This dissertation has been 63–7250
microfilmed exactly as received

DECKERT, Kenneth Laurel, 1931–
SOLUTIONS FOR NONLINEAR DIFFUSION EQUATIONS WITH INTEGRAL TYPE BOUNDARY CONDITIONS.

Iowa State University of Science and Technology
Ph.D., 1963
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

SOLUTIONS FOR NONLINEAR DIFFUSION EQUATIONS WITH INTEGRAL TYPE BOUNDARY CONDITIONS

by

Kenneth Laurel Deckert

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1963

TABLE OF CONTENTS

ABSTRACT

In this paper we study the solutions of diffusion equations of the type $u_{xx} - u_t = F(u,x,t)$ which satisfy an integral condition of the type $\int_0^1 u(x,t)dx = h(t)$ where $h(t)$ is a given continuous function and $0 \leq x \leq 1$, $t \geq 0$ is the domain of definition of $u(x,t)$.

For the linear homogeneous case, $F \equiv 0$, we show that; if (i) $g(x)$ is a continuous function represented by its Fourier cosine series on $0 \leq x \leq 1$, (ii) $f(t)$ is continuous for $t \geq 0$ and (iii) $h(t)$ is continuous for $t \geq 0$ and there is a continuous function $H(t)$ such that $h(t) - h(0) = \int_0^t H(r)R(t-r)dr$ where $R(t) = \sum_{k=1}^{\infty} \exp[-(2k-1)^2\pi^2 t]$, then there is a unique solution of $u_{xx} - u_t = 0$, continuous on $0 \leq x \leq 1$, $t \geq 0$ such that $u(x,0) = g(x)$, $u(0,t) = f(t)$ and $\int_0^1 u(x,t)dx = h(t)$. Similar results are obtained for the linear non-homogeneous case; $F(u,x,t) = f(x,t)$ with $f(x,t)$ continuous. We also give explicit integral representations of the solutions.

For the nonlinear problem with classical boundary conditions, we develop an equivalent integral equation and show that the integral equation has a unique solution provided that F(u,x,t) satisfies a uniform Lipschitz condition in u. We then apply these results to a particular nonlinear problem which arises in the study of bioelectrodes.

I. INTRODUCTION

This thesis grew out of a study of a problem considered by Victor W. Bolie[1] in his as yet unpublished paper; An Impedance Theory for Bioelectrodes. In that work, Bolie develops the following system of nonlinear diffusion equations to describe the electric field E and average ionic concentration g between two electrodes immersed in a weak aqueous salt solution and driven by a sinusoidal source:

1.1

$$\frac{ps}{f}\frac{\partial^2 g}{\partial x^2} - \frac{1}{r}\frac{\partial g}{\partial t} = \frac{e}{4f}\frac{\partial^2}{\partial x^2}(E^2) \quad ,$$

$$\frac{ps}{f}\frac{\partial^2 E}{\partial x^2} - \frac{1}{r}\frac{\partial E}{\partial t} = \frac{2f}{e}Eg + \frac{u_o}{re}\sin wt \quad .$$

Here p, s, f, r, e, u_o and w are known constants, t is time and the electrodes are thought of as two plates perpendicular to the x-axis at $x = -b/2$ and $x = b/2$. The following initial and boundary conditions were found to be appropriate.

1.2 $\quad E(x,0) = 0 \; , \; g(x,0) = g_o \; , \; -b/2 \le x \le b/2 \; ,$

---

[1]Chairman of the Biomedical Electronics department, Iowa State University, Ames, Iowa.

1.3 $$E(-x,t) = E(x,t)\ ,\ g(-x,t) = g(x,t)\ ,$$

$$-b/2 \leq x \leq b/2\ ,\ t \geq 0\ ,$$

1.4 $$E_t(b/2,t) = u_o/e\ ,\ t > 0\ ,$$

1.5 $$\int_{-b/2}^{b/2} g(x,t)\,dx = bg_o\ ,\ t \geq 0\ .$$

An interesting feature of this problem is the use of the Integral Condition 1.5 on g rather than the usual boundary condition. The author knows of no publications dealing with this type of condition for parabolic equations. We then ask ourselves the question, "does this problem have a solution, and if so, is it unique?" It is the purpose of this thesis to answer this question for the linear problem and certain types of quasilinear problems. At first it was hoped that this question could be answered for the above problem, but the term, $\partial^2/\partial x^2(E^2)$, in the right hand side of the first of Equations 1.1 gives a system which does not seem amenable to our method.

In what follows, we develop conditions under which unique solutions exist for the linear diffusion equation with an integral type boundary condition similar to 1.5. The results obtained show that this condition is, in a sense, stronger than the classical boundary condition. We also develop expressions for the solutions of a large class of such problems. In order to treat quasilinear problems similar to 1.1, we first develop methods of treating quasilinear equations of this general type with classical initial and boundary conditions and then modify them to apply to problems with integral type boundary conditions. The method we use makes use of the so called "fundamental" or "source" solution

1.6 $$K(x,t) = \frac{\exp(-x^2/4t)}{\sqrt{(4\pi t)}}$$

of the linear homogeneous heat equation

1.7 $$u_{xx} - u_t = 0 .$$

The function $K(x,t)$ is used as a kernel in an integral equation equivalent to the partial differential equation and associated initial and boundary conditions. Integral representations of solutions for the heat equation are well known and have been studied extensively by such authors as Fulks (5), Gevrey (6), Rothe (10) and Widder (13). A difficulty with our method is that solutions of the integral equations are easily seen to be solutions of the partial differential equation and associated conditions, but the converse is not immediately apparent. In case solutions of the partial differential equation and associated conditions are known to be unique, this difficulty is removed. Solutions of the integral equations which we use are unique but the proof of uniqueness of solutions of the nonlinear partial differential equation and associated conditions is quite elusive. We accomplish this is Theorem 4.2 by showing that under suitable conditions, any solution of the partial differential equation and associated conditions is also a solution of our integral equation.

II. BASIC THEOREMS AND NOTATION

In this chapter some well known results are collected for easy reference. Certain other results of a basic nature, some possible new, are also given.

In order to make the results more readily available for applications, we use only the Riemann integral. Of course more general results may be obtained by using more powerful tools such as the Lebesgue integral. Unless stated otherwise, we consider problems in one dependent and one space variable. This helps to simplify notation and makes the results easier to comprehend. Corresponding results for multi-dimension problems can be obtained, usually by simple changes in notation.

In what follows, R_c is the set of points for which $0 < x < 1$, $0 < t \leq c$ and $\overline{R_c}$ is the closure of R_c. We always use R for the domain $0 < x < 1$, $t > 0$ and $\overline{R}$ for its closure. If u is of class C^2 in a domain D and satisfies the linear homogeneous heat equation 1.7 in D, then we say that u belongs to H in D.

Theorem 2.1. If

(i) $f(x) \exp(-cx^2)$ is integrable on $(-\infty,\infty)$ for some $c > 0$,

(ii) $F(x,t) = \int_{-\infty}^{\infty} f(y)K(x-y,t)\,dy$, then F is defined and belongs to H for all x and $0 < t < 1/4c$; and at points y_o of continuity of f, $\lim_{x \to y_o,\, t \to 0^+} F(x,t) = f(y_o)$.

A proof of this theorem is given by Widder (13). It is also known that the integral representation of F(x,t) may be differentiated any number of times, each such differentiation resulting in an integral which converges uniformly in a neighborhood of the point (x,t), $t > 0$. Note that if f(y) is bounded and integrable, then the constant c in hypothesis (i) of Theorem 2.1 may be taken as small as we please so in that case, F(x,t) belongs to H for all $t > 0$.

Theorem 2.2. If

(i) f(x,t) is a bounded function of x for each $t \geq 0$,

(ii) f(x,t) is continuous for $t \geq 0$ except for at most finitely many values of x in every finite x interval,

(iii) $F(x,t) = \int_0^t \int_{-\infty}^{\infty} -f(y,r)K(x-y,t-r)\,dy\,dr$,

then $F_{xx} - F_t = f$ for $t \geq 0$ and x not on a line of discontinuity of f.

Proof. Let S be the set of values x for which f is continuous and for all x and $0 < r < t$, let G be the function defined by $G(x,t,r) = \int_{-\infty}^{\infty} f(y,r)K(x-y,t-r)\,dy$. Then by use of Theorem 2.1 we see that for each fixed $r > 0$, G is in H for all x and $t > r$. Furthermore $\lim_{t \to r^+} G(x,t,r) = f(x,r)$ for x in S. We can also show from the definition of a two dimensional limit that if $a \geq 0$, then for all x in S, $\lim_{t \to a,\ r \to a} G(x,t,r) = f(x,a)$. Hence if we define $G(x,t,t) = f(x,t)$ for $t \geq 0$ and x in S, then G is continuous for $0 \leq r \leq t$ and x in S. In view of the above, we see that $F(x,t) = -\int_0^t G(x,t,r)\,dr$ where the integral is proper for x in S. Therefore by a direct calculation, using the fact that G is in H, the conclusion follows.

The above representation for the solution of the non-homogenous heat equation on a finite range of x values was studied extensively by Gevrey (6). We give the essential result in the following corollary.

Corollary 2.2.1. If

(i) $f(x,t)$ is continuous and bounded for $a < x < b$ and $t \geq 0$,

(ii) $F(x,t) = -\int_0^t \int_a^b f(y,r)K(x-y,t-r)\,dy\,dr$,

then $F_{xx} - F_t = f$ for $a < x < b$ and $t \geq 0$, and $F(x,t) = 0$ if $x < a$ or $x > b$.

Proof. If we define a function f^* for $t \geq 0$ by $f^*(x,t) = f(x,t)$ for $a < x < b$ and $f^*(x,t) = 0$ for $x \leq a$ or $x \geq b$, then f^* satisfies the hypotheses of Theorem 2.2. If we substitute f^* for f in hypothesis (ii) above, the value of the integral will not change when the range of integration is extended from $-\infty$ to $+\infty$, so the result follows from Theorem 2.2.

In order to establish the next theorem, we need to know conditions under which the solution of the homogeneous heat Equation 1.7 on the whole x axis is unique. Theorem 2.1 shows that if $f(x)$ is continuous for all x and of order $\exp(cx^2)$ for some $c > 0$, then

$$2.1 \qquad u(x,t) = \int_{-\infty}^{\infty} f(y)K(x-y,t)\,dy$$

is a solution of $u_{xx} - u_t = 0$ which satisfies the initial condition $\lim_{t \to 0^+} u(x,t) = f(x)$. Now the function $v(x,t) = (x/t)K(x,t)$ is a solution which approaches zero with t for every fixed x, hence $u(x,t) + kv(x,t)$ is a solution of the initial value problem for every constant k. This shows that it is impossible to impose uniqueness conditions on $f(x)$ alone. However certain authors have been able to impose uniqueness conditions on u with respect to its behavior at $x = \pm \infty$. For example Goursat (7; 311) proves that if the functions $u(x,t)$ and $u_x(x,t)$ are of order $\exp(cx^2)$ for all $t > 0$ and some $c > 0$ then 2.1 is the unique solution of the problem. Titchmarsh (11; 282-283) shows that these two conditions can be replaced by the single condition that $u(x,t)$ is of order $\exp(c|x|)$; while Tychonoff, as quoted by Widder (13; 87-88), shows that $|x|$ can be replaced by x^2 but no higher power of x. More recently Cooper (2) has proved the same result under considerably weaker hypotheses. We use the result of Tychonoff and state it in the following theorem without proof.

Theorem 2.3. If

(i) $u(x,t)$ is in H for all x and $0 < t \leq c$,

(ii) $\lim_{x \to x_0,\ t \to 0^+} u(x,t) = 0$ for all x_0,

(iii) $\max_{0<t\leq c} |u(x,t)| = O(\exp ax^2)$ as $|x| \to \infty$ for some $a > 0$,

then $u(x,t) = 0$ in the strip $0 \leq t \leq c$.

We also need the following lemma.

Lemma 2.1. If

(i) $f(x)$ is bounded and integrable on any finite interval,

(ii) $f(x) = O(\exp ax^2)$ as $|x| \to \infty$ for some $a > 0$,

(iii) $u(x,t) = \int_{-\infty}^{\infty} f(y)K(x-y,t)\,dy$,

then $\max_{0<t\leq 1/8a} |u(x,t)| = O(\exp 2ax^2)$ as $|x| \to \infty$.

Proof. The constant 1/8a of the conclusion is chosen for convenience since a similar result holds for any constant less than 1/4a. Hypotheses (i) and (ii) imply the existence of a constant $M > 0$ such that $|f(x)| \leq M \exp(ax^2)$ for all x. Theorem 2.1 tells us that $u(x,t)$ exists for all x and $0 < t < 1/4a$. Thus we have the following estimate:

$$2.2 \qquad |u(x,t)| \leq M \int_{-\infty}^{\infty} \exp(ay^2)K(x-y,t)\,dy \quad .$$

If we use the explicit form of K given by Equation 1.6 and complete the square in the exponent of the integrand, we obtain the following result:

2.3 $$\frac{\exp(ay^2)\exp[-(x-y)^2/4t]}{\sqrt{(4\pi t)}} =$$

$$\frac{\exp(ax^2/(1-4at))}{\sqrt{(4\pi t)}}\exp\left[-\frac{1-4at}{4t}\left(y-\frac{x}{1-4at}\right)^2\right].$$

If we substitute 2.3 into 2.2, we get

2.4 $$|u(x,t)| \leq$$

$$\frac{M}{\sqrt{(4\pi t)}}\exp\frac{ax^2}{1-4at}\int_{-\infty}^{\infty}\exp\left[-\frac{1-4at}{4t}\left(y-\frac{x}{1-4at}\right)^2\right]dy \quad ,$$

which is valid for $0 < t < \frac{1}{4a}$. If we now make the change of variable $v^2 = [(1-4at)/4t][y-(x/1-4at)]^2$ and use the fact that $\int_{-\infty}^{\infty}\exp(-v^2)dv = \sqrt{\pi}$, we get

2.5 $$|u(x,t)| \leq \frac{M}{\sqrt{(1-4at)}}\exp\frac{ax^2}{1-4at} \quad .$$

For each x, the function on the right side of 2.5 is an increasing function of t for $0 < t < \frac{1}{4a}$, hence

2.6 $$\max_{0<t\leq 1/8a} |u(x,t)| \leq (\sqrt{2})M \exp(2ax^2) \text{ for all } x.$$

This is equivalent to the desired conclusion.

Theorem 2.4. If

(i) $u(x,t)$ is continuous for all x and $t \geq 0$,

(ii) $u(x,t) = O(\exp 2ax^2)$ as $|x| \to \infty$ for some $a > 0$ uniformly in t for $0 \leq t \leq 1/8a$,

(iii) $u_{xx} - u_t$ is continuous for all x and $t \geq 0$,

(iv) $u_{xx} - u_t = O(\exp bx^2)$ as $|x| \to \infty$ for some $b > 0$ uniformly in t for $0 \leq t \leq 1/8b$,

(v) $w(x,t) = \int_{-\infty}^{\infty} u(y,0)K(x-y,t)\, dy$,

then $-\int_0^t \int_{-\infty}^{\infty} [u_{yy}(y,r) - u_r(y,r)]K(x-y,t-r)\, dy\, dr = u(x,t) - w(x,t)$ for all x and $0 \leq t \leq 1/8c$ where $c = \max(a,b)$.

Proof. If we define $F(x,t) = -\int_0^t \int_{-\infty}^{\infty} [u_{yy}(y,r) - u_r(y,r)]K(x-y,t-r)\, dy\, dr + w(x,t)$, then, by Theorems 2.1 and 2.2, $F_{xx} - F_t = u_{xx} - u_t$ for all x and $0 \leq t < 1/4c$. From Lemma 2.1 we see that

2.7 $$\max_{0<t\leq 1/8c} |F(x,t)| = O(\exp 2cx^2) \text{ as } |x| \to \infty$$

and hypothesis (ii) says that

2.8 $$\max_{0<t\leq 1/8c} |u(x,t)| = O(\exp 2cx^2) \text{ as } |x| \to \infty.$$

Thus if we define $G(x,t) = F(x,t) - u(x,t)$, we see that G satisfies all the conditions of Theorem 2.3 in the strip $0 \leq t \leq 1/8c$ and hence $G(x,t) = 0$ in this same strip which completes the proof.

We conclude this chapter by listing some properties of solutions of the linear homogeneous heat equation. These are well known results readily available in the works of such authors as Churchill (1), Doetsch (3) and Epstein (4).

Theorem 2.5. If

(i) u belongs to H in R_c $(0 < x < 1,\ 0 < t \leq c)$,

(ii) u is continuous on $\overline{R_c}$,

then the maximum of u occurs on one of the three boundaries of $\overline{R_c}$ which are not in R_c.

In the classical boundary value problem associated with the linear homogeneous heat equation, we seek a function u continuous on $\bar{R}$ (R is the set $0 < x < 1$, $0 < t$) such that the following set of conditions are satisfied:

$$
\text{A} \quad
\begin{aligned}
u_{xx} - u_t &= 0 && \text{on } R, \\
u(x,0) &= g(x) && 0 \leq x \leq 1, \\
u(0,t) &= f_0(t) && 0 \leq t, \\
u(1,t) &= f_1(t) && 0 \leq t,
\end{aligned}
$$

g, f_o and f_1 continuous, $g(1) = f_1(0)$ and $g(0) = f_o(0)$.

The existence of a solution for Problem A is well known and Theorem 2.5 shows that this solution is unique.

In the sequel we use the solutions of the following problems.

$$A_0 \quad \begin{array}{ll} u_{xx} - u_t = 0 & \text{on } R, \\ u(x,0) = 0 & 0 \leq x < 1, \\ u(1,t) = 0 & 0 \leq t, \\ u(0,t) = f_0(t) & 0 < t. \end{array}$$

$$A_1 \quad \begin{array}{ll} u_{xx} - u_t = 0 & \text{on } R, \\ u(x,0) = 0 & 0 \leq x < 1, \\ u(0,t) = 0 & 0 \leq t, \\ u(1,t) = f_1(t) & 0 < t. \end{array}$$

In general the solutions of these problems are continuous on all of $\bar{R}$ only if $f_0(0) = 0$ for Problem A_0 and $f_1(0) = 0$ for Problem A_1. In case $f_0(0)$ and $f_1(0)$ are not zero, then the solutions are continuous on $\bar{R} - (0,0)$ for Problem A_0 and $\bar{R} - (1,0)$ for Problem A_1.

Theorem 2.6. If

(i) $f_1(t)$ is continuous for $t \geq 0$,

(ii) $Q(x,t) = \sum_{k=1}^{\infty} 2k\pi(-1)^{k+1}\exp(-k^2\pi^2 t)\sin k\pi x$

(iii) $u(x,t) = \int_0^t f_1(r)Q(x,t-r)\,dr$,

then u is in H on R, $u_1(0,t) = 0$ for $t \geq 0$, $\lim_{t \to 0^+} u(x,t) = 0$ for $0 \leq x < 1$ and $\lim_{x \to 1^-} u(x,t) = f_1(t)$ for $0 < t$. In case $f_1(0) = 0$, the above limits also hold at (1,0).

The proof of this theorem is well known and will not be given here. The details are given in Epstein (4; 227-231). In view of this theorem, if we define a function u_1 which is equal to u on R and has the limiting values of u on $\bar{R}$, then u_1 is the solution of Problem A_1. For simplicity, we adopt the convention that if a function u is defined by the equation $u(x,t) = \int_0^t f(r)Q(x,t-r)dr$, then $u(x,0) = 0$ and $u(1,t) = \lim_{x \to 1^-} \int_0^t f(r)Q(x,t-r)dr$ since u thus defined is continuous on $\bar{R}$.

Corollary 2.6.1. If

(i) $f_o(t)$ is continuous for $t \geq 0$,

(ii) $Q(x,t)$ is as in Theorem 2.6,

(iii) $U_o(x,t) = \int_0^t f_o(r)Q(1-x,t-r)dr$,

then u_o is in H on R, $u_o(1,t) = 0$ for $t \geq 0$, $\lim_{t \to 0^+} u_o(x,t) = 0$ for $0 < x \leq 1$ and $\lim_{x \to 0^+} u_o(x,t) = f_o(t)$ for $0 < t$. In case

$f_o(0) = 0$ the above limits also hold at $(0,0)$.

The function Q used above is related to the well known Jacobi theta functions. In fact $Q(x,t) = - 1/2\ \partial/\partial x\ \theta_3[(1-x/2),t]$ where θ_3 is the theta function of type 3 in the notation of Doetsch (3; 307). Because of the highly singular nature of Q at $t = 0$, we must use care when differentiating or integrating the integral, $\int_0^t f(r)Q(x,t-r)dr$. The following lemma shows that it is permissible to integrate this integral and then change the order of integration.

Lemma 2.2. If

(i) $f(t)$ is continuous for $t \geq 0$ and $f(0) = 0$,

(ii) $u(x,t) = \int_0^t f(r)Q(x,t-r)dr$,

then $\int_0^1 u(x,t)\ dx = \int_0^t f(r) \left[\int_0^1 Q(x,t-r)\ dx\right] dr$ for $t \geq 0$.

Proof. We proceed indirectly since the usual conditions for interchange of orders of integration do not hold. First we establish the result for the case where $f(t) = t^n$ for any positive integer n. If we let $v_n(x,t) = \int_0^t r^n Q(x,t-r)dr$, then v_n is the solution of Problem A_1 with $f_1(t) = t^n$. By the maximum principle, Theorem 2.5, $|v_n(x,t)| \leq t^n$ for $0 \leq x \leq 1$; hence $v_n(x,t)$ has a Laplace transform converging uniformly in x for $0 \leq x \leq 1$. If we define $V_n(x,s) = \int_0^\infty e^{-st}\, v_n(x,t)dt$, then an application of the convolution theorem gives

$$2.9 \qquad V_n(x,s) = n!/s^{n+1}\ (\sinh x\sqrt{s}/\sinh\sqrt{s}) ,$$

where $\sinh x\sqrt{s}/\sinh\sqrt{s}$ is the transform of $Q(x,t)$. The transforms of the theta functions and their derivatives are discussed in detail in Doetsch (3; 307-308, 141-144, 354-362). Now since the transform of $v_n(x,t)$ is uniformly convergent in x, we have the fact that $\int_0^1 V_n(x,s)\, dx =$ $\int_0^\infty e^{-st} \left[\int_0^1 v_n(x,t)\, dx\right] dt$ and hence by integrating 2.9 we obtain

$$2.10 \qquad \int_0^1 V_n(x,s)\, dx = n!/s^{n+1}\ [(\cosh\sqrt{s} - 1)/\sqrt{s}\sinh\sqrt{s}] .$$

Next if we use the results worked out in Doetsch (3, 141-144), we see that $\cosh\sqrt{s}/\sqrt{s}\sinh\sqrt{s}$ is the transform of $\theta_3(0,t)$ and $1/\sqrt{s}\sinh\sqrt{s}$ is the transform of $\theta_3(\frac{1}{2},t)$; hence inversion of Equation 2.10 using convolution gives

$$2.11 \qquad \int_0^1 v_n(x,t)\, dx = \int_0^t r^n R(t-r)\, dr ,$$

where $R(t) = \theta_3(0,t) - \theta_3(\frac{1}{2},t)$. Thus since $\int_0^1 Q(x,t)\, dx =$

$- \frac{1}{2} \int_0^1 \partial/\partial x\ \theta_3[(1-x)/2),t]\ dx = R(t)$, we have established the result for $f(t) = t^n$, n a positive integer, and in fact for any polynomial in t which vanishes at $t = 0$.

To extend the result to any continuous function f(t) which vanishes for $t = 0$, we can use the Weierstrass approximation theorem to uniformly approximate, for any $c > 0$, f(t) in the interval $0 \leq t \leq c$ by a sequence of polynomials each of which also vanishes at $t = 0$. Thus given any $\epsilon > 0$, there is a polynomial $p_\epsilon(t)$ such that $|f(t) - p_\epsilon(t)| < \epsilon$ for $0 \leq t \leq c$. We wish to show that

$$2.12 \qquad \int_0^1\int_0^t f(r)Q(x,t-r)\ dr\ dx \ = \int_0^t f(r)R(t-r)\ dr$$

is true for $0 \leq t \leq c$. To do this, we consider the following expression:

$$2.13 \qquad \left|\int_0^1\int_0^t f(r)Q(x,t-r)\ dr\ dx - \int_0^t f(r)R(t-r)\ dr\right| ,$$

which, since $\int_0^1\int_0^t p_\epsilon(r)Q(x,t-r)\ dr\ dx = \int_0^t p_\epsilon(r)R(t-r)\ dr$, may be written as

$$2.14 \qquad \left|\int_0^1\int_0^t (f(r)-p_\epsilon(r))Q(x,t-r)drdx+\int_0^t (p_\epsilon(r)-f(r))R(t-r)dr\right|.$$

The integral, $\int_0^t (f(r)-p_\varepsilon(r))Q(x,t-r)dr$, is the solution of Problem A_1 with $f_1(t) = f(t) - p_\varepsilon(t)$, consequently by the maximum principle we have $|\int_0^t (f(r)-p_\varepsilon(r))Q(x,t-r)dr| \leq |f(t)-p_\varepsilon(t)|$. Thus for $0 \leq t \leq c$, the absolute value 2.14 is not greater than

$$2.15 \qquad \int_0^1 |f(t)-p_\varepsilon(t)|\ dx + \int_0^t |f(r)-p_\varepsilon(r)||R(t-r)|\ dr\ .$$

The proof will be complete if we can show that $\int_0^t |R(t-r)|\ dr$ is bounded. From the definition of θ_3,

$$2.16 \qquad \theta_3(x,t) = 1 + 2\sum_{k=1}^{\infty} \exp(-k^2\pi^2 t) \cos 2k\pi x \quad ,$$

we find that

$$2.17 \qquad R(t) = 4\sum_{k=1}^{\infty} \exp[-(2k-1)^2\pi^2 t]\ .$$

Now $\int_0^t R(r)dr = \lim_{a\to 0} \int_a^t R(r)dr$ and since the series 2.17 is uniformly convergent for $0 < a \leq t \leq c$, we may integrate term-wise to get

$$2.18 \qquad \lim_{a\to 0} \int_0^t R(r)dr =$$

$$= \lim_{a \to 0} 4 \sum_{k=1}^{\infty} \frac{\exp[-(2k-1)^2\pi^2 a] - \exp[-(2k-1)^2\pi^2 t]}{(2k-1)^2\pi^2}$$

$$= 4 \sum_{k=1}^{\infty} \frac{1 - \exp[-(2k-1)^2\pi^2 t]}{(2k-1)^2\pi^2} \quad .$$

Now for any $t \geq 0$ and $k = 1, 2, \ldots,$ we have $1 - \exp[-(2k-1)^2\pi^2 t] \leq 1$ which reduces 2.18 to

2.19 $$\int_0^t R(r)dr \leq 4/\pi^2 \sum_{k=1}^{\infty} 1/(2k-1)^2 \leq 2/3 \quad .$$

Therefore the expression 2.15 is not greater than $5\epsilon/3$ for $0 \leq t \leq c$ which implies that 2.13 is less than any preassigned positive number which in turn establishes Equation 2.12.

III. THE LINEAR PROBLEM

In this chapter we develop the properties of linear diffusion problems with an integral type boundary condition of the form $\int_0^1 u(x,t)dx = h(t)$ where h is a given continuous function. This integral condition together with a given initial condition of a solution u is not sufficient for a unique solution. For example, if we let u be the solution of Problem A_0 with $f_0(t) = t$ and v the solution of A_1 with $f_1(t) = t$, then clearly $u \neq v$. On the other hand u and v have the same initial values and $\int_0^1 u(x,t)dx = \int_0^1 v(x,t)dx$. To see this we need only refer to Theorem 2.6 and Corollary 2.6.1 which show that u and v are both continuous on $\bar{R}$, $u(x,0) = v(x,0) = 0$ and $u(x,t) = v(1-x,t)$. A simple change of variables shows that $\int_0^1 v(x,t)dx = \int_0^1 v(1-y,t)dy = \int_0^1 u(y,t)dy$.

In what follows, we show that if in addition to the above integral and initial condition we specify the values of the solution along one boundary, then it is unique. Thus we are led to the consideration of the following problem; find a function u, continuous on $\bar{R}$, which satisfies the conditions:

$$u_{xx} - u_t = 0 \qquad \text{on } R,$$

$$\text{B} \qquad u(x,0) = g(x) \qquad 0 \leq x \leq 1,$$

$$u(0,t) = f(t), \quad \int_0^1 u(x,t)dx = h(t), \quad 0 \leq t,$$

f, g and h continuous for $0 \leq t$ such that

$$f(0) = g(0) \text{ and } h(0) = \int_0^1 g(x)dx \ .$$

We first show that if there is a solution of Problem B, then it is unique. It is sufficient to establish uniqueness for the following homogeneous problem:

$$u_{xx} - u_t = 0 \qquad \text{on } R,$$

$$B_0 \quad u(x,0) = 0 \qquad 0 \leq x \leq 1,$$

$$u(0,t) = \int_0^1 u(x,t)dx = 0 \qquad t \geq 0 \ .$$

The proof of this uniqueness depends on a famous theorem due to E. C. Titchmarsh (12). We quote the theorem below without proof.

Theorem (Titchmarsh). If

(i) $f(t)$ and $g(t)$ are Lebesgue integrable functions,

(ii) $\int_0^t f(r)g(t-r)dr = 0$ almost everywhere for

$0 < t < k$,

then $f(t) = 0$ almost everywhere for $0 < t < a$ and $g(t) = 0$ almost everywhere for $0 < t < b$ where $a + b \geq k$. The integral used in the above theorem is a Lebesgue integral, however the theorem is applicable to Riemann integrals when the functions involved are such that their absolute values are Riemann integrable.

Theorem 3.1. The only continuous solution of Problem B_0 is the trivial solution $u(x,t) = 0$ on $\bar{R}$.

Proof. Suppose that there is a non-zero solution v. Then by the maximum principle, the continuous function $v(1,t)$ is not identically zero. Now v is the unique solution of Problem A_1 with $f_1(t) = v(1,t)$, hence $v(x,t) = v(1,t)*Q(x,t)$. (We make use of the standard notation for convolution whenever convenient; that is, $f(t)*g(t) = \int_0^t f(r)g(t-r)dr$.) By Lemma 2.2, we have $\int_0^1 v(x,t)dx = v(1,t)*R(t)$, which by hypothesis vanishes for all $t \geq 0$. $R(t)$ is a non-zero, positive function, continuous for $t > 0$, such that $\int_0^t R(r)dr$ exists for all t, as shown in the proof of Lemma 2.2, and $v(1,t)$ is continuous; hence the theorem of Titchmarsh implies that $v(1,t) = 0$ for $t \geq 0$ which contradicts the assumption that $v(x,t)$ is not identically zero. Thus $v(x,t) = 0$ on $\bar{R}$ and the theorem is proved.

We turn now to existence of solutions of Problem B. We make use of the linearity of the partial differential equation to build up a solution using superposition. To this end we consider the following three problems:

$$B_1 \quad \begin{cases} u_{xx} - u_t = 0 & \text{on } R, \\ u(x,0) = 0 & 0 \le x \le 1, \\ u(0,t) = 0 & 0 \le t, \\ \int_0^1 u(x,t)dx = h(t) & 0 \le t,\ h(0) = 0. \end{cases}$$

$$B_2 \quad \begin{cases} u_{xx} - u_t = 0 & \text{on } R, \\ u(x,0) = 0 & 0 \le x \le 1, \\ u(0,t) = f(t) & 0 \le t,\ f(0) = 0, \\ \int_0^1 u(x,t)dx = 0 & 0 \le t. \end{cases}$$

$$B_3 \quad \begin{cases} u_{xx} - u_t = 0 & \text{on } R, \\ u(x,0) = g(x) & 0 \le x \le 1, \\ u(0,t) = g(0) & 0 \le t, \\ \int_0^1 u(x,t)dx = \int_0^1 g(x)dx & 0 \le t. \end{cases}$$

One would expect that continuity of f, g and h is sufficient for existence of solutions of the above problems, however this is not the case for Problem B_1.

Theorem 3.2. There is a function u, continuous on $\bar{R}$, which satisfies Problem B_1 if and only if there exists a continuous function $H(t)$ such that $h = H*R$.

Proof. First suppose u is a continuous solution of Problem B_1. We can represent u by $u(x,t) = u(1,t)*Q(x,t)$. Then Lemma 2.2 implies that $\int_0^1 u(x,t)dx = u(1,t)*R(t)$ and hence we may take $H(t) = u(1,t)$.

Conversely, suppose $H(t)$ is continuous and such that $h = H*R$. Then $u(x,t) = H(t)*Q(x,t)$ is a solution of Problem B_1 since by Theorem 2.6, $u(x,0) = u(0,t) = 0$; and by Lemma 2.2, $\int_0^1 u(x,t)dx = H(t)*R(t) = h(t)$.

We next ask, what class of functions is representable by $H*R$ for continuous H? This is a difficult question and cannot be simply answered. Work on this subject may be found in Hirschman and Widder (8; 146-162). The following theorem gives a partial answer to this question.

Theorem 3.3. If $h(t)$ is such that $h(0) = 0$ and its first derivative is bounded and integrable on every finite interval, then there is a continuous function H such that $h = H*R$.

Proof. We show that $H(t) = \int_0^t h'(r)[\theta_3(0,t-r) + \theta_3(1/2,t-r)]dr$ is such that $h = H*R$. For notational convenience we let $\theta_3(0,t) = a(t)$ and $\theta_3(1/2,t) = b(t)$, so that $R = a - b$ and $H = h'*(a+b)$. From the properties of the convolution integral, we have $H*R = h'*(a+b)*(a-b) = h'*(a*a-b*b)$. If we let $a*a = c$, then $L(c) = L(a)L(a)$ where

L(c) and L(a) are the Laplace transforms of c and a respectively. From Doetsch (3; 141-144) we see that $L(a) = \cosh\sqrt{s}/\sqrt{s}\sinh\sqrt{s}$ and $L(b) = 1/\sqrt{s}\sinh\sqrt{s}$, hence $L(a)L(a) = \cosh^2\sqrt{s}/s\sinh^2\sqrt{s} = 1/s + 1/s\sinh^2\sqrt{s}$, which shows that $L(c) = L(1) + L(b)L(b)$. Therefore $c(t) = 1 + b(t)*b(t)$ and we have shown that $H*R = h'*1$. Finally we note that $h'(t)*1 = \int_0^t h'(r)dr = h(t) - h(0) = h(t)$ and the proof is complete.

From the previous two theorems, we see that if h has a first derivative which is bounded and integrable on every finite sub-interval and if $h(0) = 0$, then the solution of Problem B_1 is $u(x,t) = h'(t)*[a(t)+b(t)]*Q(x,t)$.

To solve Problem B_2, we observe that if u is the solution of Problem A_0 with $f_0(t) = f(t)$, then $v(x,t) = u(1-x,t) - u(x,t)$ satisfies the initial and boundary condition of Problem B_2 and $\int_0^1 v(x,t)dx = 0$ since $\int_0^1 u(1-x,t)dx = \int_0^1 u(x,t)$. Therefore $v(x,t)$ is the solution of Problem B_2.

To solve Problem B_3, we assume that g in addition to being continuous is representable in a Fourier cosine series,

$g(x) = \sum_{k=0}^{\infty} a_k \cos k\pi x$, on the interval $0 \leq x \leq 1$. Now the function $w(x,t) = \sum_{k=0}^{\infty} a_k \exp(-k^2\pi^2 t) \cos k\pi x$ satisfies the heat equation and initial condition $w(x,0) = g(x)$. In addition, because the Fourier series may be integrated termwise, we have $\int_0^1 w(x,t)dx = \sum_{k=0}^{\infty} a_k \exp(-k^2\pi^2 t)\int_0^1 \cos k\pi x\, dx = a_0$.

By the definition of a Fourier cosine series, $a_0 = \int_0^1 g(x)dx$

hence $u(x,t) = w(x,t) - v(x,t)$ is the solution of Problem B_3 where $v(x,t)$ is the solution of Problem B_2 with $f(t) = -g(0) + \sum_{k=0}^{\infty} a_k \exp(-k^2\pi^2 t)$.

The solution of Problem B can now be constructed as a superposition of solutions of problems similar to B_1, B_2 and B_3. More precisely, we have proved the following theorem.

Theorem 3.4. If

(i) $g(x)$ is represented by its Fourier cosine series for $0 \leq x \leq 1$,

(ii) there is a continuous function H such that $f = H*R$,

then there exists a solution of Problem B.

We close this chapter with a consideration of the following non-homogeneous problem:

$$B_4 \quad \begin{array}{ll} u_{xx} - u_t = f(x,t) & \text{on } R, \\ u(x,0) = u(0,t) = 0 & 0 \leq x \leq 1,\ 0 \leq t, \\ \int_0^1 u(x,t)dx = 0 & 0 \leq t. \end{array}$$

Lemma 3.1. If

(i) $f(y)$ is a continuous even periodic function of period 2 for all y,

(ii) $g(y) = \int_0^1 f(x-y)dx$,

then $h(y) = g(y) - \int_0^1 f(x)dx$ is an odd periodic function of period 2.

Proof. The periodicity of h follows from the periodicity of g which in turn follows obviously from that of f. To show that h is odd, we let $G(y) = h(y) + h(-y) = g(y) + g(-y) - 2\int_0^1 f(x)dx$. Then if we let $u = y - x$ in the definition of g, we get $g(y) = \int_{y-1}^{y} f(u)du$ which shows that $g'(y) = f(y) - f(y-1)$. Therefore, $G'(y) = f(y) - f(y-1) - f(-y) + f(-y-1) = f(y-1) + f(y+1) = f(y-1) + f(y-1+2) = 0$, which implies that G is a constant and since $G(0) = 0$, G must be identically zero and the lemma is proved.

Lemma 3.2. If

(i) $f(x,t)$ is continuous for all x and $t \geq 0$,

(ii) $f(x,t)$ is an even periodic function of period 2 in x for each $t \geq 0$,

(iii) $u(x,t) = -\int_0^t \int_{-\infty}^{\infty} f(y,r)K(x-y),t-r)dy\, dr$ then

$$\int_0^1 u(x,t)dx = -\int_0^t \int_0^1 f(x,r)dx\, dr .$$

Proof. First we note that by the change of variable

$x - y = z$, $u(x,t) = -\int_0^t \int_{-\infty}^{\infty} f(x-z,r)K(z,t-r)dz\, dr$ and since f is bounded and K is positive that this integral is absolutely convergent. This together with the fact that u is continuous allows us to apply Fubini's theorem to get

$\int_0^1 u(x,t)dx = -\int_0^t \int_{-\infty}^{\infty} \left[\int_0^1 f(x-z,r)dx\right] K(z,t-r)dz\, dr$. Now by Lemma 3.1, $\int_0^1 f(x-z,r)dx = g(z,r) + \int_0^1 f(x,r)dx$ where $g(z,r)$ is an odd function of z for each $r \geq 0$. Thus since g is an odd and K is an even function of z, the integral, $\int_{-\infty}^{\infty} g(z,r)K(z,t-r)dz$ vanishes and we finally get the fact that

$\int_0^1 u(x,t)dx = -\int_0^t \int_{-\infty}^{\infty} \left[\int_0^1 f(x,r)dx\right] K(z,t-r)dz\, dr =$
$-\int_0^t \int_0^1 f(x,r)dx\, dr$.

Theorem 3.5. If $f(x,t)$ is continuous on $\bar{R}$, then there exists a unique solution of Problem B_4.

Proof. The solution is unique by Theorem 3.1. If we let $f^*(x,t)$ be the even extension on x of $f(x,t)$ for each t to the interval $-1 \leq x \leq 1$, and $F(x,t)$ the periodic extension of $f^*(x,t)$ for all x, then $F(x,t)$ satisfies the conditions of Lemma 3.1. Moreover, if we let $h(t) = \int_0^t \int_0^1 F(x,r)dx\, dr$, then h satisfies the conditions of Theorem 3.3 and hence there is a solution $w(x,t)$ of Problem B_1 such that $\int_0^1 w(x,t)dx = h(t)$. Therefore the function u defined by $u(x,t) = -\int_0^t \int_{-\infty}^{\infty} F(y,r)K(x-y,t-r)dy\,dr + w(x,t)$ is the solution of Problem B_4.

IV. A QUASI-LINEAR PROBLEM

In this chapter we develop an integral equation equivalent to the following quasi-linear problem with classical boundary conditions:

$$\text{C}\quad\begin{cases} u_{xx} - u_t = F(u,x,t) & \text{for } x,t \text{ in } R_c, \\ u(x,0) = g(x) & 0 \leq x \leq 1, \\ u(0,t) = f_0(t) & 0 \leq t \leq c, \\ u(1,t) = f_1(t) & 0 \leq t \leq c. \end{cases}$$

Then by use of the techniques of functional analysis, we show that the integral equation possesses a unique solution thereby establishing existence and uniqueness of the solution of Problem C.

The integral equation we use is

$$4.1 \qquad u(x,t) = -\int_0^t\int_0^1 F(u(y,r),y,r)K(x-y,t-r)\,dy\,dr$$
$$+ Q(x,t)*\int_0^t\int_0^1 F(u(y,r),y,r)K(1-y,t-r)\,dy\,dr$$
$$+ Q(1-x,t)*\int_0^t\int_0^1 F(u(y,r),y,r)K(-y,t-r)\,dy\,dr$$
$$+ w(x,t)\ ,$$

where $w(x,t)$ is the solution of Problem A and $Q(x,t)$ is as in Theorem 2.6.

By differentiation of Equation 4.1 and application of the known properties of K, Q and w, we can easily establish that any continuous solution of 4.1 is also a solution of Problem C. To establish the converse, we suppose that u is a continuous solution of Problem C such that u_{xx} and u_t are continuous and integrable on R_c. Then we replace $F(u,x,t)$ by $u_{xx} - u_t$ in Equation 4.1 to get a function H defined by:

4.2
$$\begin{aligned} H(x,t) = & -\int_0^t\int_0^1 (u_{yy}-u_r)K(x-y,t-r)\,dy\,dr \\ & + Q(x,t)*\int_0^t\int_0^1 (u_{yy}-u_r)K(1-y,t-r)\,dy\,dr \\ & + Q(1-x,t)*\int_0^t\int_0^1 (u_{yy}-u_r)K(-y,t-r)\,dy\,dr \\ & + w(x,t) \;. \end{aligned}$$

By Corollary 2.2.1 we have $H_{xx} - H_t = u_{xx} - u_t$ on R_c, and the properties of Q give us the facts that $H(x,0) = 0$, $H(0,t) = w(0,t)$ and $H(1,t) = w(1,t)$. If we recall that w is the solution of Problem A and u is a solution of Problem C, then $u(0,t) = w(0,t)$ and $u(1,t) = w(1,t)$, so that the function G defined by $G(x,t) = H(x,t) - u(x,t)$ satisfies the homogeneous linear heat equation with homogeneous initial and boundary conditions. Hence by the maximum principle (Theorem 2.5) G is identically zero on $\overline{R_c}$ which implies that $H(x,t) = u(x,t)$ on $\overline{R_c}$. We state these results in the next theorem.

Theorem 4.1. Any continuous solution of Equation 4.1 is a solution of Problem C and if u is a continuous solution of Problem C such that $u_{xx} - u_t$ is continuous and bounded on R_c, then u is a solution of Equation 4.1.

Now that we know under what conditions solutions of Problem C and Equation 4.1 correspond, we look for conditions under which there is a unique solution.

Theorem 4.2. If

(i) $F(u,x,t)$ is continuous for $0 \leq x \leq 1$, $t \geq 0$ and all values of u,

(ii) For each $c > 0$ there is a constant M such that $|F(u_1,x,t) - F(u_2,x,t)| \leq M|u_1-u_2|$ for (x,t) in $\overline{R_c}$ and any pair u_1, u_2,

then for each $c > 0$ there is a unique function u, continuous on $\overline{R_c}$ which satisfies Equation 4.1 there.

Proof. Let B_c be the real Banach space of functions $u(x,t)$ defined and continuous on $\overline{R_c}$ with the usual definitions of addition and scalar multiplication and norm given by $||u|| = \max_{(x,t) \text{ in } \overline{R_c}} |u(x,t)|$. For any u in B_c define the mapping T of B_c into B_c by

4.3 $$Tu(x,t) = -\int_0^t\int_0^1 F(u(y,r),y,r)K(x-y,t-r)\,dy\,dr$$

$$+ Q(x,t)*\int_0^t\int_0^1 F(u(y,r),y,r)K(1-y,t-r)\,dy\,dr$$

$$+ Q(1-x,t)*\int_0^t\int_0^1 F(u(y,r),y,r)K(-y,t-r)\,dy\,dr$$

$$+ w(x,t) .$$

In view of the previously established properties of K, Q and w, the mapping T is clearly a continuous transformation of B_c into B_c. We wish to show that there is a unique element u of B_c such that Tu = u.

We note that for any u and v in B_c,

$$\left|\int_0^t\int_0^1 [F(u(y,r),y,r)-F(v(y,r),y,r)]K(x-y,t-r)\,dy\,dr\right|$$

$$\leq M\int_0^t\int_0^1 |u(y,r)-v(y,r)|K(x-y,t-r)\,dy\,dr \leq M||u-v||t ,$$

where we have used hypothesis (ii) and the fact that $\int_0^1 K(x-y,t-r)dy \leq \int_{-\infty}^{\infty} K(x-y,t-r)dy = 1$. If we use the fact that $Q(x,t)*g(t)$ is the solution of Problem A_1 with $f_1(t) = g(t)$ and thus by the maximum principle $|Q(x,t)*g(t)| \leq |g(t)|$ we can show that $|Q(x,t)*\int_0^t\int_0^1 [F(u(y,r),y,r)-F(v(y,r),y,r]K(x-y,t-r)$

dy dr$|\leq M||u-v||t$. Therefore for any u and v in B_c we obtain the estimate

4.4 $$|Tu(x,t)-Tv(x,t)| \leq 3M||u-v||t \quad \text{for } (x,t) \text{ in } \overline{R_c} .$$

By induction we find that for any positive integer n,

4.5 $$|T^n u(x,t)-T^n v(x,t)| \leq [(3Mt)^n / n!]||u-v|| ,$$

and consequently

4.6 $$||T^n u-T^n v|| \leq [(3Mc)^n / n!]||u-v|| .$$

Since there is an integer N such that $[(3Mc)^N / N!] < 1$, we see that T^N is a contraction on B_c and by the generalized contraction mapping theorem (9; 50-51) there is a unique point u in Bc such that Tu = u. This completes the proof.

We note that any continuous solution u Equation 4.1 has the further property that $u_{xx} - u_t$ is continuous on $\overline{R_c}$. This follows by differentiation of Equation 4.1 and from the assumption that F(u,x,t) is continuous for all u and (x,t) in $\overline{R_c}$. Thus in the class of solutions satisfying the conditions of Theorem 4.1, there is a unique solution of Problem C.

V. AN APPLICATION

We return now to the nonlinear diffusion problem with integral condition given in Chapter I, Equations 1.1 - 1.5.

To simplify the notation in Equations 1.1, we replace x by $(bx/2)\surd(ps/f)$ and t by $b^2t/4r$ to get the equations:

$$u_{xx} - u_t = a(v^2)_{xx} ,$$

5.1

$$v_{xx} - v_t = cuv + d \sin kt ,$$

where $u(x,t) = g[bx/2)\surd(ps/f), b^2t/4r]$, $v(x,t) = E[(bx/2)\surd(ps/f), b^2t/4r]$, $a = b^2e/16f$, $c = b^2f/2e$, $d = b^2u_o/4re$ and $k = b^2w/4r$. The associated conditions 1.2 = 1.5 are then transformed to:

5.2 $v(x,0) = 0,\ u(x,0) = g_o,\ -1 \leq x \leq 1,$

5.3 $v(-x,t) = v(x,t),\ u(-x,t) = u(x,t),\ -1 \leq x \leq 1,\ t \geq 0,$

5.4 $v(1,t) = u_ot/e \quad t \geq 0,$

5.5 $\int_{-1}^{1} u(x,t)\, dx = 2g_o,\ t \geq 0,$

where we have integrated Equation 1.4 to get 5.4.

When this study began, the author had hoped to be able to demonstrate existence and uniqueness of a solution of Equations 5.1 - 5.5 by use of an equivalent set of integral equations. Such a set of integral equations does exist, but they are not amenable to our methods due to the presence of the term $(v^2)_{xx}$ in the first of Equations 5.1. We study instead a linearized version of these equations.

From the work of Bolie mentioned previously, we find the following orders of magnitude of the constants involved in Equations 5.1: $a \approx 10^{-21}$, $c \approx 10^8$ and $d \approx 10^{12}$. This leads us to consider the case where a = 0, since a is so much smaller than c and d. The resulting equations are:

$$\text{5.1}' \quad \begin{aligned} u_{xx} - u_t &= 0 \\ v_{xx} - v_t &= cuv + d \sin kt \,. \end{aligned}$$

Before we study these equations, we need to know some properties of the solutions of the following problem:

$$\text{D} \quad \begin{aligned} u_{xx} - u_t &= 0 && -1 < x < 1,\ t > 0 \,, \\ u(-x,t) &= u(x,t) && -1 \le x \le 1,\ t \ge 0 \,, \\ u(1,t) &= f(t) && t > 0 \,, \\ u(x,0) &= 0 && -1 \le x \le 1 \,. \end{aligned}$$

Lemma 5.1. If

(i) $f(t)$ is continuous for $t \geq 0$,

(ii) $P(x,t) = \partial/\partial x\ \theta_2[(1-x)/2,t]$,

then the unique solution of Problem D is $u(x,t) = \int_0^t f(r)P(x,t-r)dr$ where it is understood that $u(1,t) = \lim_{x \to \pm 1} \int_0^t f(r)P(x,t-r)dr$.

Proof. The form of the solution can easily be deduced by use of the Laplace transform. If we use the formal rules of the transform calculus, Problem D can be transformed to the following:

$$U_{xx}(x,s) - sU(x,s) = 0 \ ,$$

$$D' \quad U(-x,s) = U(x,s) \ ,$$

$$U(1,s) = F(s) \ ,$$

where U and F are the transforms of u and f respectively. The solution of D' is easily seen to be $U(x,s) = F(s) \cosh x\sqrt{s}/\cosh \sqrt{s}$. From Doetsch (3; 402) we find that $\cosh x\sqrt{s}/\cosh \sqrt{s}$ is the transform of $\partial/\partial x\ \theta_2[1-x)/2,t]$ valid for $-1 < x < 1$. Thus by convolution, the formal solution of D is seen to be $u(x,t) = f(t)*P(x,t)$. Of course the above calculations may not be valid in case f does not have a transform. Even so the formal solution obtained above can be validated indirectly by an appeal to the Weierstrass approximation theorem, since the formal calculations above are valid when f is a polynomial. We omit the details.

In Theorem 3.1 we showed that the integral condition, an initial condition and one boundary conditions are sufficient for a unique solution of the linear homogeneous diffusion equation. In the problem we are now considering, notice that an eveness condition, Equation 5.3, is used instead of the boundary condition. The following lemma shows that this condition is also sufficient for uniqueness.

Lemma 5.2. If

(i) $u_{xx} - u_t = 0, \quad -1 < x < 1,\ t > 0,$

(ii) $u(x,0) = 0, \quad -1 \leq x \leq 1,$

(iii) $u(-x,t) = u(x,t), \ -1 \leq x \leq 1,\ t \geq 0,$

(iv) $\int_{-1}^{1} u(x,t)dx = 0 \quad t \geq 0,$

then $u(x,t) = 0$ for $-1 \leq x \leq 1$ and $t \geq 0$.

Proof. Suppose that there is a function v satisfying hypotheses (i) - (iv) which is not identically zero, then by the maximum principle, $v(1,t)$ is not identically zero. Now v is also a solution of Problem B_2 with $f(t) = v(1,t)$. We have seen that the solution of Problem B_2 is given by $u(x,t) = v(1,t)*[Q(1-x,t)-Q(x,t)]$, hence by uniqueness we also have the fact that

$$5.6 \qquad v(x,t) = v(1,t)*[Q(1-x,t)-Q(x,t)], \; b \le x \le 1, \; t \ge 0.$$

By hypothesis v is an even function of x such that v_x is continuous at x = 0 and thus we also have $v_x(0,t) = 0$ for $t \ge 0$. Now if we differentiate Equation 5.6 we get

$v_x(0,t) = -\int_0^t v(1,r)[Q_x(1,t-r)+Q_x(0,t-r)]dr$, which by an application of Titchmarsh's theorem quoted in Chapter III is not identically zero, thus giving a contradiction which proves the lemma.

We now return to Equations 5.1' and the associated conditions 5.2 - 5.5. In view of Lemma 5.2, $u(x,t) = g_o$ is clearly the unique solution of the first of Equations 5.1' which satisfies the conditions of Equations 5.2, 5.3 and 5.5. If we substitute this into the second of Equations 5.1', we have the following linear equation whose steady state solution has already been studied by Bolie.

$$5.7 \qquad v_{xx} - v_t = cg_o v + d \sin kt .$$

This equation satisfies the conditions of Theorem 4.2 so a unique solution exists which is easily obtained by use of the Laplace transform. If we let V(x,s) be the transform of v(x,t), then Equation 5.7 and Conditions 5.2, 5.3 and 5.4 can be transformed to the following problem:

$$V_{xx}(x,s) - sV(x,s) = cg_oV(x,s) + kd/(s^2+k^2) ,$$

$$V(-x,s) = V(x,s) ,$$

$$V(1,s) = u_o/es^2 .$$

The solution of this problem is easily seen to be

$$5.8 \qquad V(x,s) = \frac{u_o}{es^2} + \frac{kd}{(s^2+k^2)(s+cg_o)} \cdot \frac{\cosh x\surd(s+cg_o)}{\cosh \surd(s+cg_o)} .$$

Thus using the convolution theorem, the solution of Equation 5.7 which satisfies Conditions 5.2 - 5.4 is

$$5.9 \qquad v(x,t) = \left[\frac{u_ot}{e} + (d \sin kt)*e^{-cg_ot}\right] * \left[e^{-cg_ot}P(x,t)\right] ,$$

where $P(x,t)$ is the function defined in Lemma 5.1.

The above considerations suggest a method of attacking Equations 5.1 - 5.5. We might define a sequence of approximations as follows. Assume that an n'th approximation $(u^{(n)},v^{(n)})$ has been found and define the n+1'st approximation as the solutions of the following equations

$$5.10 \qquad u_{xx}^{(n+1)} - u_t^{(n+1)} = a \frac{\partial^2}{\partial x^2} (v^{(n)})^2 ,$$

$$5.11 \qquad v_{xx}^{(n+1)} - v_t^{(n+1)} = cu^{(n)}v^{(n)} + d \sin kt \, .$$

If $\partial^2/\partial x^2(v^{(n)})^2$ is continuous on $\bar{R}$, then by Theorem 3.5, we know that the solution of 5.10 is continuous. If we assume $u^{(n)}$ and $v^{(n)}$ are continuous on $\bar{R}$, then Theorem 4.2 tells us that there is a continuous solution of 5.11. The difficulty with this technique is that it is difficult to determine the class properties of $\partial^2/\partial x^2(v^{(n)})^2$ if $v^{(n)}$ is the solution of 5.11. Furthermore, even if $\partial^2/\partial x^2(v^{(n)})^2$ has the required class properties, we do not know as yet whether the above procedure converges.

Another possibility would be to prove that the solution of Equations 5.1 approaches the solution of 5.1' as a approaches 0. Results along this line would be particularly valuable for numerical calculations.

The above ideas need much further study and will form the basis for more work on this problem.

VI. BIBLIOGRAPHY

1. Churchill, Ruel V. Operational mathematics. New York, N. Y., McGraw-Hill Book Company, Inc. 1958.

2. Cooper, J. L. B. The uniqueness of the solution of the equation of heat conduction. Journal of the London Mathematical Society 25: 173-180. 1950.

3. Doetsch, Gustav. Theorie und Anwendung der Laplace-Transformation. Berlin, Julius Springer. 1937.

4. Epstein, Bernard. Partial differential equations. New York, N. Y., McGraw-Hill Book Company, Inc. 1962.

5. Fulks, W. On the solutions of the heat equation. Proceedings American Mathematical Society 2: 973-979. 1951.

6. Gevrey, M. Sur les equations aux derivées partielles du type parabolique. Journal de Mathematiques Pures et Appliquées. Series 6, 9: 305-371. 1913.

7. Goursat, E. Cours d'analyse mathematique. Vol. 3. Paris, Gauthier-Villars. 1923.

8. Hirschman, I. I. and Widder, D. V. The convolution transform. Princeton, New Jersey, Princeton University Press. 1955.

9. Kolmogorov, A. N. and Fomin, S. V. Elements of the theory of functions and functional analysis. Vol. 1. Rochester, New York, Graylock Press. 1957.

10. Rothe, Erich. Über die Grundösung bei parabolischen Gleichungen. Mathematische Zeitschrift 33: 488-504. 1931.

11. Titchmarsh, E. C. The theory of Fourier integrals. Oxford, England, Clarendon Press. 1937.

12. Titchmarsh, E. C. The zeros of certain integral functions. Proceedings London Mathematical Society 25: 283-302. 1926.

13. Widder, D. V. Positive temperatures on an infinite rod. Transactions American Mathematical Society 55: 85-95. 1944.

VII. ACKNOWLEDGEMENT

The author wishes to express his appreciation to Dr. C. G. Maple for help and encouragement given during the preparation of this thesis.