GRADUAL MATCHING WITH AFFIRMATIVE ACTION[1]

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May 14, 2024

Abstract

Admissions to technical colleges in India feature a multi-period matching process and are subject to a complex affirmative action policy. At each period, an assignment is produced, and applicants decide whether to finalize their assignments or participate in the next period with the possibility of updating their submitted *rank-ordered lists* (ROLs). Building on Haeringer and Iehlé (2021), we formulate a multi-period college admissions problem where institutions' choice rules incorporate affirmative action restrictions and are updated consistently. We show that it is safe for applicants to participate in additional periods when their updating rule on ROLs satisfies a mild regularity condition. We also introduce a backward-looking notion of stability for multi-period matching mechanisms that consider affirmative action constraints. We use our results to analyze the multi-stage mechanism currently used in admissions to engineering colleges in India.

JEL CODES: $C78$, $D02$, $D47$, $I38$.
KEYWORDS: Market design, affirmative action, gradual matching, India.

[1]Kriti Manocha is grateful to her advisors, Arunava Sen and Debasis Mishra, for their guidance during her graduate studies. We thank Orhan Aygün, Inácio Bó, Battal Doğan, Vincent Iehlé, Bumin Yenmez, and participants of the 2024 Asian Meeting of the Econometric Society (AMES-CSW) and the 22nd annual SAET Conference for their constructive feedback.

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1 Introduction

The technical colleges in India—Indian Institutes of Technology (IITs) and the non-IIT Centrally Funded Technical Institutes (non-IITs)—have adopted a joint multi-stage matching procedure since 2016. The matching forms between students and programs, whereby each IIT and non-IIT consists of multiple programs. Students rank programs (not universities) according to their preferences. Programs use students' merit scores in relevant score categories subject to mandated affirmative action policy to select students from any given set of students. Approximately 1.5 million applicants each year compete to obtain one of the roughly 57,000 slots.

Prior to 2015, IITs conducted admissions separately from the non-IIT CFTIs. The fragmented approach caused several hundred candidates receiving offers from both categories of institutions, resulting in vacancies in one set of institutions when the candidate chose a program. These vacant seats were either left unfilled or allocated through an ad-hoc, decentralized process, leading to inefficiencies, unfairness, and logistical challenges for candidates. An interdisciplinary team of computer scientists, operations researchers, and administrators designed a novel "combined" seat allocation mechanism to address these issues. Since 2015, a new joint seat allocation process—the Multi-Round Deferred Acceptance mechanism—has been implemented by the Joint Seat Allocation Authority (JoSAA).[4]

The Multi-Round Deferred Acceptance (MRDA) is a novel procedure designed to allocate candidates to programs based on a set of input parameters, including the capacity of each program, candidates' submitted Rank Order Lists (ROLs) over programs, and merit lists of candidates for each program. The MRDA algorithm can accommodate complex reservation quotas and eligibility criteria, which are detailed in Section 5.

In each round of the MRDA, candidates are presented with the opportunity to "freeze" their allotted seat, indicating their acceptance of the assigned program and removing themselves from consideration in subsequent rounds. Alternatively, candidates may choose to be considered for upgrades in later rounds by selecting one of the available options to update their ROLs:

[4]Baswana et al. (2019a) summarize the details of the new design and reform process. Their approach incorporated a variant of the iterative deferred acceptance algorithm introduced in Manjunath and Turhan (2016) and Turhan (2019).

Float: Candidates accept the seat allotted but want to be upgraded as high as possible on their ROL.

Slide: Candidates want to remain in the institute to which they are currently allotted but want the most desirable program available at that institute.

Reject: Candidates reject the seat offered and exit the system.

Withdraw: Candidates who accepted a seat in some past round, withdraw, and exit the system.

Haeringer and Iehlé (2021) (henceforth H&I) introduced and formulated *gradual* matching problems of this nature where applicants are allowed to update their ROLs from one stage to the next[5]. However, the foundational assumption of *responsive*[6] preferences of institutions in their model precludes the applicability of their framework to real-world applications with affirmative action constraints. This paper examines such gradual matching problems with complex diversity constraints where individuals can update their ROLs in each stage, motivated by Indian technical college admissions. Institutions fulfill affirmative action mandates via carefully designed choice rules.

Admissions to technical colleges in India operate under a comprehensive affirmative action program, implemented through a reservation system comprising both *vertical* and *horizontal* reservations. Specifically, each program sets aside 15%, 7.5%, 27%, and 10% of its available slots to applicants from Scheduled Castes (SC), Scheduled Tribes (ST), Other Backward Classes (OBC), and Economically Weaker Sections (EWS), respectively. These specific designations are collectively termed as *vertical reserve categories*. Applicants who do not affiliate with any of these vertical reserved categories are categorized under the *General Category* (GC).

The positions earmarked for SC, ST, OBC, and EWS are cumulatively called *vertically reserved positions*.[7] The remaining positions are designated as *open-category*, accessible to all applicants, including those belonging to vertical reserve categories[8]. Vertical reservations adhere to the "*over-and-above*" principle, in which open-category positions are allocated to the

[5]They focused on a gradual mechanism for French college admissions problem which is no more in use.

[6]See Roth (1985) and Chambers and Yenmez (2018).

[7]Applicants from vertically reserved categories must submit verifiable membership information if they receive a position in their respective reserved category.

[8]Those who do not formally declare their membership and those categorized under GC are exclusively considered for open-category slots.

highest-scoring applicants subject to its capacity. Moreover, within the framework of technical college admissions, a minimum number of positions in each category is earmarked explicitly for female applicants (Baswana et al. (2019a)). This provision is considered *horizontal reservations*, which operate under the "*minimum guarantees*" principle by law.[9] Positions allocated for horizontal reservations are filled before the horizontally unreserved positions.

Institutions' selection criteria are formally characterized through choice rules to accommodate complicated affirmative action and diversity constraints. Specifically, we model institutions' choice rules via Position-Specific Priorities (PSP) structure, introduced by Kominers and Sönmez (2016) (henceforth K&S). Open-category positions are allocated before vertically reserved positions to provide extra boost to reserved category applicants who may not able to obtain these positions in the absence of the reservation policy. Open-category positions subject to horizontal reservations for women systematically prioritize female candidates over their male counterparts, albeit adhering to intra-group merit-score rankings. Conversely, open-category positions —not reserved horizontally for women —are allocated according to a pure merit-score ranking. For positions reserved for specific vertical category, those subject to horizontal reservations for women prioritize women within the same category over men, again abiding by merit-score rankings within each respective group. Such positions deem applicants not belonging to this vertical category unacceptable. Similarly, positions within a vertical category that are not horizontally reserved use a merit order, exclusively ranking individuals based on their merit scores and excluding applicants from other vertical categories.

In this paper, we extend the results of H&I to a more general framework in which institutions' complicated affirmative action constraints are embedded in their choice rules. Not only is this extension theoretically non-trivial, but it is also motivated by the admissions to technical colleges in India under complicated affirmative action. Similar to H&I, we study conditions imposed on individuals' ROLs across consecutive stages that yield "monotone" outcomes that makes it safe for applicants to participate in additional stages while preserving the pre-determined priority rankings. A matching outcome is monotone[10] if each individual is

[9]See the historic Supreme Court decision in Indra Shawney vs. Union of India (1992). It can be accessed at https://indiankanoon.org/doc/1363234/.

[10]H&I refers to this notion as *gradual safety*.

matched to an institution that is weakly better than the match of the previous stage with respect to their updated ROLs. Such preservation ensures that individuals tentatively matched in earlier rounds are not adversely affected.[11] We demonstrate that applicants cannot be worse off in subsequent stages when active candidates make fewer claims on objects superior to their previous assignment (see Theorem 1). Importantly, since the model allows individuals to finalize their assignments at any stage, choice rules of institutions must be *updated* from one stage to the next in a *consistent* manner. A choice rule update is consistent if it is not contingent on the active agents' proposals in the preceding period. Updating a structure of this kind is central to our analysis.

We also introduce a "backward-looking" notion of stability for gradual matching mechanisms adjusted for affirmative action constraints. We consider individual rationality, non-wastefulness, and elimination of justified envy across different periods. Theorem 2 establishes a relationship between this notion of stability—which we refer to as *gradual stability* following H&I—and monotone outcomes.

These results generalize the findings presented in H&I under responsive preferences to a setting with more general choice rules that accommodate a complex affirmative action constraints. Moreover, we relate our findings to the characterization result of Kojima and Manea (2010) for the *individually optimal deferred acceptance algorithm* (IODA). We apply our theoretical findings to analyze the *MRDA mechanism* implemented in engineering college admissions in India since 2015.

Similar to H&I, our model abstracts from the strategic interactions regarding individuals' updates to their ROLs. We consider the submitted ROLs and the corresponding stages at which participants finalize their assignments as exogenous. Sequential matching mechanisms are inherently susceptible to strategic manipulation. This vulnerability persists even when strategy-proof stage mechanisms are used. Such mechanisms can be subject to strategic behavior by the individuals. We refer the reader to Doğan and Yenmez (2023), Dur and Kesten (2019), Dur et al. (2021), Andersson et al. (2018), and Manjunath and Turhan (2016) for an exhaustive

[11] There are real-world applications in which institutions are legislatively constrained from altering their priority rankings across successive iterations of their sequential mechanisms. For a specific illustration pertaining to admissions in Indian technical colleges, see Section 5.

treatment of the non-existence of strategy-proof sequential mechanisms.

2 Base Model

We consider a finite set of institutions $S = \{s_1, ..., s_m\}$ and a finite set of individuals $I = \{i_1, ..., i_n\}$. Each individual $i \in I$ has a rank-ordered list (ROL) P_i—an asymmetric, complete, and transitive binary relation—over $S \cup \{\emptyset\}$, where $\emptyset$ denotes remaining unmatched. We write $sP_i\emptyset$ to mean that institution s is *acceptable* for individual i. Similarly, $\emptyset P_i s$ means s is *unacceptable* for individual i. We let $A_{P_i} = \{s \in S \mid sP_i\emptyset\}$ denote the set of acceptable institutions for a given rank-ordered list P_i. We denote the profile of individuals' ROLs by $P = (P_i)_{i\in\mathcal{I}}$. We let $\mathcal{P}$ denote the set of all strict ROLs. We denote by R_i the weak relation associated with P_i and by $R = (R_i)_{i\in\mathcal{I}}$ the profile of weak relations.

An institution s has a set of $\overline{q}_s$ positions denoted by $\mathcal{B}_s \equiv \{p_s^1, ..., p_s^{\overline{q}_s}\}$. Each position $p_s^j \in \mathcal{B}_s$ has a linear order $\succ_s^j$ over $I \cup \{\emptyset\}$. Each position p_s^j ranks the option $\emptyset$ that represents being unassigned. We write $\emptyset \succ_s^j i$ to mean that individual i is unacceptable for position p_s^j. We denote the profile of positions' priority orders by $\succ_s = (\succ_s^j)_{j=1}^{\overline{q}_s}$. For each institution $s \in S$, positions in $\mathcal{B}_s$ are ordered according to a linear order of precedence $\rhd_s$. For any $p_s^j, p_s^k \in \mathcal{B}_s$, we write $p_s^j \rhd_s p_s^k$ to mean that institution s fills position p_s^j before filling p_s^k, whenever possible. For convenience, if $j < k$ for any p_s^j, p_s^k, it indicates $p_s^j \rhd_s p_s^k$.

Given $(\succ_s, \rhd_s)$, the *position-specific priorities choice rule* (PSP) of institution s selects applicants from a given set $A \subseteq I$, as follows:

- First, position p_s^1 is assigned to the individual who is $\succ_s^1$–maximal among the individuals in A. Call this individual i_1.
- Then, position p_s^2 is assigned to the individual who is $\succ_s^2$–maximal among the individuals in $A \setminus \{i_1\}$. Call this individual i_2.
- This process continues with each position p_s^k is being assigned to the individual who is $\succ_s^k$–maximal among the remaining individuals in $A \setminus \{i_1, ..., i_{k-1}\}$.

If no individual is assigned to a position $p_s^l \in \mathcal{B}_s$, then p_s^l is assigned $\emptyset$. We denote the set

of chosen individuals by $C_s(A, \succ_s, \rhd_s)$[12]. For the sake of brevity, we denote this set as $C_s(A)$ for a given $(\succ_s, \rhd_s)$.

These are often used choice structures that have many appealing properties. For instance, PSP choice rules satisfy path-independence[13] for matching without contracts. In the context of college admissions, this property ensures that the set of admitted students does not depend on the order in which applications are reviewed, not allowing for malpractice or favoritism. It is well-known that a choice rule is path-independent if, and only if, it satisfies the substitute[14] condition and is consistent[15] Aizerman and Malishevski (1981)[16].

LEMMA **1** (Kominers and Sönmez (2016),Aygün and Sönmez (2013))**.** *Every PSP choice rule satisfies the substitute condition and is consistent.*[17]

A stage problem is denoted as $\Xi = (I, S, (P_i)_{i \in I}, (\succ_s, \rhd_s, \overline{q}_s)_{s \in S})$, where $(\succ_s, \rhd_s)$ represents the PSP choice rule of the institution s. A **stage matching** for a stage problem Ξ is a mapping $\mu : I \cup S \to 2^I \cup S$ such that, for each $i \in I$ and $s \in S$, (i) $\mu(i) \in S \cup \{\emptyset\}$, (ii) $\mu(s) \subseteq I$, and (iii) $\mu(i) = s$ if and only if $i \in \mu(s)$. A stage matching is **feasible** if $|\mu(s)| \leq \overline{q}_s$ for all $s \in S$.

DEFINITION 1. A feasible stage matching μ is **stage stable** if for all $i \in I$ and $s \in S$,

(S1) $\mu(i) R_i \emptyset$,

(S2) $C_s(\mu(s)) = \mu(s)$, and

(S3) $s P_i \mu(i)$ implies $i \notin \mu(s)$.

The first condition (S1), *individual rationality* for individuals, guarantees that no individual is assigned to an institution they find unacceptable. Condition (S2) ensures that institutions'

[12]PSP choice rules were introduced in K&S in a matching with contracts framework. A special case when each individual has a single contract and $\succ_s^i = \succ_s^j$ for all positions $p_s^i, p_s^j \in \mathcal{B}_s$ is referred to as *responsive* (choice rule) preference.

[13]A choice rule C is *path-independent* if $C(A \cup B) = C(C(A) \cup B)$ for any $A, B \subseteq I$ (Plott (1973)).

[14]A choice rule C satisfies the substitute condition if for every set $A, B \subset I$, $a \in A \subseteq B$, $a \in C(B)$ implies $a \in C(A)$.

[15]A choice rule C is *consistent* if for every set $A, B \subset I$ such that $C(B) \subseteq A \subseteq B$, $C(A) = C(B)$. Note that this consistency condition on choice rules is different from the consistency condition imposed on choice rule updates introduced in this paper.

[16]For further characterization results on path-independent choice rules, see Yokote et al. (2023)

[17]Kominers and Sönmez (2016) proved that PSP choice rules satisfy the bilateral substitutes condition.In settings without contracts, the bilateral substitutes condition is equivalent to the substitutes condition.

selection procedures are respected and guarantees implementing the diversity constraints when encoded into institutions' choice rules. The last condition is the standard *no blocking pair* condition.

A **stage matching mechanism** φ maps every stage problem Ξ to a feasible stage matching μ. The mechanism φ is stable if $\varphi(\Xi)$ is stable for every stage problem.

In the context of college admissions in India, outcomes are announced as a pair of programs and seat categories for each applicant. The Program-Specific Priorities (PSP) choice rules enable us to define an assignment, which maps each individual to an institution-position pair.[18] Each assignment induces a matching between applicants and programs.

In college admissions settings that incorporate affirmative action policies, the eligibility criteria for applicants to qualify for different types of positions and the order in which these positions are filled play a crucial role in the assignment procedure. The allotment of positions to applicants is a significant component of the assignment process. Specifically, the notion of gradual stability, which we introduce in Section 4, takes individuals' assignments into account when evaluating the stability of the matching outcome. This consideration is particularly relevant in the presence of affirmative action policies, as it ensures that the assignment procedure adheres to the established criteria and priorities while simultaneously accounting for the preferences of applicants.

Let $\mathcal{B} = \bigcup_{s \in S} \mathcal{B}_s$ denote the set of all positions.

DEFINITION 2. An **assignment** is a function $v: \ I \cup \ S \to (S \times \mathcal{B}) \cup 2^{I \times \mathcal{B}} \cup \{\emptyset\}$ such that

(i) for any $i \in I$, $v(i) \in (S \times \mathcal{B}) \cup \{\emptyset\}$,

(ii) for any $s \in S$, $v(s) \subseteq 2^{I \times \mathcal{B}_s}$ such that $|v(s)| \leq \overline{q}_s$,

(iii) for every $i \in I$ and $s \in S$, $v(i) = (s, p)$ if and only if $(i, p) \in v(s)$.

Let $\mu(v)$ be the matching induced by the assignment v. We denote by $\mu_i(v)$ and $\mu_s(v)$ the institution with which individual i is matched and the set of individuals matched with institution s, respectively.

[18] K&S utilize this fact to create an associated agent-slot market for a given agent-institution market. Agents' preferences are extended to encompass institution-slot pairs straightforwardly to accomplish this.

3 Multi-period Matching

Let $(\Xi^t)_{1\leq t\leq T}$ be a sequence of stage problems where $\Xi^t = (I^t, S^t, (P_i^t)_{i\in I}, (\succ_s^t, \rhd_s^t, \overline{q}_s^t)_{s\in S})$ is the problem at stage t, and T is the final stage. At stage t, for a given institution s, the set of positions, the profile of linear priority orders, and the precedence order are denoted by $\mathcal{B}_s^t$, $\succ_s^t$, and $\rhd_s^t$, respectively.

Our framework allows individuals to leave the matching process at any stage $t \leq T$ with their stage t assignments. Therefore, the set of individuals and available positions at institutions are (weakly) shrinking. Following the terminology of H&I, we refer to such a sequence of stage problems as *nested*.

DEFINITION 3. A sequence of T stage problems Ξ^1, Ξ^2,..., Ξ^T is **nested** if for all $t = 1, ..., T-1$ and $s \in S$,

1. $I^{t+1} \subseteq I^t$,
2. $S^t = S^{t+1} = S$,
3. $\mathcal{B}_s^{t+1} \subseteq \mathcal{B}_s^t$.

Individual $i \in I^t$ means that i is **active** at stage t. Individual $i \in I^t \setminus I^{t+1}$ $(t < T)$ means that i **finalize** her assignment at stage t before the final stage. We denote by $t_i := \arg\max_{1\leq t\leq T}\{i \in I^t\}$, the stage at which individual i finalizes her assignment. At this period, the individual i is last active. Once the individual finalizes her assignment, she can not be active in further stages.

A sequence $((\Xi^t), \mu^t)_{1\leq t\leq T}$ is feasible if for each stage $t \in \{1, \ldots, T\}$, μ^t is a feasible stage matching for the stage problem Ξ^t.

3.1 Updating Institutions' Choice Rules

Institutions update their choice rules based on the matching "proposed" in the preceding stages. Given a feasible matching μ^t for the stage problem Ξ^t and $(\succ_s^t, \rhd_s^t)$, the next stage's choice rule is ascertained by excluding positions allocated to individuals who have finalized their assignments.[19] The underlying idea is that once an individual solidifies her assignment, she departs

[19] As discussed in Section 2, for institutions, μ^t maintains individual rationality, ensuring that every individual matched to an institution is accorded a specific position. This condition is implicitly assumed when referencing a

with the assigned position. No additional positions are either created or eliminated during this transition. The relative precedence between the two positions and the priority sequence concerning the cohort of active individuals remain unaltered. We refer to such a methodology for updating choice rules as *consistent*.

DEFINITION 4. A choice rule update is **consistent** if for all $s \in S$ and $t = 1, \ldots, T-1$, given μ^t,

1. $\rhd_s^{t+1} = \rhd_s^t$.
2. For all $p \in \mathcal{B}_s^{t+1}$ and $i, j \in I^{t+1}$, $i(\succ_s^p)^t j$ implies $i(\succ_s^p)^{t+1} j$.
3. if $i \in I^t \setminus I^{t+1}$ and $v^t(i) = (s, p)$ for some $p \in \mathcal{B}_s^t$, then $p \notin \mathcal{B}_s^{t+1}$.

Suppose the choice rule of the institution is responsive. Then updating it consistently reduces to the following: $P_s^t = P_s^1|_{I^t}$ where $= P_s^1|_{I^t}$ is the restriction of P_s^t to the set of active individuals I^t. This way of updating thus ensures that no change in the priority structure is observed from the institution's end.[20] Note that the choice rule derived by this procedure also has a PSP structure.

A consistent update of a PSP choice rule implies that the capacity of an institution at stage $t+1$ is the number of unassigned positions at stage t plus the number of active individuals from stage t. That is, for all $t \leq T-1$,

$$\overline{q}_s^{t+1} = (\overline{q}_s^t - |\mu^t(s)|) + |\{i \in I^{t+1} : \mu^t(i) = s\}|.$$

Consistency of this form not only gives a relationship between the capacities across successive rounds but also gives a relationship between the set of individuals chosen across successive rounds. It implies that if an individual is chosen from a set A in some period t, then she must be chosen from its subset in the next period if she is still active in that period. We refer to this property as *set inclusion*.

DEFINITION 5 (Set inclusion property). Let $(C_s^t)_{t \leq T}$ be a sequence of choice rules. Then, for all $A, B \subseteq I^t$ such that $A \subseteq C_s^t(B)$, we have

feasible matching derived from the PSP structure.

[20] A restriction that is also observed in H&I.

$$A \cap I^{t+1} \subseteq C_s^{t+1}(B \cap I^{t+1}).$$

Proposition 1. *A consistent update of a PSP choice rule implies the set-inclusion property.*

This is an important property for the existence of our results. When considering more general choice rules, such as path-independent choice rules, consistent updates[21] need not fulfill this condition, unlike PSP choice rules where this property comes for free.

3.2 Gradual Matching Mechanisms

The multi-period matching problem that we study has the following properties:

(1) The sequence of stage problems $(\Xi^t)_{1 \leq t \leq T}$ is nested.

(2) Institutions' PSP choice rules are updated consistently.

In other words, individuals participate in a sequential assignment problem by submitting their ROL. Each individual decides to either finalize their match, allocated to them via the stage mechanism, or to participate in the next stage. The sequential assignment problem then comprises active individuals with updated ROLs and institutions' consistently updated choice rules. All the individuals finalize their match at some period and then exit the mechanism. Section 5 examines an application of this type of sequential mechanism in the context of Indian technical colleges.

The nestedness of matching problems and the consistent update of institutions' choice rules enable us to reduce this sequential problem to a simpler framework. Instead of individuals deciding to continue or finalize the match at every period, this centralized framework only requires the first stage problem Ξ^1 of the sequence $(\Xi^t)_{1 \leq t \leq T}$ and a list of ROLs $(P_i^1, P_i^2, \ldots, P_i^{t_i})$ for every $i \in I^1$, denoted by $\mathbf{P}_i = (P_i^t)_{t \leq t_i}$. We refer to this class of multi-period matching problems as **gradual matching problems** and is represented as $\mathbf{\Xi} = (I^1, S, (\mathbf{P}_i)_{i \in I}, (\succ_s^1, \rhd_s^1, \overline{q}_s^1)_{s \in S})$.

[21] A consistent update of path-independent choice rules can be defined similarly to that of PSP choice rules by employing their well-established mathematical representation provided by Aizerman and Malishevski (1981). Each path-independent choice rule can be associated with a collection of preference relations. From any set of individuals, the choice rule chooses the union of best individuals with respect to each preference relation. It needs to be ensured that the updated choice rule also satisfies path-independence, a characteristic inherent in consistent update of PSP choice rule.

An outcome of a gradual matching problem consists of a sequence $(\Xi^t, \mu^t)_{t \leq T}$ that associates a feasible matching to every stage problem at every $t = 1, \ldots, T$. This outcome implicitly defines a matching $\nu : I \cup S \to 2^I \cup S$ such that $\nu(i) = \mu^{t_i}(i)$ for all $i \in I^1$. We refer to this sequence as a **gradual outcome**.

We impose restrictions on updating the ROLs of individuals across different periods (till they are active) to ensure desirable properties of feasible matchings. H&I propose the concept of a *refitting rule* that defines the set of admissible ROLs based on both the ROL submitted by the individual and the match proposed to her in the preceding period. Formally, a refitting rule $\Gamma : \mathcal{P} \times (S \cup \{\emptyset\}) \to \mathcal{P}$ is a selection correspondence that ensures consistency in the updating process as follows: for every $(P, s) \in \mathcal{P} \times (S \cup \{\emptyset\})$,

- If $s \in A_P$, then $A_{P'} \neq \emptyset$ for some $P' \in \Gamma(P, s)$;
- For each $P' \in \Gamma(P, s)$, we have $A_{P'} \subseteq A_P$.

These mild conditions require that an unacceptable institution at previous stages can not be expressed as acceptable in an updated ROL[22]. These conditions are also applicable in the case of Indian technical college admissions, where each candidate submits a single preference list over all available programs in the initial period. Candidates are not permitted to change their preference list midway through the process.

We take individuals' decisions to finalize their match as given, similar to H&I. It is reflected in the length of the sequence of ROLs submitted. Thus, the *strategic behavior* of the individuals is outside the scope of our model.

For a given refitting rule Γ and a stage mechanism φ, a **gradual matching mechanism**, denoted by $\mathcal{M}^{\varphi}_{\Gamma}$, maps every gradual matching problem to a gradual outcome $\mathcal{M}^{\varphi}_{\Gamma}(\boldsymbol{\Xi}) \equiv (\Xi^t, \mu^t)_{t \leq T}$ such that

1. $\mu^t = \varphi(\Xi^t)$ for all $t = 1, \ldots, T$, and
2. $P_i^t = \Gamma(P_i^{t-1}, \mu^{t-1}(i))$ for all $i \in I^1$ and $t = 2, \ldots, T$.

We restrict our attention to stable stage mechanisms that play a crucial role in the matching

[22] H&I provides a detailed discussion on refitting rules.

literature, the *individual-optimal deferred acceptance* (IODA) mechanism, whose outcome can be found by the following iterative procedure.

Step 1: Every individual applies to her highest ranked acceptable institution under P_i (if any). Let $\hat{I}_s^1$ be the set of agents applying to institution s. The institution tentatively accepts the set $I_s^1 = C_s\left(\hat{I}_s^1\right)$ and rejects the applicants in $\hat{I}_s^1 \backslash I_s^1$.

Step t: $(t \geq 2)$. Every individual not rejected at step $t-1$ applies to her next ranked acceptable institution. Let $\hat{I}_s^t$ be the new set of individuals applying to institution s. The institution tentatively accepts the set $I_s^t = C_s\left(I_s^{t-1} \cup \hat{I}_s^t\right)$ and rejects the applicants in $(I_s^{t-1} \cup \hat{I}_s^t) \backslash I_s^t$.

This algorithm terminates when every individual—not tentatively accepted by some institution—is rejected by all institutions acceptable to her.

LEMMA **2** (Hatfield and Milgrom (2005), Aygün and Sönmez (2013))**.** *When each institution's choice rule satisfies the substitutes condition and is consistent, individual-optimal stable matching exists and is unique. Moreover, it is the outcome of the IODA.*

Lemma 1 and 2 thus implies that the IODA outputs the individual-optimal stable matching in each stage.

3.3 Monotone Outcomes

A crucial property of a gradual matching outcome is that, for each period when an individual is active, they are assigned to an institution ranked weakly higher than the institution to which they were previously assigned with respect to their updated ROLs. We refer to such gradual matching outcomes as *monotone*.[23]

DEFINITION 6. A gradual outcome $(\Xi^t, \mu^t)_{t \leq T}$ is **monotone** if, for all $2 \leq t \leq T$ and $i \in I^t$,

$$\mu^t(i) R_i^t \mu^{t-1}(i).$$

A mechanism $\mathcal{M}_\Gamma^\varphi$ is monotone if for every sequential matching problem $\boldsymbol{\Xi}$, the sequential outcome $(\Xi^t, \mu^t)_{t \leq T}$ is monotone.

[23] We adopt the term "monotone" from Baswana et al. (2019a).

One approach to ensure that every individual is matched to a weakly higher institution than the one proposed in the previous period is that individuals adhere to the proposal made in the preceding period to update their ROLs for the next stage. More precisely, individuals are required to update their ROLs so that no institution that was ranked lower than the proposal in the previous stage—if acceptable in the current stage—is now ranked higher. Formally,

DEFINITION 7. A refitting rule Γ is **regular** if for all $(P, v) \in \mathcal{P} \times (S \cup \{\emptyset\})$ and $s \in S \setminus \{v\}$, if $P' \in \Gamma(P, v)$ then,

$$sP'v \text{ and } sP'\emptyset \implies sPv.$$

There is a wider class of refitting rules that satisfy these mild conditions, including identity mapping[24] and truncation mapping[25], among others.

We are now ready to state our first result.

Theorem 1. *Let $\mathcal{M}_\Gamma^\varphi$ be the gradual mechanism where φ is the individual-optimal stable stage mechanism, and Γ is the refitting rule. $\mathcal{M}_\Gamma^\varphi$ is monotone if and only if Γ is regular.*

This equivalence result holds significant policy implications, particularly in designing admission procedures under affirmative action considerations—such as admissions to technical colleges in India—where applicants are offered several options after every round to participate in subsequent rounds.

Note that an individual-optimal stable matching is not guaranteed to exist when employing general choice rules. However, when institutions' choice functions are PSP choice rules, stable matching exists (Kominers and Sönmez (2016)).[26]

4 Gradual Stability

In this section, we introduce a stability notion that generalizes the concept of *gradual stability* introduced in H&I. When institutions' choice rules are responsive, our definition reduces to theirs.

[24] An identity mapping is a singleton-valued selection correspondence such that for all $(P, s) \in \mathcal{P} \times (S \cup \{\emptyset\})$, $\Gamma(P, s) = \{P\}$.

[25] A truncation mapping is a singleton-valued selection correspondence discussed in Manjunath and Turhan (2016) such that $\Gamma(P, s) = \{P'\}$ where $sPs' \implies \emptyset P's'$ and $s'Ps''Ps \implies s'P's''P's$

[26] See also Hatfield and Milgrom (2005) and Aygün and Sönmez (2013).

DEFINITION 8. A gradual outcome $(\Xi^t, \mu^t)_{t \leq T}$ is **gradually stable** if, for all $i \in I$

(GS1) $\mu^t(i) R_i^t \emptyset$ for all $t \leq t_i$,

(GS2) $\mu^t(i) R_i^t \mu^{t'}(i)$ for all $t' \leq t \leq t_i$,

(GS3) For all t' and t such that $t' \leq t \leq t_i$, if $|\mu^{t'}(s)| < \overline{q}_s^{t'}$ for some $s \in S$, then $s P_i^t \mu^t(i)$ implies $\emptyset \succ_s^p i$ for all positions $p \in \mathcal{B}_s^{t'}$ that are unassigned at t', and

(GS4) For all $j \in I$ such that $t_j \leq t_i$ and t such that $t_j \leq t \leq t_i$, if $v^{t_j}(j) = (s, p)$ for $p \in \mathcal{B}_s^{t_j}$, then $s P_i^t \mu^t(i)$ implies $j \succ_s^p i$.

The gradual stability incorporates the final assignment and the evolving claims individuals make throughout the sequential process, contingent upon their participation at each stage. This nuanced approach allows for a more comprehensive understanding of stability in sequential matching environments where applicants can update their ROLs from one stage to the next.

The first condition (GS1) requires each individual's assignment to be *individually rational* at each stage until they leave the problem. (GS2) states that it is gradually safe to participate in the next stage. Note that, in the sequential problem, proposals from previous periods also serve as an outside option.

The third condition is a sequential version of *non-wastefulness*. If an individual i prefers institution s to her stage t assignment, this condition ensures that all positions in s deemed acceptable by individual i must be assigned to someone else in the current period and the previous periods.

The last condition is a *no justified envy* condition adapted to our sequential environment with institutions' PSP choice structure. Consider two individuals, i and j, so j finalizes her assignment before i. Then, justified envy by i against j is checked for all periods $t_j \leq t \leq t_i$ using the priority order of the position assigned to j when she finalizes her assignment.

The theorem we propose in this section establishes a relationship between gradual stability, stage stability, and the monotonicity of gradual outcomes.

Theorem 2. *Let Γ be a regular refitting rule and φ be a stage mechanism. Then, $\mathcal{M}_\Gamma^\varphi$ is gradually stable if and only if $\mathcal{M}_\Gamma^\varphi$ is monotone and φ is stage stable.*

5 Admissions to Engineering Colleges in India

Let $\mathcal{R} = \{SC, ST, OBC, EWS\}$ denote the set of reserved categories. Students who belong to no reserved category are members of the General Category (GC). We denote the set of position categories in institutions by $\mathcal{C} = \{o, SC, ST, OBC, EWS\}$, where o denotes the open category. We denote the capacity of institution s by $\overline{q}_S$. The vector $q_s = \left(q_s^o, q_s^{SC}, q_s^{ST}, q_s^{OBC}, q_s^{EWS}\right)$ describes the *initial distribution* of positions over $\mathcal{C}$, where $q_s^o = \overline{q}_s - q_s^{SC} - q_s^{ST} - q_s^{OBC} - q_s^{EWS}$. The profile of vectors for institutions' initial distribution of positions over categories is denoted by $\mathbf{q} = (q_s)_{s \in \mathcal{S}}$.

The function $t : I \to \mathcal{R} \cup \{GC\}$ denotes the category membership of individuals. For every individual $i \in I$, $t(i)$, or t_i, denotes the category individual i belongs to. We denote a profile of reserved category membership by $T = (t_i)_{i \in \mathcal{I}}$, and let $\mathcal{T}$ be the set of all possible reserved category membership profiles.

For each category $c \in \mathcal{C}$, let κ_s^c denote the number of positions that are horizontally reserved for women in category c. Let $\kappa_s = (\kappa_s^o, \kappa_s^{SC}, \kappa_s^{ST}, \kappa_s^{OBC}, \kappa_s^{EWS})$ denote the vector of horizontally reserved positions in institution s with $\kappa_s^r \leq q_s^r$ for all $r \in \mathcal{C}$.

Merit scores induce strict meritorious ranking of individuals at each institution s, denoted by $\succ_s$, which is a linear order over $\mathcal{I} \cup \{\emptyset\}$. The profile of institutions' priorities is denoted $\succ = (\succ_{s_1}, ..., \succ_{s_m})$. The merit ordering for individuals of type $r \in \mathcal{R}$, denoted by $\succ_s^r$, is obtained from $\succ_s$ in a straightforward manner. Individuals who do not belong to category-r are removed and $\succ_s$ is maintained among category-r individuals.

The *over-and-above* implementation of vertical reserves requires filling open-category positions before the reserved categories. Moreover, the *minimum guarantee* implementation of horizontal reserves for women in each vertical category requires filling positions that are horizontally reserved for women before horizontally unreserved slots.

Formally, given an initial distribution of positions q_s over vertical categories, a vector of horizontal reservations κ_s, a set of applicants $A \subseteq I$, and a category membership profile $T \in \mathcal{T}$ for the members of A, the set of chosen applicants $C_s^{\text{Eng}}(A, q_s)$, is computed as follows:

Step 1.1: Consider open-category positions and women applicants only. Choose κ_s^o highest

merit-score women if there are more than κ_s^o women in A. Otherwise, choose all of them.

Step 1.2: Consider open-category positions and the remaining applicants. Individuals are chosen one at a time following $\succ_s$ up to the capacity q_s^o. Let $C_s^o(A, q_s^o)$ be the set of chosen applicants in Steps 1.1 and 1.2.

Step 2.1: Among the remaining applicants $A' = A \backslash C_s^o(A, q_s^o)$, for each reserve category $r \in \mathcal{R}$, consider women applicants only. Choose κ_s^r highest merit-score women for category r if there are more than κ_s^r women in A'. Otherwise, choose all of them.

Step 2.2: Among the remaining applicants, for each reserve category $r \in \mathcal{R}$, individuals are chosen one at a time following $\succ_s^r$ up to the capacity q_s^r. Let us call the set of chosen applicants in steps 2.1 and 2.2 for reserve category r as $C_s^r(A', q_s^r)$.

Then, $C_s^{Eng}(A, q_s)$ is defined as the union of the applicants chosen in Steps 1.1, 1.2, 2.1, and 2.2. That is,

$$C_s^{Eng}(A, q_s) = C_s^o(A, q_s^o) \cup \bigcup_{t \in \mathcal{R}} C_s^t\left(A', q_s^t\right)$$

One can establish an outcome-equivalent PSP choice rule to C_s^{Eng} by adjusting the priorities of positions as follows:

- Open-category positions subject to horizontal reservations for women systematically prioritize female candidates over their male counterparts, albeit adhering to intra-group merit-score rankings.

- Open-category positions—not reserved horizontally for women—ranks individuals according to a pure merit-scores.

- Positions reserved for specific vertical category $r \in \mathcal{C} \setminus \{o\}$ that are subject to horizontal reservations for women prioritize women within the same category over men, again abiding by merit-score rankings within each respective group. Such positions deem applicants not belonging to category r unacceptable.

- Category-r positions that are not horizontally reserved for women exclusively rank individuals within category-r based on their merit scores and categorically excluding applicants from other categories.

5.1 The Joint Seat Allocation Mechanism via MRDA

Baswana et al. (2019a) proposed and implemented the MRDA mechanism for centralized admissions for technical colleges. The algorithmic details of the MRDA mechanism is given in Baswana et al. (2019b).

Students submit a single application for both IITs and non-IITs, indicating their *ROLs for programs* across all participating institutions. The matching is between applicants and programs, where each technical college has multiple programs. Students' exam scores from a common entrance test, Joint Entrance Examination (JEE), are used for to rank applicant for both IITs and non-IITs. Different programs may use different weighted average of the test scores across different subject areas such as math, chemistry, and physics.

In each round of the MRDA, the Deferred Acceptance (DA) algorithm is employed to allocate students to programs based on their submitted Rank Order Lists (ROLs) and the programs' rankings, while adhering to affirmative action constraints. This process is executed concurrently for both IITs and non-IITs within the same round. Consequently, an applicant may be offered seats from both an IIT and a non-IIT program in a single round. The primary determinant in seat allocation is the candidate's integrated ROL. If the candidate has ranked an IIT seat higher than a non-IIT seat in their ROL, the algorithm will assign the IIT seat to the candidate. Conversely, if the candidate has ranked a non-IIT seat higher than an IIT seat in their ROL, the algorithm will assign the non-IIT seat to the candidate. This ensures that the allocation process prioritizes the candidate's preferences, as indicated in their submitted ROL.

Once a student is assigned a seat, her integrated ROL is updated depending on the option she chooses as follows:.

- Freeze: The candidate's preference list is truncated after the allotted program.
- Float: The candidate's preference list remains unchanged.
- Slide: The candidate's preference list is modified to include only programs from the same institute as the current allocation, ranked above the allotted program.
- Withdraw/Reject: The candidate is removed from the system, and their allotted seat (if

any) becomes vacant.[27]

If seats become available at either an IIT or a non-IIT program due to rejections or withdrawals, they are filled in the subsequent round by students who choose to participate in.

With the updated inputs, the DA algorithm is run again to compute the allocation for the next round. These steps are repeated for a predetermined number of rounds or until no further changes occur.

LEMMA **3.** *The permitted refitting rule in MRDA is regular.*

We can use Lemma 3 with Theorems 1 and 2 to state our final result.

Theorem 3. *The MRDA mechanism implemented in India is monotone and gradually stable.*

6 Related Literature

We generalize the framework of H&I to a setting where institutions' choice rules need not be responsive. This is not just a mere theoretical generalization. This generalization is necessary to analyze sequential matching procedures under affirmative action policies. We present admissions to technical universities in India as our main application, while H&I study the no-longer-in-use French college admission system as their primary application. Our gradual stability notion generalizes the one introduced in H&I. A related stability concept was introduced in Pereyra (2013) for seniority-based allocation rules.

Baswana et al. (2019a) observe that applicants' welfare improved in every round of the MRDA and wrote

> "New semi-centralized, multi-period matching mechanism enjoys monotonicity across runs. The options available to candidates are only enhanced in going from one period to the next..."

We validate the above claim that the algorithm they introduced is monotone. Moreover, it is gradually stable.

[27]Note that the 'reject' and 'withdraw' options have the same effect on the ROLs. The difference is about the timing. A candidate who previously accepted an offered program may withdraw in later stages.

The PSP framework of K&S provides a powerful tool for market designers to handle diversity and affirmative action constraints in two-sided matching models. Aygün and Bó (2021) design PSP choice rules for Brazilian college admission problem. Avataneo and Turhan (2021) extend the PSP framework to a more general one by defining PSP choice rules that allow transfers as in many real-world applications there are restrictions on slots' priorities.

Our paper is also related to Kojima and Manea (2010) that provide characterization results for the IODA mechanism. The authors' central axiom is built on the notion of *individually rational monotonic transformation* (i.r.m.t) of a preference relation. Their first characterization result establishes that the IODA mechanism satisfies IR monotonicity when the choice rules are substitutable and acceptant. Theorem 1 of H&I and our Theorem 1 can be seen as an extension of Kojima and Manea (2010)'s result in a sequential framework.

This paper adds to the growing literature on matching market design with comprehensive affirmative action constraints in India. Echenique and Yenmez (2015) is the first paper to consider object allocation problems in India from a market design perspective. Aygün and Turhan (2017) and Aygün and Turhan (2020) shed light on IIT admissions, emphasizing the issues regarding the allocation mechanisms implemented. Aygün and Turhan (2022) introduce a novel choice rule with backward transfers and the individual-optimal DA mechanism coupled with these choice rules to jointly implement reservations and de-reservations.[28] Aygün and Turhan (2024) consider a general matching model with contracts to encompass vertical and horizontal reservations and the de-reservation policy in a setting where individuals' preferences are expressed over institution-vertical category pairs.

Finally, our paper contributes to the matching literature that studies diversity and affirmative action in real-world resource allocation problems. Noteworthy papers in this domain include Abdulkadiroğlu (2005), Hafalir et al. (2013), Ehlers et al. (2014), Kamada and Kojima (2015), Doğan (2016), Bó (2016), Kamada and Kojima (2017),Imamura (2023), Correa et al. (2021), Doğan and Yıldız (2022), Hafalir et al. (2022), and Doğan et al. (2022) and among others.

[28]Meanwhile, Aygün and Turhan (2023) introduces a simpler choice rule that is outcome equivalent to the Backward Transfer choice rule introduced in Aygün and Turhan (2022).

7 Conclusion

Gradual assignment procedures have been implemented in recent years, particularly in school admissions and allocation of government jobs in countries such as India and Brazil, where complicated affirmative action policies are in effect. The multifaceted nature of these multi-round procedures, especially under diversity requirements, demands a nuanced understanding and analysis. Building on H&I and K&S, this paper attempts to do so.

Baswana et al. (2019a) designed and implemented a new admissions mechanism for India's technical colleges by collaborating with policymakers, which affects the assignment of thousands of applicants. In this paper, among our other contributions, we showed that the gradual mechanism they introduced is gradually stable and monotone. Our results are external validation for their multi-period admissions mechanism.

Appendix

Proof of Proposition 1

Proof. Let C_s^t be a choice rule that is generated by $(\succ_s^t, \rhd_s^t)$ and C_s^{t+1} is updated as per Definition 4. Let $i \in A \cap I^{t+1}$ such that $v^t(i) = (s, p_s^j)$ for some $p_s^j \in \mathcal{B}_s^t$. As i is active at stage $t+1$, $p_s^j \in \mathcal{B}_s^{t+1}$. Since i is $\succ_s^j$-maximal at stage t of the surviving set of individuals at Step j, it must be that i is $\succ_s^j$-maximal at stage $t+1$ as no individual is added to the set. Thus, $v^{t+1}(i) = (s, p_s^k)$ for some $k \leq j$. This completes the proof. □

Proof of Theorem 1

Proof. Let $\Xi = (I^1, S, (\mathbf{P}_i)_{i \in I}, (\succ_s^1, \rhd_s^1, \overline{q}_s^1)_{s \in S})$ be a gradual matching problem with the outcome $\mathcal{M}_\Gamma^\varphi(\Xi) \equiv (\Xi^t, \mu^t)_{1 \leq t \leq T}$ such that φ is individual-optimal stage stable mechanism.

To prove the "only if" part, let $\mathcal{M}_\Gamma^\varphi$ be monotone and Γ be not regular. That is, there exists a Ξ such that for some $t \leq T-1$, and $i \in I^t$, $P_i^{t+1} \notin \Gamma(P_i^t, \mu^t(i))$. The counter-example provided in Proposition 1 of H&I suffices as responsive choice functions are a special case of PSP choice rules.

To prove the "if" part, let Γ be a regular refitting rule. Consider Ξ^t for some $t \geq 2$. Let $\mu^{t-1} = \varphi(\Xi^{t-1})$ and $\mu^t = \varphi(\Xi^t)$ be stage matchings at periods $t-1$ and t, respectively. We now establish that for all $i \in I^t$, $\mu^t(i) R_i^t \mu^{t-1}(i)$, which is based on the proof of Theorem 1 of Kojima and Manea (2010)[29].

Let x_0 be the allocation μ^{t-1} restricted to the set of active players I^t. That is, for each institution s,

$$x_0(s) = \mu^{t-1}(s) \cap I^t.$$

Now, for all $i \in I^t$, let

$$x_1(i) = \begin{cases} x_0(i), & \text{if } x_0(i) P_i^t \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We define the sequence $(x_k)_{k \geq 1}$ as follows: for all $k \geq 1$ and $i \in I^t$,

$$x_{k+1}(i) = \begin{cases} s_k, & \text{if } i \in C_{s_k}^t \left(x_k(s_k) \cup \left\{ j \in I^t \mid s_k P_j^t x_k(j) \right\} \right), \\ x_k(i), & \text{otherwise.} \end{cases}$$

where s_k is an arbitrary institution that forms a blocking pair (s_k, i), if allocation x_k is blocked at $(P_{I^t}^t, (C_s^t)_{s \in S})$. If x_k cannot be blocked, then $x_{k+1} = x_k$.

If x_1 is a stable matching at $(P_{I^t}^t, (C_s^t)_{s \in S})$, then using the fact that φ generates individual optimal stable matching, $\mu^t(i) R_i^t x_1(i)$ for all $i \in I^t$ holds and we are done.

Our following lemma establishes that each x_k is well-defined for all $t \geq 0$. This sequence is a variant of the vacancy chain dynamics of Blum et al. (1997).

LEMMA **4.** *The sequence $(x_k)_{k \geq 0}$ satisfies for every $k \geq 0$:*

(I) x_k is a feasible stage matching.

(II) $x_k(i) R_i^t x_{k-1}(i)$ for all $i \in I^t$.

(III) $x_k(s) \subseteq C_s^t \left(x_k(s) \cup \left\{ j \in I^t \mid s P_j^t x_k(j) \right\} \right)$, for all $s \in S$.

[29]The'only if' part of Theorem 1 proves that ϕ^C satisfies IR monotonicity, where C is acceptant and substitutable and ϕ is deferred acceptance rule.

Proof of Lemma 4. We prove Lemma 4 using induction with the base case $k = 1$.

For $k = 1$, (I) and (II) hold by definition of x_1. Consider an institution $s \in S$. We now prove that $x_1(s) \subseteq C_s^t\left(x_1(s) \cup \left\{j \in I^t \mid sP_j^t x_1(j)\right\}\right)$.

By definition, we have $x_1(s) \subseteq x_0(s) \subseteq \mu^{t-1}(s)$. Also, as Γ is regular, it must be that

$$\{j \in I^t \mid sP_j^t x_1(j)\} \subseteq \{j \in I^t \mid sP_j^{t-1} x_0(j)\}.$$

Thus, we have

$$x_1(s) \cup \{j \in I^t \mid sP_j^t x_1(j)\} \subseteq x_0(s) \cup \{j \in I^t \mid sP_j^{t-1} x_0(j)\}. \tag{1}$$

As the period $t-1$ outcome is stable at $(P_{I^{t-1}}^{t-1}, (C_s^{t-1})_{s \in S})$, we have

(i) $C_s^{t-1}(\mu^{t-1}(s)) = \mu^{t-1}(s)$ for all $s \in S$, and

(ii) $i \notin C_s^{t-1}(\mu^{t-1}(s) \cup \{i\})$, for all $i \in \{j \in I^t \mid sP_j^{t-1} x_0(j)\}$.

By consistency of C_s^{t-1}, we have

$$C_s^{t-1}(\mu^{t-1}(s) \cup \{j \in I^t \mid sP_j^{t-1} x_0(j)\}) = \mu^{t-1}(s).$$

Let $A = \mu^{t-1}(s)$ and $B = \mu^{t-1}(s) \cup \{j \in I^t \mid sP_j^{t-1} x_0(j)\}$. Since, $A \subseteq C_s^{t-1}(B)$, by Proposition 1, we have

$$x_0(s) \subseteq C_s^t(x_0(s) \cup \{j \in I^t \mid sP_j^{t-1} x_0(j)\}).$$

Because C_s^t satisfies the substitutes condition, we also get

$$x_1(s) \subseteq C_s^t\left(x_1(s) \cup \left\{j \in I^t \mid sP_j^t x_1(j)\right\}\right).$$

This concludes our proof for the base case.

Assuming the conclusions of step $k \geq 1$ hold, we now prove it for $k+1$ (the only case to prove is when $x_k \neq x_{k+1}$).

Let us prove (I) first. Consider $s \neq s_k$. Observe that $x_{k+1}(s) \subseteq x_k(s)$ by construction. As x_k is an allocation, by the inductive hypothesis, we have $|x_{k+1}(s)| \leq |x_k(s)| \leq q_s^t$.

By the inductive hypothesis, $x_k(s_k) \subseteq C^t_{s_k}\left(x_k(s_k) \cup \left\{j \in I^t \mid s_k P^t_j x_k(j)\right\}\right)$ holds for institution s_k at k. Then using definition of $x_{k+1}(i)$,

$$x_{k+1}(s_k) = C^t_{s_k}\left(x_k(s_k) \cup \left\{j \in I^t \mid s_k P^t_j x_k(j)\right\}\right).$$

Feasibility of choice rule $C^t_{s_k}$ thus guarantees that $|x_{k+1}(s_k)| \leq q^t_{s_k}$.

We now prove (II). Observe that

$$x_{k+1}(s_k) \backslash x_k(s_k) \subseteq \left\{j \in I^t \mid s_k P^t_j x_k(j)\right\}. \tag{2}$$

Thus, for $j \in x_{k+1}(s_k) \backslash x_k(s_k)$, we get

$$s_k = x_{k+1}(j) P^t_j x_k(j). \tag{3}$$

Each agent outside of $x_{k+1}(s_k) \backslash x_k(s_k)$ is assigned the same institution under x_{k+1} and x_k. Therefore, $x_{k+1}(i) R^t_i x_k(i)$ for all $i \in I^t$.

We now show (III) for all $s \neq s_k$. By construction, we have $x_{k+1}(s) \subseteq x_k(s)$. and by Equation 3, we have $\left\{j \in I^t \mid s P^t_j x_{k+1}(j)\right\} \subseteq \left\{j \in I^t \mid s P^t_j x_k(j)\right\}$. Therefore,

$$x_{k+1}(s) \cup \{j \in I^t \mid s P^t_j x_{k+1}(j)\} \subseteq x_k(s) \cup \{j \in I^t \mid s P^t_j x_k(j)\}. \tag{4}$$

Now, because C^t_s satisfies the substitutes condition, inductive hypothesis for k (condition (II)), and Equation 4 implies

$$x_{k+1}(s) \subseteq C^t_s\left(x_{k+1}(s) \cup \left\{j \in I^t \mid s P^t_j x_{k+1}(j)\right\}\right).$$

Let us now consider institution s_k. By Equation 2, agents in $x_{k+1}(s_k) \backslash x_k(s_k)$ prefer s_k over their allocation in x_k. Individuals not chosen from this set in this iteration still prefer s_k over their allocation in x_{k+1}. This is because $x_{k+1}(i) R^t_i x_k(i)$ for all $i \in I^t$. This implies

$$x_{k+1}(s_k) \backslash x_k(s_k) = \left\{j \in I^t \mid s_k P^t_j x_k(j)\right\} \setminus \left\{j \in I^t \mid s_k P^t_j x_{k+1}(j)\right\}.$$

Equivalently,

$$x_{k+1}(s_k) \cup \left\{j \in I^t \mid s_k P^t_j x_k(j)\right\} = x_k(s_k) \cup \left\{j \in I^t \mid s_k P^t_j x_{k+1}(j)\right\}. \tag{5}$$

Using substitutability of C_s^t, (III) for k and Equation 5, we obtain

$$x_{k+1}(s_k) \subseteq C_{s_k}^t \left(x_{k+1}(s_k) \cup \left\{ j \in I^t \mid s_k P_j^t x_{k+1}(j) \right\} \right).$$

This concludes our proof of Lemma 4.

As $x_{k+1}(i) R_i^t x_k(i)$ for all $i \in I^t$, the sequence $(x_k)_{k \geq 0}$ converges to a matching x_K in a finite number of steps K. Each iteration results in a different allocation if the initial matching within the iteration is unstable. Hence, x_K is stable at $(P_{I^t}^t, (C_s^t)_{s \in S})$.

Because $x_K(i) R_i^t x_1(i) R_i^t \emptyset$ for all $i \in I^t$, the matching x_K is individually rational for agents. Also, as $\mathcal{M}_F^{\varphi}(\Xi)$ is individual optimal among all the stable outcomes, it must be true that

$$\mu^t(i) R_i^t x_K(i) R_i^t \mu^{t-1}(i).$$

□

Proof of Theorem 2

We refer to the conditions of stage stability in Definition 1 as S1, S2, and S3, and the conditions of gradual stability in Definition 8 as GS1, GS2, GS3, and GS4, respectively.

We first show, given a regular Γ, if $\mathcal{M}_\Gamma^{\varphi}$ is gradually stable, then $\mathcal{M}_\Gamma^{\varphi}$ is monotone and φ is stage stable. Condition C1' implies the monotonocity of $\mathcal{M}_\Gamma^{\varphi}$. Therefore, we only need to prove that φ is stage stable. We will prove it by showing that Definition 8 implies Definition 1.

S1 directly follows from GS1.

Toward a contradiction, suppose S2 is not true. That is, there exists an $s \in S$ such that $\mu^t(s) \subsetneq C_s(\mu^t(s))$. As $|\mu^t(s)| \leq \overline{q}_s$, this implies $|C_s(\mu^t(s))| < \overline{q}_s$. Suppose $i \notin C_s(\mu^t(s))$. Since C_s is a PSP choice rule, at each position $p_k \in \left\{ p_1, \ldots, p_{\overline{q}_s} \right\}$, either $\emptyset \succ_s^{p_k} i$ or there exists some other individual j such that $\hat{\mu}(j) = (s, p_k)$ and $j \succ_s^{p_k} i$. Both cases contradicts with our supposition that $(\Xi^t, \mu^t)_t$ is gradually stable. Thus, S2 holds.

Toward a contradiction, suppose S3 is not true. That is, there exists an individual-institution pair (i, s) such that $s P_i^t \mu^t(i)$ and $i \in C_s^t(\mu^t(s) \cup \{i\})$. Let i be assigned to the position $p_s^k \in \mathcal{B}_s$. If p_s^k is unassigned at μ, then GS3 is violated and if $\hat{\mu}(j) = (s, p_s^k)$, then GS4 is violated. Thus, S3 holds, and φ is stage stable.

We now prove that if $\mathcal{M}_\Gamma^\varphi$ is monotone and φ is a stage stable mechanism, then $\mathcal{M}_\Gamma^\varphi$ is gradually stable. Definition 1 implies GS1. Moreover, monotonicity of $\mathcal{M}_\Gamma^\varphi$ implies GS2.

Toward a contradiction, suppose GS3 is violated. Then, there exists an institution s and individual i such that $sP_i^t\mu^t(i)$. Also, for some unassigned position $p \in \mathcal{B}_s$ with $i \succ_s^p \emptyset_s$. This implies $i \in C_s^t(\mu^t(s) \cup \{i\})$. It is a contradiction. Thus, GS3 holds.

Finally, toward a contradiction, suppose GS4 fails to hold. Then, there exists $j \neq i$ such that (1) $t_j \leq t_i$, (2) $\hat{\mu}^{t_j}(j) = (s, p)$ for some $p \in \mathcal{B}_s^{t_j}$, (3) $\mu_j^t(i) \neq s$ for t such that $t_j \leq t \leq t_i$, and (4) $i \succ_s^p j$. This implies $i \in C_s^t(\mu^t(s) \cup \{i\})$. If $sP_i^t\mu^t(i)$, C3 is violated at stage t. This is a contradiction. Thus, Definition 8 is true for all $t_j \leq t \leq t_i$.

Proof of Lemma 3

This lemma is validated by formulating the "options" available to the students.

Freeze: $\Gamma(P, s) = P'$ where $\emptyset P' s'$ for all $s' \in S$ such that $s'Ps,$ and PandP' agree for the rest.

Float: $\Gamma(P, s) = P$.

Slide: $\Gamma(P, s) = P'$ where $\emptyset P' s'$ for all $s' \notin U(s)$ such that $s'Ps,$ and P, P' agree for the rest.

Reject/Withdraw: $\Gamma(P, s) = P',$ where $\emptyset P' s'$ for all $s' \in S$.

For each option, the induced refitting rule $\Gamma(P, \cdot)$ is regular.

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