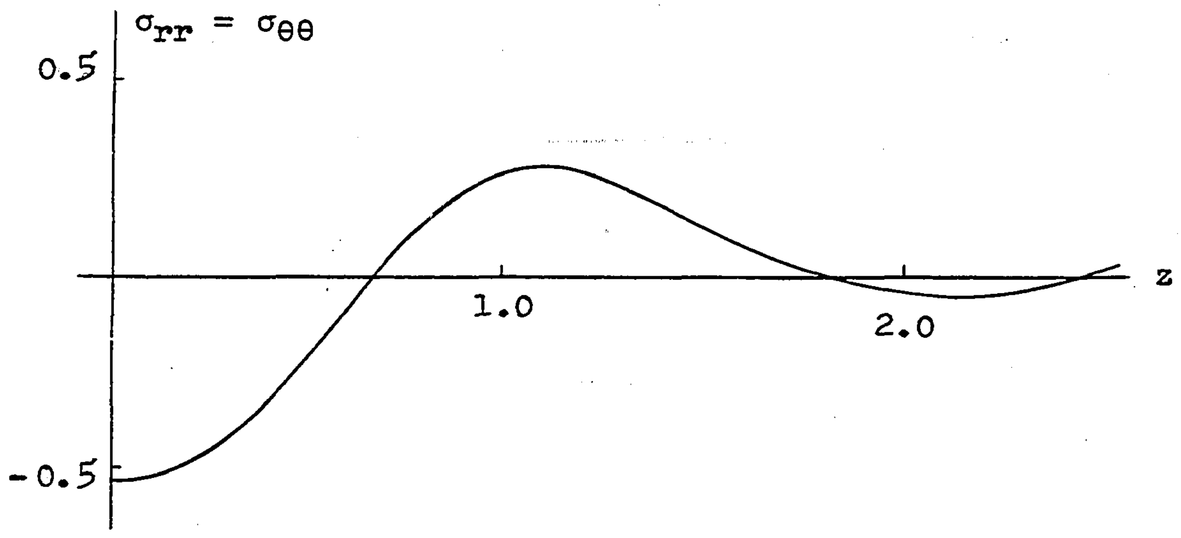

Figure 1. Distribution of stresses σ_{rr} and $\sigma_{\theta\theta}$ for r = 0

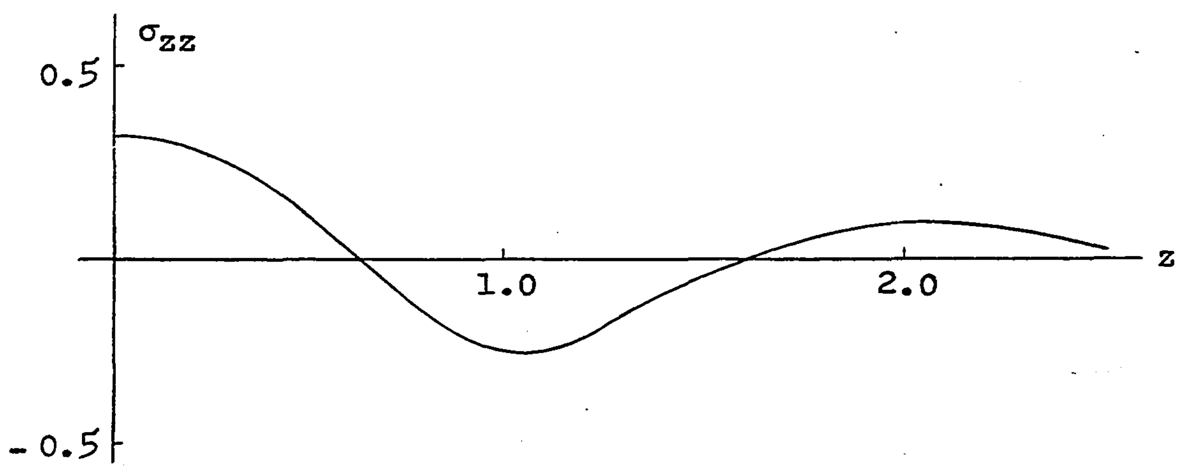

Figure 2. Distribution of stress σ_{zz} for r = 0

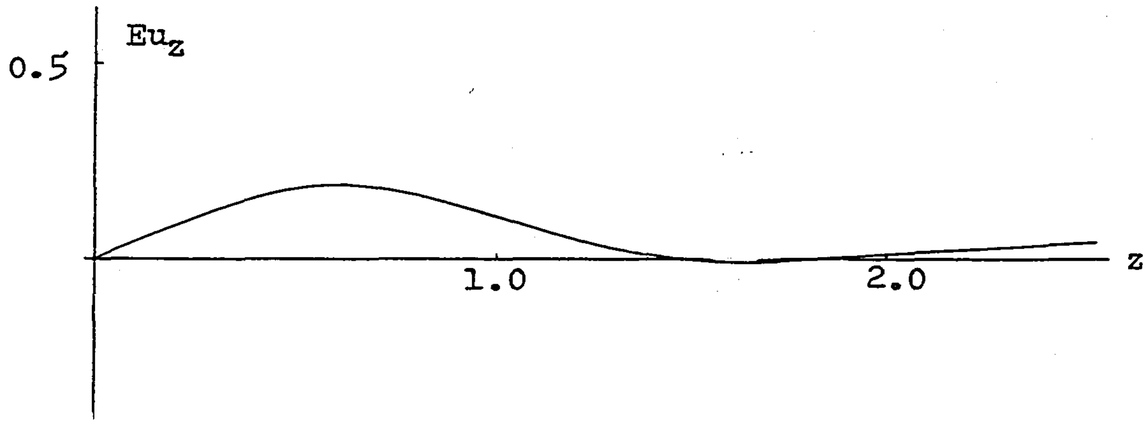

Figure 3. Distribution of axial displacement Eu_z for $r = 0$

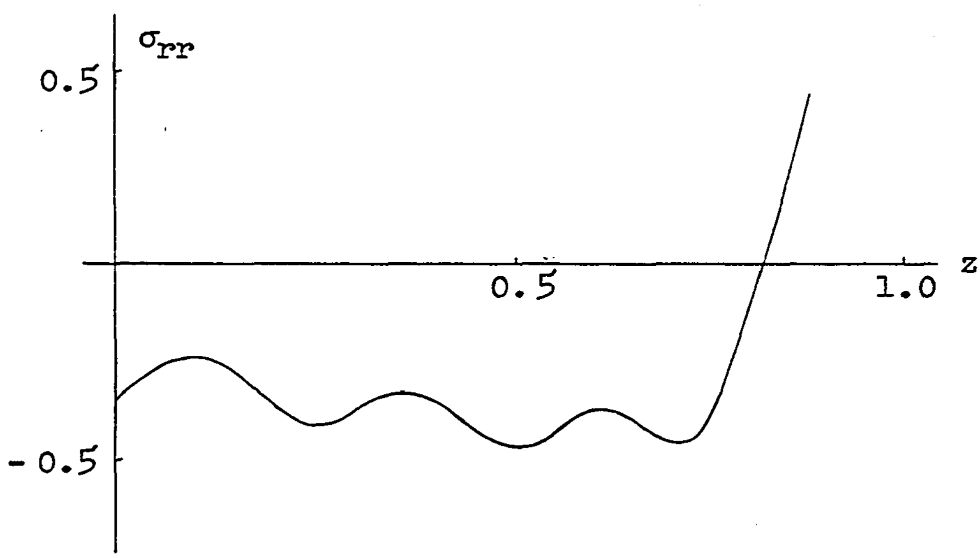

Figure 4. Distribution of normal stress σ_{rr} for $r = 1$

It seems rather obvious from Figures 1 - 4, especially Figure 4, that the results obtained are qualitative rather than quantitative descriptions of the true nature of the stress and displacement distributions. For example, it seems physically impossible that u_z could become negative as indicated in Figure 3; however, the rapid decrease of the curve indicates qualitatively the nature of the axial displacement. Similar remarks apply to the curves in Figures 1, 2 and 4.

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