

Accelerated Destructive Degradation Tests Robust to Distribution Misspecification

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Abstract

Accelerated repeated-measures degradation tests (ARMDTs) take measurements of degradation or performance on a sample of units over time. In certain products, measurements are destructive leading to accelerated destructive degradation test (ADDT) data. For example, the test of a adhesive bond needs to break the test specimen to measure the strength of the bond. Lognormal and Weibull distributions are often used to describe the distribution of product characteristics in life and degradation tests. When the distribution is misspecified, the lifetime quantile, often of interest to the practitioner, may differ significantly between these two distributions. In this study, under a specific ADDT, we investigate the bias and variance due to distribution misspecification. We suggest robust test plans under the criteria of minimizing the approximate mean square error.

Index Terms

Mean Square Error; Distribution Misspecification; Lognormal; Reliability; Robust Test Plan; Weibull.

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ACRONYMS

ADDT	accelerated destructive degradation test
ADT	accelerated degradation test
AMSE	approximate mean square error
ALT	accelerated life test
ARMDT	Accelerated repeated-measures degradation test
ML	maximum likelihood
QML	quasi ML

NOTATION

a	number of accelerated temperature
β_j	the parameters of model M_j
β_j^\square	planning value for β_j
\mathcal{D}_f	critical level for the degradation distribution
γ_j	stable parameter reparametrization for β_j
γ_j^\square	planning value for γ_j
$\hat{\gamma}_j$	the quasi ML estimator of γ_j when the working model is M_j
γ_j^*	the value of γ_j that maximizes the expected log-likelihood of the working model M_j with respect to the true model M_i .
$\mathcal{L}_j(\gamma_j, \xi)$	log-likelihood of the working model M_j and the test plan ξ
M_1	lognormal model
M_2	Weibull model
n_{ij}	number of units at level i of observed time and level j of accelerated temperature
Φ	location-scale cumulative distribution function (no unknown parameters)
σ_j	scale parameter of distribution of model M_j
σ_j^\square	planning value of σ_j .
T	the time at which the observed degradation crosses the critical level \mathcal{D}_f
τ	transformed time
$t_{j,p}(\hat{\gamma}_j)$	the quasi ML estimator of the p quantile at x_u under the working model M_j
t_p	the p quantile of the life-time distribution
w	number of observed time
x	transformed accelerating variable level
ξ	a test plan which specifies the possible choices of temperatures, times and the allocation of the test units
x_u	transformed accelerating variable level at the use condition
Y	transformed degradation

I. INTRODUCTION

A. *Background*

With a short product development period, reliability tests must be conducted with severe time constraints. Most modern products are designed to operate without failure for years, decades, or longer. Frequently no failures occur during such tests. Thus it is difficult to assess reliability with traditional life tests that record only failure time. Moreover, there are few failures even with accelerated tests. Hence, degradation data have been shown to be a superior alternative to lifetime data in many situations (e.g., [4], [1], [6], and [2]).

For certain applications, it may be difficult, costly, or impossible to obtain continued degradation measures from some components or materials. For example, a manufacturer wants to assess the long term strength of an adhesive bond. An automobile company wants to measure the degree of expansion of air bags as a function of age. Engineers need to test insulation to understand deterioration of strength over time. Taking degradation measures for such products will often require destructive measurements. That is, the degradation measurement process destroys or changes the physical/mechanical characteristics of test units so that only one meaningful measurement can be taken on each unit. This is called an accelerated destructive degradation test (ADDT). More details about destructive degradation tests can be found in [3].

The properties of ADDT plans depend on the underlying model and the parameters of the model. The form of the underlying model and at least some of the parameters are generally unknown. Hence experimenters need to obtain some information about these unknowns. Sources of such planning information include previous experience with similar products and failure modes, expert opinion, and the engineering information or judgment. As discussed in [11], an important input for planning an ADDT is the assumed distribution for the model.

The lognormal and Weibull distributions are the two common distributions used in fitting lifetime models. These distributions are also used to describe the degradation response in a ADDT. The degradation model, for a given failure definition, induces a life-time distribution. The quantiles of the lifetime distribution may depend importantly on whether the lognormal or the Weibull distribution is used to describe degradation.

The goal of this study is to build a planning procedure for ADDTs when degradation model distribution may be lognormal or Weibull. This study provides evaluation of the bias and variance of the ML estimators of the distribution quantile when we use wrong distribution as working model. Test plans are evaluated under the criterion of minimizing the large-sample approximate mean square error (AMSE). This criterion will help practitioners to choose an appropriate ADDT plan.

B. Literature Review

Much research has been conducted on accelerated life test (ALT), accelerated degradation test (ADT) and ADDT plans. Most of this work has been described in [8] and [9]. We will review some of the work which is most closely related to our study.

Unsuitable test plans cost not only time but money and may not even yield the desired information. Experiments should provide the most efficient use of available resources and the amount of extrapolation should be kept to a minimum. Nelson [7] presented concepts of traditional, optimum, and compromise accelerated life test plans. He showed how to evaluate the precision of estimates from such plans and guidance on how many specimens to test. He also pointed out that traditional test plans generally require 25 to 50% more specimens for the same precision as statistically optimum test plans and indicated that compromise plans run more specimens at lower stress than at high stress.

Accelerated Destructive Degradation Tests (ADDT's) provide reliability information quickly. Nelson [7] described applications for degradation data with the Arrhenius rate model and used a specific example to show how to analyze degradation data with only one degradation reading per specimen. Escobar et al. [3] provided an application for accelerated destructive degradation data and used likelihood-based methods for inference on both the degradation and the induced life-time distribution.

Shi, Meeker, and Escobar [11] described methods to find good ADDT plans for an important class of destructive degradation models. They derived the optimum plans minimizing the large-sample approximate variance of the ML estimator of a specified quantile of the induced life-time distribution. Because the optimum plans can be sensitive to misspecification of model assumptions, they also proposed a more robust and useful compromise plan.

Pascual and Montepiedra [10] derived expressions for the asymptotic distribution of maximum likelihood estimators of model parameters in an ALT when the model distribution is misspecified. They proposed test plan criteria based on approximate bias and approximate mean squared error to provide control over estimation bias and variance when the model distribution is misspecified. Our study will extend their work to ADDT planning.

C. Motivating Example

Here we use the example given in [3] to illustrate some of the general characteristics of ADDT's considered in this study. This example is based on an application of an ADDT to evaluate an adhesive bond (Adhesive Bond B). The objective of the experiment was to assess the strength of the adhesive bond over time. The measurement process was destructive because the strength of a test unit could be measured only once by applying stress until the bond broke. There was a special interest in estimating the time at which 1% of the devices would have a strength less than 40 Newtons assuming storage and operation at 25°C (i.e., the 0.01 quantile of the life-time distribution). In the ADDT, 8 baseline units with no aging were measured at the start of the experiment. A total of 80 additional units were aged and measured according to the temperature and time schedule presented in Table I.

TABLE I
ORIGINAL ADDT PLAN.

Temperature (°C)	Weeks						Totals
	0	2	4	6	12	16	
-	8						8
50		8	0	8	8	7	31
60		6	0	6	6	6	24
70		6	6	4	9	0	25
Totals	8	20	6	18	23	13	88

D. Overview

The remainder of this paper is organized as follows. Section 2 derives the large-sample approximate mean square error (AMSE) of the ML estimator of distribution quantile when

the distribution is misspecified in a degradation model. Section 3 presents the asymptotic distribution of observed bias for the lognormal and Weibull models when the wrong distribution is used as the working model. Test constraints and planning values are described in Section 4. In Section 5, we evaluate the test plans with proper levels of testing temperature and sample size allocation by minimizing the AMSE of the ML estimator of distribution quantile. Section 6 contains concluding remarks.

II. Asymptotic Properties of Estimators under Distribution Misspecification

A. Models and Lifetime Distribution

In the following, we use the ADDT model from [3] to investigate the asymptotic properties of estimators under distribution misspecification. The procedure and derivations could also be applied to other similar models. The following description of the ADDT model is similar to that given in [11] and is given here for completeness. The degradation for a typical observational unit at time t and accelerating variable level AccVar (e.g., temperature, humidity) is denoted by $\mathcal{D}(\tau, x, \beta)$, where $\tau = h_t(t)$ and $x = h_a(\text{AccVar})$ are known monotone increasing transformations of t and AccVar respectively, and β is a vector of unknown parameters. For the class of degradation models used here, the transformed degradation Y for a unit at the transformed time τ and the transformed accelerating variable level x is

$$Y = \mu(\tau, x) + \epsilon = \beta_0 + \beta_1 e^{\beta_2 x} \tau + \epsilon \quad (2.1)$$

where $\mu(\tau, x) = h_d(\mathcal{D})$, a monotone increasing transformation of \mathcal{D} , is a location parameter for distribution of Y that depends on the unknown parameters in $\beta = (\beta_0, \beta_1, \beta_2)'$, and ϵ is a residual deviation that describes unit-to-unit variability with $(\epsilon/\sigma) \sim \Phi(z)$ where $\Phi(z)$ is a cumulative distribution function that has no unknown parameters.

For the adhesive bond B application, the accelerating variable is temperature, and

$$\begin{aligned} Y &= h_d(\text{Strength in Newtons}) = \log(\text{Strength in Newtons}), \\ \tau &= h_t(\text{Time}) = \sqrt{\text{Time in Weeks}}, \\ x &= h_a(\text{Temperature}) = -\frac{11605.45}{\text{Temperautre in } ^\circ\text{C} + 273.15}, \end{aligned}$$

where 11605.45 is the reciprocal of Boltzmann's constant in unit of K/eV .

In order to improve the convergence properties of the ML estimation algorithm, a reparameterization is used. Let \bar{x} denote the sample mean of the stress variable and let $\bar{\tau}$ denote the sample mean transformed observed time. Then the model in (2.1) can be reparameterized as

$$Y = \gamma_0 + \gamma_1 \{e^{\gamma_2(x-\bar{x})}\tau - \bar{\tau}\} + \epsilon \quad (2.2)$$

with $\epsilon/\sigma \sim \Phi(z)$, and

$$\gamma_0 = \beta_0 + \beta_1 e^{\beta_2 \bar{x} \bar{\tau}}, \quad \gamma_1 = \beta_1 e^{\beta_2 \bar{x}}, \quad \gamma_2 = \beta_2. \quad (2.3)$$

Some products experience a gradual loss of performance with time. For example, the strength of an adhesive bond will decrease with time. Then we can define a soft failure to occur at a specified degradation level. Here, we use \mathcal{D}_f to denote the critical level for the degradation distribution at which the soft failure is assumed to occur. The lifetime, T , is defined as the time at which the observed degradation crosses the critical level \mathcal{D}_f .

For decreasing degradation, lifetime T being less than t is equivalent to an observed degradation being less than the critical level \mathcal{D}_f at time t [i.e., the event $T \leq t$ is equivalent to the event $Y \leq d_f$, where $d_f = h_d(\mathcal{D}_f)$]. Then the lifetime cdf is

$$\begin{aligned} F_T(t; x) &= P(T \leq t) = P(Y \leq d_f) = F_Y(d_f; \tau, x) \\ &= \Phi\left(\frac{d_f - \mu(\tau, x)}{\sigma}\right) = \Phi\left(\frac{h_t(t) - \nu}{\varsigma}\right), \quad \text{for } t \geq 0, \end{aligned} \quad (2.4)$$

where

$$\nu = -\frac{(\gamma_0 - \gamma_1 \bar{\tau} - d_f)e^{\gamma_2(\bar{x}-x)}}{\gamma_1} \quad \text{and} \quad \varsigma = -\frac{\sigma e^{\gamma_2(\bar{x}-x)}}{\gamma_1}.$$

With a time transformation, $h_t(t)$, for which $\tau = 0$ when $t = 0$, the lifetime distribution for decreasing degradation is a mixture distribution with a spike $\Pr(T = 0) = \Phi(-\nu/\varsigma)$ at $t = 0$. This spike corresponds to the probability of failure for a new unit that experiences no aging and it some times called the dead-on-arrival probability [11]. Hence, for $t > 0$ the

lifetime cdf in (2.4) is continuous and agrees with the cdf of a log-location-scale variable with standardized cdf $\Phi(\cdot)$, location parameter ν , and scale parameter ς . Inverting (2.4), the p quantile of lifetime distribution for decreasing degradation is

$$t_p = \begin{cases} h_t^{-1}[\nu + \varsigma\Phi^{-1}(p)] & \text{if } p \geq \Phi(-\nu/\varsigma), \\ 0 & \text{otherwise.} \end{cases} \quad (2.5)$$

B. AMSE of Quantile Estimator

In this study, we consider two candidate distributions for the strength of the adhesive bond: lognormal and Weibull. Hence, after the log transformation of the strength, the two distributions are normal and SEV distributions, respectively. We denote the two degradation models M_1 and M_2 by

$$Y = \gamma_{10} + \gamma_{11} \left\{ e^{\gamma_{12}(x-\bar{x})} \tau - \bar{\tau} \right\} + \epsilon_1, \quad (M_1)$$

where ϵ_1 follows a normal distribution with mean parameter 0 and scale parameter σ_1 , and

$$Y = \gamma_{20} + \gamma_{21} \left\{ e^{\gamma_{22}(x-\bar{x})} \tau - \bar{\tau} \right\} + \epsilon_2, \quad (M_2)$$

where ϵ_2 follows a SEV distribution with location parameter 0 and scale parameter σ_2 .

Let $\gamma_j = (\gamma_{j0}, \gamma_{j1}, \gamma_{j2})$, and $\mathcal{L}_1(\gamma_1, \xi)$, $\mathcal{L}_2(\gamma_2, \xi)$ be the log-likelihoods under test plan ξ and models M_1 and M_2 , respectively. The test plan ξ specifies the possible choices of temperatures, times and the allocation of the test units to the combinations of temperature and time. Denote the true model by M_i and denote the working model by M_j . Then let $\hat{\gamma}_j$ be the quasi ML (QML) estimator when the working model M_j is fitted to data under a test plan ξ . The definition of the quasi ML estimator is due to White [12] who also defines γ_j^* be the value of γ_j that maximizes the expected log-likelihood with respect to M_i . We refer to γ_j^* as the *working-model parameter*. That is, the QML estimator is

$$\hat{\gamma}_j = \arg \max_{\gamma_j} \mathcal{L}_j(\gamma_j, \xi), \quad (2.6)$$

and the working-model parameter is

$$\gamma_j^* = \arg \max_{\gamma_j} E_{M_i} [\mathcal{L}_j(\gamma_j, \xi)]. \quad (2.7)$$

Following similar notation to that used in [10], we define three matrices:

$$\begin{aligned} A(\boldsymbol{\gamma}_i : \boldsymbol{\gamma}_j) &= E_{M_i} \left(\frac{\partial^2}{\partial \boldsymbol{\gamma}_{jr} \partial \boldsymbol{\gamma}_{js}} \mathcal{L}_j(\boldsymbol{\gamma}_j, \boldsymbol{\xi}) \right), \\ B(\boldsymbol{\gamma}_i : \boldsymbol{\gamma}_j) &= E_{M_i} \left(\frac{\partial}{\partial \boldsymbol{\gamma}_{jr}} \mathcal{L}_j(\boldsymbol{\gamma}_j, \boldsymbol{\xi}) \times \frac{\partial}{\partial \boldsymbol{\gamma}_{js}} \mathcal{L}_j(\boldsymbol{\gamma}_j, \boldsymbol{\xi}) \right), \end{aligned} \quad (2.8)$$

and

$$C(\boldsymbol{\gamma}_i : \boldsymbol{\gamma}_j) = [A(\boldsymbol{\gamma}_i : \boldsymbol{\gamma}_j)]^{-1} B(\boldsymbol{\gamma}_i : \boldsymbol{\gamma}_j) [A(\boldsymbol{\gamma}_i : \boldsymbol{\gamma}_j)]^{-1}, \quad (2.9)$$

where $\boldsymbol{\gamma}_{jk}$ is the k^{th} element of $\boldsymbol{\gamma}_j$. By theorem 3.2 in [12], $\sqrt{n}(\hat{\boldsymbol{\gamma}}_j - \boldsymbol{\gamma}_j^*)$ is asymptotically normal with mean vector $\mathbf{0}$ and variance covariance matrix $C(\boldsymbol{\gamma}_i : \boldsymbol{\gamma}_j = \boldsymbol{\gamma}_j^*)$. Suppose that our interest is the p quantile of lifetime distribution at x_u . From (2.5), the p lifetime quantile at the use condition x_u is

$$\begin{aligned} t_{i,p}(\boldsymbol{\gamma}_i) &= h_t^{-1} \left(-\frac{e^{\boldsymbol{\gamma}_{i2}(\bar{x}-x_u)}}{\boldsymbol{\gamma}_{i1}} (\boldsymbol{\gamma}_{i0} - \boldsymbol{\gamma}_{i1}\bar{\tau} - d_f + \Phi_i^{-1}(p)\sigma_i) \right) \quad \text{under true model,} \\ t_{j,p}(\boldsymbol{\gamma}_j) &= h_t^{-1} \left(-\frac{e^{\boldsymbol{\gamma}_{j2}(\bar{x}-x_u)}}{\boldsymbol{\gamma}_{j1}} (\boldsymbol{\gamma}_{j0} - \boldsymbol{\gamma}_{j1}\bar{\tau} - d_f + \Phi_j^{-1}(p)\sigma_j) \right) \quad \text{under working model.} \end{aligned}$$

Moreover, $t_{j,p}(\hat{\boldsymbol{\gamma}}_j)$ is the quasi ML estimator of the p quantile at x_u . Therefore, $\sqrt{n}(t_{j,p}(\hat{\boldsymbol{\gamma}}_j) - t_{j,p}(\boldsymbol{\gamma}_j^*))$ is asymptotically normal with mean vector $\mathbf{0}$ and approximate variance

$$\text{AVar}[t_{j,p}(\hat{\boldsymbol{\gamma}}_j)|M_i] = \left[\frac{\partial}{\partial \boldsymbol{\gamma}} t_{j,p}(\boldsymbol{\gamma}_j^*) \right] C(\boldsymbol{\gamma}_i : \boldsymbol{\gamma}_j = \boldsymbol{\gamma}_j^*) \left[\frac{\partial}{\partial \boldsymbol{\gamma}} t_{j,p}(\boldsymbol{\gamma}_j^*) \right]'. \quad (2.10)$$

This means that for a large sample size, the average value of $t_{j,p}(\hat{\boldsymbol{\gamma}}_j)$ tends to be near $t_{j,p}(\boldsymbol{\gamma}_j^*)$ with approximate variance $\text{AVar}[t_{j,p}(\hat{\boldsymbol{\gamma}}_j)|M_i]$.

We refer to the observed bias (OBias) as the difference between an estimator and the true value of the quantile. The OBias results from using M_j to estimate the p quantile at x_u when M_i is the true model. That is,

$$\text{OBias}[t_{j,p}(\hat{\boldsymbol{\gamma}}_j)|M_i] = t_{j,p}(\hat{\boldsymbol{\gamma}}_j) - t_{i,p}(\boldsymbol{\gamma}_i). \quad (2.11)$$

Notice that the OBias defined in 2.11 is a random variable. Moreover, $\text{OBias}[t_{j,p}(\hat{\gamma}_j)|M_i]$ is asymptotically normal with large-sample approximate mean

$$\text{ABias}[t_{j,p}(\hat{\gamma}_j)|M_i] = t_{j,p}(\gamma_j^*) - t_{i,p}(\gamma_i), \quad (2.12)$$

and large-sample approximate variance

$$\text{AVar}\{\text{OBias}[t_{j,p}(\hat{\gamma}_j)|M_i]\} = \text{AVar}[t_{j,p}(\hat{\gamma}_j)|M_i], \quad (2.13)$$

as is given by (2.10). From (2.7) we can see that the value of γ^* does not depend on the sample size n . This suggests that, unlike AVar , ABias is the same for any sample size. Finally, the large-sample approximate mean square error (AMSE) of $t_{j,p}(\hat{\gamma}_j)$ is

$$\text{AMSE}[t_{j,p}(\hat{\gamma}_j)|M_i] = \text{AVar}[t_{j,p}(\hat{\gamma}_j)|M_i] + \{\text{ABias}[t_{j,p}(\hat{\gamma}_j)|M_i]\}^2. \quad (2.14)$$

We will use AMSE as a criterion to determine test plans that will control the bias and variance under distribution misspecification.

III. Working Model Definition

A. Lognormal Working Model for Weibull Data

This subsection gives expressions for approximate mean and variance of the observed bias in estimating the p lifetime quantile at transformed time x when lognormal distribution (M_1) is used as a working model for data which actually comes from Weibull distribution (M_2). Suppose that there will be w levels of observed time, a levels of the accelerating variable, and let π_{ij} denote the corresponding proportional allocation of level i observed time and level j accelerating variable. Then the log-likelihood for the lognormal distribution for model M_1 is

$$\mathcal{L}_1(\gamma_1) = \sum_{i=1}^w \sum_{j=1}^a \sum_{k=1}^{n_{ij}} \left\{ -\ln \sqrt{2\pi} - \ln \sigma_1 - \frac{1}{2\sigma_1^2} [y_{ijk} - \gamma_{10} - \gamma_{11} (e^{\gamma_{12}(x_j - \bar{x})} \tau_i - \bar{\tau})]^2 \right\},$$

where n_{ij} represents the sample size observed at time level i and accelerating variable level j . Let $\gamma_1^* = (\gamma_{10}^*, \gamma_{11}^*, \gamma_{12}^*)'$ denote the working-model parameter, i.e., the value of γ_1 that maximizes $E_{M_2}[\mathcal{L}_1(\gamma_1)]$, where

$$E_{M_2} [-\mathcal{L}_1(\gamma_1)] = n \sum_{i=1}^w \sum_{j=1}^a \pi_{ij} \left\{ \ln \sqrt{2\pi} + \ln \sigma_1 + \frac{1}{2\sigma_1^2} \left[\frac{\pi^2}{6} \sigma_2^2 + \left(\gamma_{20} + \gamma_{21} (e^{\gamma_{22}(x_j - \bar{x})} \tau_i - \bar{\tau}) - 0.5772 \sigma_2 - \gamma_{10} - \gamma_{11} (e^{\gamma_{12}(x_j - \bar{x})} \tau_i - \bar{\tau}) \right)^2 \right] \right\}.$$

Hence the approximate mean of OBias of the p lifetime quantile $t_{1,p}(\hat{\gamma}_1)$ at transformed temperature x under distribution misspecification as in (2.12) becomes

$$\begin{aligned} \text{ABias} [t_{1,p}(\hat{\gamma}_1) | M_2] &= t_{1,p}(\gamma_1^*) - t_{2,p}(\gamma_2) \\ &= h_t^{-1} \left(-\frac{e^{\gamma_{12}^*(\bar{x}-x)}}{\gamma_{11}^*} (\gamma_{10}^* - \gamma_{11}^* \bar{\tau} - d_f + \Phi_1^{-1}(p) \sigma_1) \right) - \\ &\quad h_t^{-1} \left(-\frac{e^{\gamma_{22}(\bar{x}-x)}}{\gamma_{21}} (\gamma_{20} - \gamma_{21} \bar{\tau} - d_f + \Phi_2^{-1}(p) \sigma_2) \right). \end{aligned}$$

From (2.10) and (2.13), the approximate variance of the OBias is

$$\text{AVar} \{ \text{OBias} [t_{1,p}(\hat{\gamma}_1) | M_2] \} = \left[\frac{\partial}{\partial \gamma_1} t_{1,p}(\gamma_1^*) \right] C(\gamma_2 : \gamma_1^*) \left[\frac{\partial}{\partial \gamma} t_{1,p}(\gamma_1^*) \right]'$$

Further details on the calculation of these terms are given in Appendix A.1.

B. Weibull Working Model for Lognormal Data

In this subsection, we switch the roles of the working and the true distribution presented in the previous subsection. We also give expressions for the approximate mean and variance of the observed bias in estimating the p lifetime quantile at transformed accelerating variable x when a Weibull distribution is a working model for Lognormal data. The log-likelihood of the Weibull distribution for model M_2 with lognormal data is

$$\begin{aligned} \mathcal{L}_2(\gamma_2) &= \sum_{i=1}^w \sum_{j=1}^a \sum_{k=1}^{n_{ij}} \left\{ -\ln \sigma_2 + \frac{1}{\sigma_2} (y_{ijk} - \gamma_{20} - \gamma_{21} (e^{\gamma_{22}(x_j - \bar{x})} \tau_i - \bar{\tau})) - \right. \\ &\quad \left. \exp \left\{ \frac{1}{\sigma_2} (y_{ijk} - \gamma_{20} - \gamma_{21} (e^{\gamma_{22}(x_j - \bar{x})} \tau_i - \bar{\tau})) \right\} \right\}, \end{aligned}$$

where n_{ij} represents the sample size at observed time level i and accelerating variable level j . Let $\boldsymbol{\gamma}_2^* = (\gamma_{20}^*, \gamma_{21}^*, \gamma_{22}^*)'$ denote the working-model parameter (i.e., the value of $\boldsymbol{\gamma}_2$ that maximizes $E_{M_1}[\mathcal{L}_2(\boldsymbol{\gamma}_2)]$), where

$$E_{M_1}[-\mathcal{L}_2(\boldsymbol{\gamma}_2)] = n \sum_{i=1}^w \sum_{j=1}^a \pi_{ij} \left\{ \ln \sigma_2 - \frac{1}{\sigma_2} \left[\gamma_{10} + \gamma_{11} (e^{\gamma_{12}(x_j - \bar{x})} \tau_i - \bar{\tau}) - \gamma_{20} - \gamma_{21} (e^{\gamma_{22}(x_j - \bar{x})} \tau_i - \bar{\tau}) \right] + \Delta_{ij} \right\},$$

and where

$$\Delta_{ij} = \exp \left\{ \frac{\mu_{1ij}}{\sigma_2} - \frac{\mu_{2ij}}{\sigma_2} + \frac{\sigma_1^2}{2\sigma_2^2} \right\}.$$

The approximate mean of the observed bias of the p lifetime quantile $t_{2,p}(\hat{\boldsymbol{\gamma}}_2)$ at the transformed temperature x under distribution misspecification (2.12) is

$$\begin{aligned} \text{ABias} [t_{2,p}(\hat{\boldsymbol{\gamma}}_2) | M_1] &= t_{2,p}(\boldsymbol{\gamma}_2^*) - t_{1,p}(\boldsymbol{\gamma}_1) \\ &= h_t^{-1} \left(-\frac{e^{\gamma_{22}^*(\bar{x}-x)}}{\gamma_{21}^*} (\gamma_{20}^* - \gamma_{21}^* \bar{\tau} - d_f + \Phi_2^{-1}(p)\sigma_2) \right) - \\ &\quad h_t^{-1} \left(-\frac{e^{\gamma_{12}(\bar{x}-x)}}{\gamma_{11}} (\gamma_{10} - \gamma_{11} \bar{\tau} - d_f + \Phi_1^{-1}(p)\sigma_1) \right). \end{aligned}$$

By (2.10) and (2.13), the approximate variance of the OBias becomes

$$\text{AVar} \{ \text{OBias} [t_{2,p}(\hat{\boldsymbol{\gamma}}_2) | M_1] \} = \left[\frac{\partial}{\partial \boldsymbol{\gamma}_2} t_{2,p}(\boldsymbol{\gamma}_2^*) \right] C(\boldsymbol{\gamma}_1 : \boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_2^*) \left[\frac{\partial}{\partial \boldsymbol{\gamma}_2} t_{2,p}(\boldsymbol{\gamma}_2^*) \right]'$$

Further details on the calculation of these terms are given in Appendix A.2.

IV. Test Constraints and Planning Values

In the planning stage of a test, it could happen that the practitioner selects a lognormal (Weibull) ADDT working model and that data after the test are better described by Weibull (lognormal) distribution. If the distribution used for the model turns out to be incorrect, then estimates of our quantities of interest will be biased and possibly less precise. In this section, we explore robust test plans that provide a control over the resulting bias and variance of the lifetime quantile estimator when the ADDT model distribution is misspecified.

A. Test Constraints

An ADDT generally has several test constraints such as a specified critical degradation level, and the ranges of the accelerating variables available for the experiment. There also will be constraints on the maximum test time and the number of units available for testing.

In this study, we used the adhesive bond B data in Section 1.4 to obtain relative information for an ADDT. The degradation model is described in (2.1). The critical degradation level is specified as $\mathcal{D}_f = 40$ Newtons. The maximum temperature that can be used is 70°C (higher temperatures would cause the model to break down) and only 16 weeks are available for testing.

An ADDT plan will provide factor levels of transformed time τ_i and transformed temperature x_j , and the corresponding allocation sample sizes n_{ij} . Not all possible combinations of factors give reasonable plans. A compromise plan proposed by [11] is convenient in implementation during the test as well as informative for model checking after the test. The plan allocates some test units at the beginning of the experiment and some units at each of nine equally spaced factor-level combinations. The nine combinations have three equally spaced time levels and three equally spaced temperature levels, as presented in Table II.

TABLE II
COMPROMISE ADDT PLAN.

Temperature ($^\circ\text{C}$)	Weeks				Total
	0	τ_L	τ_M	τ_H	
25	n_{00}				
x_L		n_{11}	n_{12}	n_{13}	
x_M		n_{21}	n_{22}	n_{23}	
x_H		n_{31}	n_{32}	n_{33}	
Total					n

B. Planning Values

In the planning stage, planning values for the model parameters are needed. We use the data in Escobar et al. (2003) to estimate the parameters of the working model (lognormal or Weibull) under the test constraints described above to obtain the planning values. The planning values will be denoted by β_{i0}^\square , β_{i1}^\square , β_{i2}^\square and σ_i^\square , depending on the working model M_i . Here the superscript \square is used to denote a planning value of a population or process quantity.

We now list the steps that we follow for test plan evaluation. Suppose that distribution M_j is used in planning stage and the data actually come from distribution M_i . Because the evaluation criteria depend on unknown parameter values, we use the planning values in their place.

- Select a criterion for the test plan evaluation. We will minimize the AMSE of the ML estimator of a particular lifetime quantile.
- Determine planning values of β_{i0}^\square , β_{i1}^\square , β_{i2}^\square and σ_i^\square in (2.1) corresponding to model M_i . We use (2.3) to calculate γ_{i0}^\square , γ_{i1}^\square and γ_{i2}^\square . We assume that ϵ in (2.2) has same standard deviation under M_i and M_j . Then we can determine σ_j^\square , the planning value for the scale parameter of M_j .
- Determine the working-model parameters γ_{j0}^* , γ_{j1}^* , and γ_{j2}^* for each test plan ξ . These working-model parameters are obtained by finding those values of γ_j that maximize the expected log-likelihood $E_{M_i}[\mathcal{L}_j(\gamma_j, \xi)]$ with respect to M_i .
- Finally, use γ_{i0}^\square , γ_{i1}^\square , γ_{i2}^\square , σ_i^\square , γ_{j0}^* , γ_{j1}^* , γ_{j2}^* , and σ_j^\square to compute ABias, AVar, AMSE in (2.12), (2.13), and (2.14).

V. Evaluation of Test Plans

A. Plans with Unconstrained Parameters

The planning values in Table III are derived by using the adhesive bond B data mentioned in Section 1.4 to estimate the parameters in models M_1 and M_2 . The first set is obtained by fitting a lognormal distribution model (M_1) and the second set is obtained by fitting a Weibull distribution model (M_2). These two sets of values are obtained separately, without any other constraints. Constrained parameters are obtained under a restriction which will be explained in the next subsection.

Figure 1 shows the mean transformed degradation paths versus time at 4 selected levels of temperature for the lognormal and Weibull distributions. Note that, the strength axis is in a log scale and the time axis is in a square root scale, so that the mean transformed degradation paths are linear with respect to the transformed time.

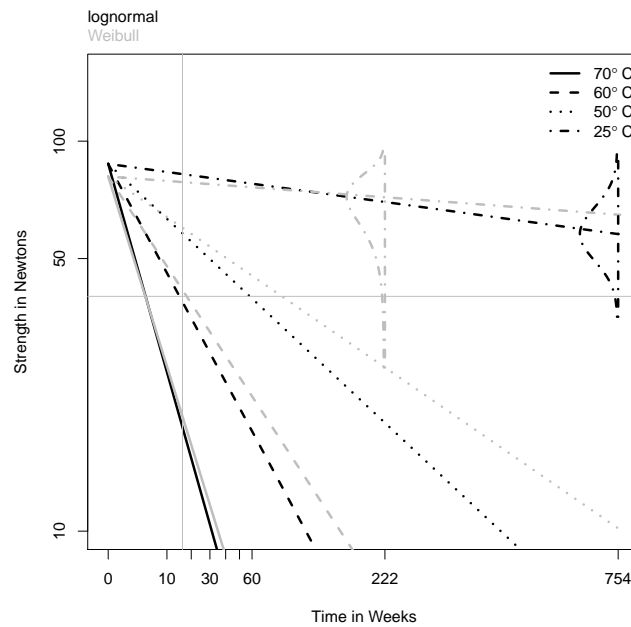


Fig. 1. Models for the mean transformed degradations evaluated for the lognormal and Weibull distributions at four different temperatures. The lines for each temperature indicate mean transformed degradation paths $\mu(\tau, x)$ as a function of time. Black and gray lines are for the lognormal and Weibull distributions, respectively.

In this study, we have special interest in the estimation of the 0.01 quantile of the lifetime distribution at 25°C. The Weibull and lognormal distribution 0.01 quantiles evaluated at planning parameters, given in Table III, are 222 weeks and 754 weeks, respectively. We now consider the effect of changing test temperature levels.

TABLE III
UNCONSTRAINED PLANNING VALUES FOR TWO DEGRADATION
MODELS BASED ON THE ADHESIVE BOND B DATA.

M_1	β_{10}^{\square}	β_{11}^{\square}	β_{12}^{\square}	σ_1^{\square}	\mathcal{D}_f^{\square}
	4.4710	-8.643×10^8	0.6364	0.1580	40
M_2	β_{20}^{\square}	β_{21}^{\square}	β_{22}^{\square}	σ_2^{\square}	\mathcal{D}_f^{\square}
	4.4804	-2.405×10^{10}	0.7375	0.1455	40

TABLE IV
SAMPLE SIZE ALLOCATION FOR TABLE V.

Temperature (°C)	Weeks				Total
	0	12	14	16	
25	7				
x_L		9	9	9	
$(x_L + 70)/2$		9	9	9	
70		9	9	9	
Total					88

Table V gives optimized compromise plans for minimizing AMSE of the estimator of the 0.01 lifetime quantile at 25°C, based on the sample size allocation in Table IV. The second column in Table V gives the lowest level x_L , and the highest and middle levels which are $x_H = 70$ and $x_M = (x_L + 70)/2$, respectively. The last four columns give the ABias, \sqrt{AVar} , \sqrt{AMSE} and the values of the 0.01 lifetime quantiles for the underlying models with planning values.

TABLE V
 THE LOWER STRESS LEVELS, x_L , FOR THE OPTIMIZED COMPROMISE
 PLANS WHICH MINIMIZE THE AMSE OF THE ML ESTIMATOR OF THE
 .01 LIFETIME QUANTILE UNDER THE UNCONSTRAINED PLANNING
 VALUES.

$M_j M_i$	x_L	ABias	\sqrt{AVar}	\sqrt{AMSE}	$t_{i,.01}(\gamma_i)$
$M_1 M_2$	55.24	17.07	11.24	20.44	4.26
$M_1 M_1$	54.09	0	5.53	5.53	14.50
$M_2 M_1$	45.13	-10.29	2.55	10.60	14.50
$M_2 M_2$	43.96	0	1.59	1.59	4.26

Table V shows that when a lognormal distribution (working M_1) is used to make the test plan that minimizes AMSE under distribution misspecification (true M_2), the lowest level of temperature is around 55°C. When there is no misspecification, the minimum AMSE (also minimum AVar because ABias=0) results in a similar temperature 54°C. An analogous result can also be observed when the working model is the Weibull distribution. This results in a smaller lowest level near 44°C both with and without distribution misspecification. This result provides a guideline for choosing lower temperature levels once we decide on the working distribution.

B. Plans with Constrained Parameters

Because the shape of the SEV distribution has a longer lower tail than the normal distribution, the true values of the 0.01 lifetime quantile at the use condition of the two distributions, evaluated by parameters in Table III, are different. Because of these differences, it is not easy to compare the bias and variance for a specific target quantile. To provide a clear comparison of these two distributions we constraint the two sets of planning values in Table VI such that these two distributional models will give the same values of the 0.01 lifetime quantile at the use condition.

TABLE VI
 CONSTRAINED PLANNING VALUES FOR TWO DEGRADATION MODELS
 BASED ON THE ADHESIVE BOND B DATA.

$M_1 :$	β_{10}^{\square}	β_{11}^{\square}	β_{12}^{\square}	σ_1^{\square}	\mathcal{D}_f^{\square}
	4.4710	-8.643×10^8	0.6364	0.1580	40
$M_2 :$	β_{20}^{\square}	β_{21}^{\square}	β_{22}^{\square}	σ_2^{\square}	\mathcal{D}_f^{\square}
	4.5355	-2.405×10^{10}	0.7275	0.1117	40

Figure 2 indicates the mean transformed degradation paths versus time corresponding to parameters in Table VI at four selected levels of temperature for the lognormal and Weibull distributions. In the following examples, we use the parameters in Table VI and obtain numerical values by doing evaluations similar to those done in the previous section.

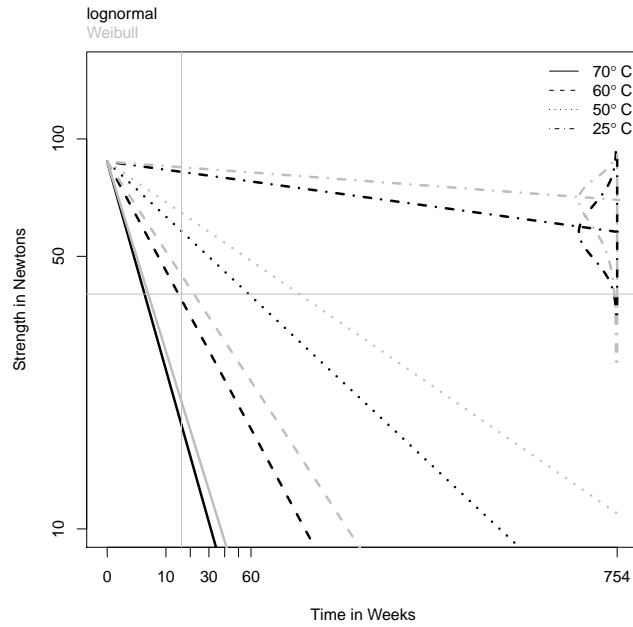


Fig. 2. The mean transformed degradation paths of the lognormal and Weibull distributions defined by the constrained planning values in Table VI. Black and gray lines are for lognormal and Weibull distributions, respectively. The lognormal and Weibull distributions at the use condition (25°C) have the same 0.01 quantile value.

Table VII gives x_L , the lower levels of temperature of the optimized compromise plans that minimize the AMSE of the ML estimator of the 0.01 lifetime quantile at 25°C, according to the sample size allocation given in Table IV. From Table VII, when we use Weibull distribution as the working model (M_2), the lowest level of temperature at about 45°C will minimize the AMSE under distribution misspecification (true M_1). For the other model combinations in Table VII, the lowest level of temperature at about 55°C will minimize the AMSE. Therefore, when the working model is lognormal, we can set the lower temperature at about 55°C for robust estimation. Also, the lower temperature can be set half way between 45°C and 55°C when we use the Weibull distribution for test planning.

TABLE VII
THE LOWER STRESS LEVELS, x_L , FOR THE OPTIMIZED COMPROMISE
PLANS WHICH MINIMIZE THE AMSE OF THE ML ESTIMATOR OF THE
.01 LIFETIME QUANTILE UNDER THE CONSTRAINED PLANNING
VALUES.

$M_j M_i$	x_L	ABias	\sqrt{AVar}	\sqrt{AMSE}	$t_{.01}(\gamma_i)$
$M_1 M_2$	55.70	11.89	7.98	14.32	14.50
$M_1 M_1$	54.09	0	5.53	5.53	14.50
$M_2 M_1$	45.13	-10.29	2.55	10.60	14.50
$M_2 M_2$	54.13	0	3.01	3.01	14.50

Now we consider the allocation of sample size to minimize the AMSE of the ML estimator of the 0.01 lifetime quantile at 25°C. The possible sample size allocation for a compromise plan is shown in Table II. However, we consider a simpler scheme. The allocation for time (12, 14, 16) is set to be $(p_1, p_2, 1 - p_1 - p_2)$ and allocation for temperature (54, 62, 70) is set to be $(p_3, p_4, 1 - p_3 - p_4)$ where $0 \leq p_1 + p_2 \leq 1$ and $0 \leq p_3 + p_4 \leq 1$. This setting reduces the search dimensions of sample allocation into 4. The restriction is described in Table VIII.

TABLE VIII
ALLOCATIONS OF SAMPLE SIZE WHICH IS USED TO OBTAIN THE COMPROMISE PLANS IN TABLE IX.

(°C)	Weeks				Total
	0	12	14	16	
25	7				
54		$81p_1p_3$	$81p_2p_3$	$81(1 - p_1 - p_2)p_3$	$81p_3$
62		$81p_1p_4$	$81p_2p_4$	$81(1 - p_1 - p_2)p_4$	$81p_4$
70		$81p_1(1 - p_3 - p_4)$	$81p_2(1 - p_3 - p_4)$	$81(1 - p_1 - p_2)(1 - p_3 - p_4)$	$81(1 - p_3 - p_4)$
Total		$81p_1$	$81p_2$	$81(1 - p_1 - p_2)$	88

Table IX shows the test plans that result from the restriction in Table VIII. When the Weibull distribution is used for the working model in the $M_2|M_2$ and $M_2|M_1$ cases, allocating more units to the lower temperature and some units to the higher temperature will minimize the AMSE. On the other hand, when we select the lognormal distribution as the working model and the data also come from the lognormal distribution (the $M_1|M_1$ case), the optimum allocation is similar to the optimum allocation under the Weibull distribution. If the data actually come from a Weibull distribution (the $M_1|M_2$ case), however, we should increase the allocation to the middle temperature to obtain a smaller AMSE.

In order to provide guidance for sample allocation when the lognormal is selected as the working model, we do the following check. The sample allocation given in the top left of Table IX (optimum allocation for the case $M_1|M_2$) are used to calculate $\sqrt{\text{AMSE}}$ for the case $M_1|M_1$. The corresponding $\sqrt{\text{AMSE}}$ is 5.16, which is only slightly larger than the optimum AMSE value 4.9 for the case $M_1|M_1$. Hence we suggest to allocate half of the available test units to the middle level of temperature (similar to the sample allocation for the case $M_1|M_2$) when we use the lognormal distribution as the working model.

C. Adjusted Compromise Plans

The optimized compromise plans in the previous sections use either plans in Table IV in which the sample allocations are fixed or plans in Table VIII in which the levels of temperature are fixed. Now we explore an adjusted compromise plan which has some flexibility in the test unit allocation as well as the levels of temperature. Every factor

TABLE IX
OPTIMAL SAMPLE ALLOCATION FOR THE COMPROMISE PLANS FOR ESTIMATING THE 0.01 LIFETIME QUANTILE
WITH THE CONSTRAINED PLANNING VALUES IN TABLE VI AND WITH THE RESTRICTION OF ALLOCATIONS IN TABLE
VIII.

$M_1 M_2$	ABias	\sqrt{AVar}	\sqrt{AMSE}
$M_1 M_2$	11.89	7.29	13.95

Weeks				
Temp $^{\circ}$ C	0	12	14	16
25	7			
54		0	0	18
62		0	0	43
70		0	0	20

$M_1 M_1$	ABias	\sqrt{AVar}	\sqrt{AMSE}
$M_1 M_1$	0	4.90	4.90

Weeks				
Temp $^{\circ}$ C	0	12	14	16
25	7			
54		0	0	66
62		0	0	0
70		0	0	15

$M_2 M_1$	ABias	\sqrt{AVar}	\sqrt{AMSE}
$M_2 M_1$	-5.49	3.23	6.37

Weeks				
Temp $^{\circ}$ C	0	12	14	16
25	7			
54		0	0	55
62		0	0	0
70		0	0	26

$M_2 M_2$	ABias	\sqrt{AVar}	\sqrt{AMSE}
$M_2 M_2$	0	2.63	2.63

Weeks				
Temp $^{\circ}$ C	0	12	14	16
25	7			
54		0	0	66
62		0	0	1
70		0	0	14

level combination has a basic number of test units. This will assure that there are at least some units at each combination of time and temperature. This will allow the checking of the adequacy of the regression relationship after the test. The adjusted compromise plans, however, also allow certain factor level combinations to have more test units than the others in order to reduce the AMSE of the ML estimator of the lifetime quantiles. One possible setting is the following: n_{00} , n_{23} , and $n_{33} \geq n/20$ (shaded in the Table X) with $n_{ij} \approx n/20$ for the other factor level combinations.

TABLE X
 ADJUSTED COMPROMISE ADDT PLAN. THERE ARE
 MORE UNITS AT THE SHADED LEVEL COMBINATIONS
 THAN THE OTHERS.

Temp (°C)	Weeks				Total
	0	12	14	16	
25	n_{00}				
x_L		n_{11}	n_{12}	n_{13}	
$(x_L + 70)/2$		n_{21}	n_{22}	n_{23}	
70		n_{31}	n_{32}	n_{33}	
Total					n

Table XI shows the adjusted compromise test plans. When the working model is lognormal, there is some improvement of AMSE when compared with the plans in the previous section. The amount of improvement is large when the working model is Weibull but the true model is lognormal. The square root of AMSE is 6.36 which is 60% of the value for the optimized compromise plan in Table VII. When the working model is wrong, the resulting AMSE will be much larger than the AMSE in the case of correct model. To obtain a conservative plan, one could use a test plan that will result in a smaller AMSE than other plans when the working distributions are assumed to be wrong for all the plans in comparison.

VI. CONCLUDING REMARKS

The lognormal and Weibull distribution are the two common distributions used to fit data from ADDT experiments. If an incorrect distribution is used in planning stage, the resulting estimate of the lifetime quantile after the test may be biased. Our primary goal in this study is to provide robust test plans that engineers can use to plan ADDT's that provide some protection against bias and variance when the goal is to estimate the quantile of the lifetime distribution.

We derived expressions for asymptotic distribution of the quasi ML estimator of the p quantile of the lie-time distribution when there is distribution misspecification. We find

TABLE XI

OPTIMAL SAMPLE ALLOCATION FOR THE ADJUSTED COMPROMISE PLANS FOR ESTIMATING THE 0.01 LIFETIME QUANTILE WITH THE CONSTRAINED PLANNING VALUES IN TABLE VI AND WITH THE RESTRICTION OF ALLOCATIONS IN TABLE X.

$M_1 M_2$					$M_1 M_1$				
	ABias	\sqrt{AVar}	\sqrt{AMSE}		ABias	\sqrt{AVar}	\sqrt{AMSE}		
	11.89	6.99	13.79		0	4.77	4.77		
	Weeks					Weeks			
Temp ^o C	0	12	14	16	Temp ^o C	0	12	14	16
25	15				25	11			
45.70		4	4	4	42.35		4	4	4
57.85		4	4	37	56.18		4	4	36
70		4	4	6	70		4	4	5

$M_2 M_1$					$M_2 M_2$				
	ABias	\sqrt{AVar}	\sqrt{AMSE}		ABias	\sqrt{AVar}	\sqrt{AMSE}		
	-5.84	2.52	6.36		0	2.70	2.70		
	Weeks					Weeks			
Temp ^o C	0	12	14	16	Temp ^o C	0	12	14	16
25	6				25	8			
49.80		4	4	4	45.34		4	4	4
59.90		4	4	46	57.67		4	4	42
70		4	4	4	70		4	4	7

that the Weibull distribution provides smaller estimate of p quantile than the lognormal distribution. If a conservative estimate is preferred, the Weibull distribution is a better working model than lognormal distribution in this ADDT setup. Through the criterion of minimizing the AMSE, we give several guidelines for temperature and sample allocation settings for two different working models.

An adjusted compromise plans is a feasible option to preserve the compromise property of a test plan as well as to allocate sample units for somewhat better estimation efficiency. Conservatively, we may use the test plan which minimizes the AMSE assuming the wrong working model. An alternative would be to choose a compromise between the plans suggested under the two different working models.

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APPENDIX

APPENDIX A.1

Here are the formulas for deriving the AMSE of the ML estimators of the p lifetime quantile when the working model is lognormal and the data is from a Weibull distribution.

$$\frac{\partial}{\partial \gamma_{10}} t_{1,p}(\gamma_1^*) = \frac{2}{|\gamma_{11}^*|^2} e^{-2\gamma_{12}^*(x-\bar{x})} (\gamma_{10}^* - \gamma_{11}^* \bar{\tau} - d_f + \Phi_1^{-1}(p)\sigma_1),$$

$$\frac{\partial}{\partial \gamma_{11}} t_{1,p}(\gamma_1^*) = \frac{2}{|\gamma_{11}^*|^3} e^{-2\gamma_{12}^*(x-\bar{x})} (\gamma_{10}^* - \gamma_{11}^* \bar{\tau} - d_f + \Phi_1^{-1}(p)\sigma_1) (\gamma_{10}^* - d_f + \Phi_1^{-1}(p)\sigma_1),$$

$$\frac{\partial}{\partial \gamma_{12}} t_{1,p}(\gamma_1^*) = \frac{2}{|\gamma_{11}^*|^2} (x - \bar{x}) e^{-2\gamma_{12}^*(x-\bar{x})} (\gamma_{10}^* - \gamma_{11}^* \bar{\tau} - d_f + \Phi_1^{-1}(p)\sigma_1)^2,$$

and by (2.8) and (2.9), $C(\gamma_2 : \gamma_1^*) = [A(\gamma_2 : \gamma_1^*)]^{-1} B(\gamma_2 : \gamma_1^*) [A(\gamma_2 : \gamma_1^*)]^{-1}$.

For simplifying the expression of the components of **A** and **B**, we define some notation.

Let

$$\begin{aligned} \tau_{1ij} &= e^{\gamma_{12}(x_j - \bar{x})} \tau_i, \\ \tau_{2ij} &= e^{\gamma_{22}(x_j - \bar{x})} \tau_i, \\ \mu_{1ij} &= \gamma_{10} + \gamma_{11} (e^{\gamma_{12}(x_j - \bar{x})} \tau_i - \bar{x}), \\ \mu_{2ij} &= \gamma_{20} + \gamma_{21} (e^{\gamma_{22}(x_j - \bar{x})} \tau_i - \bar{x}), \\ m_{1ij} &= E_{M_1}(Y_{ijk}) = \mu_{1ij}, \\ m_{2ij} &= E_{M_2}(Y_{ijk}) = \mu_{2ij} - 0.5772\sigma_2. \end{aligned}$$

Then the components of **A** and **B** can be represented in the following way:

$$\begin{aligned}
A_{11} &= -\frac{n}{\sigma_1^2}, \\
A_{12} &= -n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} (\tau_{1ij} - \bar{\tau}), \\
A_{13} &= -n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} \gamma_{11} (x_j - \bar{x}) \tau_{1ij}, \\
A_{22} &= -n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} (\tau_{1ij} - \bar{\tau})^2, \\
A_{23} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} (x_j - \bar{x}) \tau_{1ij} (m_{2ij} - 2m_{1ij} + \gamma_{10}), \\
A_{33} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} \gamma_{11} (x_j - \bar{x})^2 \tau_{1ij} (m_{2ij} - m_{1ij} - \gamma_{21} \tau_{1ij}), \\
\\
B_{11} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^4} \left\{ \frac{\pi^2}{6} \sigma_2^2 + (m_{2ij} - m_{1ij})^2 \right\}, \\
B_{12} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^4} (\tau_{1ij} - \bar{\tau}) \left\{ \frac{\pi^2}{6} \sigma_2^2 + (m_{2ij} - m_{1ij})^2 \right\}, \\
B_{13} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^4} \gamma_{11} (x_j - \bar{x}) \tau_{1ij} \left\{ \frac{\pi^2}{6} \sigma_2^2 + (m_{2ij} - m_{1ij})^2 \right\}, \\
B_{22} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^4} (\tau_{1ij} - \bar{\tau})^2 \left\{ \frac{\pi^2}{6} \sigma_2^2 + (m_{2ij} - m_{1ij})^2 \right\}, \\
B_{23} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^4} \gamma_{11} (x_j - \bar{x}) \tau_{1ij} (\tau_{1ij} - \bar{\tau}) \left\{ \frac{\pi^2}{6} \sigma_2^2 + (m_{2ij} - m_{1ij})^2 \right\}, \\
B_{33} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^4} \gamma_{11}^2 (x_j - \bar{x})^2 \tau_{1ij}^2 \left\{ \frac{\pi^2}{6} \sigma_2^2 + (m_{2ij} - m_{1ij})^2 \right\}.
\end{aligned}$$

APPENDIX A.2

Here we provide the formulas for deriving the AMSE of the ML estimators of the p lifetime quantile when working model is Weibull and data is from lognormal distribution.

$$\frac{\partial}{\partial \gamma_{20}} t_{2,p}(\gamma_2^*) = \frac{2}{|\gamma_{21}^*|^2} e^{-2\gamma_{22}^*(x-\bar{x})} (\gamma_{20}^* - \gamma_{21}^* \bar{\tau} - d_f + \Phi_2^{-1}(p)\sigma_2),$$

$$\frac{\partial}{\partial \gamma_{21}} t_{2,p}(\gamma_2^*) = \frac{2}{|\gamma_{21}^*|^3} e^{-2\gamma_{22}^*(x-\bar{x})} (\gamma_{20}^* - \gamma_{21}^* \bar{\tau} - d_f + \Phi_2^{-1}(p)\sigma_2) (\gamma_{20}^* - d_f + \Phi_2^{-1}(p)\sigma_2),$$

$$\frac{\partial}{\partial \gamma_{22}} t_{2,p}(\gamma_2^*) = -\frac{2}{|\gamma_{21}^*|^2} (x - \bar{x}) e^{-2\gamma_{22}^*(x-\bar{x})} (\gamma_{20}^* - \gamma_{21}^* \bar{\tau} - d_f + \Phi_2^{-1}(p)\sigma_2)^2,$$

and by (2.8) and (2.9), the structure of $C(\gamma_1 : \gamma_2 = \gamma_2^*) = [A(\gamma_1 : \gamma_2^*)]^{-1} B(\gamma_1 : \gamma_2^*) [A(\gamma_1 : \gamma_2^*)]^{-1}$, where the components of \mathbf{A} and \mathbf{B} can be expressed in the following way:

$$\begin{aligned} A_{11} &= -n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} \Delta_{ij}, \\ A_{12} &= -n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} (\tau_{2ij} - \bar{\tau}) \Delta_{ij}, \\ A_{13} &= -n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} \gamma_{21} (x_j - \bar{x}) \tau_{2ij} \Delta_{ij}, \\ A_{22} &= -n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} (\tau_{2ij} - \bar{\tau})^2 \Delta_{ij}, \\ A_{23} &= -n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_2} \left\{ (x_j - \bar{x}) \tau_{2ij} - \frac{1}{\sigma_2} (x_j - \bar{x}) \tau_{2ij} [\sigma_2 - \gamma_{21} (\tau_{2ij} - \bar{x})] \Delta_{ij} \right\}, \\ A_{33} &= -n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_2} \left\{ \gamma_{21} (x_j - \bar{x})^2 \tau_{2ij} - \frac{1}{\sigma_2} \gamma_{21} (x_j - \bar{x})^2 \tau_{2ij} (\sigma_2 - \gamma_{21} \tau_{2ij}) \Delta_{ij} \right\}, \end{aligned}$$

$$\begin{aligned}
B_{11} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} (1 - 2\Delta_{ij} + \Gamma_{ij}), \\
B_{12} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} (\tau_{2ij} - \bar{\tau}) (1 - 2\Delta_{ij} + \Gamma_{ij}), \\
B_{13} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} \gamma_{21} (x_j - \bar{x}) \tau_{2ij} (1 - 2\Delta_{ij} + \Gamma_{ij}), \\
B_{22} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} (\tau_{2ij} - \bar{\tau})^2 (1 - 2\Delta_{ij} + \Gamma_{ij}), \\
B_{23} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} \gamma_{21} (x_j - \bar{x}) \tau_{2ij} (1 - 2\Delta_{ij} + \Gamma_{ij}), \\
B_{33} &= n \sum_{i=1}^w \sum_{j=1}^a \frac{\pi_{ij}}{\sigma_1^2} \gamma_{21}^2 (x_j - \bar{x})^2 \tau_{2ij}^2 (1 - 2\Delta_{ij} + \Gamma_{ij}),
\end{aligned}$$

where

$$\Gamma_{ij} = \exp \left\{ \frac{2\mu_{1ij}}{\sigma_2} - \frac{2\mu_{2ij}}{\sigma_2} + \frac{2\sigma_1^2}{\sigma_2^2} \right\}.$$

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