APPLICATION OF GEOMETRICAL DIFFRACTION THEORY TO SCATTERING BY CRACKS

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ABSTRACT

At high frequencies, the geometrical theory of diffraction provides useful and relatively simple approximations to diffracted fields. In this paper the theory is applied to the diffraction of time-harmonic longitudinal waves by a penny-shaped crack in an elastic solid.

Introduction

Methods to calculate the far fields when ultrasonic waves are diffracted by a crack, are essential to the scattering approach to QNDE. For diffraction of elastic waves by a penny-shaped crack, a considerable literature exists, which was summarized in a recent review article by Kraut[1]. Particularly interesting are simple approximate solutions which are useful for either high or low frequencies. A number of these approximations, including the Kirchhoff approximation, the adaptation of the Keller theory to shear-free media, and a new quasi-static approximation, were recently discussed by Domany, Krumhansl and Teitel[2].

In this paper, we will apply the geometrical theory of diffraction to ultrasonic diffraction problems. We will show that this theory provides useful and relatively simple approximations to diffracted fields.

The geometrical theory of diffraction is based on ray theory. Just as in other wave propagation phenomena, wavefronts are defined as surfaces of constant phase and the rays are normal to the wavefronts. In elastodynamic ray theory the amplitude of a high frequency mechanical disturbance is traced as the disturbance propagates along a ray. When a ray strikes an interface, reflected and refracted rays are generated. However, when a ray carrying a high frequency body wave strikes the edge of a crack, two cones of diffracted rays are generated. The surfaces of the inner and outer cones consist of rays of longitudinal and transverse motion, respectively. The half-angles of the cones are related by Snell's law. The groundwork for a three-dimensional geometrical diffraction theory for crack-like obstacles in an elastic solid has been presented in a paper by Achenbach and Gautesen[3].

For diffraction phenomena which are governed by a single wave equation, geometrical diffraction theory for a planar obstacle was developed by Keller[4]. The formulation of Ref. 4, and subsequent papers, e.g., Ref. 5, is, however, not applicable to diffraction by cracks in solids, since wave motions in solids are governed by two wave equations, which are coupled by the boundary conditions on the diffracting obstacle. Thus, although in first approximation it is useful to neglect this coupling, as was done in Refs. 6 and 7, it is appropriate to consider the problem in a mathematically more rigorous manner by taking the coupling into account.

For <u>plane</u> longitudinal and transverse waves, which are under arbitrary angles of incidence with a traction-free <u>semi-infinite</u> crack, the fields on the diffracted rays can be obtained by asymptotic considerations, as shown in Ref. 3. The results can be expressed in terms of diffraction coefficients which relate the diffracted fields to the incident fields. Geometrical diffraction theory provides modifications to the semi-infinite crack results, to account for curvature of incident wave-fronts and curvature of crack edges, and finite dimensions of the crack. In the usual terminology the results for diffraction of plane waves by a semi-infinite crack are the canonical solutions.

Geometrical diffraction theory is applicable if $\omega d/c_L >> 1$, where d is a characteristic dimension of the crack and c_L is the velocity of longitudinal waves. For some special problems where comparisons with mathematically exact solutions were possible, we found good agreement for $\omega d/c_L > 2$. For a crack with a cross-sectional dimension of 1 mm in steel ($c_L \sim 6.10^6$ mm/sec), the frequency should then be larger than about 1 MHz. This seems to be within experimental capability, see e.g. Ref. 6 and 7.

It should be noted that a slightly different interpretation renders the results of geometrical diffraction theory useful as wavefront results for propagating pulses.

Geometrical Theory of Diffraction-Basic Ideas

The basic ideas of geometrical diffraction can best be explained by means of an example. Figure 1 shows a planar crack with a smoothly curving edge in an elastic solid. The origin of a cartesian coordinate system (x, y, z) is in the plane of the crack, and the z-axis is normal to that plane. A time harmonic wave of the form

$$u_z = A_0\, e^{i\omega(z/c_L - t)} \qquad (1)$$

is incident on the crack. This is a case of normal incidence, i.e., the wavefronts are parallel to the z-axis. When a ray strikes the point P_2 on the edge of the crack, a fan of diffracted body wave rays is generated as shown in Fig. 1. Both longitudinal and transverse motions are carried by the diffracted rays. One ray of each kind emanating from two or more points on the boundary will pass through any point of observation. To keep the figure compact the point Q is shown closer to the crack than would actually be the case.

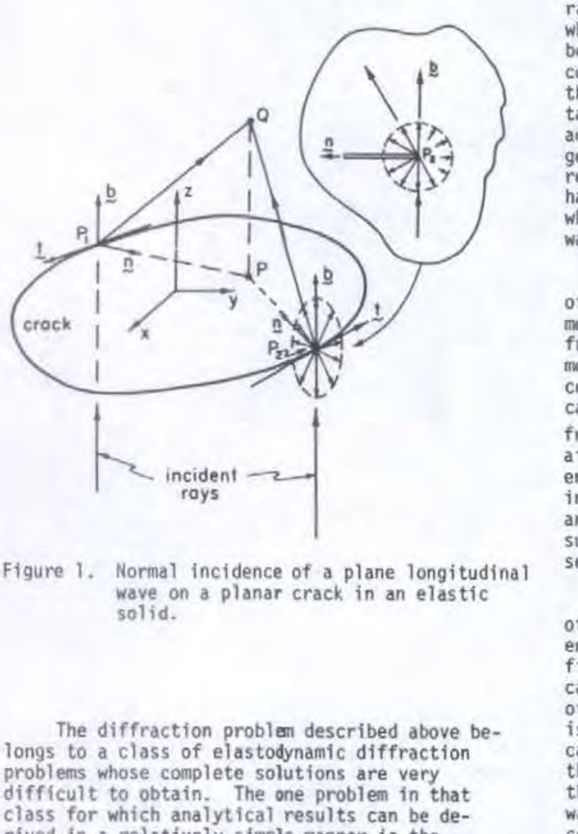

Figure 1. Normal incidence of a plane longitudinal wave on a planar crack in an elastic solid.

The diffraction problem described above belongs to a class of elastodynamic diffraction problems whose complete solutions are very difficult to obtain. The one problem in that class for which analytical results can be derived in a relatively simple manner is the diffraction of a plane incident wave by a traction free semi-infinite crack. In particular, for large values of $\omega r/c_L$ the fields on the diffracted rays emanating from the tip of the semi-infinite crack can be obtained by asymptotic considerations. These results can be expressed in terms of diffraction coefficients which relate the diffracted fields to the incident field. Geometrical diffraction theory for cracks of arbitrary shape is based on the semi-infinite crack results, in that it immediately provides first-order corrections to the semi-infinite crack results, to account for a curvature of incident wavefronts and curvature of crack edges. In the usual terminology the results for diffraction of plane waves by a semi-infinite crack are the canonical solutions. With some additional effort corrections for the finite dimensions of a crack can now be obtained, as described in this paper.

The total fields at point Q are, indeed, not just comprised of the fields on the "primary diffracted body wave rays" P_1Q and P_2Q. At the edge of the crack there are also rays of crack-face motion generated. These rays intersect the crack edges again and generate additional diffracted body wave rays. Some of these "secondary diffracted rays" will pass through point Q. This system of reflected and diffracted rays can become rather extensive. Fortunately, on the faces of the crack the main contributions to the diffracted fields are not coming from the diffracted rays of longitudinal and transverse motion, but rather from rays of surface waves. These rays, which have not been studied before, are important, because in the first approximation the diffraction coefficients for the body wave motions vanish on the crack faces, except for diffracted horizontally polarized transverse wave motions. In addition, surface wave motions suffer less geometrical decay than body wave motions. In a recent paper Gautesen, Achenbach and McMaken[8] have presented a theory for surface wave rays which are generated by the diffraction of body wave rays.

When a surface wave ray intersects the edge of a crack, a ray of reflected surface wave motion is generated, as well as cones of diffracted rays of longitudinal and transverse motions. The reflection coefficients can be computed. The cones of diffracted body rays can also be analyzed, and the associated diffraction coefficients can be obtained. With the aid of these results the stepwise radiation of energy which is temporarily trapped by the crack in the form of surface motion can be analyzed, and the scattered field can be computed as the sum of primary diffractions and a system of secondary diffractions.

For curved wavefronts and for curved edges of diffraction, the cones of diffracted rays have envelopes, at which the rays coalesce and the fields become singular. The envelopes are called caustics. The results of the geometrical theory of diffraction are not valid near caustics. It is, however, possible to extend the theory to the caustics. In the next section this is shown for the case of a penny-shaped crack which is under the normal incidence of a plane longitudinal wave. For this case the normal axis through the center point of the crack is a caustic. Uniformly valid expressions for the field on the body wave rays are obtained.

Some General Results for Diffraction of Longitudinal Waves

In this section we briefly summarize some pertinent expressions. The details can be found in a paper by Gautesen, Achenbach and McMaken[8].

Primary diffracted body wave rays. For an incident longitudinal wave the displacement fields on the diffracted body wave rays are

$$\underset{\sim}{u}^{\beta}_{d} = e^{i\omega S_\beta/c_\beta} \left[S_\beta \; (1+S_\beta/\rho^L_\beta)\right]^{-\frac{1}{2}} D^L_\beta \; \underset{\sim}{i}^L_\beta \; U^L \qquad (2)$$

Here U^L defines the incident wave at the point of diffraction. In Eq.(2) the superscript β denotes the nature of the wave motion on the diffracted rays. Thus we will use β = L and β = T for longitudinal and transverse waves, respectively. The distances S_β are along the diffracted rays from the point of diffraction 0 to the point of observation. The unit vectors $\underline{i}^L_\beta$ relate the displacement directions of the diffracted fields to those of the incident fields. The symbols ρ^L_β define the distance from 0 to the caustics and D^L_β are the diffraction coefficients. For an incident longitudinal wave we have

$$\rho^L_\beta = - a \sin^2\phi_\beta \left[a \; (d\phi_\beta/ds) \sin\phi_\beta + \cos\delta_\beta\right]^{-1} \quad (3)$$

where a is the radius of curvature of the edge at the point of diffraction, s is the distance measured along the edge, and δ_β are the angles between the relevant diffracted rays and the normal to the crack. The angles ϕ_L and ϕ_T are related by

$$c_T \cos\phi_L = c_L \cos\phi_T \quad (4)$$

The diffraction coefficients are obtained from the canonical problem of diffraction of a plane longitudinal wave by a semi-infinite crack.

Diffracted surface wave rays. For normal incidence, only symmetric surface wave motions are generated on the faces of the crack. The displacements on the diffracted surface wave rays then are

$$\underline{u}^R_d = e^{i\omega S_R/c_R} \left(1+S_R/\rho_R\right)^{-\frac{1}{2}} D^L_R \; \underline{i}^L_R \; U^L \quad (5)$$

The principal difference between Eqs.(5) and (2) is the additional term $S_\beta^{-\frac{1}{2}}$ in Eq. (2). This term reflects three dimensional (spherical) growth and decay in Eq. (2) versus two dimensional (cylindrical) growth and decay in Eq. (5). In Eq.(5) we have

$$\rho_R = - a \sin\phi_R \; (a \; d\phi_R/ds + 1)^{-1} , \quad (6)$$

where ϕ_R is related to ϕ_L by

$$c_L \cos\phi_R = c_R \cos\phi_L \quad (7)$$

Reflection of surface wave rays. A surface wave ray which intersects the edge of a crack gives rise to a ray of reflected surface waves, and to two cones of diffracted body rays. For a surface wave incident on the edge of a semi-infinite crack these reflection and diffraction processes were studied by Freund[9]. In the spirit of geometrical diffraction theory, we can immediately introduce the appropriate corrections for curvature of the incident wavefront and for curvature of the edge of the crack.

A surface wave ray is reflected such that the angle between the reflected ray and the tangent to the edge is just the same as the angle of incidence, ϕ_R, between the incident ray and the tangent to the edge. Moreover, rays of symmetric (antisymmetric) surface waves are reflected as rays of symmetric (antisymmetric) surface waves.

The incident field is defined by Eq. (5). Quantitatively, the fields on the reflected surface rays are given by

$$\underline{u}^R_r = e^{i\omega S_R/c_R} \left(1 + S_R/\rho_R\right)^{-\frac{1}{2}} \mathcal{R} \; \underline{i}^R_r \; u^R \quad (8)$$

The nature of the motions on the incident rays is the same as on the reflected rays. In Eq. (8) S_R is the distance from the point of reflection to the point of observation, $\mathcal{R}$ is the reflection coefficient, and ρ_R is the distance to the caustic where

$$\rho_R = - a \sin\phi_R \; (a \; d\phi_R/ds + 1)^{-1} \quad (9)$$

This is the same formula as given by Eq. (6), but here $\phi_R(s)$ is the given angle of incidence, while in Eq. (6), ϕ_R was computed from Eq. (7).

Body wave rays generated by diffraction of surface way rays. For this case the displacement fields are of the general form

$$\underline{u}^\beta_d = e^{i\omega S_\beta/c_\beta} \left[S_\beta \; (1 + S_\beta/\rho^R_\beta)\right]^{-\frac{1}{2}} D^R_\beta \; \underline{i}^R_\beta \; u^R . \quad (10)$$

where β = L or β = T for diffracted rays of longitudinal and transverse motion, respectively. Also D^R_β is the pertinent diffraction coefficient which can be computed, see Ref. 8. The distances to the caustics are given by ρ^R_L and ρ^R_T, respectively.

Diffraction by a penny-shaped crack. For normal incidence of a plane longitudinal wave of the kind given by Eq. (1), the incident and diffracted fields are axially symmetric with respect to the z-axis. Thus, only the radial distance $r = (x^2+y^2)^{\frac{1}{2}}$ and the axial distance z enter in the results. The geometry is shown in Fig. 2. The diffracted field at any point Q away from the crack can now easily be computed on the basis of the semi-infinite crack solution, as explained in the previous section. In cylindrical coordinates the radial and axial displacements for the diffracted field are obtained as

$$(u_d)_r = A_o \sum_{j=1}^{2} Q_j \; (-1)^j \; \times$$

$$\left[G_L(\theta_j) \cos\theta_j \; e^{i\omega R_j/c_L} - G_T(\theta) \sin\theta_j \; e^{i\omega R_j/c_T}\right], \quad (11)$$

$$(u_d)_z = A_o \sum_{j=1}^{2} Q_j \; \times$$

$$\left[G_L\ (\theta_j)\ \sin\theta_j\ e^{i\omega R_j/c_L} + G_T(\theta)_j\ \cos\theta_j\ e^{i\omega R_j/c_T}\right], \quad (12)$$

where j = 1 and j = 2 correspond to the contributions from P_1 (the point closer to Q) and P_2, respectively, while

$$G_L(\theta_j) = D_L^L\left(\theta_j;\tfrac{\pi}{2},\tfrac{\pi}{2}\right) + u_{inc}^{RS} D_L^{RS}\left(\theta_j;\tfrac{\pi}{2}\right) \quad (13)$$

$$G_T(\theta_j) = -D_T^L\left(\theta_j;\tfrac{\pi}{2},\tfrac{\pi}{2}\right) - u_{inc}^{RS} D_T^{RS}\left(\theta_j;\tfrac{\pi}{2}\right) \quad (14)$$

$$R_j = \left\{z^2 + \left[a + (-1)^j r\right]^2\right\}^{1/2}, \quad (15)$$

$$Q_j = R_j^{-1/2}\left[1 - (R_j/a)\cos\theta_j\right]^{-1/2}, \quad (16)$$

$$\cos\theta_j = a + (-1)^j r\ /R_j \quad (17)$$

In Eqs. (4) and (5), D_L^L and D_T^L are the diffraction coefficients for the primary diffractions, and the second terms in Eqs. (4) and (5) represent the contributions due to the motions of the crack faces. Explicit expressions are given in Ref. 8.

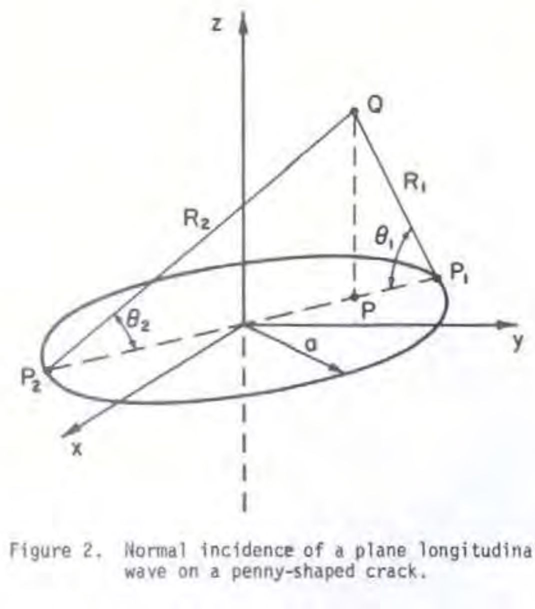

Figure 2. Normal incidence of a plane longitudinal wave on a penny-shaped crack.

The diffracted fields become singular at the caustic r = 0, and at the edge of the shadow zone. It is, however, not difficult to obtain fields which are also valid near r = 0, by the use of appropriate multiplication constants. The details are given in Ref. 8. The results are, for $r/a \ll 1$:

$$(u_d)_r = A_0\left(\frac{2\pi a}{rR}\right)^{1/2} e^{i\pi/4}\left[g_L^{1/2} G_L(\theta) J_1(g_L)\cos\theta\ e^{i\omega R/c_L} - g_T^{1/2} G_T(\theta) J_1(g_T)\ \sin\theta\ e^{i\omega R/c_T}\right] \quad (18)$$

$$(u_d)_z = A_0\left(\frac{2\pi a}{rR}\right)^{1/2} e^{-i\pi/4}\left[g_L^{1/2} G_L(\theta) J_0(g_L)\sin\theta\ e^{i\omega R/c_L} + g_T^{1/2} G_T(\theta) J_0(g_T)\ \cos\theta\ e^{i\omega R/c_T}\right] \quad (19)$$

where

$$\cos\theta = a/R, \quad R = (a^2 + z^2)^{1/2}, \quad (20)$$

$$g_L = (\omega r/c_L)\cos\theta, \quad g_T = (\omega r/c_T)\cos\theta. \quad (21)$$

In Eqs. (18) and (19), $G_L(\theta)$ and $G_T(\theta)$ are defined by Eqs. (13) and (14), and $J_0(g_T$ and $J_1(g_T)$ are Bessel functions of the first kind.

Some results for the penny-shaped crack. For the case of normal incidence the amplitudes of the axial and radial displacement components have been computed from Eqs. (11) and (12). The computations were carried out for fixed frequency and for fixed radial distance $\overline{OQ}$, as functions of the angle χ. The results are shown in Figs. 3 and 4, where the angle χ is defined in the insert in Fig. 3. The following numerical values were chosen: Poisson's ration: $\upsilon = 0.25$; frequency $\omega a/c_L = 3.5$; radial distance: $\overline{OQ}/a = 5$. The variation of the amplitudes with angle χ shows qualitative agreement with experimental results in the literature.

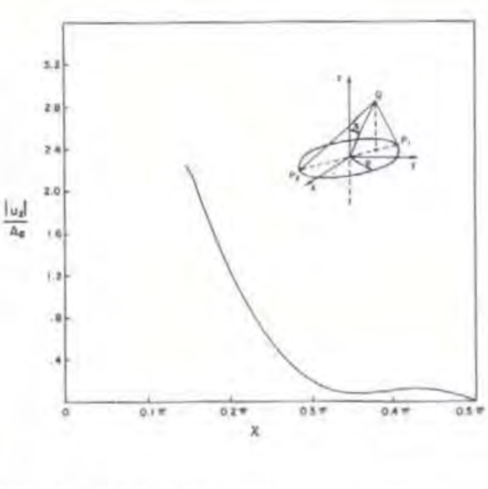

Figure 3. Axial displacement versus angle χ for R/a = 5 and $\omega a/c_L = 3.5$.

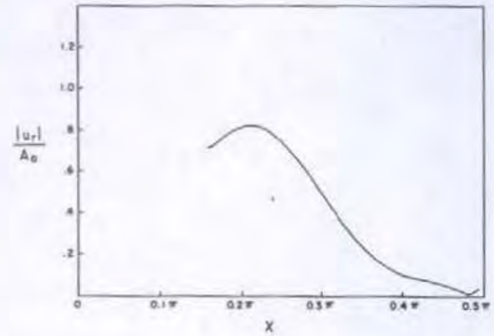

Figure 4. Radial displacement versus angle χ for $R/a = 5$ and $\omega a/c_L = 3.5$.

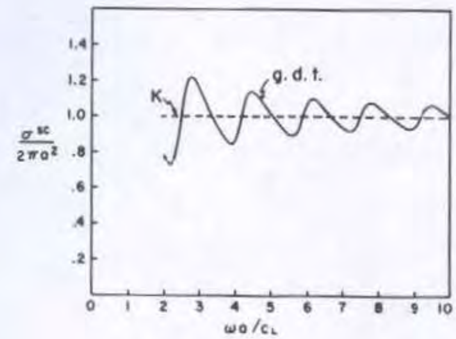

Figure 5. Scattering cross-section versus dimensionless frequency according to geometrical diffraction theory (g.d.t.), as compared with the Kirchoff (K) approximation.

A useful quantity to compute for diffraction of a longitudinal wave by a penny-shaped crack is the scattering cross section σ^{sc}. This quantity is defined as the ratio of the time-averaged scattered power over the intensity of the incident wave, I. For an incident wave of the form given by Eq. (1) we have:

$$I = \tfrac{1}{2}\omega^2 \rho c_L A_0^2 \ . \qquad (22)$$

The far field may in general terms be expressed as:

$$u_d(x_Q) = A^L(\hat{x}_Q)\,\frac{\exp(i\omega|x_Q|/c_L)}{4\pi|x_Q|} + A^T(\hat{x}_Q)\,\frac{\exp(i\omega|x_Q|/c_T)}{4\pi|x_Q|} \qquad (23)$$

The interesting result, which was shown by Tan[10], now is that only the far-field diffracted wave amplitude which is of the same type as the incident wave, and computed for x_Q as a unit vector in the direction of wave incidence, occurs in the expression for σ^{sc}. For the problem at hand the simple result is:

$$\sigma^{sc} = \frac{c_L}{\omega}\,\frac{1}{A_0}\,\mathrm{Im}\,[A^L(i_z)] \qquad (24)$$

It is easy to compute the right-hand side from Eq. (19). For $\nu = 0.25$ the result has been plotted in Fig. 5 versus the dimensionless frequency $\omega a/c_L$.

The result shown in Fig. 5 is valid for $\omega a/c_L$ sufficiently larger than unity. With results available in the literature, it is also possible to compute the corresponding result for $\omega a/c_L \ll 1$. With the low frequency approximations available in the literature, and with the high frequency approximations of the work proposed here, it seems very likely that enough information will be available to cover the whole frequency range.

Finally we present some computations for the amplitude of the axial displacement on the center axis of the crack for fixed z/a, as a function of $\omega a/c_L$. The results were computed from Eq. (19). The results are shown in Fig. 6.

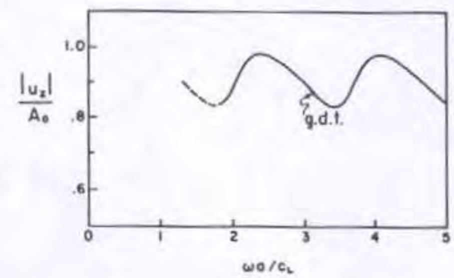

Figure 6. Axial displacement at r = 0 versus dimensionless frequency according to geometrical diffraction theory (g.d.t.) for z/a = 5.

References

1. E. A. Kraut, "Review of Theories of Scattering of Elastic Waves by Cracks," IEEE Trans. SU-23, 162 (1976).

2. E. Domany, J. A. Krumhansl, and S. Teitel, "Quasi-Static Approximation to the Scattering of Elastic Waves by a Circular Crack," to be published.

3. J. D. Achenbach and A. K. Gautesen, "Geometrical Theory of Diffraction for Three-D Elastodynamics," J. Acoust. Soc. Amer., 61, 413 (1977).

4. J. B. Keller, "A Geometrical Theory of Diffraction," in Calculus of Variations and its Applications, McGraw Hill, New York (1958).

5. J. B. Keller, "Diffraction by an Aperture," J. Appl. Phys., 28, 426 (1957).

6. L. Adler and D. K. Lewis, "Scattering of Broadband Ultrasonic Beams from Circular Disks," J. Acoust. Soc. Amer., 57, 59 (1975).

7. L. Adler and D. K. Lewis, "Scattering of a Broadband Ultrasonic Pulse by Discontinuities, IEEE Trans. Sonics and Ultrasonics, SU-23, 351 (1976).

8. A. K. Gautesen, J. D. Achenbach, and H. McMaken, "Surface Wave Rays in Elastodynamic Diffraction Theory," submitted for publication.

9. L. B. Freund, "The Oblique Reflection of a Rayleigh Wave from a Crack Tip," Int. J. Solids Structures, 7, 1199 (1971).

10. T. H. Tan, "Diffraction Theory for Time-Harmonic Elastic Waves," Doctoral Dissertation, Delft University of Technology, September 1975.