COMMUTATIVE FINITELY GENERATED ALGEBRAS
SATISFYING $((yx)x)x = 0$ ARE SOLVABLE

IVAN CORREA AND IRVIN ROY HENTZEL

1. Introduction. We study commutative, nonassociative algebras satisfying the identity

$$(1) \qquad\qquad ((yx)x)x = 0.$$

We show that finitely generated algebras over a field K of characteristic $\neq 2$ satisfying (1) are solvable. For x in an algebra A, define the multiplication operator R_x by $yR_x = yx$, for all $y \in A$. Our identity is then that $R_x^3 = 0$.

Our interest in this problem arose from attempts to prove the Albert-Gerstenhaber conjecture. This conjecture asks if every commutative, power-associative, finite-dimensional, nil algebra is solvable. In such algebras the multiplication operator R_x is nilpotent for each x. Our result forms part of the solution for the Albert-Gerstenhaber conjecture in the particular case of nilindex four. In fact, from Gerstenhaber [**3**, Theorem 1], it follows that in this case the algebra satisfies $R_x^5 = 0$ for every element x. Since there exists an x with $x^3 \neq 0$, one of the following cases occurs: i) $R_x^3 = 0$ for all x, or ii) $R_x^4 = 0$ for all x, and there exists an x with $R_x^3 \neq 0$, or iii) there exists an x with $R_x^4 \neq 0$. Our result gives the solution for case i). The Albert-Gerstenhaber conjecture for nilindex four and dimension ≤ 4 has been studied by Gerstenhaber and Myung [**4**] and a generalization without the hypotheses of power associativity was obtained by Correa, Hentzel and Labra [**2**].

We chose to study commutative algebras satisfying $R_x^n = 0$ for some fixed n. In the case $n = 1$, the algebra squares to zero. In the case $n = 2$ and characteristic $\neq 2$, the algebra cubes to zero. In the case $n = 2$ and characteristic $= 2$, the algebra is associative and if it is finitely

generated it will be nilpotent. But if it is not finitely generated, it may not be solvable.

This brings us to the case $n = 3$ which is the topic of this paper.

Our proof requires characteristic $\neq 2$. But this is the only restriction we need on the characteristic. Our proofs have to be a bit more complicated to avoid use of the characteristic $\neq 3$ and the characteristic $\neq 5$.

The goal to have the least restriction on the characteristic as possible was motivated by the related problem of a commutative algebra satisfying $x^3 = 0$. In this case finitely generated algebras are nilpotent, and there is no restriction of characteristic at all.

2. Preliminary results. Because our proofs will involve some fine distinctions based on linearization and characteristic, we will give a brief review (see [**7**, page 9] for a discussion of linearization). Let A be a commutative, nonassociative algebra over a field of characteristic $\neq 2$. Let $f(x_1, x_2, x_3)$ be a multi-linear function. Suppose that W is a subalgebra. We will consider the following three statements.

(i) $f(a, a, a)$ in W, for all a in A.

(ii) $f(a, a, b) + f(a, b, a) + f(b, a, a)$ in W, for all a, b in A.

(iii) $f(a, b, c) + f(b, c, a) + f(c, a, b) + f(b, a, c) + f(a, c, b) + f(c, b, a)$ in W, for all a, b, c in A.

Then, in characteristic $\neq 2$, the underlying field has necessarily at least three elements and we have that (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) but, in characteristic 3, neither (ii) nor (iii) implies (i).

In this paper we will be working with a subalgebra W such that $(aa)(a(aa))$ is in W for all a in A. It will turn out that all of the partial linearizations of $(aa)(a(aa))$ are also in W. These linearizations are in W because the proof that showed $(aa)(a(aa))$ is in W can be linearized. The reason that the proof can be linearized is because it only requires linearizations of degree three. When we say that an expression and all of its linearizations are in W, we mean that the expression evaluates to an element W for any choice of arguments from A, and also all of its partial linearizations evaluate into A as well.

Since (1) is degree 3 all of its linearizations are also identities. It follows that this partial linearization is an identity.

(2) $$((yz)x)x + ((yx)z)x + ((yx)x)z = 0.$$

It also follows from (1) that all linearizations of

(3) $$((xx)x)x = 0$$

are identities as well.

From (2) with $y = x$ and (1) we obtain that all linearizations of

(4) $$(zx^2)x + zx^3 = 0$$

are identities.

In the following we denote the Jacobi operator by $J(x, y, z) = (xy)z + (xz)y + x(yz)$. It is easily seen that $J(x, y, z)$ is symmetric on its three variables by writing it in its equivalent form $J(x, y, z) = (xy)z + (yz)x + (zx)y$.

Lemma 1. *Let A be a commutative algebra of characteristic $\neq 2$ satisfying (1). Then A satisfies the identity*

(5) $$J(x, y, zw) = J(xy, z, w).$$

Proof. Replacing y by z in (2), we obtain:

(6) $$(z^2 x)x + ((zx)z)x + ((zx)x)z = 0.$$

Interchanging x with z in (6), we obtain $(x^2 z)z + ((xz)x)z + ((xz)z)x = 0$. From this equation and (6), we obtain $(z^2 x)x = (x^2 z)z$. Linearizing this identity we obtain:

$$((wz)x)y + ((wz)y)x = ((xy)z)w + ((xy)w)z.$$

We add $(xy)(zw)$ to both sides of this identity and get (5). □

Corollary 2. *Let A be as in Lemma 1. Then the set $I = \{x \in A \mid J(A, A, x) = \{0\}\}$ is an ideal of A.*

Proof. It is clear that I is closed under addition and scalar multiplication. Let x in I and a in A. Then, using (5) we obtain that $J(A, A, ax) \subseteq J(A^2, a, x) = \{0\}$. This proves the corollary.   □

Lemma 3. *Let A be as in Lemma 1. Then all the linearizations of $J(A, x^2, x^3)$ are zero.*

Proof. Notice that this proof uses (1), (4) and (5). All linearized forms of these three identities are valid. Therefore, this proof will be valid for all linearized forms of $J(a, x^2, x^3)$ as well.

$$
\begin{aligned}
J(a, x^2, x^3) &= J(ax^2, x, x^2) && \text{Symmetry of } J \\
&= J((ax^2)x, x, x) && \text{Using (5)} \\
&= -J(ax^3, x, x) && \text{Using (4)} \\
&= -J(a, x^3, x^2) && \text{Using (5)} \\
&= -J(a, x^2, x^3) && \text{Symmetry of } J.
\end{aligned}
$$

Therefore, $2J(a, x^2, x^3) = 0$, and by characteristic $\neq 2$ we conclude that all of the linearizations of $J(a, x^2, x^3)$ are zero.

An immediate consequence of the identity (5) and Lemma 3 is the following:

Corollary 4. *Let A be as in Lemma 1. Then all of the linearizations of $J(A, A, x^3 x^2)$ are zero.*

Lemma 5. *Let A be a commutative algebra satisfying identity (1) as well as all linearizations of $x^3 x^2$. Then all linearizations of $J(A, A, x^2 x^2)$ are zero.*

Proof. We first show that all linearizations of $4J(y, x, x^3) + J(y, x^2, x^2)$ are zero. The proof only uses (1) and (4) so the proof is valid for all linearizations.

$$4J(y, x, x^3) + J(y, x^2, x^2)$$
$$= 2J(y, x, x^3) + 2J(y, x, x^3) + J(y, x^2, x^2) \qquad \text{Splitting the}$$
$$\text{first expression}$$
$$= 2J((yx)x, x, x) + 2J(y, x, x^3) + J(yx^2, x, x) \quad \text{by (5)}$$
$$= \begin{cases} 4(((yx)x)x)x + 2((yx)x)x^2 \\ +2(yx)x^3 + 2(yx^3)x + 2y(xx^3) \quad \text{by expansion} \\ +2((yx^2)x)x + (yx^2)x^2 \end{cases}$$
$$= \begin{cases} 0 + 2((yx)x)x^2 & \text{by (1)} \\ +2x^3(yx) + 2(yx^3)x + 0 & \text{by commutativity and (1)} \\ -2(yx^3)x + (yx^2)x^2 & \text{by (4) and commutativity} \end{cases}$$
$$= 2((yx)x)x^2 + 2x^3(yx) + (yx^2)x^2 \quad \text{collecting terms}$$
$$= 0 \qquad\qquad\qquad\qquad\qquad \text{linearization of } x^3 x^2.$$

Now:

$$J(a, b, x^2 x^2) = J(ab, x^2, x^2) \qquad \text{by (5)}$$
$$= -4J(ab, x, x^3) \quad \text{by first part of this proof}$$
$$= -4J(a, b, xx^3) \quad \text{by (5)}$$
$$= 0 \qquad\qquad\qquad \text{by (1).}$$

Lemma 6. *Let A be a commutative algebra of characteristic $\neq 2$ satisfying* (1) *and all linearizations of $x^2 x^2$. Then all linearizations of $J(A, A, x^3)$ are zero.*

Proof. We have:

$$J(a, b, x^3)$$
$$= J(ab, x, x^2) \qquad\qquad \text{by (5)}$$
$$= J((ab)x, x, x) \qquad\quad \text{by (5)}$$
$$= ((ab)x)x)x + ((ab)x)(xx) \quad \text{by expansion.}$$
$$= 0 \qquad\qquad\qquad\qquad \text{The first term is zero by (1) and}$$
$$\text{the second is a linearization of } x^2 x^2.$$

3. Solvability. In this section A will be a commutative, nonassociative algebra satisfying (1). We have three nonassociative polynomials which will be of interest to us. These are x^3, x^2x^2 and x^3x^2. We will linearize these polynomials in all possible ways. We will then evaluate these linearizations using all possible choices of elements in the algebra A for the arguments. Let M_3 be the subset of A obtained by first linearizing x^3 and then evaluating these linearizations on A. Similarly, M_4 is the subset of A obtained from x^2x^2 and M_5 is the subset of A obtained from x^3x^2. We use pointy brackets to represent the ideal generated. This means that $\langle M \rangle$ is the ideal generated by the set M. Let $I_3 = \langle M_3 \rangle$, $I_4 = \langle M_4 \rangle$ and $I_5 = \langle M_5 \rangle$.

Lemma 7. a) $J(M_3, A, A) \subseteq \langle M_4 \rangle = I_4$.

b) $J(M_4, A, A) \subseteq \langle M_5 \rangle = I_5$.

c) $J(M_5, A, A) = \{0\}$.

Proof. a) is from Lemma 6.

b) is from Lemma 5.

c) is from Corollary 4.   □

Lemma 8. *Let M be any subset of A. Then:*

a) *The set $K = \{k \in A \mid \langle M \rangle k \subseteq \langle J(M, A, A) \rangle\}$ is an ideal of A,*

b) $I_3 \subseteq K$.

Proof. a) From identity (5) we have that

$$(7) \qquad\qquad J(\langle M \rangle, A, A) \subseteq J(M, A, A).$$

Let t be an element in $\langle M \rangle$, let a be any element of A, and let k be any element in K. We have: $J(t, a, k) = (ta)k + (ak)t + (kt)a$. Thus, $t(ak) = J(t, a, k) - (ta)k - (kt)a$. So, using (7), we get:

$$\begin{aligned}
\langle M \rangle (AK) &\subseteq J(\langle M \rangle, A, K) + (\langle M \rangle A)K + (\langle M \rangle K)A \\
&\subseteq J(\langle M \rangle, A, A) + \langle M \rangle K + (\langle M \rangle K)A \\
&\subseteq J(M, A, A) + \langle J(M, A, A) \rangle + \langle J(M, A, A) \rangle A \\
&\subseteq \langle J(M, A, A) \rangle.
\end{aligned}$$

This shows that $AK \subseteq K$ and so K is an ideal of A.

b) Let x be an element in A. If a and b are also elements of A, then $x(ab) = -(xa)b - (xb)a + J(x,a,b)$. Letting x be in $\langle M \rangle$ gives:

$$\langle M \rangle (ab) \subseteq (\langle M \rangle a)b + (\langle M \rangle b)a + J(\langle M \rangle, A, A).$$

Using this three times gives us:

$$\langle M \rangle x^3 \subseteq ((\langle M \rangle x)x)x + J(\langle M \rangle, A, A) + J(\langle M \rangle x, A, A) + J(\langle M \rangle, A, A)x.$$

Now, using (1) and (7), we have:

$$\langle M \rangle x^3 \subseteq \langle J(M, A, A) \rangle. \qquad \square$$

Lemma 9. a) $J(I_3, A, A) \subseteq I_4$,

b) $J(I_4, A, A) \subseteq I_5$,

c) $J(I_5, A, A) = \{0\}$.

Proof. Using (7) and Lemma 7, we get:

a) $J(I_3, A, A) = J(\langle M_3 \rangle, A, A) \subseteq J(M_3, A, A) \subseteq \langle M_4 \rangle = I_4$.

b) $J(I_4, A, A) = J(\langle M_4 \rangle, A, A) \subseteq J(M_4, A, A) \subseteq \langle M_5 \rangle = I_5$.

c) $J(I_5, A, A) = J(\langle M_5 \rangle, A, A) \subseteq J(M_5, A, A) = 0$.

Lemma 10. a) $I_3 I_3 \subseteq I_4$,

b) $I_4 I_3 \subseteq I_5$,

c) $I_5 I_3 = \{0\}$.

Proof. a) $I_3 I_3 = \langle M_3 \rangle I_3 \subseteq \langle J(M_3, A, A) \rangle \subseteq < M_4 > = I_4$.

b) $I_4 I_3 = < M_4 > I_3 \subseteq \langle J(M_4, A, A) \rangle \subseteq < M_5 > = I_5$.

c) $I_5 I_3 = < M_5 > I_3 \subseteq \langle J(M_5, A, A) \rangle = \{0\}$. $\qquad \square$

Lemma 11. I_3 *is solvable.*

Proof. Using Lemma 10 we have:

$$((I_3 I_3)(I_3 I_3))((I_3 I_3)(I_3 I_3)) \subseteq ((I_3 I_3)I_3)I_3 \subseteq (I_4 I_3)I_3 \subseteq I_5 I_3. \quad \square$$

The following two results will be useful in the proof of our main result.

Proposition 12 [**6**, Proposition 2.2, page 18]. *If an algebra A contains a solvable ideal I, and if A/I is solvable, then A is solvable.*

Theorem 13. *Let A be a commutative finitely generated algebra over a field of characteristic $\neq 2$, satisfying the polynomial identity $((yx)x)x = 0$. Then A is solvable.*

Proof. A/I_3 is a commutative finitely generated algebra satisfying the identity $x^3 = 0$. From [**7**, Exercise 4, page 114] (see also [**1**, Proposition page 2]) A/I_3 is nilpotent. From Lemma 11, I_3 is solvable. It follows from Proposition 12 that A is solvable.   □

REFERENCES

1. I. Correa, *Finitely generated Bernstein algebras*, Algebras Groups Geom. **13** (1996), 380–383.

2. I. Correa, I.R. Hentzel and A. Labra, *On nilpotence of commutative right nilalgebras of low dimension*, Int. J. Math. Game Theory Algebra **3** (2003), 199–202.

3. M. Gerstenhaber, *On nilalgebras and linear varieties of nilpotent matrices* II, Duke Math. J. **27** (1960), 21–31.

4. M. Gerstenhaber and H.C. Myung, *On commutative power-associative nilalgebras of low dimension*, Proc. Amer. Math. Soc. **48** (1975), 29–32.

5. D.P. Jacobs, S.V. Muddana, A.J. Offut, D. Lee and T. Whiteley, *Albert user's guide*, Department of Computer Science, Clemson University, 1993.

6. R.D. Schafer, *An introduction to nonassociative algebras*, Academic Press, New York, 1966.

7. K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov, I.P. and A.I. Shirshov, *Rings that are nearly associative*, Academic Press, New York, 1982.

Departamento de Matemática, Universidad Metropolitana de Ciencias de la Educación, J.P. Alessandri 774, Santiago, Chile
Email address: ivan.correa@umce.cl

Department of Mathematics, Iowa State University, Ames, IA 50011-2064
Email address: hentzel@iastate.edu