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ASYMPTOTIC FORMULAS FOR ELLIPTIC INTEGRALS

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Asymptotic formulas

for elliptic integrals

by

John Leroy Gustafson

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ABSTRACT

Asymptotic formulas are derived for incomplete elliptic integrals of all three kinds when the arguments are real and tend to infinity or to zero. Practical error bounds are found for the asymptotic formulas. Several techniques are used, including a method recently discovered by R. Wong for finding asymptotic expansions with remainder terms for integral transforms. Most of the asymptotic formulas and all of the error bounds appear to be new.

We use incomplete elliptic integrals which possess a high degree of permutation symmetry in the function arguments. The asymptotic formulas are applicable to complete elliptic integrals as a special case; some of the error bounds are treated separately in the complete case.

Numerical examples are given to demonstrate the typical accuracy which can be expected from the formulas, as well as the closeness of the error bounds.

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INTRODUCTION

Elliptic integrals are classically defined as integrals of the form

$$\int r(x,y)\,dx \tag{1.1}$$

where r is a rational function of x and y, and y^2 is a cubic or quartic polynomial in x. If y^2 is linear or quadratic in x, then the integral may be evaluated using logarithms and rational functions of x and y, but if y^2 is cubic or quartic then the integral is said to be elliptic and is not in general expressible in terms of elementary functions. Legendre [9] showed that only three nonelementary functions are needed to express (1.1) in the elliptic case.

We will choose as our three basis functions R_F, R_D, and R_J, which have the integral definitions

$$R_F(x,y,z) = \frac{1}{2}\int_0^\infty (x+t)^{-1/2}(y+t)^{-1/2}(z+t)^{-1/2}\,dt \tag{1.2a}$$

$$R_D(x,y,z) = \frac{3}{2}\int_0^\infty (x+t)^{-1/2}(y+t)^{-1/2}(z+t)^{-3/2}\,dt \tag{1.2b}$$

$$R_J(x,y,z,p) = \frac{3}{2}\int_0^\infty (x+t)^{-1/2}(y+t)^{-1/2}(z+t)^{-1/2}(p+t)^{-1}dt. \tag{1.2c}$$

We assume x, y, z, and p are positive and real. The R_F function is the elliptic integral of the first kind and is symmetric in x, y, and z. The elliptic integral of the second kind, R_D, is symmetric in x and y only; it is related to R_F by

$$R_D(x, y, z) = -6 \frac{\partial}{\partial z} R_F(x, y, z). \tag{1.3}$$

The R_J function is symmetric in x, y, and z only. The constant in front of each integral is chosen so that

$$\begin{aligned} R_F(x, x, x) &= x^{-1/2} \\ R_D(x, x, x) &= x^{-3/2} \\ R_J(x, x, x, x) &= x^{-3/2} \end{aligned} \tag{1.4}$$

Furthermore, the functions are homogeneous of degrees -1/2, -3/2, and -3/2; that is,

$$\begin{aligned} R_F(Cx, Cy, Cz) &= C^{-1/2} R_F(x, y, z) \\ R_D(Cx, Cy, Cz) &= C^{-3/2} R_D(x, y, z) \\ R_J(Cx, Cy, Cz, Cp) &= C^{-3/2} R_J(x, y, z, p) \end{aligned} \tag{1.5}$$

These basis functions are not the classical ones used by Legendre which became established as standard after his work was published in 1825. Legendre's basis functions are denoted by $F(\varphi,k)$, $E(\varphi,k)$ and $\Pi(\varphi,k,n)$; however, this notation hides the underlying permutation symmetry in the variables, introduces unnecessary linear transformations, and makes the quadratic Gauss and Landen transformations as well as other identities more cumbersome than with the present choice of basis functions. Carlson [3], [4] has shown elliptic integrals to be hypergeometric functions of several variables and has shown many classical results concerning properties of elliptic integrals to be special cases of the general properties of R functions. References [2], [3], [4], and [14] provide methods of converting between the classical basis functions and the ones used here. In particular,

$$F(\varphi,k) = (\sin\varphi)\, R_F(\cos^2\varphi,\ 1-k^2\sin^2\varphi, 1), \tag{1.6}$$

$$E(\varphi,k) = F(\varphi,k) - \frac{1}{3}k^2(\sin^3\varphi)R_D(\cos^2\varphi, 1-k^2\sin^2\varphi, 1), \tag{1.7}$$

$$\Pi(\varphi,k,n) = F(\varphi,k) - \frac{n}{3}(\sin^3\varphi)R_J(\cos^2\varphi, 1-k^2\sin^2\varphi, 1, 1+n\sin^2\varphi). \tag{1.8}$$

We will also require the following definitions of complete elliptic integrals:

$$R_K(x,y) = \frac{1}{\pi}\int_0^\infty t^{-1/2}(x+t)^{-1/2}(y+t)^{-1/2}dt \tag{1.9a}$$

$$R_Q(x,y) = \frac{2}{\pi}\int_0^\infty t^{-1/2}(x+t)^{-1/2}(y+t)^{-3/2}dt \tag{1.9b}$$

$$R_M(x,y,p) = \frac{2}{\pi}\int_0^\infty t^{-1/2}(x+t)^{-1/2}(y+t)^{-1/2}(p+t)^{-1}dt, \tag{1.9c}$$

where the constants in front of the integrals are chosen so that properties analogous to (1.4) are satisfied. These complete cases are related to the incomplete cases by

$$R_K(x,y) = \frac{2}{\pi} R_F(x,y,0) \tag{1.10a}$$

$$R_Q(x,z) = \frac{4}{3\pi} R_D(x,0,z) \tag{1.10b}$$

$$R_M(x,y,p) = \frac{4}{3\pi} R_J(x,y,0,p). \tag{1.10c}$$

We will find it convenient to state some of the formulas in terms of the elementary function R_C:

$$R_C(x,y) = \frac{1}{2}\int_0^\infty (x+t)^{-1/2}(y+t)^{-1}\,dt$$
$$= \begin{cases} (x-y)^{-1/2}\log((x^{1/2}+(x-y)^{1/2})/y^{1/2}), & (0<y<x), \\ (y-x)^{-1/2}\arccos(x/y)^{1/2}, & (0\le x<y) \end{cases} \tag{1.11}$$

This is simply a special case of the R_F function, as can be seen from (1.2a):

$$R_C(x, y) = R_F(x, y, y). \tag{1.12}$$

There is a similar relationship between R_D and R_J, evident from (1.2b) and (1.2c):

$$R_D(x, y, z) = R_J(x, y, z, z). \tag{1.13}$$

Despite the intense activity which surrounded elliptic integrals and their inverses (elliptic functions) during the last century, much was left undiscovered concerning the behavior of elliptic integrals as the arguments tend to zero or to infinity. Kaplan [8, page 13] gives an asymptotic series for $F(\varphi,k)$ which implies the formula

$$R_F(x, y, z) \sim z^{-1/2}\log(4z^{1/2}/(x^{1/2}+y^{1/2})) \text{ as } z \longrightarrow \infty. \quad (1.14)$$

Kaplan also gives an asymptotic series for $E(\varphi,k)$ which can be shown to imply the one-term asymptotic formula for R_D derived in the sections which follow. Both formulas also appear in reference [11, page 228]. No series is given in either work for the elliptic integral of the third kind, and no error bounds are given for truncations of the $F(\varphi,k)$ and $E(\varphi,k)$ series.

Convergent series expansions for functions $F(\varphi,k)$, $E(\varphi,k)$, and $\Pi(\varphi,k,n)$, for $0 < \varphi < \pi/2$ and $k^2 < 1$ can also be found in [1]. These are power series in k^2, $1-k^2$, and n which do not converge for arguments in the neighborhoods being considered here. Asymptotic series have been long established for the complete elliptic integrals $K(k)$ and $E(k)$, and these series are equivalent to series for R_K and R_Q; see, for example, reference [7].

Tables 1 through 4 indicate the asymptotic approximations and error bounds for incomplete and complete elliptic integrals as the arguments tend to infinity or to zero. Where there is symmetry in the arguments, only the case for one argument is shown.

TABLE 1

Incomplete Elliptic Integrals

Behavior for Large Arguments

Function	Argument	Asymptotic Formula	Error Bound
$R_F(x,y,z)$	$z \to \infty$	$z^{-1/2}\log\left[\frac{4\,z^{1/2}}{x^{1/2}+y^{1/2}}\right] + r$	$0 < r < \frac{1}{8}\,z^{-3/2}(x+y)\left[\log(1+\frac{4z}{x+y}) + 1\right]$
$R_D(x,y,z)$	$z \to \infty$	$3\,z^{-3/2}\left[\log\left[\frac{4\,z^{1/2}}{x^{1/2}+y^{1/2}}\right] - 1\right] + r$	$0 < r < \frac{9}{8}\,z^{-5/2}(x+y)\left[\log(1+\frac{4z}{x+y}) + 1\right]$
$R_D(x,y,z)$	$y \to \infty$	$y^{-1/2}\left[\frac{3}{(xz)^{1/2} + z}\right] + r$	$-\frac{3}{4}\,y^{-3/2}\left[\log\left[1 + \frac{2y}{x^{1/4}z^{3/4}}\right] + \frac{1}{2}\right] < r < 0$
$R_J(x,y,z,p)$	$p \to \infty$	$3\,p^{-1}\,R_F(x,y,z) - \frac{3}{2}\pi\,p^{-3/2} + r$	$0 < r < 3(2^{1/2})\,p^{-2}(x+y+z)^{1/2}$
$R_J(x,y,z,p)$	$z \to \infty$	$3\,z^{-1/2}R_C(X^2, Y^2) + r$, where $X=(xy)^{1/2}+p$, $Y=(xp)^{1/2}+(yp)^{1/2}$	$-\frac{3}{4}\,z^{-3/2}\left[\log\left[1+\frac{2z}{(xy)^{1/4}p^{1/2}}\right] + \frac{1}{2}\right] < r < 0$

TABLE 2

Incomplete Elliptic Integrals

Behavior for Small Arguments

Function	Argument	Asymptotic Formula	Error Bound
$R_F(x,y,z)$	$z \to 0^+$	$\frac{\pi}{2} R_K(x,y) + r$	$-z^{1/2}\left[\frac{\pi}{(8xy)^{1/2}}\right] < r < 0$
$R_D(x,y,z)$	$z \to 0^+$	$3(xyz)^{-1/2} + r$	$-6\,\frac{(x+y)^{1/2}}{xy} < r < 0$
$R_D(x,y,z)$	$y \to 0^+$	$\frac{3\pi}{4} R_Q(x,z) + r$	$-y^{1/2}\left[\frac{3\pi}{(2z)^{3/2}x^{1/2}}\right] < r < 0$
$R_J(x,y,z,p)$	$p \to 0^+$	$3R_C(\alpha,\beta) + 2R_J(x+\lambda,y+\lambda,z+\lambda,\lambda) + r$ where $\lambda=(xy)^{1/2}+(xz)^{1/2}+(yz)^{1/2}$, $\alpha=[p(x^{1/2}+y^{1/2}+z^{1/2})+(xyz)^{1/2}]^2$, and $\beta = p(p+\lambda)^2$	$-\frac{6}{5}\,p\,\lambda^{-5/2} < r < 0$
$R_J(x,y,z,p)$	$z \to 0^+$	$\frac{3\pi}{4} R_M(x,y,p) + r$	$-z^{1/2}\left[\frac{3\pi}{(8xy)^{1/2}p}\right] < r < 0$

TABLE 3

Complete Elliptic Integrals

Behavior for Large Arguments

Function	Argument	Asymptotic Formula	Error Bound
$R_K(x,y)$	$y \to \infty$	$\frac{1}{\pi} y^{-1/2} \log\left[\frac{16\,y}{x}\right] + r$	$0 < r < \frac{1}{4\pi} y^{-3/2} x \left[\log(1 + \frac{4y}{x}) + 1\right]$
$R_D(x,y)$	$y \to \infty$	$\frac{2}{\pi} y^{-3/2}\left[\log\left[\frac{16y}{x}\right] - 2\right] + r$	$0 < r < \frac{3}{2\pi} y^{-5/2} x \left[\log(1 + \frac{4y}{x}) + 1\right]$
$R_D(x,y)$	$x \to \infty$	$\frac{4}{\pi x^{1/2} y} + r$	$-\frac{1}{\pi} x^{-3/2}\left[\log\left[1 + \frac{2x}{y}\right] + \frac{2}{3}\right] < r < 0$
$R_M(x,y,p)$	$p \to \infty$	$2\,p^{-1} R_K(x,y) - 2\,p^{-3/2} + r$	$0 < r < \frac{4(2^{1/2})}{\pi} p^{-2} (x + y)^{1/2}$
$R_M(x,y,p)$	$y \to \infty$	$\frac{4}{\pi} y^{-1/2} p^{-1/2} R_C(p,x) + r$	$-\frac{1}{\pi} y^{-3/2}\left[\log\left[1 + \frac{2y}{x^{1/3} p^{2/3}}\right] + \frac{2}{3}\right] < r < 0$

TABLE 4

Complete Elliptic Integrals

Behavior for Small Arguments

Function	Argument	Asymptotic Formula	Error Bound
$R_K(x,y)$	$y \to 0^+$	$\frac{1}{\pi} x^{-1/2} \log\left[\frac{16x}{y}\right] + r$	$0 < r < \frac{1}{4\pi} x^{-3/2} y \left[\log(1+\frac{4x}{y}) + 1\right]$
$R_D(x,y)$	$y \to 0^+$	$\frac{4}{\pi x^{1/2} y} + r$	$-\frac{1}{\pi} x^{-3/2}\left[\log\left[1+\frac{2x}{y}\right] + \frac{2}{3}\right] < r < 0$
$R_D(x,y)$	$x \to 0^+$	$\frac{2}{\pi} y^{-3/2}\left[\log\left[\frac{16y}{x}\right] - 2\right] + r$	$0 < r < \frac{3}{2\pi} y^{-5/2} x \left[\log(1+\frac{4y}{x}) + 1\right]$
$R_M(x,y,p)$	$p \to 0^+$	$\frac{4}{\pi} R_C(\alpha,\beta) + \frac{8}{3\pi} R_D(x+\lambda, y+\lambda, \lambda) + r$ where $\lambda = (xy)^{1/2}$, $\hat{p} = p(p+\lambda)^2$, and $\alpha = [p(x^{1/2} + y^{1/2})]^2$	$-\frac{8}{5\pi} p\,(xy)^{-5/4} < r < 0$
$R_M(x,y,p)$	$y \to 0^+$	$\frac{2}{\pi} p^{-1}\left[x^{-1/2}\log(\frac{16x}{y}) - 2R_C(x,p)\right]$ $+ r$	$0 < r <$ $\frac{1}{2\pi} y \frac{p+2x}{x^{3/2} p^2}\left[\log\left[1 + \frac{4xp}{y(2x+p)}\right] + 1\right]$

PRELIMINARY THEOREMS

Before proving the formulas summarized in Tables 1 through 4, we require several preliminary results.

Consider functions which can be defined by

$$I(\lambda) = \int_0^\infty f(t)h(\lambda t)\,dt \tag{2.1}$$

where $f(t)$ and $h(t)$ are locally integrable functions on $(0, \infty)$ which have asymptotic expansions

$$f(t) \sim \sum_{k=0}^{n-1} a_k t^{k+u-1}, \qquad \text{as} \quad t \to 0^+, \tag{2.2}$$

where $0 < u \leq 1$, and

$$h(t) \sim \sum_{k=0}^{n-1} b_k t^{-k-v}, \qquad \text{as} \quad t \to +\infty, \tag{2.3}$$

where $0 < v \leq 1$. Wong [13] has found a method for finding the asymptotic expansion for $I(\lambda)$ which also provides an explicit error term. Define the generalized Mellin transform of $h(t)$ by

$$M[h;z] = \int_0^\infty t^{z-1}h(t)\,dt + \int_c^\infty t^{z-1}h(t)\,dt. \tag{2.4}$$

Near the origin, we assume that

$$h(t) = O(t^b), \quad u + b > 0. \tag{2.5}$$

The first integral in (2.4) is analytic for $\mathrm{Re}\, z > -b$, and the second integral is analytic for $\mathrm{Re}\, z < v$. By analytic continuation of the second integral, the Mellin transform $M[h;z]$ can be extended to a meromorphic function in the half-plane $\mathrm{Re}\, z > -b$. For $n = 1, 2, \ldots$ define

$$h_n(t) = h(t) - \sum_{k=0}^{n-1} b_k t^{-k-v} \tag{2.6}$$

and

$$\varphi_n(t) = \begin{cases} h(t), & 0 < t < 1, \\ h_n(t), & 1 \le t < \infty. \end{cases} \tag{2.7}$$

With these definitions, the following result is easily established:

Lemma 1. Let $h(t)$ be a locally integrable function on $(0,\infty)$ satisfying (2.3) and (2.5). Then for $-b < \mathrm{Re}\, z < n + v$,

$$M[h;z] = M[\varphi_n;z] - \sum_{k=0}^{n-1} b_k/(z - k - v). \tag{2.8}$$

Analogous results are needed for $f(t)$. Define

$$M[f;1-z] = \int_0^1 t^{-z} f(t)\, dt + \int_1^\infty t^{-z} f(t)\, dt. \tag{2.9}$$

As $t \to +\infty$, we assume that

$$f(t) = O(t^{-a}), \quad a + v > 1. \tag{2.10}$$

The first integral in (2.9) is analytic for $\operatorname{Re} z < u$, and the second integral is analytic for $\operatorname{Re} z > 1 - a$. By analytic continuation of the first integral, $M[f;1-z]$ can be extended to a meromorphic function in the half plane $\operatorname{Re} z > 1 - a$. For $n = 1, 2, \ldots$ define

$$f_n(t) = f(t) - \sum_{k=0}^{n-1} a_k t^{k+u-1} \tag{2.11}$$

and

$$\psi_n(t) = \begin{cases} f_n(t), & 0 < t < 1, \\ f(t), & 1 \leq t < \infty. \end{cases} \tag{2.12}$$

Lemma 2. Let $f(t)$ be a locally integrable function on $(0,\infty)$ satisfying (2.2) and (2.10). Then for $1 - a < \operatorname{Re} z < n + u$,

$$M[f;1-z] = M[\psi_n;1-z] + \sum_{k=0}^{n-1} a_k/(k + u - z). \tag{2.13}$$

We can now give the asymptotic expansion of $I(\lambda)$ and the remainder term.

Theorem 1 (Wong). Let $f(t)$ satisfy (2.2) and (2.10). Let $h(t)$ satisfy (2.3) and (2.5). Then for any $n \geq 1$,

$$I(\lambda) = \sum_{k=0}^{n-1} a_k M[h;k+u]\,\lambda^{-k-u} + \sum_{k=0}^{n-1} b_k M[f;1-k-v]\,\lambda^{-k-v} + \delta_n(\lambda) \tag{2.14}$$

for the case $u \neq v$, and

$$I(\lambda) = \sum_{k=0}^{n-1} c_k(v)\, \lambda^{-k-v} + \log \lambda \sum_{k=0}^{n-1} a_k b_k \lambda^{-k-v} + \delta_n(\lambda) \tag{2.15}$$

for the case u = v, where

$$c_k(v) = a_k M[\varphi_n;k+v] + b_k M[\psi_n;1-k-v] - \sum_{\substack{j=0\\ j\neq k}}^{n-1} (a_k b_j + a_j b_k)/(k-j). \tag{2.16}$$

In both cases, the remainder term is given by

$$\delta_n(\lambda) = \int_0^\infty f_n(t) h_n(\lambda t)\, dt. \tag{2.17}$$

<u>Proof.</u> We write

$$I(\lambda) = I_1(\lambda) + I_2(\lambda), \tag{2.18}$$

where I_1 and I_2 correspond to the intervals $(0,1)$ and $(1,\infty)$, respectively. We break up the remainder term similarly:

$$\delta_n(\lambda) = \delta_{n,1}(\lambda) + \delta_{n,2}(\lambda). \tag{2.19}$$

By applying (2.6) and (2.11) to I_1 and I_2:

$$I_1(\lambda) = \sum_{k=0}^{n-1} a_k \int_0^1 t^{k+u-1} h(\lambda t)\, dt + \sum_{k=0}^{n-1} b_k \lambda^{-k-v} \int_0^1 t^{-k-v} f_n(t)\, dt + \delta_{n,1}(\lambda)$$

and

$$I_2(\lambda) = \sum_{k=0}^{n-1} a_k \int_1^\infty t^{k+u-1} h_n(\lambda t)\, dt + \sum_{k=0}^{n-1} b_k \lambda^{-k-v} \int_1^\infty t^{-k-v} f(t)\, dt + \delta_{n,2}(\lambda).$$

Assume for the moment that $u \neq v$. Since $\psi_n(t) = f_n(t)$ for $0 < t < 1$ and $\psi_n(t) = f(t)$ for $1 < t < \infty$, adding the last two identities together gives

$$I(\lambda) = \sum_{k=0}^{n-1} a_k \left[\int_0^1 t^{k+u-1} h(\lambda t)\, dt + \int_1^\infty t^{k+u-1} h_n(\lambda t)\, dt \right] + \sum_{k=0}^{n-1} b_k\, M[\psi_n; 1-k-v]\, \lambda^{-k-v} + \delta_n(\lambda). \tag{2.20}$$

The presence of λ in the quantity in square brackets prevents us from similarly combining h and h_n into a Mellin transform of φ_n; however, we can rewrite that quantity as a Mellin transform of φ_n plus other terms if we substitute $s = \lambda t$:

$$\begin{aligned}
[\ldots] &= \lambda^{-k-u} \int_0^\lambda s^{k+u-1} h(s)\,ds + \lambda^{-k-u} \int_\lambda^\infty s^{k+u-1} h_n(s)\,ds \\
&= \lambda^{-k-u} M[\varphi_n; k+u] + \lambda^{-k-u} \left[\int_1^\lambda t^{k+u-1} h(t)\,dt - \int_1^\lambda t^{k+u-1} h_n(t)\,dt \right] \\
&= \lambda^{-k-u} M[\varphi_n; k+u] + \lambda^{-k-u} \int_1^\lambda t^{k+u-1} \sum_{j=0}^{n-1} b_j t^{-j-v}\, dt \\
&= \lambda^{-k-u} M[\varphi_n; k+u] + \lambda^{-j-v} \sum_{j=0}^{n-1} b_j/(k+u-j-v) - \lambda^{-k-u} \sum_{j=0}^{n-1} b_j/(k+u-j-v).
\end{aligned} \tag{2.21}$$

By Lemma 1, the sum of the first term and the third term in (2.21) is equal to $\lambda^{-k-u} M[h; k+u]$. So (2.20) now becomes

$$I(\lambda) = \sum_{k=0}^{n-1} a_k M[h;k+u]\lambda^{-k-u} + \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} \lambda^{-j-v} a_k b_j/(k+u-j-v)$$
$$+ \sum_{k=0}^{n-1} b_k M[\psi_n;1-k-v]\lambda^{-k-v} + \delta_n(\lambda). \qquad (2.22)$$

By applying Lemma 2 to (2.22), the double sum is completely cancelled, leaving

$$I(\lambda) = \sum_{k=0}^{n-1} a_k M[h;k+u]\,\lambda^{-k-u} + \sum_{j=0}^{n-1} b_k M[f;1-k-v]\,\lambda^{-k-v} + \delta_n(\lambda), \qquad (2.23)$$

which proves the case $u \neq v$. For the case $u = v$, we apply a limiting process. Since

$$\lim_{u\to v} (\lambda^{-k-u}-\lambda^{-k-v})/(u-v) = \lambda^{-k-v}(-\log\lambda), \qquad (2.24)$$

this limit applies to the terms of the sums in (2.21) for which $k = j$:

$$\lim_{u\to v} [\int_0^1 t^{k+u-1} h(\lambda t)dt + \int_1^\infty t^{k+u-1} h_n(\lambda t)dt]$$
$$= \lambda^{-k-v} M[\varphi_n;k+v] + b_k(\log\lambda)\lambda^{-k-v} + \sum_{\substack{j=0\\ j\neq k}}^{n-1} b_j[\lambda^{-j-v}-\lambda^{-k-v}]/(k-j). \qquad (2.25)$$

Therefore, instead of (2.22) and (2.23), we obtain the second case of the theorem:

$$I(\lambda) = \sum_{k=0}^{n-1} c_k(v)\,\lambda^{-k-v} + \log\lambda \sum_{k=0}^{n-1} a_k b_k \lambda^{-k-v} + \delta_n(\lambda),$$

where

$$c_k(v) = a_k M[\varphi_n; k+v] + b_k M[\psi_n; 1-k-v] - \sum_{\substack{j=0 \\ j\neq k}}^{n-1} (a_k b_j + a_j b_k)/(k-j). \blacksquare$$

Both cases will be of considerable value to us in deriving asymptotic formulas and error bounds for elliptic integrals.

PROOFS OF THE FORMULAS FOR LARGE ARGUMENTS

We now prove the asymptotic formulas and error bounds summarized in Tables 1 and 3, and give more detailed results for certain cases. In attempting to bound the error term, we seek the following features in a bound:

1. The bound should be simple, i.e. an elementary function of the argument in question. It should not involve limits or integrals.

2. There should be little "waste" in the bound. The bound should not exceed the actual error by more than, say, a factor of three. Ideally, the bound should asymptotically agree with the actual error.

These features tend to compete directly with one another, since the actual error always involves an integral which is not expressible using elementary functions, and all simplifications of that integral introduce a difference between the actual error and the bound. Hence, the bounds presented here are the result of compromise. All satisfy the first condition in that they involve only logarithms and rational powers of the argument; the second condition is generally satisfied also, based on numerical tests (see Appendices).

<u>Theorem 2.</u> If $x, y \geq 0$ and $x+y, z > 0$, then

$$R_F(x,y,z) = z^{-1/2} \log[4z^{1/2}/(x^{1/2}+y^{1/2})] + r, \tag{3.1}$$

where the error term, r, is bounded by

$$0 < r < \frac{1}{8} z^{-3/2}(x+y)[\log(1 + 4z/(x+y)) + 1]. \tag{3.2}$$

<u>Proof.</u> If we let $s = 1/t$ in the integral representation (1.2a) of $R_F(x,y,z)$,

$$\begin{aligned} R_F(x,y,z) &= \frac{1}{2}\int_0^\infty [(x+t)(y+t)(z+t)]^{-1/2}\,dt \\ &= \frac{1}{2}\int_\infty^0 [(x+1/s)(y+1/s)(z+1/s)]^{-1/2}(-1/s^2)ds \\ &= \frac{1}{2}\int_0^\infty [s(1+xs)(1+ys)(1+zs)]^{-1/2}\,ds. \end{aligned}$$

Hence, R_F has the alternative representation

$$\bar{R}_F(x,y,z) = \frac{1}{2}\int_0^\infty [t(1+xt)(1+yt)]^{-1/2}(1+zt)^{-1/2}\,dt, \tag{3.3}$$

which is in the form (2.1) for which Theorem 1 applies, with $z = \lambda$. Let

$$f(t) = t^{-1/2}[(1+xt)(1+yt)]^{-1/2}$$

and

$$h(zt) = (1+zt)^{-1/2} = (zt)^{-1/2}[1 + 1/(zt)]^{-1/2}.$$

The functions f and h have asymptotic formulas

$$f(t) \sim t^{-1/2}[1 - ((x+y)/2)\, t] \quad \text{as} \quad t \to 0,^{+}$$

$$h(t) \sim t^{-1/2}[1 - (1/2)\, t^{-1}] \quad \text{as} \quad t \to +\infty, \tag{3.4}$$

which means that $u = v = 1/2$, and the second case of Theorem 1 applies. Here, $a_0 = 1$, $b_0 = 1$, $a_1 = -(x+y)/2$, and $b_1 = -1/2$. Observe that $f(t) = O(t^{-3/2})$ as $t \to \infty$, and $h(t) = O(1)$ as $t \to 0$, so conditions (2.5) and (2.10) are satisfied. Using (2.15) with $n = 1$, we obtain the first term of the asymptotic expansion:

$$R_F(x,y,z) = \frac{1}{2} c_0(1/2)\, z^{-1/2} + \log(z^{1/2}) z^{-1/2} + \frac{1}{2}\delta_1(z), \tag{3.5}$$

where c_0 and $\delta_1(z)$ will be derived as follows: By (2.7) and (2.12),

$$\varphi_1(t) = \begin{cases} h(t), & 0 < t < 1, \\ h(t) - t^{-1/2}, & t \geq 1, \end{cases} \quad (3.6)$$

and

$$\psi_1(t) = \begin{cases} f(t) - t^{-1/2}, & 0 < t < 1, \\ f(t), & t \geq 1. \end{cases} \quad (3.7)$$

The fact that $a_0 = b_0$ permits considerable simplification of the expression (2.16) for $c_0(1/2)$:

$$c_0(1/2) = M[\varphi_1;1/2] + M[\psi_1;1/2] = M[\varphi_1+\psi_1;1/2]$$
$$= \int_0^\infty (h(t) + f(t) - t^{-1/2})\, t^{-1/2}\, dt.$$

This integral can be evaluated using elementary functions. By writing $(1+xt)(1+yt) = 1 + (x+y)t + xyt^2$ and expressing the improper integral as a limit, we find

$$c_0(1/2) = \int_0^\infty \{t^{-1/2}(1+t)^{-1/2} + t^{-1}[1+(x+y)t+xyt^2]^{-1/2} - 1/t\}\, dt$$
$$= \lim_{\substack{b\to\infty \\ \epsilon\to 0}} \{\log[(t(1+t))^{1/2}+t+1/2]$$
$$- \log[(1+(x+y)t+xyt^2)^{1/2}/t+1/t+(x+y)/2] - \log t \,\Big|_\epsilon^b\}$$
$$= 2\log[4/(x^{1/2} + y^{1/2})]. \quad (3.8)$$

(See [6], entries 2.261 and 2.266). Combining (3.8) with (3.5) provides the first term of the asymptotic expansion:

$$R_F(x,y,z) = z^{-1/2}\log(4z^{1/2}/(x^{1/2}+y^{1/2}) + \delta_1(z)/2.$$

As has already been mentioned, this first term is not new to the literature; however, we now have an exact expression for the error:

$$r = \delta_1(z)/2 = \frac{1}{2}\int_0^\infty (f(t)-t^{-1/2})(h(zt)-(zt)^{-1/2})\,dt$$

$$= \frac{1}{2}\, z^{-1/2} \int_0^\infty (1/t)[1-(1+(x+y)t+xyt^2)^{-1/2}][1-(1+1/zt)^{-1/2}]dt. \quad (3.9)$$

This integral is of course not elementary, but the integrand may be bounded rather closely by a function which has an elementary integral. Consider the factors in square brackets in (3.9); letting $w = 1/zt$,

$$1-(1+w)^{-1/2} = ((1+w)^{1/2}-1)/(1+w)^{1/2}$$

$$= w \;/\; (1+w)^{1/2}(1+(1+w)^{1/2}), \quad (3.10)$$

where we have multiplied top and bottom by $(1+(1+w)^{1/2})$ to rationalize the numerator. The denominator of (3.10) may be rewritten and bounded from below by

$$(1+w) + (1+w)^{1/2} > (1+w) + 1 = 2 + w.$$

Here we use the fact that $w = 1/(zt)$ is positive for t in the range of integration, $(0,\infty)$. For the same reason, the quantity in (3.10) is positive. Hence, $1-(1+w)^{-1/2} < w/(2+w)$, or

$$0 < 1 - (1 + 1/zt)^{-1/2} < 1 / (1 + 2zt). \qquad (3.11)$$

It will turn out to be convenient to specialize further this inequality according to whether t is small or large. Since $1/(1+2zt) < 1/2zt$,

$$\begin{aligned} &1 - (1 + 1/zt)^{-1/2} < 1 / (1 + 2zt) \quad \forall\, t>0, \text{ close for small } t, \\ &1 - (1 + 1/zt)^{-1/2} < 1 / 2zt \quad \forall\, t>0, \text{ close for large } t. \end{aligned} \qquad (3.12)$$

The term "close" here means that the ratio of the terms being compared approaches unity as t becomes small or large, appropriately.

The first term in square brackets in (3.9) is bounded differently; we use

$$\begin{aligned} &1 - (1 + (x+y)t + xyt^2)^{-1/2} < 1 \quad \forall\, t>0, \text{ close for large } t, \\ &1 - (1 + (x+y)t + xyt^2)^{-1/2} < (x+y)t/2 \quad \forall\, t>0, \text{ close for small } t. \end{aligned} \qquad (3.13)$$

The first inequality is obvious; the second follows from the inequality of arithmetic and geometric means:

$$(xy)^{1/2} \le (x+y)/2 \qquad \text{for } x,y \ge 0. \tag{3.14}$$

As a result,

$$1 - (1 + (x+y)t + xyt^2)^{-1/2} \le 1 - (1 + (x+y)t + (x+y)^2t^2/4)^{-1/2}$$

$$= 1 - (1 + (x+y)t/2)^{-1}$$

$$< 1 - (1 - (x+y)t/2) = (x+y)t/2.$$

By partitioning the interval of integration into $(0,s)$ and (s,∞) for any $s>0$, we may bound the remainder term r defined by (3.9):

$$r < \frac{1}{2}z^{-1/2}\int_0^s [(x+y)/2][(1/(1+2zt)] \, dt + \frac{1}{2}z^{1/2}\int_s^\infty 1/(2zt^2) \, dt$$

$$= \frac{1}{4} z^{-3/2}[((x+y)/2)\log(1+2zs) + 1/s]. \tag{3.15}$$

As s tends to infinity, the log term dominates and the bound in (3.15) tends to infinity; as s tends to zero, the $1/s$ term dominates and the bound tends to infinity. In either case, the bound is not useful. The minimum occurs for a particular value of s which is a function of x, y, and z. Minimizing the quantity in square brackets in (3.15) by differentiation,

$$(x+y)z/(1+2zs) - 1/s^2 = 0,$$

which implies

$$x+y = 1/zs^2 + 2/s. \tag{3.16}$$

Since we are mainly interested in large z, drop the $1/zs^2$ term to obtain $s_{min} \sim 2/(x+y)$. By using this value of s in (3.15), the stated bound (3.2) is obtained. ∎

Since R_F is $O(z^{-1/2}\log z)$ and the error bound is $O((x+y)z^{-3/2}\log z)$, the relative error is $O((x+y)/z)$. Therefore, one expects to be able to approximate R_F to about three significant figures if $z/(x+y)$ is on the order of 1000. Numerical tests bear this out; see Appendix A. Since there is complete symmetry in the arguments x, y, and z, one can always permute the arguments so that z is the largest argument.

As a corollary to Theorem 2, we can find an asymptotic formula and error bound for the complete case $R_K(y,z)$ as $z \to \infty$. Using (1.10a) gives

$$R_K(y,z) = \frac{1}{\pi} z^{-1/2}\log(16z/y) + r, \tag{3.17}$$

where

$$0 < r < \frac{1}{4\pi} z^{-3/2} y[\log(1+4z/y)+1]. \tag{3.18}$$

The second-order asymptotic formula ($n = 2$ in Theorem 1) is obtainable by methods similar to those above, and is stated here without proof:

$$R_F(x,y,z) = z^{-1/2}\log(4z^{1/2}/(x^{1/2}+y^{1/2}))[1+(x+y)/(4z)]$$
$$+ z^{-3/2}[(xy)^{1/2} - x - y]/4 + r, \qquad (3.19)$$

where r is $O(z^{-5/2}\log z)$. A simple bound for the second-order error is less easy to find because of the complexity of the integrand in the remainder term. The complete asymptotic series for R_F appears to be very complicated, and the complexity lies in the evaluation of $c_k(1/2)$ in (2.16) for $k>0$. The logarithmic term in (2.16), however, is tractable as n approaches infinity. By using [4, (6.1-1), (6.1-4)], it can be shown to converge to

$$z^{-1/2}\log(z^{1/2})\, R_K(1-x/z,1-y/z), \quad \text{for} \quad 0 \le x,y < z.$$

We now prove the asymptotic formulas given for R_D. The proofs are simpler than that for R_F since they build on the results for R_F.

<u>Theorem 3.</u> If $x, y \ge 0$, and $x+y, z > 0$, then

$$R_D(x,y,z) = 3z^{-3/2}[\log(4z^{1/2}/(x^{1/2}+y^{1/2})) - 1] + r, \qquad (3.20)$$

where the error term, r, is bounded by

$$0 < r < \frac{9}{8} z^{-5/2}(x+y)\{\log[1 + 4z/(x+y)] + 1\}. \qquad (3.21)$$

<u>Proof.</u> By (1.3), R_D can be obtained through differentiation of R_F with respect to z. Hence, we obtain the first-order term and remainder by applying $-6\,\partial/\partial z$ to the expressions for R_F and r, (3.1) and (3.9):

$$R_D(x,y,z) = -3z^{-3/2}\log[4z^{1/2}/(x^{1/2}+y^{1/2})] - 3z^{-3/2}$$
$$- 3\int_0^\infty [t^{-1/2}-f(t)](\partial/\partial z)[(zt)^{-1/2}-h(zt)]\,dt \qquad (3.22)$$

To justify the differentiation of (3.9) under the integral sign, we need to majorize the integrand of (3.22) for z in any closed interval $[K,L]$, $0 < K < L$, by a function which is integrable and does not depend on z; see [4, Appendix B.3]. This will be done below in the process of bounding the integral.

To bound the error term, we need to bound

$$-3\frac{\partial}{\partial z}[(zt)^{-1/2}-(1+zt)^{-1/2}] = \frac{3}{2}t^{-1/2}z^{-3/2}\{1-(1+1/(zt))^{-3/2}\}. \qquad (3.23)$$

We rewrite the quantity in curly brackets above by substituting $w = 1/(1+zt)$:

$$1-(1+1/zt)^{-3/2} = 1 - [(1+zt)/((1+zt)-1)]^{-3/2}$$
$$= 1 - (1-w)^{3/2}, \text{ where } 0 < w < 1.$$

This function has the same value and derivative as $(3/2)w$ at $w = 0$, but is concave down since its second derivative is negative for $0 < w < 1$. Hence, $1 - (1-w)^{3/2} < (3/2)w$, or

$$1-(1+1/zt)^{-3/2} < (3/2)/(1+zt).$$

The above inequality can be used to bound (3.23) for $0 < K \le z \le L$ by $(3/2)t^{-1/2}K^{-3/2}(3/2)/(1+Kt)$. This provides an integrable majorizing function for the integrand of (3.22) which is independent of z, as required to justify differentiation under the integral sign. The inequality is specialized for large and small values of zt;

$$1 - (1 + 1/zt)^{-3/2} < (3/2)/(1+zt) \quad \forall\ t>0, \text{ close for small } t,$$

$$1 - (1 + 1/zt)^{-3/2} < (3/2)/zt \quad \forall\ t>0, \text{ close for large } t. \tag{3.24}$$

By combining inequalities (3.13) from the previous proof and (3.24),

$$0 < r < \frac{9}{4}\int_0^s ((x+y)/2)z^{-3/2}/(1+zt)\,dt + \frac{9}{4}\int_s^\infty t^{-2}z^{-5/2}\,dt$$

$$= \frac{9}{4}z^{-5/2}[((x+y)/2)\log(1+zs) + 1/s] \tag{3.25}$$

The quantity in square brackets in (3.25) is the same as that in (3.15), so for large z we approximate s_{min} by $2/(x+y)$ as before. Using this value of s in (3.25) gives the error bound stated in Theorem 3. ∎

The relative error here is $O(1/z)$. By applying (1.10b), the complete case of Theorem 3 is seen to be

$$R_Q(y,z) = \frac{2}{\pi} z^{-3/2}[\log(16z/y)-2] + r, \tag{3.26}$$

where

$$0 < r < \frac{3}{2\pi} z^{-5/2} y[\log(1+4z/y)+1] \tag{3.27}$$

By applying the $-6\,\partial/\partial z$ operator to (3.20), we can obtain an approximation for $R_D(x,y,z)$ with an error which is $O(z^{-7/2}\log z)$:

$$R_D(x,y,z) \sim 3z^{-3/2}[\log(4z^{1/2}/(x^{1/2}+y^{1/2})) - 1]$$

$$+\ 3z^{-5/2}[3((x+y)/4)\log(4z^{1/2}/(x^{1/2}+y^{1/2}))-(x+y)+3(xy)^{1/2}/4] \tag{3.28}$$

By making use of the permutation symmetry of the arguments of $R_F(x,y,z)$, the asymptotic behavior of $R_D(x,y,z)$ as $y \to \infty$ can be derived by a method similar to that used to obtain Theorem 3.

Theorem 4. If x, y, and z are positive, then

$$R_D(x,y,z) = 3y^{-1/2}/[(xz)^{1/2} + z] + r, \tag{3.29}$$

where

$$0 < -r < \frac{3}{4} y^{-3/2}\{\log[1 + 2yx^{-1/4}z^{-3/4}] + 1/2\}. \tag{3.30}$$

Proof. Since R_F is symmetric in x, y, and z, we can permute the arguments in Theorem 2 and write

$$R_F(x,y,z) = y^{-1/2} \log[4y^{1/2}/(x^{1/2}+z^{1/2})] + r, \tag{3.31}$$

where

$$r = \frac{1}{2} y^{-1/2}\int_0^\infty (1/t)[1-(1+xt)^{-1/2}(1+zt)^{-1/2}][1-(1+1/(yt))^{-1/2}]dt. \tag{3.32}$$

The first term, (3.29), is obtained by applying $-6\ \partial/\partial z$ to equation (3.31). Applying it to the error term as well yields

$$-r = \frac{3}{2} y^{-1/2}\int_0^\infty (1+xt)^{-1/2}(1+zt)^{-3/2}[1-(1+1/(yt))^{-1/2}]dt, \tag{3.33}$$

where r now represents the R_D error. To justify the differentiation under the integral sign, observe that for z in any closed interval $[K,L]$, $0 < K < L$, $(1+zt)^{-3/2}$ can be majorized by $(1+Kt)^{-3/2}$ in (3.33) to yield an integrand which is integrable and independent of z; see [4, Appendix B.3]. The factor $[1-(1+1/(yt))^{-1/2}]$ is bounded as it was in the proof of Theorem 2:

$$\begin{aligned} &1 - (1 + 1/(yt))^{-1/2} < 1/(1+2yt) \quad \forall\ t>0, \text{ close for small } t, \\ &1 - (1 + 1/(yt))^{-1/2} < 1/(2yt) \quad \forall\ t>0, \text{ close for large } t. \end{aligned} \tag{3.34}$$

In bounding the other term, we rely on the fact that $x > 0$, which means that the following bound is not useful for the complete case R_Q:

$$(1+xt)^{-1/2}(1+zt)^{-3/2} < x^{-1/2}z^{-3/2}t^{-2} \quad \forall\ t>0, \text{ close for large } t,$$

$$(1+xt)^{-1/2}(1+zt)^{-3/2} < 1 \qquad \forall\ t>0, \text{ close for small } t. \tag{3.35}$$

Combining these inequalities in (3.32) gives

$$0 < -r < \frac{3}{2}y^{-1/2}\int_0^s 1/(1+2yt)\,dt + \frac{3}{4}y^{-3/2}x^{-1/2}z^{-3/2}\int_s^\infty t^{-3}\,dt$$

$$= \frac{3}{4}y^{-3/2}[\log(1+2ys) + \frac{1}{2}x^{-1/2}z^{-3/2}s^{-2}]. \tag{3.36}$$

The quantity in square brackets in (3.36) can be minimized by differentiation with respect to s:

$$2y/(1+2ys) - x^{-1/2}z^{-3/2}s^{-3} = 0. \tag{3.37}$$

As $y \to \infty$, $s_{min} \sim x^{-1/4}z^{-3/4}$, which when applied to (3.36) yields the bound (3.30). ∎

As $x \to 0$, the bound in (3.30) approaches infinity, and hence a slightly different approach is needed for the complete case.

<u>Theorem 5.</u> If y and z are positive, then

$$R_Q(y,z) = \frac{4}{\pi}y^{-1/2}z^{-1} + r, \tag{3.38}$$

where

$$0 < -r < \frac{1}{\pi} y^{-3/2}[\log(1 + 2y/z) + 2/3]. \qquad (3.39)$$

<u>Proof.</u> Equation (3.38) is easily obtained from (1.10b) and (3.29). To obtain a bound, apply (1.10b) and (3.33) with $x = 0$:

$$r = \frac{-2}{\pi} y^{-1/2} \int_0^\infty (1+zt)^{-3/2}[1-(1+1/yt)^{-1/2}]dt. \qquad (3.40)$$

Here we use a slightly different bound on the first factor:

$$\begin{aligned} &(1+zt)^{-3/2} < z^{-3/2}t^{-3/2} && \forall\ t>0, \text{ close for large } t, \\ &(1+zt)^{-3/2} < 1 && \forall\ t>0, \text{ close for small } t. \end{aligned} \qquad (3.41)$$

So the bound on r here takes the form

$$\begin{aligned} -r &< \frac{2}{\pi} y^{-1/2}\left[\int_0^s 1/(1+2yt)\,dt + \int_s^\infty 1/(2yz^{3/2}t^{5/2})\,dt\right] \\ &= \frac{1}{\pi} y^{-3/2}[\log(1+2ys) + \frac{2}{3} z^{-3/2}s^{-3/2}]. \end{aligned} \qquad (3.42)$$

As $y \to \infty$, $s_{min} \sim 1/z$, which gives the bound stated in (3.39). ∎

The relative error here is $O(y^{-1/2}\log y)$, so the approximation is less accurate for the complete case than for the incomplete case.

We now turn our attention to elliptic integrals of the third kind, R_J. Since R_J is symmetric in x, y, and z, we need only treat the cases $p \to \infty$ and $z \to \infty$.

<u>Theorem 6.</u> If $p > 0$ and $x, y, z \geq 0$ with at most one of x, y, z equal to 0, then

$$R_J(x,y,z,p) = 3\, p^{-1}\, R_F(x,y,z) - \frac{3}{2}\pi\, p^{-3/2} + r, \qquad (3.43)$$

where

$$0 < r < 3(2^{1/2})\, p^{-2}(x + y + z)^{1/2}. \qquad (3.44)$$

<u>Proof.</u> By using the substitution $s = 1/t$ in (1.2c),

$$R_J(x,y,z,p) = \frac{3}{2}\int_0^\infty (x+t)^{-1/2}(y+t)^{-1/2}(z+t)^{-1/2}(p+t)^{-1}dt$$

$$= \frac{3}{2}\, p^{-1/2} \int_0^\infty [(1+xs)(1+ys)(1+zs)]^{-1/2}(ps)^{1/2}/(1+ps)\, ds. \qquad (3.45)$$

This is of the form $\frac{3}{2}p^{-1/2}\int_0^\infty f(t)h(pt)dt$, where

$$f(t) = [(1+xt)(1+yt)(1+zt)]^{-1/2} \sim 1 \quad \text{as} \quad t \to 0^+ \qquad (3.46)$$

and

$$h(t) = t^{1/2}/(1+t) \sim t^{-1/2} \quad \text{as} \quad t \to +\infty. \qquad (3.47)$$

So the first case of Theorem 1 applies, with $u = 1$ and $v = 1/2$. Since $f(t) = O(t^{-3/2})$ as $t \to +\infty$ and $h(t) = O(t^{1/2})$ as $t \to 0^+$, conditions (2.5) and (2.10) are satisfied. By (2.14),

$$R_J(x,y,z,p) = \frac{3}{2}\, p^{-1/2} \{M[h;1]p^{-1} + M[f;1/2]p^{-1/2}\} + r, \quad (3.48)$$

where by (3.3),

$$M[f;1/2] = \int_0^\infty t^{-1/2}[(1+xt)(1+yt)(1+zt)]^{-1/2}dt = 2\, R_F(x,y,z).$$

Using the beta function,

$$M[h;1] = \int_0^\infty (1+t)^{-1}\, t^{1/2}\, dt = B(3/2,-1/2) = -\pi,$$

we arrive at the first asymptotic term, equation (3.43). The error term, r, is by (2.17) equal to

$$\frac{3}{2}\int_0^\infty \{1 - [(1+xt)(1+yt)(1+zt)]^{-1/2}\}t^{1/2}\{1/(pt) - 1/(1+pt)\}dt. \quad (3.49)$$

where we bound each factor in curly brackets as follows:

$$1/pt - 1/(1+pt) = 1/pt(1+pt) < 1/p^2t^2 \;\forall\; t > 0, \quad (3.50)$$

and

$$1-[(1+xt)(1+yt)(1+zt)]^{-1/2} < (x+y+z)t/2 \;\forall\; t > 0, \text{ close for small } t, \quad (3.51a)$$

$$1-[(1+xt)(1+yt)(1+zt)]^{-1/2} < 1 \qquad \forall\; t > 0, \text{ close for large } t. \quad (3.51b)$$

It might at first appear that we should specialize the inequality in (3.50) by bounding $1/pt(1+pt)$ by $1/pt$ for small t; however, this turns out to yield an error bound which is only $O(p^{-7/4})$. Inequality (3.51a) follows from the following general inequality for $b, x_i > 0$:

$$\prod_{i=1}^{n} (1+x_i)^{-b} = e^{-b \sum_{i=1}^{n} \log(1+x_i)} > 1 - b \sum_{i=1}^{n} \log(1+x_i) > 1 - b \sum_{i=1}^{n} x_i. \quad (3.52)$$

Applying these bounds to (3.49) gives

$$0 < r < \frac{3}{4}(x+y+z)p^{-2} \int_0^s t^{-1/2} dt + \frac{3}{2} p^{-2} \int_s^\infty t^{-3/2} dt$$

$$= \frac{3}{2} p^{-2} [(x+y+z)s^{1/2} + 2s^{-1/2}]. \quad (3.53)$$

The minimum value of the quantity in square brackets occurs exactly when $s = 2/(x+y+z)$, resulting in the bound stated in (3.44). ∎

The complete case, $R_M(x,y,p)$, may be easily obtained as a corollary to Theorem 6 by letting $z = 0$ and multiplying the formula and the error bound by $4/3\pi$ (see (1.10c).

The first term of Theorem 6 is in a sense not a convenient asymptotic approximation because it involves the nonelementary function R_F. However, the variable p is involved only as a polynomial in $p^{-1/2}$. In using Theorem 1 to find higher-order approximations to R_J, one finds that the coefficients of that polynomial are nonelementary R functions. Of the five formulas for large arguments, R_J as $p \to \infty$ has the most complicated approximation but the simplest error bound. The approximation for R_J as $z \to \infty$ involves only elementary functions.

Theorem 7. If x, y, z, and p are positive, then

$$R_J(x,y,z,p) = 3\ z^{-1/2}\ R_C(X^2,Y^2) + r, \qquad (3.54)$$

where

$$X = (xy)^{1/2} + p, \quad Y = (xp)^{1/2} + (yp)^{1/2},$$

and

$$-\frac{3}{4} z^{-3/2}\{\log[1+2z(xy)^{-1/4}p^{-1/2}] + 1/2\} < r < 0. \qquad (3.55)$$

Proof. As was done in the proof of Theorem 6, we may substitute $s = 1/t$ in (1.2c) to obtain

$$R_J(x,y,z,p) = \frac{3}{2}\int_0^\infty \{t^{1/2}[(1+xt)(1+yt)]^{-1/2}(1+pt)^{-1}\}(1+zt)^{-1/2}\, dt. \quad (3.56)$$

Rather than use Theorem 1, simply rewrite (3.56) using $(1+zt)^{-1/2} = (zt)^{-1/2} + [(1+zt)^{-1/2} - (zt)^{-1/2}]$:

$$R_J(x,y,z,p) = \frac{3}{2}z^{-1/2}\int_0^\infty [(1+xt)(1+yt)]^{-1/2}(1+pt)^{-1}dt$$
$$+ \frac{3}{2}\int_0^\infty t^{1/2}[(1+xt)(1+yt)]^{-1/2}(1+pt)^{-1}[(1+zt)^{-1/2}-(zt)^{-1/2}]dt. \quad (3.57)$$

The first integral is elementary, being expressible as $3z^{-1/2}R_C(X^2,Y^2)$, where R_C is defined by equation (1.11) and X, Y are defined above. A derivation of this may be found in [4, pp. 290, 314]. The second integral is the error term, for which we seek a bound. Bounding each factor, we find

$$t^{1/2}[(1+xt)(1+yt)]^{-1/2}(1+pt)^{-1} < t^{1/2} \qquad \forall\ t>0,\ \text{close for small } t,$$
(3.58)
$$t^{1/2}[(1+xt)(1+yt)]^{-1/2}(1+pt)^{-1} < t^{-3/2}(xy)^{-1/2}p^{-1} \quad \forall\ t>0,\ \text{close for large } t,$$

and

$$(zt)^{-1/2} - (1+zt)^{-1/2} < (zt)^{-1/2}/(1+2zt) \qquad \forall\ t>0,\ \text{close for small } t,$$
(3.59)
$$(zt)^{-1/2} - (1+zt)^{-1/2} < (zt)^{-3/2}/2 \qquad \forall\ t>0,\ \text{close for large } t.$$

The latter inequality follows from multiplying (3.12) by $(zt)^{-1/2}$. Applying these inequalities to the error term, for any $s > 0$, gives

$$0 < -r < \frac{3}{2}z^{-1/2}\int_0^s 1/(1+2zt)\,dt + \frac{3}{4}z^{-3/2}(xy)^{-1/2}p^{-1}\int_s^\infty t^{-3}\,dt,$$
$$= \frac{3}{4}\,z^{-3/2}\,[\log(1+2zs) + \frac{1}{2}\,(xy)^{-1/2}p^{-1}s^{-2}]. \qquad (3.60)$$

We can minimize the quantity in square brackets by differentiation:

$$2z/(1+2zs) - (xy)^{-1/2}p^{-1}s^{-3} = 0, \qquad (3.61)$$

which implies that as $z \to \infty$, $s_{min} \to (xy)^{-1/4}p^{-1/2}$. Using this value in (3.60) gives the inequality (3.61). ∎

The asymptotic formula and bound given for $R_D(x,y,z)$ as $y \to \infty$ are special cases of the preceding theorem, in accordance with (1.13). Hence, this gives an independent proof of Theorem 4.

In the complete case, $y = 0$, the error bound of Theorem 7 is not useful. An error bound for $R_M(x,z,p)$ may be found by a slight modification of inequality (3.58).

Theorem 8. If $x, z \geq 0$ and $x+z, p > 0$, then

$$R_M(x,z,p) = \frac{4}{\pi} z^{-1/2} p^{-1/2} R_C(p,x) + r, \qquad (3.62)$$

where

$$0 < -r < \frac{1}{\pi} z^{-3/2} [\log(1 + 2zx^{-1/3} p^{-2/3}) + 2/3]. \qquad (3.63)$$

Proof. The first term on the right side of (3.62) is obtained by setting $y = 0$ in (3.54) and then using (1.10c) and homogeneity to bring the $p^{-1/2}$ outside the R_C function. To bound the error term, we use

$$\begin{aligned} t^{1/2}(1+xt)^{-1/2}(1+pt)^{-1} &< t^{1/2} \quad \forall t>0, \text{ close for small } t; \\ t^{1/2}(1+xt)^{-1/2}(1+pt)^{-1} &< t^{-1} x^{-1/2} p^{-1} \quad \forall\, t>0, \text{ close for large } t. \end{aligned} \qquad (3.64)$$

Inequality (3.59) is still useful in this case; by combining it with (3.64), we obtain a variation on (3.60):

$$0 < -r < \frac{2}{\pi} z^{-1/2} \int_0^s 1/(1+2zt)\, dt + \frac{1}{\pi} z^{-3/2} x^{-1/2} p^{-1} \int_s^{\infty} t^{-5/2}\, dt,$$

$$= \frac{1}{\pi} z^{-3/2} [\log(1+2zs) + \frac{2}{3} x^{-1/2} p^{-1} s^{-3/2}], \qquad (3.65)$$

We can minimize the quantity in square brackets by differentiation:

$$2z/(1+2zs) - x^{-1/2} p^{-1} s^{-5/2} = 0, \qquad (3.66)$$

which implies that as $z \to \infty$, $s_{min} \to x^{-1/3} p^{-2/3}$. Using this value in (3.65) gives the bound on the error, (3.63). ∎

This completes the proofs of formulas for large arguments.

PROOFS OF THE FORMULAS FOR SMALL ARGUMENTS

The formulas for arguments tending to zero are of a different character from those for arguments tending to infinity. In four of the five incomplete cases, setting an argument equal to zero gives a finite value which can be used as an approximation. In three of the cases, that first term is simply the complete version of the elliptic integral. In addition to Theorem 1, a variety of approaches is used to obtain approximations and error bounds.

Theorem 9. If $x, y > 0$ and $z \geq 0$, then

$$R_F(x,y,z) = \frac{\pi}{2} R_K(x,y) + r, \tag{4.1}$$

where

$$0 < -r < \pi\, z^{1/2} 2^{-3/2} (xy)^{-1/2}. \tag{4.2}$$

Proof. By (1.10a), the first term is the complete case, $R_F(x,y,0)$. By using the alternative representation (3.3), the error term can be expressed as

$$-r = \frac{1}{2}\int_0^\infty t^{-1/2}[(1+xt)(1+yt)]^{-1/2}\{1 - (1+zt)^{-1/2}\}. \tag{4.3}$$

We bound $t^{-1/2}[(1+xt)(1+yt)]^{-1/2}$ by $(xy)^{-1/2}t^{-3/2}$ for all $t>0$. Using inequality (3.11), we bound $1 - (1+zt)^{-1/2}$ by $zt/(2+zt)$ for all $t>0$.

Combining these yields an elementary integral as a bound:

$$0 < -r < \frac{1}{2} z(xy)^{-1/2} \int_0^\infty t^{-1/2}(2+zt)^{-1}\, dt$$

$$= (xy)^{-1/2} R_C(0,2/z)$$

$$= \pi z^{1/2} 2^{-3/2} (xy)^{-1/2}. \blacksquare$$

The asymptotic formula for the complete case $R_K(x,y)$ as $y \to 0$ is the same as the formulas for $R_K(x,y)$ as $x \to \infty$, (3.17) and (3.18). This follows from homogeneity, since $R_K(x,y) = y^{-1/2} R_K(x/y,1)$. Hence, by letting $z = x$ in (3.17) and (3.18), we obtain a formula and an error bound which are $O(\log y)$ and $O(y \log y)$, respectively:

$$R_K(x,y) = \frac{1}{\pi} x^{-1/2} \log(16x/y) + r, \tag{4.4}$$

where

$$0 < r < \frac{1}{4\pi} x^{-3/2} y[\log(1+4x/y)+1]. \tag{4.5}$$

For the same reason, the asymptotic formula and bound for $R_Q(x,y)$ as $y \to 0$ are the same as those for $x \to \infty$, derived in Theorem 5. The formula is $O(y^{-1/2})$ and the bound is $O(\log y)$. The behavior of $R_Q(x,y)$ as $x \to 0$ is described by the formulas for $y \to \infty$, (3.26) and (3.27); the formula is $O(\log x)$, and the bound is $O(x \log x)$. The first three cases shown in Table 4, therefore, are equivalent to the first three cases in Table 3.

In the formula for $R_D(x,y,z)$ as $z \to 0$, we require that z be the smallest of the three arguments; this allows considerable simplification of the error bound.

Theorem 10. If x, y, and z are positive, and $z = \min(x,y,z)$, then

$$R_D(x,y,z) = 3(xyz)^{-1/2} + r, \tag{4.6}$$

where

$$0 < -r < 6\,(x+y)^{1/2}/xy. \tag{4.7}$$

Proof. Let $g(z) = z^{1/2}R_D(x,y,z)$. By homogeneity (1.5),

$$g(z) = R_D(x/z,y/z,1)/z = \frac{3}{2}\int_0^\infty (x+zt)^{-1/2}(y+zt)^{-1/2}(1+t)^{-3/2}dt. \tag{4.8}$$

Hence, $g(0) = 3(xy)^{-1/2}$, and this gives the term $3(xyz)^{-1/2}$ in (4.6). The error term can be bounded by using

$$z^{1/2}r = g(z)-g(0) = \frac{3}{2}(xy)^{-1/2}\int_0^\infty \{[(1+zt/x)(1+zt/y)]^{-1/2}-1\}(1+t)^{-3/2}dt. \tag{4.9}$$

We can bound the quantity in curly brackets in (4.9) by the method used in Theorem 2:

$$1 - [(1+zt/x)(1+zt/y)]^{-1/2} < \quad 1 \qquad \forall\ t>0, \text{ close for large } t,$$
(4.10)
$$1 - [(1+zt/x)(1+zt/y)]^{-1/2} < \quad Ct/2 \qquad \forall\ t>0, \text{ close for small } t,$$

where $C = z/x+z/y$. So for any positive s,

$$0 < -z^{1/2}r < \frac{3}{2}(xy)^{-1/2}[\ C\int_0^s (t/2)(1+t)^{-3/2}dt + \int_s^\infty (1+t)^{-3/2}dt\ ]$$
$$= \frac{3}{2}(xy)^{-1/2}[\ C((1+s)^{1/2}+(1+s)^{-1/2}) + 2(1+s)^{-1/2}], \qquad (4.11)$$

If we let $v = (1+s)^{-1/2}$ and minimize (4.11) with respect to v, we obtain $v^2 = C/(2+C)$. Substituting this value of v in the quantity in square brackets in (4.11) gives

$$[...] = 2\,C^{1/2}(2+C)^{1/2} < 4\,C^{1/2},$$

where the latter inequality follows from the assumption that $z = \min(x,y,z)$, which implies $C < 2$. The bound on r, therefore, is

$$0 < -r < \frac{3}{2}(xy)^{-1/2}4[(1/x+1/y)]^{1/2}$$
$$= 6\,(x+y)^{1/2}/xy. \ \blacksquare$$

Hence, the bound in this case is actually independent of the argument tending to zero. But since the approximation is $O(z^{-1/2})$, the relative error is $O(z^{1/2})$. A similar relative error is obtained for the asymptotic approximation of $R_D(x,y,z)$ as y tends to zero; that approximation is found below as a special case of $R_J(x,y,z,p)$ as z tends to zero.

The behavior of $R_J(x,y,z,p)$ as p tends to zero requires a different approach. If x, y, and z are all nonzero, then $R_J(x,y,z,p)$ has a logarithmic singularity at $p = 0$; specifically, it is shown below that $R_J(x,y,z,p) \sim -(3/2)(xyz)^{-1/2}\log p$ as $p \to 0$. However, the integral which defines the error of this approximation is very difficult to bound. Instead we make use of the Duplication Theorem, trading a more complicated approximation for a simpler error bound.

<u>Theorem 11.</u> If $p > 0$ and $x, y, z, \geq 0$ with at most one of x, y, z equal to 0, then

$$R_J(x,y,z,p) = 3R_C(\alpha,\beta) + 2R_J(x+\lambda,y+\lambda,z+\lambda,\lambda) + r, \qquad (4.12)$$

where

$$\lambda = (xy)^{1/2}+(xz)^{1/2}+(yz)^{1/2},$$

$$\alpha = [p(x^{1/2}+y^{1/2}+z^{1/2})+(xyz)^{1/2}]^2,$$

$$\beta = p(p+\lambda)^2,$$

and the error term, r, is bounded by

$$0 < -r < \frac{6}{5} p\, \lambda^{-5/2}. \qquad (4.13)$$

<u>Proof.</u> References [13] and [14] give proofs of the Duplication Theorem in the notation used here; it says that

$$R_J(x,y,z,\beta) = 3R_C(\alpha,p) + 2R_J(x+\lambda,y+\lambda,z+\lambda,p+\lambda) \qquad (4.14)$$

where λ, α, and β are defined above. If we let $p = 0$ in the last term in (4.14), we obtain the first term of the approximation in (4.12). Observe that this meets our requirement that the approximation be an elementary function of p, since λ does not depend on p, and R_C is an elementary function. The error term, then, is

$$\begin{aligned} r &= 2R_J(x+\lambda,y+\lambda,z+\lambda,p+\lambda) - 2R_J(x+\lambda,y+\lambda,z+\lambda,\lambda) \\ &= 3\int_0^\infty [(x+\lambda+t)(y+\lambda+t)(z+\lambda+t)]^{-1/2}[1/(p+\lambda+t) - 1/(\lambda+t)]\,dt \\ &= -3p\int_0^\infty [(x+\lambda+t)(y+\lambda+t)(z+\lambda+t)]^{-1/2}/[(p+\lambda+t)(\lambda+t)]\,dt. \qquad (4.15) \end{aligned}$$

The integrand in (4.15) is positive and is easily bounded above by dropping x, y, z, and p:

$$0 < -r < 3p\int_0^\infty (\lambda+t)^{-7/2}dt = \tfrac{6}{5}p\,\lambda^{-5/2},$$

This is the error bound claimed in (4.13). ∎

If xyz is nonzero, then as p tends to zero, α tends to xyz, and β tends to $p\lambda^2$; it is easy to show from (1.11) that the R_C term thus agrees with the $-(3/2)(xyz)^{-1/2}\log p$ term mentioned earlier, and that the difference is $O(1)$. Since the R_J function on the right hand side of (4.12) is also $O(1)$ as $p \to 0$,

$$R_J(x,y,z,p) = -\frac{3}{2}(xyz)^{-1/2}\log p + O(1), \qquad (4.16)$$

which may be more convenient to use than Theorem 11 in situations where an error bound is not required.

If one of x, y, z is zero, then we have the complete case, R_M. Theorem 11 yields a result for the complete case R_M if we let $z = 0$ and use (1.10c):

$$R_M(x,y,p) = \frac{4}{\pi}R_C(\alpha,\beta) + \frac{8}{3\pi}R_D(x+\lambda,y+\lambda,\lambda) + r \qquad (4.17)$$

where

$$\lambda = (xy)^{1/2},$$

$$\alpha = [p(x^{1/2}+y^{1/2})]^2,$$

$$\beta = p(p+\lambda)^2;$$

and the error term, r, is bounded by

$$0 < -r < \frac{8}{5\pi} p \lambda^{-5/2}. \qquad (4.18)$$

If $p = \min(x,y,p)$, then the R_C function in (4.17) is an arccosine (see (1.11)) and can be written as

$$R_C(\alpha,\beta) = [p(x-p)(y-p)]^{-1/2} \arccos\{[(px)^{1/2}+(py)^{1/2}]/[p+(xy)^{1/2}]\}. \qquad (4.19)$$

Therefore, the complete case of (4.16) is

$$R_M(x,y,p) = 2(pxy)^{-1/2} + O(1), \quad p \to 0. \qquad (4.20)$$

We now derive the behavior of $R_J(x,y,z,p)$ as $z \to 0$.

Theorem 12. If $x, y, p > 0$ and $z \geq 0$, then

$$R_J(x,y,z,p) = \frac{3\pi}{4} R_M(x,y,p) + r, \qquad (4.21)$$

where

$$0 < -r < 3(2^{-3/2}) \pi z^{1/2} (xy)^{-1/2} p^{-1}. \qquad (4.22)$$

Proof. The R_M term follows from (1.10c), with corresponding error

$$- r = R_J(x,y,0,p) - R_J(x,y,z,p)$$

$$= \frac{3}{2}\int_0^\infty \{t^{1/2}[(1+xt)(1+yt)]^{-1/2}(1+pt)^{-1}\}\{1 - (1+zt)^{-1/2}\}\, dt. \quad (4.23)$$

The first factor in brackets can be bounded by $(xy)^{-1/2}p^{-1}t^{-3/2}$, and the second factor can be bounded by $zt/(2+zt)$, as was done in the proof of Theorem 9. Also as in Theorem 9, the bound involves $R_C(0,2/z)$:

$$0 < -r < \frac{3}{2}(xy)^{-1/2}p^{-1}\int_0^\infty t^{-1/2}(t+2/z)^{-1}dt$$

$$= \frac{3}{2}(xy)^{-1/2}p^{-1}\,\pi\,(z/2)^{1/2}.$$

This is the bound claimed in (4.22). ∎

Since $R_D(x,y,z) = R_J(x,y,z,z)$ and $R_Q(x,z) = R_M(x,z,z)$, Theorem 12 has the following corollary:

$$R_D(x,y,z) = \frac{3\pi}{4} R_Q(x,z) + r, \quad (4.24)$$

where

$$0 < -r < 3\,\pi\,(2z)^{-3/2}x^{-1/2}y^{1/2}. \quad (4.25)$$

Finally, we show the behavior of $R_M(x,y,p)$ as $y \to 0$. It does not appear that the following theorem is a special case of any previous theorem, and its proof provides an illustration of how Wong's method can be used for small arguments as well as large arguments.

<u>Theorem 13.</u> If x, y, and p are positive, then

$$R_M(x,y,p) = \frac{2}{\pi} p^{-1} [x^{-1/2}\log(16x/y) - 2R_C(x,p)] + r, \quad (4.26)$$

where

$$0 < r < \frac{1}{2\pi}\, y\, x^{-3/2} p^{-2}(p+2x)[\log(1+4xp/(y(2x+p))) + 1]. \quad (4.27)$$

<u>Proof.</u> Let $w = 1/y$. Then as $y \to 0^+$, $w \to +\infty$, and we can apply Theorem 1 to $R_M(x,1/w,p)$:

$$R_M(x,1/w,p) = \frac{2}{\pi}\int_0^\infty [t(x+t)(1/w+t)]^{-1/2}(p+t)^{-1}\, dt$$
$$= \frac{2}{\pi} w^{1/2} \int_0^\infty \{[t(x+t)]^{-1/2}(p+t)^{-1}\}\{(1+wt)^{-1/2}\}\, dt. \quad (4.28)$$

The quantities in curly brackets in (4.28) are of the form required of $f(t)$ and $h(wt)$ in Theorem 1. Here, $h(t) \sim t^{-1/2}$ as $t \to +\infty$, and $f(t) \sim x^{-1/2}p^{-1}t^{-1/2}$ as $t \to 0^+$. So $u = v = 1/2$, and the second case of Theorem 1 applies:

$$R_M(x,1/w,p) = \frac{2}{\pi} [c_0(1/2) + x^{-1/2}p^{-1}\log w + w^{1/2}\delta_1(w)], \quad (4.29)$$

To find $c_0(1/2)$, terms with similar coefficients can be combined, as was done in the proof of Theorem 2. The quantity $a_0\rho_1 + b_0\psi_1$ has the same functional form for all $t > 0$, and we obtain

$$c_0(1/2) = x^{-1/2}p^{-1}\int_0^{\infty}\{(1+t/x)^{-1/2}t^{-1}(1+t/p)^{-1} + [t(1+t)]^{-1/2} - 1/t\}\, dt.$$

In the first term of the integrand, we write

$$t^{-1}(1+t/p)^{-1} = t^{-1} - p^{-1}(1+t/p)^{-1}.$$

It follows from (1.11) that

$$c_0(1/2) = x^{-1/2}p^{-1}\, J - 2p^{-1}R_C(x,p),$$

where

$$J = \int_0^{\infty}\{t^{-1}(1+t/x)^{-1/2} + [t(1+t)]^{-1/2} - 1/t\}\, dt. \tag{4.30}$$

From Gradshteyn and Ryzhik, [6, 2.224 and 2.261], we find

$$\begin{aligned} J &= \lim_{\substack{b\to\infty \\ \epsilon\to 0}} \{\log[((1+t/x)^{-1/2}-1)/((1+t/x)^{-1/2}+1)] \\ &\qquad + \log[t^{1/2}(1+t)^{1/2}+t+1/2] - \log t\}\Big|_{\epsilon}^{b} \\ &= \log(16x). \end{aligned}$$

Therefore, since $w = 1/y$,

$$c_0(1/2) + x^{-1/2}p^{-1}\log w = -2p^{-1}R_C(x,p) + x^{-1/2}p^{-1}\log(16x/y).$$

This is the approximation given in (4.25).

The last term in (4.29) is the error term, r; by Theorem 1,

$$r = \frac{2}{\pi} x^{-1/2} p^{-1} \int_0^\infty \{1-(1+t/x)^{-1/2}(1+t/p)^{-1}\}(1/t)\{1-(1+1/wt)^{-1/2}\}dt. \quad (4.31)$$

By inequality (3.52), the first term in curly brackets is bounded by $(1/2x + 1/p)t$ for all $t>0$. With this bound, the rest of the proof is similar to that for Theorem 2: in equations (3.11) - (3.16), replace z by w and $(x+y)/2$ by $1/2x + 1/p$. ∎

By using homogeneity, Theorems 2 - 13 can be used to derive asymptotic behavior when more than one variable tends to zero or infinity. For example, $R_F(x,y,z) = z^{-1/2} R_F(x/z,y/z,1)$, so the case of both x and y tending to zero with x/y fixed is essentially the same as z tends to infinity, which is the case described by Theorem 2. Many additional formulas are therefore easily derived from the theorems here together with (1.5). Still other cases of asymptotic behavior admittedly are not considered here at all, such as when two arguments tend to zero at different rates.

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APPENDIX A:

NUMERICAL EXAMPLES

Behavior for Large Arguments

The following examples illustrate use of the asymptotic formulas, letting one argument grow large and leaving the others fixed. The error and the bound on the error are scaled by an appropriate function of the argument for clarity. Examples were generated on a TRS-80 Model III microcomputer using the subroutines in Appendix B.

Example 1. $R_F(x,y,z)$ as $z \to \infty$ (See equations (3.1) and (3.2)).

z	$R_F(0.5, 1, z)$	Asymp. Approx.	Error and Bound/$z^{-3/2}\log z$	
10	0.651849021414	0.633336830635	0.2542	0.3518
100	0.316397144546	0.315407945737	0.2150	0.2683
1000	0.136192629971	0.136147817066	0.2051	0.2413
10000	[illegible]	[illegible]	0.2007	0.2278
100000	0.020896267200	0.020896195107	0.1980	0.2198
1000000	0.007759252355	0.007759249643	0.1963	0.2144
10000000	0.002817760950	0.002817760851	0.1950	0.2105
100000000	0.001006183477	0.001006183474	0.1941	0.2077

z	$R_F(1, 100, z)$	Asymp. Approx.	Error and Bound/$z^{-3/2}\log z$	
10	0.230768363604	0.044174373610	2.5626	7.3123
100	0.147803766237	0.129098418132	4.0618	7.1319
1000	0.078528034442	0.077231571363	5.9350	8.5970
10000	0.036004039969	0.035935692743	7.4207	9.5733
100000	0.015007641418	0.015004570537	8.4348	10.1812
1000000	0.005896280502	0.005896154367	9.1299	10.5883
10000000	0.002228603301	0.002228598394	9.6289	10.8792
100000000	0.000819874130	0.000819873946	10.0033	11.0974

z	$R_F(0, 1, z)$	Asymp. Approx.	Error and	Bound/$z^{-3/2}\log z$
10	0.815264309590	0.802455438869	0.1759	0.2559
100	0.369563736299	0.368887945411	0.1467	0.1898
1000	0.153090053461	0.153059677889	0.1391	0.1682
10000	0.059915893405	0.059914645471	0.1355	0.1574
100000	0.022587429752	0.022587381189	0.1334	0.1509
1000000	0.008294051464	0.008294049640	0.1320	0.1466
10000000	0.002986879526	0.002986879459	0.1310	0.1435
100000000	0.001059663476	0.001059663473	0.1302	0.1412

The last of the three examples illustrates the approximation as applied to the complete elliptic integral R_K (see equation (1.10a)).

<u>Example 2.</u> $R_D(x,y,z)$ as $z \to \infty$ (See equations (3.20) and (3.21)).

z	$R_D(0.5, 1, z)$	Asymp. Approx.	Error and	Bound/$z^{-5/2}\log z$
10	1.08671998D-01	9.51327194D-02	1.8594	3.1662
100	6.54037985D-03	6.46223837D-03	1.6968	2.4147
1000	3.13942996D-04	3.13575121D-04	1.6841	2.1715
10000	1.33715445D-05	1.33699937D-05	1.6838	2.0504
100000	5.32023656D-07	5.32017523D-07	1.6845	1.9778
1000000	2.02777722D-08	2.02777489D-08	1.6850	1.9294
10000000	7.50460011D-10	7.50459925D-10	1.6853	1.8949
100000000	2.71855045D-11	2.71855042D-11	1.6856	1.8690

z	$R_D(1, 100, z)$	Asymp. Approx.	Error and	Bound/$z^{-5/2}\log z$
10	2.10703863D-02	-6.16160177D-02	14.1025	65.8107
100	2.08792585D-03	8.72952544D-04	26.3830	64.1874
1000	1.46377845D-04	1.36826384D-04	43.7254	77.3734
10000	7.83481979D-06	7.78070782D-06	58.7513	86.1598
100000	3.55521297D-07	3.55268786D-07	69.3574	91.6305
1000000	1.46895226D-08	1.46884631D-08	76.6890	95.2943
10000000	5.73715366D-10	5.73711193D-10	81.9683	97.9128
100000000	2.15962342D-11	2.15962184D-11	85.9181	99.8768

z	$R_D(0, 1, z)$	Asymp. Approx.	Error and Bound/$z^{-5/2}\log z$	
10	1.55301288D-01	1.45868302D-01	1.2955	2.3030
100	8.12013278D-03	8.06663836D-03	1.1616	1.7086
1000	3.64560452D-04	3.64310704D-04	1.1433	1.5137
10000	1.49754418D-05	1.49743936D-05	1.1380	1.4165
100000	5.82757239D-07	5.82753106D-07	1.1354	1.3582
1000000	2.18821646D-08	2.18821489D-08	1.1336	1.3193
10000000	8.01195566D-10	8.01195508D-10	1.1324	1.2916
100000000	2.87899044D-11	2.87899042D-11	1.1315	1.2707

The latter case represents the complete elliptic integral, $R_Q(y,z)$ (see equation (1.10b)).

Example 3. $R_D(x,y,z)$ as $y \to \infty$ (See equations (3.29) and (3.30)).

y	$R_D(0.5, y, 1)$	Asymp. Approx.	Error and Bound/$y^{-3/2}\log y$	
10	5.04685378D-01	5.55725810D-01	-0.7010	-1.2220
100	1.72589209D-01	1.75735931D-01	-0.6833	-0.9789
1000	5.54196522D-02	5.55725810D-02	-0.7001	-0.9021
10000	1.75670359D-02	1.75735931D-02	-0.7119	-0.8640
100000	5.55699615D-03	5.55725810D-03	-0.7195	-0.8412
1000000	1.75734930D-03	1.75735931D-03	-0.7246	-0.8260
10000000	5.55725438D-04	5.55725810D-04	-0.7282	-0.8152
100000000	1.75735918D-04	1.75735931D-04	-0.7309	-0.8070

Example 4. $R_J(x,y,z,p)$ as $p \to \infty$ (See equations (3.43) and (3.44)).

p	$R_J(0.5,1,100,p)$	Asymp. Approx.	Error and Bound / p^{-2}	
20	2.41399893D-02	-5.22638880D-03	11.7466	42.7434
200	3.58114815D-03	3.07989107D-03	20.0503	42.7434
2000	4.28461949D-04	4.21911106D-04	26.2034	42.7434
20000	4.58664149D-05	4.57936406D-05	29.1097	42.7434
200000	4.69404058D-06	4.69328606D-06	30.1807	42.7434
2000000	4.72938770D-07	4.72931136D-07	30.5375	42.7434
20000000	4.74071122D-08	4.74070356D-08	30.6522	42.7434
200000000	4.74430685D-09	4.74430609D-09	30.6887	42.7434

<u>Example 5.</u> $R_J(x,y,z,p)$ as $z \to \infty$ (See equations (3.54) and (3.55)).

z	$R_J(0.5,1,z,200)$	Asymp. Approx.	Error and	Bound/$z^{-3/2}\log z$
10	8.34460976D-03	1.33667951D-02	-0.0690	-0.4842
100	3.58114815D-03	4.22695174D-03	-0.1402	-0.5505
1000	1.28204386D-03	1.33667951D-03	-0.2501	-0.6114
10000	4.19439696D-04	4.22695174D-04	-0.3535	-0.6456
100000	1.33511662D-04	1.33667951D-04	-0.4293	-0.6664
1000000	4.22628545D-05	4.22695174D-05	-0.4823	-0.6804
10000000	1.33665298D-05	1.33667951D-05	-0.5205	-0.6903
100000000	4.22694162D-06	1.33667951D-06	-0.5492	-0.6978

z	$R_J(0.5,1,z,0.75)$	Asymp. Approx.	Error and	Bound/$z^{-3/2}\log z$
10	5.90885029D-01	6.44795200D-01	-0.7404	-1.2536
100	2.00646440D-01	2.03902146D-01	-0.7070	-0.9966
1000	6.43230108D-02	6.44795200D-02	-0.7165	-0.9140
10000	2.03835434D-02	2.03902146D-02	-0.7243	-0.8730
100000	6.44768645D-03	6.44795200D-03	-0.7294	-0.8484
1000000	2.03901133D-03	2.03902146D-03	-0.7328	-0.8320
10000000	6.44794825D-04	6.44795200D-04	-0.7353	-0.8203
100000000	2.03902132D-04	2.03902146Z-04	-0.7371	-0.8115

Behavior for Small Arguments

<u>Example 6.</u> $R_F(x,y,z)$ as $z \to 0^+$ (See equations (4.1) and (4.2)).

z	$R_F(0.5, 1, z)$	Asymp. Approx.	Error and	Bound/$z^{1/2}$
0.10000000	1.50869516D+00	1.85407468D+00	-1.0922	-1.5708
0.01000000	1.72488576D+00	1.85407468D+00	-1.2919	-1.5708
0.00100000	1.81066075D+00	1.85407468D+00	-1.3729	-1.5708
0.00010000	1.84006621D+00	1.85407468D+00	-1.4088	-1.5708
0.00001000	1.84961600D+00	1.85407468D+00	-1.4100	-1.5708
0.00000100	1.85266181D+00	1.85407468D+00	-1.4129	-1.5708
0.00000010	1.85362760D+00	1.85407468D+00	-1.4138	-1.5708
0.00000001	1.85393327D+00	1.85407468D+00	-1.4141	-1.5708

z	$R_F(1, 100, z)$	Asymp. Approx.	Error and Bound/$z^{1/2}$	
0.10000000	3.42157748D-01	3.69563736D-01	-0.0867	-0.1111
0.01000000	3.60040400D-01	3.69563736D-01	-0.0952	-0.1111
0.00100000	3.66451218D-01	3.69563736D-01	-0.0984	-0.1111
0.00010000	3.68568783D-01	3.69563736D-01	-0.0995	-0.1111
0.00001000	3.69248015D-01	3.69563736D-01	-0.0998	-0.1111
0.00000100	3.69463787D-01	3.69563736D-01	-0.0999	-0.1111
0.00000010	3.69532119D-01	3.69563736D-01	-0.1000	-0.1111
0.00000001	3.69553737D-01	3.69563736D-01	-0.1000	-0.1111

<u>Example 7</u>. $R_D(x,y,z)$ as $z \to 0^+$ (See equations (4.6) and (4.7)).

z	$R_D(0.5, 1, z)$	Asymp. Approx.	Error and Bound	
0.10000000	8.04918446D+00	1.34164079D+01	-5.367	-14.697
0.01000000	3.54313794D+01	4.24264069D+01	-6.995	-14.697
0.00100000	1.26444772D+02	1.34164079D+02	-7.719	-14.697
0.00010000	4.16285636D+02	4.24264069D+02	-7.978	-14.697
0.00001000	1.33357699D+03	1.34164079D+03	-8.064	-14.697
0.00000100	4.23454953D+03	4.24264069D+03	-8.091	-14.697
0.00000010	1.34083080D+04	1.34164079D+04	-8.100	-14.697
0.00000001	4.24183043D+04	4.24264069D+04	-8.103	-14.697

z	$R_D(1, 100, z)$	Asymp. Approx.	Error and Bound/$y^{1/2}$	
0.10000000	7.16729210D-01	9.48683298D-01	-0.232	-0.603
0.01000000	2.72275349D+00	3.00000000D+00	-0.277	-0.603
0.00100000	9.19132357D+00	9.48683298D+00	-0.296	-0.603
0.00010000	2.96982020D+01	3.00000000D+01	-0.302	-0.603
0.00001000	9.45644869D+01	9.48683298D+01	-0.304	-0.603
0.00000100	2.99695505D+02	3.00000000D+02	-0.304	-0.603
0.00000010	9.48378596D+02	9.48683298D+02	-0.305	-0.603
0.00000001	2.99969523D+03	3.00000000D+03	-0.305	-0.603

<u>Example 8.</u> $R_D(x,y,z)$ as $y \to 0^+$ (See equations (4.21) and (4.22)).

y	$R_D(0.5, y, 1)$	Asymp. Approx.	Error and	Bound/$y^{1/2}$
0.10000000	2.07511068D+00	3.02058478D+00	-2.990	-4.712
0.01000000	2.64565967D+00	3.02058578D+00	-3.749	-4.712
0.00100000	2.89176784D+00	3.02058578D+00	-4.074	-4.712
0.00010000	2.987870761+00	3.02058578D+00	-4.188	-4.712
0.00001000	3.00722377D+00	3.02058578D+00	-4.225	-4.712
0.00000100	3.01634769D+00	3.02058578D+00	-4.237	-4.712
0.00000010	3.01924369D+00	3.02058578D+00	-4.241	-4.712
0.00000001	3.02016057D+00	3.02058578D+00	-4.242	-4.712

y	$R_D(1, y, 100)$	Asymp. Approx.	Error and	Bound/$y^{1/2}$
0.10000000	7.30147713D-03	8.12013278D-03	-0.003	-0.003
0.01000000	7.83481979D-03	8.12013278D-03	-0.003	-0.003
0.00100000	8.02679722D-03	8.12013278D-03	-0.003	-0.003
0.00010000	8.09028822D-03	8.12013278D-03	-0.003	-0.003
0.00001000	8.11066156D-03	8.12013278D-03	-0.003	-0.003
0.00000100	8.11713435D-03	8.12013278D-03	-0.003	-0.003
0.00000010	8.11918425D-03	8.12013278D-03	-0.003	-0.003
0.00000001	8.11983288D-03	8.12013278D-03	-0.003	-0.003

<u>Example 9.</u> $R_J(x,y,z,p)$ as $p \to 0^+$ (See equations (4.12) and (4.13)).

p	$R_J(0.5,1,.75,p)$	Asymp. Approx.	Error and	Bound/p
0.10000000	4.35195760D+00	4.36392074D+00	-0.1196	-0.1699
0.01000000	9.12974537D+00	9.13097569D+00	-0.1230	-0.1699
0.00100000	1.45899195D+01	1.45900429D+01	-0.1234	-0.1699
0.00010000	2.02014175D+01	2.02014298D+01	-0.1234	-0.1699
0.00001000	2.58376155D+01	2.58376167D+01	-0.1234	-0.1699
0.00000100	3.14772681D+01	3.14772682D+01	-0.1234	-0.1699

p	$R_J(0.5,1,10,p)$	Asymp. Approx.	Error and	Bound/p
0.10000000	1.44661811D+00	1.44743386D+00	-0.0082	-0.0130
0.01000000	2.80549265D+00	2.80557506D+00	-0.0082	-0.0130
0.00100000	4.31282273D+00	4.31283098D+00	-0.0082	-0.0130
0.00010000	5.85161741D+00	5.85161823D+00	-0.0082	-0.0130
0.00001000	7.39544124D+00	7.39544132D+00	-0.0082	-0.0130
0.00000100	8.93996102D+00	8.93996103D+00	-0.0082	-0.0130

<u>Example 10.</u> $R_J(x,y,z,p)$ as $z \to 0^+$ (See equations (4.21) and (4.22)).

z	$R_J(0.5,1,z,200)$	Asymp. Approx.	Error and	Bound/$z^{-1/2}$
0.10000000	2.10646527D-02	2.62404569D-02	-0.0164	-0.0236
0.01000000	2.43032154D-02	2.62404569D-02	-0.0194	-0.0236
0.00100000	2.55893114D-02	2.62404569D-02	-0.0206	-0.0236
0.00010000	2.60303364D-02	2.62404569D-02	-0.0210	-0.0236
0.00001000	2.61735775D-02	2.62404569D-02	-0.0211	-0.0236
0.00000100	2.62192640D-02	2.62404569D-02	-0.0212	-0.0236
0.00000010	2.62337507D-02	2.62404569D-02	-0.0212	-0.0236
0.00000001	2.62383358D-02	2.62404569D-02	-0.0212	-0.0236

z	$R_J(0.5,1,z,0.75)$	Asymp. Approx.	Error and	Bound/$z^{-1/2}$
0.10000000	2.52692719D+00	3.76253620D+00	-3.9073	-6.2832
0.01000000	3.26652678D+00	3.76253620D+00	-4.9601	-6.2832
0.00100000	3.59123740D+00	3.76253620D+00	-5.4169	-6.2832
0.00010000	3.70674821D+00	3.76253620D+00	-5.5788	-6.2832
0.00001000	3.74472643D+00	3.76253620D+00	-5.6319	-6.2832
0.00000100	3.75688725D+00	3.76253620D+00	-5.6490	-6.2832
0.00000010	3.76074814D+00	3.76253620D+00	-5.6544	-6.2832
0.00000001	3.76197060D+00	3.76253720D+00	-5.6561	-6.2832

<u>Example 11.</u> $R_M(x,y,p)$ as $y \to 0^+$ (See equations (4.26) and (4.27)).

y	$R_M(1,y,.5)$	Asymp. Approx.	Error and	Bound/y|log y|
0.10000000	3.64234339D+00	3.28785191D+00	1.5395	2.2099
0.01000000	6.28724318D+00	6.21959431D+00	1.4690	1.8643
0.00100000	9.16167596D+00	9.15133670D+00	1.4968	1.7708
0.00010000	1.20844781D+01	1.20830791D+01	1.5189	1.7258
0.00001000	1.50149980D+01	1.50148215D+01	1.5333	1.6989
0.00000100	1.79465852D+01	1.79465639D+01	1.5430	1.6810
0.00000010	2.08783088D+01	2.08783063D+01	1.5499	1.6683
0.00000001	2.38100490D+01	2.38100487D+01	1.5551	1.6587

y	$R_M(1,y,10)$	Asymp. Approx.	Error and	Bound/y|log y|
0.10000000	2.76318419D-01	2.70084451D-01	0.0271	0.0376
0.01000000	4.17709274D-01	4.16671571D-01	0.0225	0.0283
0.00100000	5.63405983D-01	5.63257869D-01	0.0213	0.0252
0.00010000	7.09864930D-01	7.09845810D-01	0.0208	0.0237
0.00001000	8.56435282D-01	8.56432930D-01	0.0204	0.0228
0.00000100	1.00302033D+00	1.00302005D+00	0.0202	0.0221
0.00000010	1.14960720D+00	1.14960717D+00	0.0200	0.0217
0.00000001	1.29619429D+00	1.29619429D+00	0.0199	0.0214

APPENDIX B:

PROGRAMS USED TO CREATE EXAMPLES

The following programs are based on algorithms described in [2], and provide a method of obtaining numerical values for the basis functions R_F, R_J, and R_D. In all cases, the Duplication Theorem is used until all arguments are equal within some specified tolerance. The arguments are specified as assignment statements near the beginning of each program. Since R_D is easily obtained as a special case of R_J by letting $p = z$, there is no separate program for R_D. Minor changes in the initial arguments and the column headers enable these programs to generate all of the examples in Appendix A.

At the time of writing, Microsoft BASIC is perhaps the most widely implemented computer language in existence, owing to its near-universal use in personal computers; hence it is the language chosen here. Applications requiring speed or large numbers of function values should use a version of these algorithms written in a compiled language such as FORTRAN. Such a version is available as Algorithm 577 of the ACM library [S21]; see reference [5]. Obviously, comments and spaces may be deleted from the following BASIC version for improved performance.

These programs were written and used on a TRS-80 Model III microcomputer. The functions LOG and SQR unfortunately return only single-precision values on this machine, so two short subroutines at lines 1000 and 2000 were written to return double-precision values. They assume X as the input argument and return the value of the function in variables LN or SQ.

<u>Program 1.</u>   Computation of R_F(X0,Y0,Z1) for Z1 = 10, 100, ... 100000000.

```
100 DEFDBL A-Z                      'Use double precision variables.
110 X0=.5: Y0=1                     'Initial values of x and y.
120                                 'Print column headers.
130 PRINT TAB(63)"3/2"
140 PRINT CHR$(27)CHR$(30)TAB(5)"z"TAB(15)"R (0.5,1,z)"TAB(33)
     "Asympt. Approx.   Error and Bound/(z  log z)
150 PRINT CHR$(27)CHR$(30)TAB(16)"F"
160 PRINT"-------   ----------   ----------   -----   -----"
170 FOR Z0%=1 TO 8                  'Z0% is the power of ten for z.
180   Z1=VAL("1"+STRING$(Z0%,48))
190   X1=X0: Y1=Y0
200   X=X1*Y1                       'These steps form lambda.
210   GOSUB 2000
220   LA=SQ
230   X=X1*Z1
240   GOSUB 2000
250   LA=LA+SQ
260   X=Y1*Z1
270   GOSUB 2000
280   LA=LA+SQ
290   MU=(X1+Y1+Z1)/3               'Proceed with duplication theorem.
300   X2=(X1+LA)/4: Y2=(Y1+LA)/4: Z2=(Z1+LA)/4
310   X3=1-X2/MU: Y3=1-Y2/MU: Z3=1-Z2/MU
320   S2=(X3*X3+Y3*Y3+Z3*Z3)/4
330   S3=(X3*X3*X3+Y3*Y3*Y3+Z3*Z3*Z3)/6
340   X=MU
350   GOSUB 2000
360   RF=(1+S2/5+S3/7+S2*S2/6+S2*S3*3/11+5*S2*S2*S2/26+3*S3*S3/26)/SQ
370   X1=X2: Y1=Y2: Z1=Z2           '"New" arguments become "old".
380   IF ABS((RF-R0)/RF)>1E-14 THEN R0=RF: GOTO 200
390                                 'Relative error small; stop iterating.
400   Z1=VAL("1"+STRING$(Z0%,"0"))
410   X=Z1                          'Now compute the first term
420   GOSUB 2000                    'of the asymptotic expansion.
430   T1=SQ
440   X=X0
450   GOSUB 2000
460   T2=SQ
470   X=Y0
480   GOSUB 2000
```

```
490   X=4*T1/(T2+SQ)
495   GOSUB 1000
500   X=Z1
510   GOSUB 2000
520   ES=LN/SQ
530                               'Print value, approximation, error, and bound.
540   PRINT USING"#########  ##.############  ##.############    ##.####    ##.####";
        Z1,RF,ES,(RF-ES)*Z1[1.5/LOG(Z1),(X0+Y0)/8*(LOG(1+4*Z1/(X0+Y0))+1)/LOG(Z1)
550 NEXT Z0%
560 END
570 '
1000 LN=0                         'Subroutine to compute ln(x)
1010 Q=.6931471805599453          'in double precision.
1020 IF X=1 THEN RETURN
1030 E=VARPTR(X)+7                'Extract floating point exponent.
1040 N=PEEK(E)-128
1050 POKE E,128
1060 X=1-X                        'Use Taylor series on mantissa.
1070 FOR I9%=50 TO 1 STEP -1
1080   LN=X*(LN+1#/I9%)
1090 NEXT
1100 LN=N*Q-LN
1110 RETURN
1120 '
2000 SQ=SQR(X)                    'Compute double precision square root
2010 IF SQ=0 THEN RETURN ELSE W9=0
2020 E9=(X/SQ-SQ)/2D              'Usual Newton algorithm.
2040 IF E9=0 OR E9=W9 THEN RETURN ELSE SQ=SQ+E9: W9=E9: GOTO 2030
```

The second program contains subroutines for R_C and R_F in addition to the routine for R_J.

Program 2. $R_J(X0,Y0,Z0,P1)$ for P1 = 1, 0.1, 0.01, ... , 0.000000001.

```
100 DEFDBL A-Z
110 PI=3.141592653589793
120 X0=1: Y0=.5: Z0=.75
130 PRINT: PRINT CHR$(27)CHR$(30)TAB(63)"  "
140 PRINT CHR$(27)CHR$(30)TAB(6)"p"TAB(14)"R (1,.5,.75,p)"
      TAB(33)"Asympt. Approx.   Error and Bound / p      "
150 PRINT CHR$(27)CHR$(30)TAB(15)"J"
160 PRINT"--------   -----------   -----------   ------   -------"
170 FOR N%=0 TO 7
```

```
180   P0=VAL("."+STRING$(N%,"0")+"1")
190   X1=X0: Y1=Y0: Z1=Z0: P1=P0
200   GOSUB 590: RG=RJ
210   X=X0*Y0: GOSUB 2000: L0=SQ
230   X=X0*Z0: GOSUB 2000: L0=L0+SQ
250   X=Y0*Z0: GOSUB 2000: L0=L0+SQ
270   X1=X0+L0: Y1=Y0+L0: Z1=Z0+L0: P1=L0: GOSUB 590
280   P1=P0: X=X0: GOSUB 2000
290   AL=SQ: X=Y0: GOSUB 2000
300   AL=AL+SQ: X=Z0: GOSUB 2000
310   AL=AL+SQ: X=X0*Y0*Z0: GOSUB 2000
320   AL=P0*AL+SQ: AL=AL*AL: BE=P0*(P0+L0)*(P0+L0): GOSUB 500
330   ES=2*RJ+3*RC: RJ=RG
340   PRINT USING".#########  ##.#########[[[[  ##.#########[[[[   ##.####   ##.####";
        P0,RJ,ES,(RJ-ES)/P0,-6/5*L0[-2.5
350 NEXT N%
360 END
370 'Compute RF(X1,Y1,Z1):
380 X=X1*Y1: GOSUB 2000: LA=SQ
390 X=X1*Z1: GOSUB 2000: LA=LA+SQ
400 X=Y1*Z1: GOSUB 2000: LA=LA+SQ
410 MU=(X1+Y1+Z1)/3
420 X2=(X1+LA)/4: Y2=(Y1+LA)/4: Z2=(Z1+LA)/4
430 X3=1-X1/MU: Y3=1-Y2/MU: Z3=1-Z2/MU
440 S2=(X3*X3+Y3*Y3+Z3*Z3)/4: S3=(X3*X3*X3+Y3*Y3*Y3+Z3*Z3*Z3)/6
450 X=MU: GOSUB 2000
460 RF=(1+S2/5+S3/7+S2*S2/6+S2*S3*3/11+5*S2*S2*S2/26+3*S3*S3/26)/SQ
470 X1=X2: Y1=Y2: Z1=Z2
480 IF ABS((RF-R0)/RF)>1E-15 THEN R0=RF: GOTO 380  ELSE RETURN
490 'Compute RC(AL,BE):
500 A1=AL: B1=BE
510 X=A1*B1: GOSUB 2000: LA=2*SQ+B1
530 A2=(A1+LA)/4: B2=(B1+LA)/4
540 MV=(A1+2*B1)/3: S1=(B1-A1)/3/MV: X=MV: GOSUB 2000
550 RC=(1+3*S1*S1/10+S1*S1*S1/7+3*S1*S1*S1*S1/8+9*S1*S1*S1*S1*S1/22
     +159*S1*S1*S1*S1*S1/208/SQ
560 A1=A2: B1=B2
570 IF ABS((RC-R1)/RC)>1E-15 THEN R1=RC: GOTO 510  ELSE RETURN
580 'Compute RJ(X1,Y1,Z1,P1):
590 S=0: FA=3: RJ=0: R0=1
600 X=X1*Y1: GOSUB 2000: LA=SQ
610 X=X1*Z1: GOSUB 2000: LA=LA+SQ
620 X=Y1*Z1: GOSUB 2000: LA=LA+SQ
630 MU=(X1+Y1+Z1+2*P1)/5
640 X2=(X1+LA)/4: Y2=(Y1+LA)/4: Z2=(Z1+LA)/4: P2=(P1+LA)/4
650 X3=1-X1/MU: Y3=1-Y1/MU: Z3=1-Z1/MU: P3=1-P1/MU
660 S2=(X3*X3+Y3*Y3+Z3*Z3+2*P3*P3)/4:S3=(X3*X3*X3+Y3*Y3*Y3+Z3*Z3*Z3+2*P3*P3*P3)/6:
     S4=(X3*X3*X3*X3+Y3*Y3*Y3*Y3+Z3*Z3*Z3*Z3+2*P3*P3*P3*P3)/8:
     S5=(X3*X3*X3*X3*X3+Y3*Y3*Y3*Y3*Y3+Z3*Z3*Z3*Z3*Z3+2*P3*P3*P3*P3*P3)/10
670 X=X1: GOSUB 2000: AL=SQ
```

```
680 X=Y1: GOSUB 2000: AL=AL+SQ
690 X=Z1: GOSUB 2000: AL=AL+SQ
700 AL=AL*P1: X=X1*Y1*Z1: GOSUB 2000: AL=AL+SQ
710 AL=AL*AL: BE=P1*(P1+LA)*(P1+LA): GOSUB 500
720 S=S+FA*RC: FA=FA/4: X=MU: GOSUB 2000
730 RJ=S+FA/4*(1+3*S2/7+S3/3+3*S2*S2/22+S2*S3*3/13+3*S4/11
     +3*S5/13-S2*S2*S2/10+3*S3*S3/10+3*S2*S4/5)/SQ/SQ/SQ
740 X1=X2: Y1=Y2: Z1=Z2: P1=P2
750 IF ABS((RJ-R0)/RJ)>1E-14 THEN R0=RJ: GOTO 600 ELSE RETURN
999 'Double precision F=log(X):
1000 LN=0                        'Subroutine to compute ln(x)
1010 Q=.6931471805599453         'in double precision.
1020 IF X=1 THEN RETURN
1030 E=VARPTR(X)+7               'Extract floating point exponent.
1040 N=PEEK(E)-128
1050 POKE E,128
1060 X=1-X                       'Use Taylor series on mantissa.
1070 FOR I9%=50 TO 1 STEP -1
1080   LN=X*(LN+1#/I9%)
1090 NEXT
1100 LN=N*Q-LN
1110 RETURN
2000 SQ=SQR(X)                   'Compute double precision square root
2010 IF SQ=0 THEN RETURN ELSE W9=0
2020 E9=(X/SQ-SQ)/2D             'Usual Newton algorithm.
2040 IF E9=0 OR E9=W9 THEN RETURN ELSE SQ=SQ+E9: W9=E9: GOTO 2030
```