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INFLUENCE OF SOIL VARIABILITY ON OPTIMUM SOIL
SAMPLING AND FERTILIZER USE

by

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I. INTRODUCTION

One of the prime objectives of applied research in soil fertility is to establish quantitative relations between crop yields and factors of production. Numerous factors affect crop production: some are subject to cultural control while others are not. Because yields depend on both, precise use of production resources appears to be conditioned on knowledge of environmental factors. In particular, crop responses and hence practical fertilizer application rates depend on quantities of nutrients present in the soil.

Empirical production relations are based on observations from small experimental plots selected for homogeneous soil conditions. However, it is evident that soil fertility varies from point to point in the field. Thus arises the problem of extending experimental results to applied production problems.

The object of this study is to utilize soil test information quantitatively in fertilizer recommendations for more than one nutrient, and, in addition, to extend experimental results to variable soil conditions by incorporating knowledge of soil variability. Concepts of variability are used in evaluating recommended fertilizer use and soil sampling procedures.

We introduce the subject in detail in Part II by citing previous work. Discussion there pertains generally to eco-
nomics and statistical aspects of applied research in crop production. A brief section reviews some problems encountered in soil sampling specifically for purposes of laboratory analyses.

Some pertinent analytical results from production theory are summarized in Part III. Using this as supplementary material, analytical production theory is extended specifically to cases characterized by variable conditions. The general concepts then are developed in some detail for a particular class of production relationships.

Experimental procedures followed in the study are given in Part IV. In addition, some general results obtained in Part III are written as special cases to facilitate the numerical computations.

In Part V, we present empirical results obtained by applying some of the derivations of Part III to soil test information from intensive field sampling. These results then are discussed in relation to their practical significance in applied production problems. A brief summary follows in Part VI. Finally, related materials are given in the appendices.
II. REVIEW OF LITERATURE

As an introduction to this study, we draw attention in succeeding paragraphs to references which appear to be most relevant to (a) economic use of fertilizer and (b) soil sampling for purposes of laboratory analyses for plant nutrient supplies. Restricting discussion primarily, although not exclusively, to fertilizer use in crop production, we trace recent developments in application of economic theory. In relation to other areas of agronomic interest, soil sampling has received little attention. In the second part of this section, we briefly review some work which has considered statistical as well as agronomic aspects of soil sampling.

A. Economic Use of Fertilizer

A comprehensive application of economic theory to agricultural production problems was undertaken by Heady (1952). He considered various aspects of allocating scarce resources among competing activities under conditions of perfect as well as imperfect knowledge. More recently, Heady and Dillon (1961) reviewed important concepts of economic theory, examined selected problems of experimental method, and presented several applications of theory to production of agricultural commodities. Their extensive collection and review of related references obviates that need here.
Specification of optimum resource allocation presumes a criterion. Among those discussed by Heady (1952) and Heady and Dillon (1961) are (a) unrestricted maximum profit per productive unit, and (b) restricted maximum profit under capital limitations. Pesek and Heady (1958) obtained minimum recommended fertilizer rates by another criterion: maximum dollar returns per dollar invested. It should be noted that the present study assumes the criterion (a) throughout.

After analyzing results of an experiment designed for the purpose, Heady and Pesek (1954) specified profit-maximizing rates and ratios of nitrogenous and phosphatic fertilizers. The experimental approach taken by these workers has influenced subsequent research; hence we describe it in some detail. Selected rates of nitrogen and phosphorus fertilizer materials were applied to corn planted at a rate of 18,000 plants per acre. Conducted in western Iowa, the experiment occupied a selected site on an Ida silt loam soil for one particular year. Yield observations were used in multiple regression

1In defense of the agronomic implications of this review, this writer interjects a few comments. All that one needs to borrow from the social studies is a criterion which specifies relevant goals. Relationships among yield and input factors become the proper concern of agronomists. Given a practical criterion, meaningful recommendations follow elementary analysis of production relationships. Criticism of farmers who maximize total corn yield, perhaps for a contest, and of agronomists who recommend "irrational" fertilizer rates, fails to consider the relevant goals.
analysis to estimate parameters of alternative algebraic functions which approximate the observed relationship between yield and fertilizer inputs. Marginal economic analysis then applied to an empirical yield function led to specification of nutrient combinations which maximize profit per acre for various corn and nutrient prices.

Although several experiments of similar scope have been reported (see, for example, Chapters 14 and 15 of Heady and Dillon (1961)), some researchers have recognized inherent limitations in attempts to apply the experimental results as recommendations. These limitations arise when experimental conditions fail in practice. Swanson (1957) devotes a general discussion to this problem. Before reviewing various attempts to relax restrictions on experimental results, we examine in further detail some of the strict conditions imposed by experimental procedures and analysis of results.

Consider a fertilizer rate experiment conducted at a single site in one particular year. Crop, variety, and seasonal characteristics are assumed constant for all yield observations. The experimental area usually is selected on the basis of uniform conditions of slope, color, soil type, drainage, and other observable soil characteristics; moreover, because of the small sizes of plots and experiments in relation to management units, the former are generally conceded to be homogeneous, relative to the latter, in soil nutrient and
moisture supplies and other soil properties. Stand and production factors other than fertilizer rates are supposedly held constant at arbitrary levels. By paying personal attention to such details as treatment application, the experimenter ensures rigid control of fertilizer rates, ratios, and uniformity of spreading. Finally, a particular algebraic form of input-output function is selected prior to estimation of parameters and application of marginal analyses.

We proceed to examine some approaches which have been advanced to facilitate extension of experimental results to ranges of conditions encountered in practice. We find it convenient to categorize the work of various authors for purposes of discussion.

1. Errors of marginal analysis

Among others, Anderson (1957) discussed biases in regression models which may occur when an incorrect polynomial function is selected. Taking account of estimation errors, he also obtained a confidence interval for optimum fertilizer rate as calculated from an estimated quadratic function for one variable nutrient. In one empirical study he found the intervals to be rather wide. Although biases of this nature were not investigated in the present study, we point out the relevance of Anderson's work to such problems.
2. **Year-to-year climatic variation**

In a discussion of resource use under imperfect knowledge, Heady (1952) postulated a distribution of production functions over years. Expanding on the subject, he later commented (Heady (1956, p. 3)): "The central methodological problem in fertilizer use on a single crop is prediction of the mathematical form and the probability distribution of the response function".

Tramel (1954) estimated yield-fertilizer production functions for each of 32 years in Mississippi, then obtained so-called planning functions by aggregating like years. For use in management decisions, he noted historical frequencies of year types. Brown and Oveson (1958) used a similar approach in Oregon for a ten-year series of experiments. Parks (1956), working in terms more specific than "seasonal effects", reported a family of yield functions in which response to applied nitrogen depended on moisture and stand levels.

A quantitative approach to climatic differences among years has been followed by several workers. Utilizing data from a 26-year period, Rust and Odell (1957) related yields quantitatively to weather factors, nitrogen applied in current and preceding years, and phosphorus and potassium applications. Weather variables selected by them were total precipitation and mean of daily maximum temperatures for the period July 1 through August 31. The significance of the investiga-
tion is indicated by their conclusion that more year-to-year variation was associated with weather than with all other factors measured in those experiments.

In addition to incidence of rainfall during the growing season and rates of applied nitrogen and phosphorus, Orazem and Herring (1958) used a measure of soil moisture at seeding time for yield prediction. They found that knowledge of initial soil moisture was particularly useful in reducing yield uncertainty under dry-land farming conditions where crop production is strongly dependent on soil moisture.

R. Hildreth (1957) considered a function which included fertilizer rate and rainfall as variables. He then used historical rainfall data to obtain "probabilities" for choosing between no fertilizer and an arbitrary experimental rate. Because of the forced dichotomous decision between fertilizer and no fertilizer, the utility of this approach appears to be somewhat limited.

Parks and Knetsch (1960) expressed corn yield as a function of applied nitrogen and a drought index which depended on soil moisture tension at successive stages of plant growth. These workers also made use of historical drought occurrence in presenting multiple-valued yield expectations in terms of risks. A somewhat different approach to seasonal moisture distribution was made by Gomez (1960) and Besson (1961). Both of these workers related observed fertilizer-induced yield
responses as polynomial functions of applied phosphorus, soil test results for various characteristics, and climatic factors. In order to distinguish seasonal patterns, temperature and precipitation were expressed as orthogonal polynomial functions of time.

Climatic factors have been reviewed in this section because of their relevance to crop production. In Part III, discussion is directed to problems caused by variability of environmental factors. Although it was not incorporated in this study, variability of climatic factors, particularly those of the microclimate, could be considered in the light of results of Part III.

3. Site-to-site variation in soil fertility

Several writers have recognized the need for yield estimation functions which provide for differences in soil fertility levels. Among these are Hanway and Dumenil (1955), Brown (1956), Anderson (1956), Pesek (1956) and Heady and Pesek (1957). It may be of interest to note here that current nitrogen fertilizer recommendations for corn in Iowa are based on the work of Hanway and Dumenil (1955). We discuss the latter in the following paragraph.

Data for the study were extracted from nitrogen rate experiments conducted in Iowa during the period 1943 through 1952. A Mitscherlich function was taken to represent the
relationship between total yield and total available nitrogen. From this function was derived another which expressed incremental yield as a function of total available nitrogen, where the yield increase is relative to yield with no fertilizer applied. However, the problem of scaling soil test results in units of applied nitrogen was circumvented by exploiting a particular property of the asymptotic function. Yield response then was expressed as a function of nitrogen applied, in pounds per acre, and soil nitrification rate, in parts per million per two-week laboratory incubation period. Rates of applied nitrogen, soil tests, and observed yield increases for the various experiments provided data for estimation of parameters.

Little of the work which succeeds that of Hanway and Dumenil (1955) has avoided the scaling problem when use is made of soil tests. Of the research reviewed in the following paragraphs, Gomez (1960) and Besson (1961) provide exceptions.

Concepts entertained by several writers were summarized by Heady and Pesek (1957). They visualized a yield surface over a total nutrient plane for two variables. Properly scaled, soil tests from a particular field or part thereof would serve to locate in the plane the origin of a surface considered relevant for determining fertilizer recommendations for this field.

Anderson (1956) pointed out some consequences of failure
to locate the origins of particular surfaces for both experimental and production sites. He noted that biases in total and incremental yield predictions may occur when experimental results are applied to a field with different soil fertility. Because of changes in soil fertility from one experiment to another, observed relations between yield and fertilizer inputs may appear quite diverse. Assuming a quadratic relation between yield and total quantity of a nutrient, Anderson (1956) algebraically evaluated prediction biases and went on to show that apparent differences in two yield functions may be reconciled by considering soil nutrient sources available in the individual experiments.

Hurst and Mason (1957) expanded the results of Anderson (1956) for the case of a quadratic yield function. They indicated a method by which quantity of soil nutrient, expressed in units of applied fertilizer nutrient, could be estimated from yield data at a particular experimental site without reference to soil tests.

Some aspects of resolving the scales for soil test and fertilizer nutrients were considered by Pesek (1956) and C. Hildreth (1957). Pesek (1956) postulated proportional scaling factors to be applied to soil test results, and then suggested estimating two such factors along with other parameters of a second degree polynomial function of two nutrients. This approach was elaborated for a quadratic function of one nutri-
ent by C. Hildreth (1957), who indicated a non-linear procedure for estimating the scaling factor.

The work of both Pesek (1956) and Anderson (1956) presumes that a polynomial of second degree characterizes the yield surface over the total nutrient plane.\(^1\) Attempts to extend and empirically test these concepts were made by Jensen (1957) and Jensen and Pesek (1959), Hurst and Mason (1957), and Voss (1960). These workers examined whether second-degree yield surfaces, estimated from experimental areas with different soil fertility levels, could be considered as translated portions of the same general second-degree surface over the total nutrient plane. Results of Anderson (1956), extended to two variable nutrients, indicated that coefficients of terms of degree two in the population polynomial would be invariant under translation of axes. Hence, under the hypothesis of a general second-degree function, those coefficients of second degree terms estimated for various experimental sites could be considered as independent estimates of the corresponding population parameters.

Using results of field and greenhouse experiments involving six and three fertility levels, respectively, Hurst and

\(^1\)We use this terminology in agreement with prior discussion. Although Anderson restricted his analysis to a single nutrient, this terminology is consistent if we admit a one-dimensional "plane". Similarly, we could consider \(n\) production factors and \(n\)-dimensional hyperplanes.
Mason (1957) and Jensen (1957) found little evidence against a hypothetical general second-degree surface. However, in a group of thirty bulk soil samples used in a greenhouse study, Voss (1960) obtained evidence which discounted the hypothesis for the wide range of soil conditions studied. By arbitrarily dividing the soils into three groups on the basis of phosphorus fertility as indicated by soil tests, he accepted possible existence of aggregate surfaces of somewhat restricted generality.

Another method of incorporating soil test information in yield predictors was used by Gomez (1960) and Besson (1961). They included laboratory test results in the regression model as separate variables per se, with no implied equivalence to fertilizer nutrient sources. Thus they avoided problems of scale. In relation to the total input plane visualized by Heady and Pesek (1957), this approach corresponds to a fertilizer nutrient plane with axes orthogonal to those of a soil nutrient plane. In contrast to the previous case, yields would occupy a hypersurface over the four-dimensional soil-fertilizer hyperplane for two nutrients.

The latter approach and that taken by Hanway and Dumenil (1955) appear to be advantageous in many ways. These procedures lead to a family of yield functions, the members of which depend on soil tests. Furthermore, yield prediction functions are used conditionally, i.e., yield-fertilizer
relations and marginal analyses relevant to a particular site are conditioned on the outcome of laboratory soil tests for that site. This fact contributes to the practical utility of functions which incorporate soil test information directly.

A principal advantage of using soil test results independently lies in avoidance of scaling problems, particularly since it is not now known whether linear or non-linear scales should be used or, more generally, whether scaling is a meaningful operation in this system. A final advantage is related to the types of functions considered admissible for describing a yield surface. One morphological restriction for a general yield function deduced by Heady and Pesek (1957), and later discussed in more detail in Chapter 3 of Heady and Dillon (1961), requires convergence of isoclines at the origin of the total nutrient plane. This need not be required of conditional functions, particularly in recognition of the fact that soil nutrient supplies seldom degenerate to zero.

Numerical results given later in Part V were obtained by use of an empirical function which described yield as a function of applied fertilizer and soil fertility. Because of the apparent advantages for the method used by Gomez (1960) and Besson (1961), their approach was taken in this study.

\[\text{Isoclines are curves in the input plane which correspond to all points on the yield surface for which marginal rates of substitution between two nutrients are equal.}\]
In concluding this section, a few additional comments in regard to empirical work may be in order. Results of Voss (1960) appear to refute a general second-degree polynomial function of two nutrients over a wide range of soil fertility, even though polynomials of that degree quite ably described relations between yield and fertilizer inputs for each soil fertility level. This is not surprising when polynomial regression is considered in terms of Taylor's expansion of the unknown function as outlined by Hartley and others. Local approximation with a few terms of the expansion may be quite satisfactory, whereas more terms generally are required to approximate larger portions of the unknown surface.

It appears that a useful family of yield functions may be obtainable from functions estimated by Voss (1960) for thirty levels of soil fertility. Examining trends among regression coefficients for the square of applied phosphorus, he observed that, sign ignored, size of the estimated coefficient decreased as soil phosphorus test increased. This writer found a correlation coefficient of -0.73 between these entities as they were reported by Voss. There also appears to be a relation between phosphorus soil test and regression coefficients for the first power of the applied phosphorus variable. Considering effects of phosphorus only, it appears that a one-

parameter family of yield relations might be obtained by expressing regression coefficients as functions of soil phosphorus. Member functions of the system would be quadratic.

4. Nonuniform fertilizer application

The desirability of precisely controlled fertilizer application has received quite general recognition; this is reflected in recent technological advances. Changes in design of bulk spreading equipment, for example multiple instead of single fan systems, have been directed toward uniform spatial distribution of fertilizer materials. Fertilizer manufacturers, focusing attention on stability of nutrient ratios within a given lot of mixed fertilizer, have developed aggregation methods for ensuring constant nutrient ratios among particles. In spite of these efforts, it appears that control of fertilizer application, as achieved under experimental conditions, may not be realized in practice. Hence we find it relevant to consider relaxation of these conditions on experimental results.

Fertilizer application techniques currently in use in the United Kingdom were discussed by Hebblethwaite and Pascal (1960) with particular reference to evaluation and design of equipment. These workers dismissed the problem of particle segregation by citing and encouraging the trend toward use of granular complete fertilizer materials.
Emphasizing the importance of uniform distribution as a criterion for evaluating equipment, Hebblethwaite and Pascal (1960) cited results of several field studies designed to determine effects exerted on yields by uneven fertilizer application. In addition, these workers used response curves to predict average yield from application of two fertilizer rates, low and high. They found that predicted yield averages generally decreased with distance of low and high rates from their mean, i.e., as fertilizer was applied less evenly.

Hebblethwaite and Pascal concluded:

The task of predicting the effects of uneven distribution falls within a no-man's land between the agricultural engineer and the agronomist, and there is clearly a need for further work.

Independently of the British study, Jensen and Pesek (1962) examined some consequences of nonuniform fertilizer application and segregation of nutrient fractions in mixed fertilizers. Assuming algebraic models for (a) characterizing a relation between yield and applied nutrients and (b) describing application rate in terms of distance from a line of reference, they derived quantitative effects of nonuniform fertilizer application. Yield losses were noted for cases in which diminishing returns accompany additional fertilizer inputs. This was given a graphic explanation which illustrated the fact that the yield increase associated with a given excess of fertilizer on part of a field fails to compensate for
yield sacrificed because of a corresponding deficit application.

As suggested by a regular succession of traverses of equipment across a field, periodic functions were taken to represent relations between quantity of fertilizer applied at a point and its position with respect to traverses followed by the equipment. Average yield for an area thus treated may be estimated. Applied fertilizer rate, in terms of its equivalent algebraic function, is substituted in the yield function; this step is followed by integration of the resulting expression between limits specified by the area concerned. Such analysis was applied to a quadratic yield function of one fertilizer nutrient which was assumed to follow a cosine function of lateral distance from the applicator. Yield loss resulting from such an application pattern, as compared with uniform treatment, was found to be proportional to the square of the amplitude of the cosine function. Thus yield losses increase as the uniformity of application decreases, a result which is in qualitative agreement with that obtained by Hebblethwaite and Pascal (1960).

Jensen and Pesek (1962) extended these results to yield functions of several variables, each of which may be applied nonuniformly. In a check of practical consequences of the findings, they applied derived results to empirical relationships. In addition, they assumed models and derived analogous
results for segregation of nutrient carriers in mixed fertilizers.

We conclude by noting the significance of this study in relation to the work reviewed previously. Experimental results may not apply in practice because of differences in spatial distribution of fertilizer. However, given knowledge of spreading characteristics of fertilizer equipment, experimental yield relations may be modified quantitatively to new functions which are useful commercially. It may be noted that this work attacked assumptions on variables of primary interest, instead of attempting to extend experimental results through quantitative evaluation of supplementary environmental variables.

The study of nonuniform resource application serves as an introduction to the present study. Instead of variation in a controlled production factor, however, we discuss in following sections the variation of environmental factors which may not be subject to control.

B. Soil Sampling

During the past two decades, considerable attention has been given to developing chemical and biological assays of soil nutrient supplies. Results of selected laboratory methods have been calibrated in terms of crop response to applied fertilizer in field studies. Used in conjunction with results
of laboratory analyses of field samples, these calibrations provide a basis for fertilizer recommendations. The importance of proper soil sampling procedure is punctuated by increasing numbers of managers who rely on fertilizer recommendations; however, relative to other areas of soil fertility, little research on sampling method has been undertaken. Here we briefly review some work related to soil sampling methods, with particular reference to statistical considerations.

A partial review of literature related to soil sampling was made by Cline (1944). In this and another discussion (Cline (1945)), he outlined some pertinent aspects of the problem, including sampling from soil areas which are homogeneous with respect to observable physical conditions. In particular, he recognized the possibility that chemical properties may vary widely in areas which appear to be physically uniform. Commenting on the practice of analyzing composite\(^1\) soil samples in contrast to analyses of the components, Cline stated that the former is restricted in use to cases in which (a) equal quantities of soil materials are combined, (b) interactions between subsamples with different nutrient quantities are negligible, and (c) interest lies solely in unbiased estimates of nutrient means in an area, in which case he suggested removing subsamples from a random grid superimposed on

\(^{1}\text{A composite soil sample is one composed of soil material obtained at more than one location in an area.}\)
the area. Treating a single soil core as a sample unit and assuming a normal distribution of analyses of these, Oline (1944) suggested solving for the required number of sampling units by arbitrarily choosing "maximum sampling error permissible", in terms of differences between two sample means, and using tabular values of Student's $t$-statistic. The ratio of the permissible difference to its standard error contains the number of sampling units as a factor.

Empirical work on soil sampling was reported by Rigney and Reed (1945). As they pointed out, recommendations at that time commonly suggested compositing 5 to 30 borings of soil from an area not exceeding 5 acres. In an attempt to supply objective bases for sampling instructions, these workers subdivided each of 20 adjoining 8-acre fields into 4 units; 20 borings per unit then were obtained according to a systematic, grid-like pattern which was duplicated with a random starting point. In addition, subsamples were taken from the composite samples, and all samples were analyzed for several chemical properties. The analysis of variance table for each chemical property was used to provide estimates of composite and sub-sample variances. These estimates were used for examining accuracy of the sample mean in alternative sampling schemes as numbers of samples and subsamples varied.

In a later study involving several chemical properties of soils, Reed and Rigney (1947) compared small areas (3/4-acre)
which were either homogeneous or heterogeneous with respect to
conditions of slope, soil type, color, depth, and texture.
Two areas were sampled at 20 grid positions by collecting
individual specimens at each point. Absence of row or column
effects strengthened their assumption concerning random occur-
rence of soil properties. Data for positions, borings, sub-
samples, and duplicate laboratory determinations provided
estimates of corresponding variance components; these quanti-
ties were useful for determining numbers of each sample type
required for a given "limit of accuracy" of means in parts per
million. These workers recognized the usual dilemma which
accompanies sampling for multiple characteristics, namely,
that possibly different sample schemes are required for each
of the characteristics. Noting different patterns of varia-
tion in the properties studied, they concluded that, in gen-
eral, the precision of sampling may be limited by the property
requiring the greatest sample density.

Rigney (1956) reviewed some pertinent aspects of the pre-
sent status of soil sampling. In addition to reiterating his
previous work with Reed, he cited an empirical study in which
a "zig-zag" pattern and stratified random sample were superior
to random and systematic grid samples.

Jacob and Klute (1956) contributed little to the work
which preceded them. They applied concepts previously out-
lined by Reed and Rigney (1947) to sampling experimental plots
for physical as well as chemical properties. For a given "limit of accuracy" of a treatment mean, they obtained an optimum number of plots, samples, and determinations by minimizing a cost function.

A somewhat different approach to soil sampling was suggested by Hammond et al. (1958). They suggested application of multistage sampling theory, in which the sampled units are drawn randomly at each stage. In seven fields which varied in size from two to 35 acres, three-stage sampling was compared with simple random sampling. Without exception, three-stage sampling was less efficient than simple random sampling on the basis of variances of means. However, by manipulating a cost function, these workers concluded under admittedly questionable assumptions that "efficiency" may be relatively high for multistage sampling. They attributed this to convenience of locating the larger number of samples required for multistage sampling.

In light of comments made by Cline (1944), it appears that multistage sampling is useful only if estimates of means for a field are required. Multistage sampling without stratification ignores the possibility that means may vary from stratum to stratum in a field.

In conclusion, we review an outline of soil sampling recommendations which are currently used in many states. Cooperating with the Soil Test Work Group of the National Soil
and Fertilizer Research Committee, Reed (1953) presented contemporary ideas on sampling soils for purposes of chemical tests. On the basis of published and unpublished research, the following general sampling procedures were suggested:

(a) divide each field into uniform areas not exceeding 5 to 10 acres, or 20 acres in the case of notable uniformity; (b) obtain a separate composite sample for each area; (c) prepare each composite sample by mixing cores taken from the plow layer (0 to 6 inches) at 20 sites distributed throughout the area; and (d) avoid those sites which exhibit unusual soil conditions.

Two major deficiencies may be noted in the work cited in this section; both appear to arise frequently in surveys. First, the "maximum sampling error permissible" or "limit of accuracy" is quite arbitrary. No objective basis has been used to specify magnitudes of these quantities for chemical properties of soils. In addition, by considering the factors singly, different patterns of variability may lead to specification of different sample designs for each soil characteristic. These and other sampling problems are treated in light of sampling theory and methods by Cochran (1953).

It should be pointed out that results obtained later in Parts III and IV are based on population quantities instead of sample statistics. However, consideration is given in Part V to the population characteristics obtained earlier as well as sample statistics.
III. EXTENSION OF PRODUCTION THEORY

The central theme reviewed in the preceding chapter was extension of experimental results to conditions not encountered in any one experiment. Of particular interest is the concept that, in some cases, yield functions may be modified quantitatively to account for heterogeneous conditions. In this chapter we consider environmental variability with regard to optimal resource use. In light of results obtained here, we evaluate contemporary ideas regarding the role of soil test information in fertilizer recommendations. Some results of this chapter appear to provide useful criteria for examining alternative soil sampling schemes.

To expedite the discussion, we introduce some symbols. The symbols $Y$ and $X$ denote yield output and resource input, respectively, for some production process. These are conveniently expressed in terms of physical quantities per productive unit. The symbol $x$ refers to a factor\(^1\) which affects output quantitatively, but which is observed as a concomitant environmental characteristic. Soil nutrient supplies, as well as other environmental factors discussed in the literature review, are included in this category. Prices $P_x$ and $P_Y$ refer

\(^1\)In later developments, it will be convenient to consider $X$ and $x$ as vectors.
to prices per unit quantities of resource and product, respectively. The letters \( f \), \( g \), and \( h \), and symbols such as \( Y(X) \), indicate functional relationships between variables. The expectation operator is represented by \( E \).

In some parts of this chapter, certain mathematical operations are indicated. It is apparent that validity of each would have to be established in a rigorous analysis. However, those mathematical functions used to characterize many biological relations between yield and input factors are quite arbitrary. Hence, we assume at the outset that functions can be selected which possess certain analytical properties\(^1\) and which at the same time satisfactorily approximate observed relations between yield and resources.

A. Review of Production Theory

We recall some well-known results from economics which are pertinent to the present discussion. Suppose we have the one-parameter family of input-output functions

\[
Y = f(X,x)
\] (3.1)

where the relationship between \( Y \) and \( X \) depends on the parameter \( x \). It is useful to note that a class of yield functions \( Y(X) \) is generated as the parameter \( x \) assumes admissible values

\(^1\)Some properties found useful in the following analyses are continuity, differentiability, integration in terms of elementary functions, and convergence of improper integrals.
within its domain.

From the profit equation
\[ \pi = Y P_y - X P_x \]  
(3.2)
where \( \pi \) represents profit, we maximize \( \pi \) with respect to \( X \).

This leads to the well-known condition
\[ \frac{\partial Y}{\partial X} = \lambda \]  
(3.3)
where \( \lambda \) represents the ratio \( P_x/P_y \) and \( x \) is assumed to have the constant value \( x_k \). Solving Equation 3.3 for the input rate which maximizes profit, given \( x_k \), we obtain the equality
\[ X_{ok} = g(\lambda, x_k) \]  
(3.4)
for optimum input rate \( X_{ok} \) in terms of prices and \( x_k \). This equation is valid for any admissible value of the variable \( x \).

Maximum profit may be determined by substituting the expression \( g(\lambda, x_k) \) for \( X \) in Equations 3.1 and 3.2 and evaluating the latter. Denoting by \( \pi_{ok} \) the maximum profit under condition \( x_k \), we obtain the relations
\[ \pi_{ok} = f(X_{ok}, x_k) P_y - X_{ok} P_x \]
\[ = f \left[ g(\lambda, x_k), x_k \right] P_y - g(\lambda, x_k) P_x \]  
(3.5)
\[ = h(P_x, P_y, x_k) \]

---

1Equation 3.3 may be solved for \( X \) if the function \( Y(X, x) \) satisfies the condition \( \frac{\partial^2 Y}{\partial X^2} \neq 0 \).

2This quantity depends on \( x \) if the relation \( \frac{\partial^2 Y}{\partial X \partial x} \neq 0 \) holds (as pointed out by Swanson (1957)).
which indicates that $\pi_{0k}$ is, in general, a function of resource and product prices as well as environmental level $x_k$.\footnote{Suppose, however, that environment was incorrectly specified as $x_j$ instead of $x_k$. Profit from application of rate $X_{0j}$, under environmental conditions $x_k$, would be}

These results may be extended readily to more than one resource and one environmental factor. We let the symbol $x$ refer to a vector of environmental factors of dimensions $q$, while $X$ and $P_x$ respectively represent $p$-dimensional quantity and price vectors for $p$ resources. Profit defined by Equation 3.2 may be rewritten in terms of the vector product $X'P_x$ if $X$ and $P_x$ are both column vectors. Corresponding to Equation 3.3 for maximum profit, we have the condition

$$\nabla Y = \lambda$$

(3.6)

where $\lambda$ is a $p$-dimensional vector of ratios of resource to product prices and the symbol $\nabla$ represents the operator $(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_p)$. The $p$-dimensional vector of optimum resource inputs corresponding to Equation 3.4 is obtained by simultaneous solution, under certain conditions, of Equation 3.6 for the vector $X$. Maximum profit may be obtained from a

Because of maximum properties of $\pi_{0k}$, the inequality $\pi_{jk} \leq \pi_{0k}$ follows; equality holds only when $x_j$ and $x_k$ are equal. The difference $\pi_{0k} - \pi_{jk}$ represents cost of the specification error.
relation corresponding to Equation 3.5, where the vector product $X_v P_x$ replaces a scalar product.

The above analyses are indicated for cases in which known environmental conditions characterize a homogeneous production site. For cases in which heterogeneous conditions prevail, however, problems of resource allocation generally have been ignored. In the following sections we indicate an extension of production theory for heterogeneous cases.

B. Resource Use under Heterogeneous Environmental Conditions

In this section we consider two types of environmental heterogeneity, namely, spatial trends and random occurrence. The latter suggests a framework for Section C below which forms the basis for empirical investigations reported in later sections.

1. Environmental trends

Spatial environmental trends within a productive unit may be considered in light of standard production theory and methods suggested by Jensen and Pesek (1962). Consider fertilizer use and crop production in a given field. Suppose that environmental factor $x$ may be described by some function

$$x = h(u,v)$$

(3.7)
of Cartesian coordinates $u$ and $v$ superimposed on the field.
Using Equation 3.3 as a criterion for maximum profit, we obtain optimum resource rate

\[ X_0 = g \left[ \lambda, h(u,v) \right] \]  \hspace{1cm} (3.8)

by using Equations 3.7 and 3.4. In this form, optimum rate would depend on prices and coordinates \((u,v)\) of a particular production site. We could consider Equation 3.8 as a function describing spatial trends in resource application which would be necessary for maximum profit. However, if fertilizer application is restricted to a single rate, then the relevant average input-output function is obtained by using Equations 3.7 and 3.1 in the relationship

\[ Y = \frac{1}{|A|} \int \int_{u,v \in A} f \left[ X, h(u,v) \right] \, du \, dv \]  \hspace{1cm} (3.9)

where \(|A|\) represents size of the relevant area \(A\) and the integral is evaluated between limits of \(u\) and \(v\) which depend on \(A\). The resulting function of \(X\) then may be used in conjunction with Equations 3.2 and 3.3 to obtain the input rate which maximizes profit under the restriction.

2. **Random occurrence of environmental conditions**

Here we consider cases in which environmental characteristics are randomly distributed within a production site. In this section, we consider optimal resource use when the probability density function (p.d.f.) is known for a particular production case. Later, recognizing that such functions are
not known generally, we proceed to determine those properties of a distribution of environmental characteristics which are pertinent to the problem of fertilizer recommendations for selected functions $Y(X,x)$.

The function $f(X,x)$ in Equation 3.1 characterizes yield as a function of resource input and environmental factors. Note that a random collection of yield functions is generated from a family of functions $Y(X,x)$ as the concomitant variable $x$ ranges over a subset of its admissible values. Suppose it is known that the variable $x$ is distributed within a site according to the probability density function $p(x;\beta)$ with parameters represented by $\beta$. If optimum resource rate could be applied to each member of a hypothetical population of small units, then expected maximum profit could be written as the expectation

$$E_{x_{ok}} = \int_{-\infty}^{\infty} h(P_X,P_Y,x) p(x;\beta) \, dx.$$  \hspace{1cm} (3.10)

This is recognized as the expectation of the right hand side of Equation 3.5 with respect to the variable $x$, as denoted by the symbol $E_X$.

On the other hand, suppose that resource application to an area is restricted to a single rate. The relevant yield function $Y(X;\beta)$ is obtained from Equation 3.1 by means of the expectation
\[ E_X Y(X,x) = \int_{-\infty}^{\infty} f(x,x) p(x;\beta) \, dx. \] (3.11)

The resulting function may be used in Equations 3.2 and 3.3 to give optimal resource application

\[ x_0 = g(\lambda;\beta) \] (3.12)

which depends on parameters \( \beta \) of the p.d.f. if the relations

\[ \frac{\partial^2 Y(X;\beta)}{\partial X \partial \beta_i} \neq 0 \]

are satisfied, where \( \beta_i \) is the \( i \)th parameter of the p.d.f. Maximum profit is determined from an equality which corresponds to Equation 3.5.

These concepts may be expressed more briefly. We recognize three operations: (a) profit, which we denote by \( \pi(X,x) \); (b) maximization with respect to resource \( X \), which we symbolize by \( M_X \); and (c) expectation taken with respect to the random variable \( x \), which is denoted by \( E_x \) as before. By permuting these symbols under the restriction that \( M_X \) always precedes \( \pi(X,x) \), we observe the three cases:

Case 1. \( E_X M_X \pi(X,x) \)

Case 2. \( M_X E_X \pi(X,x) \)

Case 3. \( M_X \pi(X,E_X(x)) \)

We note that Case 1, which corresponds to Equation 3.10, represents an ideal case in which profit-maximizing resource input would be applied to each small unit. Cases 2 and 3 both presume utilization of a single resource rate, which in each case is chosen as that rate which maximizes profit. However, for reasons outlined in the following paragraph, Case 3 in
general is not valid for a variable system.

Suppose the function $Y(X,x)$ is convex with respect to $x$. Then we exploit a well-known property of convex functions to deduce the inequality $E_X Y(X,x) \leq Y(X,E_X(x))$, where $E_X(x)$ represents the expected value of the variable. Convexity of $Y(X,x)$ with respect to $x$ implies convexity of the profit function with respect to $x$. This may be seen by examining the form of Equation 3.2, since the term $XP_X$ does not involve the variable $x$. Using this fact, we obtain the inequality $M_X E_X \pi(X,x) \leq M_X \pi(X,E_X(x))$. This may be recognized as the perennial problem of pre- and post-averaging. Here it is seen that incorrect use of the preaverage, as in Case 3, leads to overestimation of both yields and profits.

In light of current instructions for soil sampling, we point out that Case 3 corresponds to fertilizer recommendations which are based on average laboratory results of a sample compositive from several variable subsamples. From the results of this section, we may conclude that, in general, fertilizer recommendations based on composite soil samples are questionable in the absence of additional information.

In light of the above conclusions, plus the fact that Case 1 represents an ideal case which may be unattainable in practice, we restrict further discussion to Case 2. Here we

\[ ^1 \text{In economic terminology, it exhibits diminishing returns to increasing levels of } x. \]
introduce the concept of conditional expectations, which facilitates a later discussion of alternative sampling schemes.

If we admit stratification of a field into areas which may be treated separately, then previous comments apply to each stratum. Case 2 may be taken to represent maximum profit for a stratum when account is taken of variation within the stratum. The expectation \( E_x \) may now be written \( E_x \mid s \), indicating conditional expectation with respect to \( x \) given a particular stratum. Total expectation is the product \( E_s E_x \mid s \) of the marginal \( (E_s) \) and conditional \( (E_x \mid s) \) expectations.

Making use of this extension, we note that the Case 2 symbol \( M_x E_s E_x \mid s \pi(x, x) \) corresponds to the case in which all strata in a field are treated with a single rate. On the other hand, if each stratum receives an optimal fertilizer application, then maximum stratum profit is, as before, \( M_x E_x \mid s \pi(X, x) \). The expectation of these quantities, taken with respect to strata, may be denoted by the expression \( E_s M_x E_x \mid s \pi(X, x) \).

C. Application of Theory to Fertilizer Use

Although they serve as useful guides, analyses outlined in the preceding section were based on assumed knowledge of a true yield function and the probability density function of a population of environmental factors. Neither is known in practice. In this section we proceed to examine only those
properties of unknown distributions to which analyses of specific yield functions relate.

We state at the outset that derivations given in this section involve population quantities only; as a consequence they are mathematical rather than statistical in nature. In addition, computational procedures outlined in Part IV are based on enumeration of individuals in a finite population, which also involves population quantities only. It should be pointed out that the population variance-covariance-like terms arise as a consequence of taking expectations of squares and quadratic forms. However, a section in Part V is devoted to the statistical problem of fertilizer recommendations and profit estimates based on statistical sampling procedures.

We introduce the concepts of this section by briefly considering the case of one variable environmental factor. Results are then given in detail for the general multivariate case when the yield function is approximated by a second-degree polynomial.

1. The univariate case

The derivations of this section will be necessarily brief. The general results of the following section hold for the special case of a univariate environmental factor; derivations for the special case may be obtained from the succeeding section by considering the quantities as scalars instead of
vectors and matrices.

Consider a field, divided into strata, to which fertilizer is applied. Incorporating concepts of soil variability into fertilizer recommendations, we make the following assumptions: (a) soil supply $x_{ij}$ of a particular nutrient in the $j$th area element of the $i$th stratum is distributed about its mean $\mu_i$ with variance $\sigma^2_w$, (b) the unknown yield function is satisfactorily approximated by a second-degree polynomial, and (c) stratum means $\mu_i$ are distributed about the field mean $\mu$ with variance $\sigma^2_a$.

We find it convenient to formalize these statements. The first set of assumptions is summarized in the relations

$$x_{ij} = \mu_i + e_{ij}$$

$$E_{x|s}(e_{ij}) = 0$$

$$E_{x|s}(e_{ij}^2) = \sigma^2_w$$

(3.13)

where $i$ represents stratum number and $j$ identifies the individual production site within the $i$th stratum. The subscript $w$ designates a within-stratum property. The assumed yield function may be written in the form

$$Y(X, x_{ij}) = b_0 + b_1 x_{ij} + b_{11} x_{ij}^2 + b_2 x + b_{22} x^2 + b_{12} x_{ij} x.$$  (3.14)

The third set of assumptions may be expressed as the relationships

$$\mu_i = \mu + \gamma_i$$

$$E_s(\gamma_i) = 0$$

(3.15)
\[ E_s(Y_i^2) = \sigma_a^2. \]

Equations 3.13 facilitate taking the expectation of \( Y(X, x_{ij}) \) with respect to \( x \). Substituting \( \mu_1 + e_{ij} \) for \( x_{ij} \) in Equation 3.14 and expanding, we obtain a function \( Y(X, \mu_1 + e_{ij}) \). Applying the conditional expectation operator to the resulting function in light of the formal properties exhibited in Equations 3.13, we obtain the equality

\[
E_{x|s}Y(X,x_{ij}) = b_0 + b_1 \mu_1 + b_{11} \mu_1^2 + b_2 X + b_{22} X^2 + b_{12} \mu_1 X + b_{11} \sigma_{wi}^2. \tag{3.16}
\]

The first six terms of this function are simply the function \( Y(X, \mu_1) \) or, equivalently, \( Y(X, E_{x|s}(x_{ij})) \). Thus we have the relation

\[
E_{x|s}Y(X,x_{ij}) = Y(X, E_{x|s}(x_{ij})) + b_{11} \sigma_{wi}^2. \tag{3.17}
\]

If the function \( Y(X,x_{ij}) \) is convex with respect to \( x_{ij} \), then \( b_{11} \) of Equation 3.14 is negative and, since \( \sigma_{wi}^2 \) is non-negative, we find the inequality \( E_{x|s}Y(X,x_{ij}) \leq Y(X, E_{x|s}(x_{ij})) \) as before.

Examined in another light, Equation 3.17 provides the mechanism by which \( Y(X, \mu_1) \) may be corrected to the relevant function \( E_{x|s}Y(X,x_{ij}) \). Recall that the latter characterizes the relationship between yield and fertilizer on a heterogeneous stratum when account is taken of heterogeneity. Thus, it appears that one may obtain the appropriate function when estimates of within-stratum means and variances are available.
It is interesting to note that the difference $Y(X, E_x|x_{1j}) - E_x|x_{1j} Y(X, x_{1j})$ is proportional to the within-stratum variance of soil nutrient $x$.

Given Equation 3.16 for the $i$th stratum, we may specify the profit-maximizing input rate $X_{o1}$ by using the condition of Equation 3.3. The result is

$$X_{o1} = \frac{1}{2} b_{22}^{-1} (\Theta - b_{12} \mu_1) \quad (3.18)$$

where $\Theta$ represents the difference $\lambda - b_2$.

We conclude from the form of Equation 3.18 that optimum fertilizer rate $X_{o1}$ is not affected by incorrect use of $Y(X, \mu_1)$ instead of $E_x|x_{1j} Y(X, x_{1j})$; since these functions differ by a constant for any particular stratum, as shown in Equation 3.17, marginal analyses are unaffected. This statement does not apply to profits, which depend on yield level. The conclusion concerning optimum rates may be a source of comfort to agronomists concerned with fertilizer recommendations; however, lest comfort change to complacency, we should point out that this conclusion is a consequence of the assumed form of yield function. For example, if the term $b_{11} x^2 X$ were required in the function $Y(X, x_{1j})$ for satisfactory approximation of yield relationships, then Equation 3.16 would contain the term $b_{11} x_{o1}^2$. In this event, optimum input rate $X_{o1}$, given in Equation 3.18, would depend on the within-stratum variance.

We proceed to evaluate profits associated with use of $X_{o1}$.
units of input in the $i$th stratum. Using the definition of profit given in Equation 3.2, we see that the maximum profit $M_{X}E_{X}|_{S}^{\pi}(X,x_{ij})$ may be determined by substituting for $X_{0i}$ from Equation 3.18 in the equation

$$M_{X}E_{X}|_{S}^{\pi}(X,x_{ij}) = \left[ Y(X_{0i},\mu_{1}) + b_{ll}\sigma_{w1}^{2} \right] p_{y} - X_{0}P_{x}. \quad (3.19)$$

On substituting for $X_{0i}$ and the function $Y(X_{0i},\mu_{1})$ into Equation 3.19, then making use of the relation $\mu_{1} = \mu + \gamma_{1}$ and the properties of Equations 3.15, we find that expected profit for all strata, each optimally fertilized, is the quantity

$$E_{S}M_{X}E_{X}|_{S}^{\pi}(X,x_{ij}) = M_{X}\pi(X,E(x_{ij}))$$

$$+ \left[ b_{ll}E_{s}\sigma_{w1}^{2} + (b_{ll} - \frac{1}{4}b_{22}b_{12})\sigma_{a1}^{2} \right] p_{y}$$

where it is recalled that $E = E_{S}E_{X}|_{S}$. The expression $M_{X}\pi(X,E(x_{ij}))$ represents profit from fertilizing an ideally uniform field, characterized by constant level $\mu$, with a single optimum rate which depends on $\mu$.

Equation 3.20 appears to be useful in examining alternative sampling schemes. It is relevant to select stratification schemes for which the function $E_{S}M_{X}E_{X}|_{S}^{\pi}(X,x_{ij})$ is maximum when account is taken of costs of sampling and stratification. Also, the form of Equation 3.20 indicates that the distribution of the total variation between the within-stratum and among-stratum components, for a given kind of stratification, is of prime importance. This type of information obviously must come from empirical investigations. The best
stratification method appears to be that for which the total variation is, in some way, optimally distributed between the two components.

In the following section, we derive the results indicated in this section. In contrast to the present section, we will deal with the case of multivariate distribution of concomitant environmental variables $x_{ij}$; in addition, we consider yield to be a polynomial function of several applied factors.

2. **The multivariate case**

Here we outline the relevant theory for multivariate problems and indicate its role in subsequent numerical work. Results of the previous section immediately follow, as special cases, those presented here.

Consider a second-degree polynomial yield function $Y(X,x)$ of the form

$$Y = B_0 + B_1 x + x'B_{11}x + B_2 X + x'B_{22}X + x'B_{12}X$$  \ (3.21)

where

$$x' = [x_1, x_2, \ldots, x_q], \quad X' = [X_1, X_2, \ldots, X_p],$$

and where $B_0, B_1, B_{11}, B_2, B_{22},$ and $B_{12}$ have dimensions $1 \times 1,$ $1 \times q,$ $q \times q,$ $1 \times p,$ $p \times p,$ and $q \times p,$ respectively. The prime mark indicates a transpose of a matrix or vector. We assume, without loss of generality, that $B_{11} = B'_{11}$ and $B_{22} = B'_{22},$ i.e., these matrices are symmetric.
We determine optimal resource use within a particular stratum with variable soil conditions, then utilize this information in evaluating alternative stratification procedures.

Suppose that, within the \( i \)th stratum, the vector \( x_{ij} \) follows a \( q \)-variate distribution about the mean \( \mu_1 \) with error \( e_{ij} \). For the \( i \)th stratum, we write the conditional relationship

\[
x_{ij} = \mu_1 + e_{ij}
\]

(3.22)

where \( x_{ij} \), \( \mu_1 \), and \( e_{ij} \) are vectors of dimensions \( q \times 1 \). We assume the properties

\[
E_x|s(e_{ij}) = 0
\]

\[
E_x|s(e_{ij}e_{ij}') = \sigma^2W_1
\]

(3.23)

where \( e_{ij} \) is a \( q \times 1 \) vector, and \( \sigma^2W_1 \) is a \( q \times q \) matrix whose elements are the conditional covariance terms \( \sigma^2w_{ikl} = \text{Cov}(x_{ijk}, x_{ijl}) = E_x|s(e_{ijk}e_{ijl}) \). The symbol 0 represents the null vector; as before, the conditional expectation, given \( s \), is denoted by \( E_x|s \).

We further assume that the stratum means \( \mu_1 \) follow a \( q \)-variate distribution about their mean \( \mu \). This assumption may be represented by the equality

\[
\mu_1 = \mu + \gamma_1
\]

(3.24)

where \( \gamma_1 \) is an error with assumed properties

\[
E_s(\gamma_1) = 0
\]

\[
E_s(\gamma_1\gamma_1') = \sigma^2A
\]

(3.25)
and where $\sigma^2 A$ is a qxq matrix with entries $\sigma^2 a_{jk} = \text{Cov}(\mu_{ij}, \mu_{ik})$.

We note that the matrix $\sigma^2 A$ may be interpreted to be the variance-covariance matrix among strata. The symbol $E_s$ indicates the unconditional expectation over strata.

By using the relationship $E = E_s E_x | s$, where $E$ denotes the total expectation, we note the equalities

$$E(x_{ij}) = E_s E_x | s(x_{ij}) = E_s (\mu_i) = \mu. \quad (3.26)$$

We also find the total covariation of $x_{ij}$ in terms of $\sigma^2 W_1$ and $\sigma^2 A$ by using the relationships

$$E(x_{ij} - \mu) (x_{ij} - \mu)' = E_s E_x | s(x_{ij} - \mu) (x_{ij} - \mu)' = E_s E_x | s[(x_{ij} - \mu_1 + \mu_1 - \mu)(x_{ij} - \mu_1 + \mu_1 - \mu)']$$

$$= E_s E_x | s[(x_{ij} - \mu_1)(x_{ij} - \mu_1)' + (\mu_1 - \mu)(\mu_1 - \mu)' + (x_{ij} - \mu_1)(\mu_1 - \mu)' + (\mu_1 - \mu)(x_{ij} - \mu_1)']$$

$$= E_s E_x | s(e_{ij}' e_{ij}) + E_s (\gamma_1 y_1')$$

$$= E_s \sigma^2 W_1 + \sigma^2 A$$

because $E_x | s(\mu_i)$ equals $\mu_1$; also $E_x | s(x_{ij})$ equals $\mu_1$ so that the cross product terms vanish. From the last line of Equation 3.27 above, we note that the total covariation of $x_{ij}$'s may be written as the sum of two terms introduced previously. The first member of the sum is simply the average, over strata, of within-stratum covariance.
Given a particular stratum, it is relevant to consider in a conditional sense the optimal resource rate \( X_0 \). In terms of previous notation, we proceed to find \( M^X \mathbb{E}_X | s(\pi(X,x_{ij})) \). In order to find expected yield, and hence expected profit, for the \( i \)th stratum, we substitute the right hand side of Equation 3.22 for \( x \) in Equation 3.21. The result is

\[
Y_{ij} = B_0 + B_1(\mu_1 + e_{ij}) + (\mu_1 + e_{ij})'B_{11}(\mu_1 + e_{ij}) \\
+ B_2X + X'B_{22}X + (\mu_1 + e_{ij})'B_{12}X \\
= B_0 + B_1(\mu_1 + e_{ij}) + \mu_1B_{11}\mu_1 + 2\mu_1B_{11}e_{ij} + e_{ij}B_{11}e_{ij} \\
+ B_2X + X'B_{22}X + \mu_1B_{12}X + e_{ij}B_{12}X
\]

where \( Y_{ij} \) is equivalent to the symbol \( Y(X,x_{ij}) \). Applying properties of Equations 3.23, we obtain the equality

\[
\mathbb{E}_X|s(Y_{ij}) = B_0 + B_1\mu_1 + \mu_1'B_{11}\mu_1 + B_2X + X'B_{22}X \\
+ \mu_1'B_{12}X + \mathbb{E}_X|s(e_{ij}B_{11}e_{ij}).
\]

The last term of Equation 3.29 may be readily evaluated. Write \( M \) for \( B_{11} \). Then we obtain the equation

\[
\mathbb{E}_X|s(e_{ij}Me_{ij}) = \mathbb{E}_X|s \sum_{k,l} e_{ij}k^{m}k^{l}e_{ijl} \\
= \sum_{k,l} m_{kl}\mathbb{E}_X|s(e_{ij}k^{e_{ijl}}) \quad (3.30) \\
= \sigma^2 \sum_{k,l} m_{kl}w_{ikl} = \sigma^2\text{Tr}M\bar{W}_1
\]

where by \( \text{Tr} \) is meant the trace, i.e. the sum of diagonal elements, of the matrix product \( MW_1 \).

Substituting \( B_{11} \) for \( M \), and recognizing the first six
terms of Equation 3.29 to be, in symbols, \( Y(X, E_{x|s}(x_{1j})) = Y(X, \mu_4) \), we write Equation 3.29 in the form

\[
E_{x|s}(Y_{1j}) = Y(X, \mu_4) + \sigma^2 \text{Tr} B_{11} W_1. \tag{3.31}
\]

If the last term of Equation 3.31 is non-positive, then the inequality \( E_{x|s}(Y_{1j}) \leq Y(X, \mu_4) \) holds for the multivariate case. A sufficient condition for \( \sigma^2 \text{Tr} B_{11} W_1 \) to be non-positive is for the matrix \( B_{11} \) to be negative semi-definite, i.e., for all vectors \( z \), the inequality \( z'B_{11}z \leq 0 \) is satisfied. To show sufficiency of this condition, we assume that \( B_{11} \) is in fact negative semi-definite. Then the inequality \( z'B_{11}z \leq 0 \) is satisfied by all \( z \); in particular it is satisfied by \( z = e_{1j} \). Thus the scalar \( e_{1j}B_{11}e_{1j} \) is non-positive, and, since \( \sigma^2 \text{Tr} B_{11} W_1 \) is the conditional expectation of non-positive quantities, it follows that the inequality \( \sigma^2 \text{Tr} B_{11} W_1 \leq 0 \) is satisfied.

The optimum rate \( X_{o1} \) may be determined by applying the criterion \( \nabla Y = \lambda \) to Equation 3.31, which leads to the relation

\[
B_2' + 2B_{22}X + B_{12}' \mu_1 = \lambda \tag{3.32}
\]

from which we obtain the optimal rate

\[
X_{o1} = \frac{1}{2}B_{22}^{-1}(\theta - B_{12}' \mu_1) \tag{3.33}
\]

provided \( |B_{22}| \neq 0 \), where \( \theta = \lambda - B_2' \). Since profit is determined by the quantity \( \pi = YP_Y - X'P_X \), maximum profit \( M_{x}E_{x|s} \). \( \pi(X, x_{1j}) \) may be determined by using \( X_{o1} \) from Equation 3.33 in
the equation

\[
M_x E_x | s_{\pi_{ij}} = \left[ Y(x_0, \mu_1) + \sigma^2 Tr B_{11} W_1 \right] P_y - x_0' P_x
\]

\[= \left[ B_0 + B_1 \mu_1 + \mu_1 B_{11} \mu_1 + B_2 x_0 + x_0' B_{22} x_0 \\
+ \mu_1' B_{12} x_0 + \sigma^2 Tr B_{11} W_1 \right] P_y - x_0' P_x \tag{3.34}
\]

where \( \pi_{ij} \) is equivalent to \( \pi(x, x_{ij}) \). After substituting for \( x_{0j} \) in Equation 3.34, the latter simplifies to

\[
M_x E_x | s_{\pi_{ij}} = \left[ B_0 + \frac{1}{2} B_2 B_{22}^{-1} \theta + \frac{1}{4} \theta' B_{22}^{-1} \theta + (B_1 - \frac{1}{2} B_2 B_{22}^{-1} B_{12}) \mu_1 \\
+ \mu_1'(B_{11} - \frac{1}{4} B_{12} B_{22}^{-1} B_{12}) \mu_1 + \sigma^2 Tr B_{11} W_1 \right] P_y \tag{3.35}
\]

\[- \frac{1}{2} \left( B_{22}^{-1} (\theta - B_{12} \mu_1) \right)' P_x \]

Using Equation 3.35, we are able to determine the magnitude of maximum profit for any stratum, given \( \mu_1 \) and \( \sigma^2 W_1 \).

We eventually will be interested in examining alternative sampling schemes. Hence we continue a derivation which appears to facilitate this step. One criterion for examining a sampling scheme which makes use of stratification is the average maximum profit for all strata. In terms of previous results, this may be accomplished by taking the expectation of \( M_x E_x | s_{\pi_{ij}} \) with respect to strata, i.e., by obtaining the expression \( E_s M_x E_x | s_{\pi_{ij}} \). We apply properties of Equation 3.25 after substituting \( \mu + \gamma_1 \) for \( \mu_1 \) in Equation 3.35. In a manner similar to that of Equations 3.28, 3.29, and 3.30, we find the expectation
\[ E_s M_x E_x | s \pi_{1j} = \left[ B_0 + \frac{1}{2} B_2 B_{22}^{-1} \theta + \frac{1}{4} \theta' B_{22}^{-1} \theta + (B_1 - \frac{1}{2} B_2 B_{22}^{-1} B_{12}) \mu \right. \]
\[ + \left. \mu'(B_{11} - \frac{1}{4} B_{12} B_{22}^{-1} B_{12}') \mu \right] P_y \]
\[ - \frac{1}{2} \left[ B_{22}^{-1} (\theta - B_{12}' \mu) \right]' P_x \]
\[ + \left[ \sigma^2 \mathrm{Tr} B_{11} E_s(W_1) + \sigma^2 \mathrm{Tr}(B_{11} - \frac{1}{4} B_{12} B_{22}^{-1} B_{12}') A \right] P_y. \]

It may be verified that the terms of Equation 3.36, excluding those in the last bracket, are equivalent to maximum profit from an ideal field, with homogeneous environmental level \( \mu \), to which a single resource rate \( X_0 \) is applied. Thus we may write Equation 3.36 as the equality
\[ E_s M_x E_x | s \pi(X, x_{1j}) = M_x \pi(X, E(x_{1j})) + \left[ \sigma^2 \mathrm{Tr} B_{11} E_s(W_1) \right. \]
\[ + \left. \sigma^2 \mathrm{Tr}(B_{11} - \frac{1}{4} B_{12} B_{22}^{-1} B_{12}') A \right] P_y. \]

We may alternatively denote the first term on the right hand side by the symbol \( M_x \pi(X, \mu) \).

It appears to be relevant to select stratification schemes for which the difference \( E_s M_x E_x | s \pi_{1j} - C_s \) is a maximum, where \( C_s \) denotes costs of subdividing the field into strata. Referring back to Equations 3.27, we note that total covariation among several environmental factors in a field may be partitioned into "Average Within" and "Among" strata covariance matrices. In light of this result and standard procedures of multivariate analysis of variance, it appears that elements of the matrices \( E_s \sigma^2 W_1 \) and \( \sigma^2 A \) may be extracted from entries in a multivariate analysis of variance table for a
given type of stratification. At this point, population quantities are involved; we defer discussion of estimation procedures until Part V.

Having derived results which appear to be useful for examining alternative stratification schemes, we terminate the discussion of this section. We have given only a cursory examination of terms which have arisen in the derivation, specifically those of Equation 3.37. In Part V we discuss in further detail the practical significance and meaning of those terms.
IV. EXPERIMENTAL PROCEDURES

Research activities undertaken in this study may be divided into three categories. Major emphasis was placed on intensive field sampling as a means of obtaining empirical evidence regarding variability of selected soil properties. A supplemental study, which was incidental to the field sampling, was designed to furnish some evidence concerning similarity between single cores and samples composited from local areas surrounding particular cores. The third phase of the study was concerned with relationships between laboratory analysis of a sample composited from heterogeneous soil materials and the average of the individual analyses. In the following paragraphs, we discuss procedures which were used in each experimental phase; we conclude this part with Section C on methods of numerical analysis.

In the following section, we describe methods by which data were obtained. Since data were collected in the usual sense of statistical sampling, for example the systematic grid discussed below, we refer to sampling procedures as such. This should not be confused with the fact that other assumptions, namely enumeration of a finite population, will be made later to facilitate computations.
A. Field Sampling

1. Site selection

Sampling was limited to square, 40-acre fields arbitrarily selected in four soil association areas in Iowa. The four association groups indicate some of the differences which occur in soils throughout the state. Nicollet and Webster soils in northern Polk County developed from relatively recent glacial till. Three fields from this area were selected for sampling. Tama, Muscatine, and other loess-derived associates of these soils were present in two fields sampled in Tama County. In the southwestern part of the state, soils of eastern Cass County originated from such diverse materials as exposed ancient glacial tills and more recent loess deposits. Two fields in Cass County were chosen on sites which contained Sharpsburg, Shelby, and associated soils. Western Crawford County provided two sample sites on Ida, Monona, and related soils which developed on deep loess materials.

The descriptions given in the preceding paragraph were intentionally brief. We should point out that properties of these soil types, their relationships to each other, and geologic history of the areas are given in detail by Simonson et al. (1952).

Several criteria were employed in preliminary and final selection of particular fields. Tentative sites, found by
examination of standard soil survey maps, were selected ac­
cording to the following characteristics: (a) each site
occupied a square 40 acres of tillable land; (b) major soil
types of the association area were represented; (c) soil type
patterns within sites appeared to be those in common occur­
rence throughout the area; (d) extremes of soil conditions
were not exhibited; and (e) soil mapping units were contiguous
and generally larger than the size of the selected sample grid.

Final selections were made after inspection of the tenta­
tive sites. Causes for rejection included permanent buildings
and fences, farm roads, woodlands, and other man-made and
natural features (with the exception of terraces and natural
waterways) within a site which prevented tillage of the entire
40-acre unit. In addition, sites adjacent to lots containing
permanent buildings were rejected. These restrictions were
imposed in an attempt to minimize effects of differential
management within sites. Although preference was given to
fields cropped and managed as a unit, this condition was not
satisfied in every case. In some areas, notably western Craw­
ford County, use of square 40-acre fields as single units
appeared to be uncommon.

Throughout the remainder of this dissertation, we find it
convenient to refer to fields by number. Soil maps of the
nine fields, oriented with north at the top of the sheet, are
shown in Figures 1 through 9 with appropriate field numbers.
Figure 1. Field 1, Polk County, with Nicollet 1. (55) and Webster s.c.l. (107) soils.
Figure 2. Field 2, Polk County, containing Webster s.c.l. (107) and Nicollet 1. (55) soils.
Figure 3. Field 3, Polk County, on Nicollet 1. (55) and Webster s.c.l. (107) soils.
Figure 4. Field 4, Cass County, containing Sharpsburg s.l. (370), Shelby s. (24), and Clarinda s.c.l. (222) soils, and Colo-Judson-Zook (11) and Shelby-Adair (93) complexes
Figure 5. Field 5, Cass County, with Sharpsburg s.l. (370) and Clarinda s.c.l. (222) soils, and Colo-Judson-Zook (11) and Shelby-Adair (93) complexes
Figure 6. Field 6, Tama County, containing Muscatine s.l. (119), Garwin s.c.l. (118), and Garwin-Colo complex (11) soils
Figure 7. Field 7, Crawford County, containing Ida s.l. (1), Monona s.l. (10), and Napier s.l. (12) soils
Figure 8. Field 8, Tama County, with Tama s.l. (120), Muscatine s.l. (119), Garwin s.c.l. (118), and Sperry s.l. (122) soils
Figure 9. Field 9, Crawford County, containing Ida s.l. (1), Monona s.l. (10), and Colo-Napier-Nodaway complex (17) soils
Detailed information is given for each field in Table 4 of Appendix B; included are field location, name of operator, dates sampled, previous crops, and other pertinent items.

Code numbers appearing within the mapping units of the figures are those currently used in standard soil survey work. The first, second, and third members of the code triplet refer to soil type, slope, and erosion classes, respectively. Soil types are identified in the figure titles. In agreement with conventional soils terminology, abbreviations l., s.l., and s.c.l. are for loam, silt loam, and silty clay loam, respectively. Slope classes are given as a percent point in the range, e.g. 3, 7, 11, and 16 designate the ranges 2 to 5, 5 to 9, 9 to 14, and 14 to 20 percent slopes, respectively. Coded erosion classes are 0, 1, 2, and 3 for the qualitative attributes none, slight, moderate, and severe erosion, respectively.

2. Point sampling procedures

Within each field, a soil sample was drawn from each point of a grid which was superimposed on the field. For Fields 1, 2, and 3, a 13x13 grid provided 169 sample points at intersections of rows and columns. With borders 60 feet in width around the perimeter of the field, this resulted in a spacing of 100 feet between adjacent rows and an equal distance between adjacent columns. For the remaining six fields, a total of 144 sample points were designated as intersections
of a 12x12 grid. Under this scheme, grid lines nearest field borders were located at a distance of 55 feet from the borders, and adjacent rows (and columns) were spaced 110 feet apart. It should be noted that stub lines in the margins of Figures 1 through 9 are ends of grid lines.

The system used to identify grid points is consistent with that of the U.S. Army. After orienting the soil map with north at the top of the sheet, columns were numbered from left to right beginning with 00. Rows were numbered from bottom to top beginning with 00. The row numbers (and similarly for columns) progress from 00 to 11 for 12x12 grids and from 00 to 12 for 13x13 grids. Grid points were designated by two number pairs; the first pair refers to column and the second to row number.\(^1\)

Field techniques appear to be of interest at this point. Sample points were located in the field by a combination of measurement and pacing. Pairs of range poles used on opposite sides of a field established a pair of grid lines across the field. To facilitate spacing of these lines, members of each pair of poles were connected by a small rope with length corresponding to the grid spacing. Sample points lying on

\(^1\)The rule for locating a particular grid point is "Read right up". For example, we use the rule to locate the point 0308. Beginning at the lower left corner, we would read right to column 03, then read up column 03 to row 08.
grid lines between opposing poles were located by pacing from the field boundary. White and red flags were used to distinguish the members of each pair of poles. Traverses across the field were facilitated by pacing from a white to a red flag. After samples were drawn from each traverse across the field, the flags were rotated into position for the next cycle.

Grid lines traversed during sampling were oriented along rows or columns as dictated by visibility of the range flags from points in the field. Auxiliary poles were used when border flags were not visible. We should note here that traverses across the fields were located near the ideal grid lines by sighting on range flags. By contrast, points located by pacing across irregular topography fluctuated around the ideal grid points. However, these errors appeared to be within the accuracy attained in mapping soil units.

One soil sample was taken at each grid point. A probe with diameter of 1.5 inches was used to remove a core from the surface six inches of soil. A quantity of soil sufficient for filling a pint soil sample carton was obtained by enlarging the initial cavity. The depth of sampling was chosen in agreement with current soil sampling procedures and soil test calibrations.

Samples were submitted to the Iowa State University Soil Testing Laboratory for analyses. Single determinations were made of organic matter, unbuffered and buffered soil pH, and
excess lime. Duplicate determinations were scheduled for nitrates initially in the soil, nitrogen released by microbial nitrification processes during a two-week incubation period, phosphorus extracted by a weak ammonium fluoride-hydrochloric acid solution, and potassium which exchanges with a neutral ammonium acetate solution.¹

3. Area sampling procedures

Field sampling, as designed and executed, provided information concerning variation of soil properties from point to point on a grid. Ignoring the volume of soil removed for analyses, test results could be considered as representative of ideal geometric points. However, it is known that plants extract soil nutrients from non-negligible volumes of soil. Further, the plant in some sense integrates differences in

¹These procedures are currently in use by the Iowa State University Soil Testing Laboratory. In somewhat greater detail, we list the following; (a) the test for organic matter was visual; (b) using water extraction of samples incubated for two weeks at 35°, nitrates were determined by intensity of phenoldisulphonic acid-induced color, using a calibrated Bausch and Lomb Spectronic 20 colorimeter; (c) the pH of a 2:1 water-soil paste was determined with a Beckman glass electrode pH meter; (d) phosphorus extracted by a solution of .03 N NH₄F in .025 N HCl was determined by intensity of stannous chloride-reduced ammonium phosphomolybdate color, using a calibrated Cenco Photometer; and (e) potassium extracted with neutral, 1 N ammonium acetate solution was determined directly in a calibrated Perkin-Elmer Model 52A flame photometer.
soil fertility which occur within the root zone. For these reasons, it is relevant to consider relations between a point sample and a sample which represents a larger area.

For this study, circles with a radius of five feet were arbitrarily selected. This size was selected as one which would include lateral extension of corn roots under Iowa conditions. In Fields 4, 5, 7, 8, and 9, a total of 21 grid points were selected on the principal soil types. In addition to the point sample at these selected grid points, a second sample was obtained by compositing ten cores drawn within a radius of five feet. These were mixed prior to removal of one pint of material for laboratory analyses.

B. Composites of Heterogeneous Materials

This brief study was undertaken for the purpose of comparing analysis of a composite sample with the average of individual determinations. Thus, for considering alternative sampling schemes, we obtain evidence concerning the extent to which average analyses of separate cores taken from an area may be simulated by analysis of a composite sample.

After laboratory analyses were completed for samples from Fields 4 through 9, pairs of these samples were arbitrarily chosen for compositing. Ten samples, representing a range of conditions within a field, were selected from each of the six fields on the basis of soil test alone, regardless of other
soil characteristics. For each field, five composite samples were obtained by mixing pairs of the ten samples. Forty grams of soil from each sample were weighed, and the 80-gram composite was mixed in a sealed pint soil sample carton by shaking for a period of three minutes. These composite samples were also analyzed by the Iowa State University Soil Testing Laboratory.

C. Numerical Analyses

Numerical analyses involving point samples from the several fields were restricted essentially to examination of alternative sampling procedures in the spirit of the theoretical approach outlined in Part III. The problem of resource allocation under variable environmental conditions requires only minor revisions of the usual approach, which has been documented empirically. It appears that little would be added to the subject by choosing several resource and product prices and determining optimal fertilizer rates for all combinations of the various prices for several different strata. Instead, we presume an optimum application rate for each stratum, and proceed directly to the problem of considering alternative stratification techniques.

In the following section, we find it convenient to consider finite population theory, as a special case of the general approach of Part III, to obtain empirical sample quanti-
ties. Two subsequent sections deal with computational techniques and types of stratification considered, respectively.

1. **Computational formulas**

We recall the final results of Part III, as given in Equation 3.37. For purposes of deriving computational formulas, we consider a field with $N$ production units, where $N$ is 144 or 169. We assume complete enumeration of the finite population, i.e., laboratory results of a point sample are representative of the square from which it came. For convenient reference, some equations of Part III also appear in this section.

Suppose we have the yield function

$$Y(X, x_{1j}) = B_0 + B_1 x_{1j} + x_{1j}' B_{11} x_{1j} + B_2 x$$

$$+ x_{1j}' B_{22} x + x_{1j}' B_{12} x$$

(4.1)

where units of $Y$ are bushels per acre, and $x$ and $X$ are in the units pounds per acre. We wish to find average production, in bushels per acre, for stratum $i$, which is assumed to have $N_i$ production units. For later reference, we note that $\sum_i N_i = N$.

Using the usual definition of average, we find the result

$$\frac{1}{N_i} \sum_j Y(X, x_{1j}) = B_0 + B_1 x_{1i} + \frac{1}{N_i} \sum_j x_{1j}' B_{11} x_{1j}$$

$$+ B_2 x + x_{1i}' B_{22} x + x_{1i}' B_{12} x$$

(4.2)

where the summation over $j$ is understood to range from 1 to $N_i$. 
and where $x_{1.} = \left( \frac{1}{N_1} \right) \sum_j x_{1j}$.

Writing the identity $x_{1j} = (x_{1j} - x_{1.}) + x_{1.}$ and noting that $\left( \frac{1}{N_1} \right) \sum_j (x_{1j} - x_{1.}) = 0$, we find that the third term on the right side of Equation 4.2 reduces to the expression

$$\frac{1}{N_1} \sum_j x_{1j} B_{11} x_{1j} = \frac{1}{N_1} \sum_j \frac{1}{N_1} \left( x_{1j} - x_{1.} \right)' B_{11} \left( x_{1j} - x_{1.} \right) + x_{1.} B_{11} x_{1.}.$$  (4.3)

As in Part III, it can be shown that the first term on the right side of Equation 4.3 may be written as $\text{Tr} B_{11} S_1$, where the $q \times q$ matrix $S_1$ has elements of the form

$$\sum_{k} (x_{1jk} - x_{1k})(x_{1jl} - x_{1l}).$$

It will be noted that the subscripts $k$ and $l$ refer to particular soil factors which appear implicitly in the vector $x_{1j}$.

Alternatively, the matrix $S_1$ may be written in the form $S_1 = \left( \frac{1}{N_1} \right) \sum_j (x_{1j} - x_{1.})(x_{1j} - x_{1.})'$. Using these relationships, we find that Equation 4.2 may be expressed as

$$Y_{1.} = B_0 + B_1 x_{1.} + x_{1.}' B_{11} x_{1.} + B_2 X + X' B_{22} X + x_{1.}' B_{12} X + \text{Tr} B_{11} S_1$$  (4.4)

where $Y_{1.}$ is the average yield, in bushels per acre, for the $i$th stratum.

To find profit for the $i$th stratum, we subtract fertilizer costs from yield obtained from the stratum. We find it convenient to express yield in terms of bushels per production unit, instead of bushels per acre. Realizing that we have, by
assumption, $N$ units per 40-acre field, we apply a factor of $40/N$ to the yields expressed in bushels per acre. We immediately recall that profit-maximizing rate $x_{01}$ is equivalent to the term $\frac{1}{2}B_{22}^{-1}(\theta - B_{12}^{'x_1})$. Using these facts, we find that maximum profit for the $i$th stratum may be written as

$$M_{X_{i}} = \frac{40}{N}N_{1}Y_{1}P_{Y} - \frac{40}{N}N_{1}x_{01}'P_{X}.$$  (4.5)

Substituting for $Y_{1}$ from Equation 4.4, and using the equivalent expression for $X_{01}$, we find, after reduction and simplification, the result

$$M_{X_{i}} = \frac{40}{N}N_{1}\left[ B_{0} + B_{1}x_{i} + x_{i}'(B_{11} - \frac{1}{4}B_{12}B_{22}^{-1}B_{12}')x_{i}. \right. \right.
\left. \left. + \frac{1}{2}B_{22}^{-1}\theta + \frac{1}{4}x_{1}'B_{22}^{-1}\theta - \frac{1}{2}B_{22}B_{22}^{-1}B_{12}'x_{i}. \right. \right. \right.
\left. \left. + TrB_{11}S_{1}\right] P_{Y} - \frac{40}{N}N_{1}\left[ B_{22}^{-1}(\theta - B_{12}'x_{i}) \right]'P_{X}. \right. \right.$$

We may obtain total profit from the 40 acres by summing maximum profits $M_{X_{i}}$ over all strata. However, we wish to express it in terms of average profit per acre. Dividing Equation 4.6 by 40 and summing over strata, we obtain

$$\frac{1}{40} \sum_{i} M_{X_{i}} = \left[ B_{0} + B_{1}x_{i} + \frac{1}{N} \sum_{i} N_{1}x_{i}'B_{44}x_{i}. \right. \right.
\left. \left. + \frac{1}{2}B_{22}^{-1}\theta + \frac{1}{4}x_{1}'B_{22}^{-1}\theta - \frac{1}{2}B_{22}B_{22}^{-1}B_{12}'x_{i}.. \right. \right. \right.
\left. \left. + \frac{1}{N} \sum_{i} N_{1}TrB_{11}S_{1}\right] P_{Y} \right. \right.
\left. \right. \right.
\left. \right. - \frac{1}{2} \left[ B_{22}^{-1}(\theta - B_{12}'x_{i}..) \right]'P_{X}. \right. \right.$$

(4.7)
where \( B_{44} = B_{11} - \frac{1}{4} B_{12} B_{22}^{-1} B_{12}' \); we recall the equality \( N = \sum_i N_i \), where the index \( i \) is summed over all strata. We have also used the relations \( x_{..} = \frac{1}{N} \sum_{i,j} x_{ij} \).

In a manner similar to previous steps, we let \( x_{1..} = (x_{1..} - x_{..}) + x_{..} \) and substitute into the third term of Equation 4.7. After expanding, simplifying, and identifying the trace, we find that the third term of Equation 4.7 reduces to

\[
\frac{1}{N} \sum_i N_i x_{1..} B_{44} x_{1..} = \frac{1}{N} \text{Tr} B_{44} A + x_{..} B_{44} x_{..} \quad (4.8)
\]

where \( A = \sum_i N_i (x_{1..} - x_{..})(x_{1..} - x_{..})' \).

Examining the term \((1/N) \sum_i N_i \text{Tr} B_{11} W_i\), we expand, insert the equivalent expression for \( S_i \), note cancellation of \( N_i \), and reduce to the expression \((1/N) \text{Tr} B_{11} W\), where \( W = \sum_{i,j} (x_{ij} - x_{1..})(x_{1..} - x_{1..})' \). Further recognizing that all those terms of Equation 4.7 which do not involve \( A \) or \( W \) correspond to maximum profit for the case of an area characterized by \( x_{..} \), we obtain a result corresponding to one from Part III in the equality

\[
E_M^E X | s \pi(X, x_{1..}) = M_X \pi(X, E(x_{1..}))
\]

\[
+ \frac{1}{N} (\text{Tr} B_{11} W + \text{Tr} B_{44} A) P_Y \quad (4.9)
\]

where the expectations for the finite case are simply the averages indicated above.

We note from the above discussion that elements of \( A \) are the among-stratum sums of squares and products, corrected for means, and that \( W \) is composed of within-stratum sums of
squares and products, similarly corrected. It follows that these quantities may readily be extracted from a multivariate analysis of variance table. We note the common divisor of $N$.

As a computational device, we slightly rearrange the last term of Equation 4.9. As discussed later, numerical output was in terms of "Total" and "Among" strata sums of squares and products. By recognizing the analysis of variance identity, "Total" = "Among" + "Within", or $T = A + W$, we write the expression

$$\frac{1}{N} \left[ \text{Tr}B_{11}W + \text{Tr}(B_{11} - \frac{1}{4}B_{12}B_{22}^{-1}B_{12})A \right]P_Y$$

$$= \frac{1}{N}(\text{Tr}B_{11}T - \text{Tr}B_{33}A)P_Y$$

(4.10)

where we have substituted the original quantity for $B_{44}$, and $B_{33}$ is substituted for the matrix product $\frac{1}{4}B_{12}B_{22}^{-1}B_{12}$.

2. **Computational methods**

In order to evaluate profit associated with any stratification technique, we utilize point sample data to obtain the matrices $T$ and $A$ discussed in the previous section. Elements of these are recognized to be the "Among" and "Total" sums of squares and products lines from a multivariate analysis of variance table. Computations were directed toward this end.

Pertinent data associated with each sample point were recorded on International Business Machines (IBM) cards.
Sample identification included field number as well as column and row numbers of the grid points. Complete laboratory results were entered, including organic matter and excess lime codes in addition to quantitative results of soil and buffered soil pH, initial and final nitrogen, phosphorus, and potassium soil tests. Mean determinations were recorded for duplicate analyses. Codes were given to soil type, slope, erosion, and mapping units. The latter were included to accommodate cases in which the same soil type, slope, and erosion classes appeared in two distinct, noncontiguous areas within a field. Thus, mapping units, when nested in soil type, slope, and erosion symbols, designate the smallest units into which a field may be decomposed on natural classification bases.

Although laboratory results were recorded for those properties indicated in the preceding paragraph, computations involved only soil pH, nitrifiable nitrogen, and phosphorus test results. These are the soil factors \( x \) which were included quantitatively in the yield function \( Y(X,x) \) used in the empirical study. Estimated parameters of the function are given in Part V, while source of data and discussion of its analysis are outlined in Appendix A.

Three phases of computations were carried out for each type of stratification. In the first, stratum sums were obtained for pH, nitrogen, and phosphorus soil tests. In addition, three squares and three products of stratum sums,
divided by numbers of individuals $N_1$ within the stratum, were obtained from the pH, nitrogen, and phosphorus soil test stratum sums. When the entire field was considered as a single stratum, the products and squares, divided by $N$, provided correction factors for later computations. The final quantities obtained in the first phase were total sums of squares and products of individual observations in each field, corrected for means. These provided "Total" lines in the multivariate analysis of variance tables. Stratum sums, squares and products of sums (divided by $N_1$), the numbers $N_1$, and the total corrected sums of squares and products of individuals were listed on output sheets.

In the second phase of computations, all stratum squares and products of sums, as divided, were summed and diminished by the proper correction factors to give "Among" strata sums of squares and products, corrected for means. These quantities were listed, along with total corrected sums of squares and products, to provide essential elements of multivariate analysis of variance tables. It is apparent that "Within" stratum quantities may be obtained by difference.

The third phase of the numerical work made immediate use of results of the second phase in obtaining the quantities $TrB_{11}^T$, $TrB_{11}A$, and $TrB_{33}A$. The reason for computing $TrB_{11}A$ will become clear in Part V. Elements of the matrices $B_{11}$ and $B_{33}$ are given in Part V for the yield function $Y(X,x)$ used in
the empirical study.

In all of these steps, computations were carried out by the IBM 650 computer of the Iowa State University Statistical Laboratory. The output cards of earlier stages were used as input cards in successive stages, which facilitated the numerical analyses. The three computational phases were used in lieu of a single program because of interest in intermediate outputs per se.

3. Stratification techniques

Two bases were used for stratifying each field. The first, on which current sampling recommendations are founded, is morphological in nature. The second depends exclusively on convenient geometrical shapes, thus ignoring natural criteria.

In the first method of stratification, soil type, slope, erosion, and mapping units (as defined previously) were considered to be hierarchical in that order. Thus, the largest strata were those consisting of single soil types. For the next level of stratification, strata composed of single soil types were subdivided to accommodate slope differences within each soil type. The next level of stratification was obtained by considering possibly different erosion classes within each slope and soil type, while the terminal strata were individual mapping units within all higher categories. In terms of soil
maps shown in Figures 1 through 9, each area which is bounded but not transected by defining lines is a mapping unit stratum. In some cases, of course, a mapping unit stratum may be identical to a slope stratum if mapping unit, erosion, and slope class are indistinguishable.

In contrast to the hierarchy, slope alone, ignoring other characteristics, was used to distinguish strata. The average size of slope strata was larger than that of soil type strata when the number of slope classes was less than the number of soil types in a particular field.

In contrast to stratification based on soil and topographic differences, arbitrary square and rectangular geometric patterns were considered. Two types of these were examined. The first involved use of squares and rectangles to give 4, 6, 9, 12, and 16 strata per 40 acres; the second utilized north-south strips which spanned the length of each field. The numbers of strata in the latter case were 2, 3, 4, and 6.

Geometric strata considered in the study are shown as stratification classes in Figure 10. Note that 16 strata are contained within all lines of Class 1, while 4 strata are bounded by solid lines. Class 2 provides 9 square strata as well as three strips between solid lines. In Class 3, all lines enclose 12 rectangular strata, while solid lines delineate 6 strata. Class 4 provides 2 and 6 strips, and Class 5
Figure 10. Class designation of fields stratified into square and rectangular geometric strata
contains 4 strips.

Note that divisions of field boundaries shown in Figure 10 are factors of 12. For this reason, all geometric strata of a given type were symmetric with respect to size in Fields 4 through 9. For Fields 1, 2, and 3, the extra row and column were included in the bottom and left marginal tiers of strata for all classes with the exception of Class 4. In that case, the right-hand strip contained the extra grid column. The asymmetry was chosen as a more desirable alternative than sacrificing part of the data for these fields, and hence part of the information.
V. RESULTS AND DISCUSSION

Theoretical results derived in Parts III and IV relied on parameters of a yield function. Empirical estimates of these parameters, which were obtained from a separate study, are given in Section A. Results of applying the theory, as outlined in Part IV, are tabulated and discussed in Section B. Section C is devoted to results and conclusions of the area and compositing phases of the study.

The reader occasionally has been reminded that theoretical results and computational formulas of Parts III and IV were based on population variance-covariance quantities. In this light, stratification methods discussed in Section B below must be interpreted as alternative ways of partitioning the enumerated population for purposes of fertilizer use. In an effort to unite the previous derivations with usual statistical approaches, we include Section D. In that section, some relationships are indicated between sample estimates, in the usual statistical sense, and the population quantities used extensively throughout the study.

A. The Empirical Yield Function

We assumed satisfactory approximation of the yield function by the second-degree polynomial

\[ Y(X, x) = B_0 + B_1 x + x'B_{11} x + B_2 x + x'B_{22} x + x'B_{12} x. \]  (5.1)
From data collected by Dumenil (1958), a function of the form of Equation 5.1 was estimated using corn production, in bushels per acre, as the dependent variable. Independent variables \( x \) and \( X \) were defined by the vectors

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}
\]

where \( x_1, x_2, \) and \( x_3 \) correspond to soil tests for pH, nitrogen, and phosphorus. The treatment variables \( X_1, X_2, \) and \( X_3 \) refer to the fertilizer nutrients nitrogen and phosphorus, in units of pounds N and \( \text{P}_2\text{O}_5 \) applied per acre, and stand, in thousands of corn plants per acre, respectively. Details of the study, along with estimates of parameters of the assumed model, are given in Appendix A. The parameter estimates are given below in matrix form to indicate their role in the numerical application of theoretical results.

Entries in the following matrices and vectors were obtained from appropriate terms of the estimated function. We write the expressions

\[
B_1' = \begin{bmatrix} 27.6414300 \\ 3.0600511 \\ -21.4008080 \end{bmatrix}, \quad B_2' = \begin{bmatrix} -0.55155072 \\ -0.32258109 \\ 0.07512600 \end{bmatrix}
\]

\[
B_{11} = \begin{bmatrix} -5.08720710 & -0.54423795 & 0.75103970 \\ -0.54423795 & -0.00408782 & -0.00738404 \\ 0.75103970 & -0.00738404 & -0.11734778 \end{bmatrix}
\]
\[
B_{22} = \begin{bmatrix}
-0.01141416 & 0.00650962 & 0.01192796 \\
0.00650962 & -0.01605604 & 0.02801285 \\
0.01192796 & 0.02801285 & -8.67489990
\end{bmatrix}
\]
\[
B_{12} = \begin{bmatrix}
0.40145179 & 0.53017849 & 2.59532570 \\
-0.00403020 & 0.00081080 & 0.45134895 \\
0.11275763 & -0.36635357 & 1.23482410
\end{bmatrix}
\]

It is pointed out in Appendix A that the term \(B_0\) may be considered to be dependent on the cropping sequence. We do not discuss the point further, since the immediate results do not depend on \(B_0\).

For use in the numerical computations, the matrix \(B_{33}\) was obtained, where \(B_{33} = \frac{1}{4}B_{12}B_{22}B_{12}'\). After performing the required operations, we obtain the matrix

\[
B_{33} = \begin{bmatrix}
-15.97760900 & -0.00835726 & 3.57716130 \\
-0.00835728 & -0.00612780 & -0.00562176 \\
3.57716130 & -0.00562175 & -2.13356700
\end{bmatrix}
\]

In conjunction with matrices \(T\) and \(A\) obtained from laboratory results of field samples, the above matrices were used in evaluating the quantities \(\text{Tr}B_{11}T\), \(\text{Tr}B_{11}A\), and \(\text{Tr}B_{33}A\). In connection with a point raised in Part III, we ascertained whether \(B_{11}\) and \(B_{33}\) were definite matrices. The matrix \(B_{11}\) was found to be indefinite, while \(B_{33}\) is negative definite.

At this point perhaps we should point out assumptions made with respect to the yield function as estimated. For
among-strata analyses, where stratum yields and profits are relevant, we assume that the function adequately serves to characterize yields on all soil types considered in the study, i.e., soil pH, nitrogen, and phosphorus tests satisfactorily distinguish yield relations for the different soil types. For within-stratum analyses, for example specification of optimum fertilizer rates, the assumption may be relaxed to admit a distinct additive constant associated with each soil type.

B. Alternative Stratification Techniques

We recall derivations in Part III which led to criteria for examining various stratification methods. From the computational Equations 4.9 and 4.10, we recall the relations

\[ E_\pi X^T X|s \pi(X, x_{ij}) = M_X \pi(X, E(x_{ij})) \]

+ \[ \frac{1}{N} \left[ \text{Tr} B_{11} W + \text{Tr} (B_{11} - B_{33}) A \right] P_y \]

where \( B_{33} \) is defined as the matrix \( \frac{1}{4} B_{12} B_{22}^{-1} B_{12} \) and where elements of the matrices \( W \) and \( A \) are taken from sums of squares and products lines of the multivariate analysis of variance table corresponding to a particular stratification for any field. We recall the computational facility gained by using the identity \( W = T - A \), which led to the equation

\[ E_\pi X^T X|s \pi(X, x_{ij}) = M_X \pi(X, E(x_{ij})) \]

+ \[ \frac{1}{N} \left[ \text{Tr} B_{11} T - \text{Tr} B_{33} A \right] P_y. \]

Corresponding to each stratification technique discussed
in Part IV, there is a multivariate analysis of variance table in which "Total" sums of squares and products are partitioned into "Within" and "Among" quantities. For each type of stratification, Tables 5 through 13 of Appendix B contain "Among" sums of squares and products and, for each field, the "Total" lines. It is apparent that "Within" lines may be obtained by difference. Also included in those tables are numbers of strata of a given type.

The scalar quantities \( \frac{1}{N}(\text{Tr}B_{11}^T - \text{Tr}B_{33}A) \), obtained by indicated operations on elements of matrices \( B_{11} \) and \( B_{33} \) and "Total" and "Among" lines of Tables 5 through 13 of Appendix B, are shown in Table 1 for each type of stratification and each field. It should be pointed out that the quantities \( \text{Tr}B_{11}^T \) and \( \text{Tr}B_{33}A \) were negative throughout. In view of the negative definite properties of \( B_{33} \), this result is expected. It is interesting to note that traces involving the indefinite matrix \( B_{11} \) are also negative. The word "squares", which appears in Table 1 and later in figures, indicates the square and rectangular geometric strata obtained by subdividing the field into 4, 6, 9, 12, and 16 strata, in contrast to strips which extended across the length of the field.

We proceed to discuss some aspects of terms which appear in Equations 5.2 and 5.3. It should be pointed out parenthetically that Equation 5.3 was obtained strictly as a computational convenience. In the following, it also lends insight
Table 1. Changes in yields in bushels per acre expected from increasing numbers of strata in each of three types of stratification when within- and among-strata variability is considered

<table>
<thead>
<tr>
<th>Type of strata</th>
<th>Field number</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td><strong>Natural:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Slope only</td>
<td>-8.55</td>
<td>-10.21</td>
<td>-13.71</td>
<td>0.42</td>
<td>1.59</td>
</tr>
<tr>
<td>Soil type</td>
<td>-3.92</td>
<td>-10.36</td>
<td>-13.09</td>
<td>2.47</td>
<td>-0.18</td>
</tr>
<tr>
<td>Slope and type</td>
<td>-3.88</td>
<td>-10.36</td>
<td>-12.47</td>
<td>2.85</td>
<td>2.00</td>
</tr>
<tr>
<td>Erosion, slope, and type</td>
<td>-3.88</td>
<td>-10.36</td>
<td>-12.47</td>
<td>3.03</td>
<td>2.74</td>
</tr>
<tr>
<td>Mapping unit, erosion, slope, and type</td>
<td>8.95</td>
<td>-9.65</td>
<td>-6.04</td>
<td>4.25</td>
<td>2.75</td>
</tr>
<tr>
<td><strong>Squares:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quarters</td>
<td>14.07</td>
<td>-8.30</td>
<td>-0.07</td>
<td>0.11</td>
<td>-0.21</td>
</tr>
<tr>
<td>Sixths</td>
<td>15.18</td>
<td>-6.30</td>
<td>4.00</td>
<td>3.74</td>
<td>1.06</td>
</tr>
<tr>
<td>Ninths</td>
<td>19.09</td>
<td>-3.64</td>
<td>7.36</td>
<td>5.16</td>
<td>3.47</td>
</tr>
<tr>
<td>Twelfths</td>
<td>18.86</td>
<td>-0.43</td>
<td>11.62</td>
<td>4.90</td>
<td>3.70</td>
</tr>
<tr>
<td>Sixteenths</td>
<td>22.60</td>
<td>1.24</td>
<td>13.68</td>
<td>5.93</td>
<td>4.11</td>
</tr>
<tr>
<td><strong>Strips:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Halves</td>
<td>-8.02</td>
<td>-9.81</td>
<td>-8.65</td>
<td>-1.38</td>
<td>-1.10</td>
</tr>
<tr>
<td>Thirds</td>
<td>-6.84</td>
<td>-8.13</td>
<td>-7.24</td>
<td>-0.69</td>
<td>0.66</td>
</tr>
<tr>
<td>Fourths</td>
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<td>-6.48</td>
<td>-6.67</td>
<td>-1.14</td>
<td>0.81</td>
</tr>
<tr>
<td>Sixths</td>
<td>-3.19</td>
<td>-5.69</td>
<td>-5.38</td>
<td>-0.52</td>
<td>1.96</td>
</tr>
<tr>
<td><strong>Field:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No subdivision</td>
<td>-8.70</td>
<td>-11.16</td>
<td>-14.00</td>
<td>-1.57</td>
<td>-1.35</td>
</tr>
<tr>
<td>Complete subdivision</td>
<td>73.82</td>
<td>58.30</td>
<td>100.13</td>
<td>20.99</td>
<td>18.48</td>
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Table 1 (Continued).

<table>
<thead>
<tr>
<th>Type of strata</th>
<th>Field number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td><strong>Natural:</strong></td>
<td></td>
</tr>
<tr>
<td>Slope only</td>
<td>16.21</td>
</tr>
<tr>
<td>Soil type</td>
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</tr>
<tr>
<td>Slope and type</td>
<td>16.46</td>
</tr>
<tr>
<td>Erosion, slope, and type</td>
<td>16.46</td>
</tr>
<tr>
<td>Mapping unit, erosion, slope, and type</td>
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<tr>
<td><strong>Squares:</strong></td>
<td></td>
</tr>
<tr>
<td>Quarters</td>
<td>13.17</td>
</tr>
<tr>
<td>Sixths</td>
<td>9.52</td>
</tr>
<tr>
<td>Ninths</td>
<td>17.23</td>
</tr>
<tr>
<td>Twelfths</td>
<td>22.33</td>
</tr>
<tr>
<td>Sixteenths</td>
<td>47.96</td>
</tr>
<tr>
<td><strong>Strips:</strong></td>
<td></td>
</tr>
<tr>
<td>Halves</td>
<td>-8.98</td>
</tr>
<tr>
<td>Thirds</td>
<td>-2.07</td>
</tr>
<tr>
<td>Fourths</td>
<td>-6.91</td>
</tr>
<tr>
<td>Sixths</td>
<td>0.52</td>
</tr>
<tr>
<td><strong>Field:</strong></td>
<td></td>
</tr>
<tr>
<td>No subdivision</td>
<td>-22.79</td>
</tr>
<tr>
<td>Complete subdivision</td>
<td>287.38</td>
</tr>
</tbody>
</table>
into the meaning of terms of the equation. Recall that the left-hand side of Equation 5.3 is the correct average maximum profit for a stratified field when account is taken of within- and among-strata variability. We have found it expedient to consider this as the sum of two readily obtainable terms. As we indicated briefly in Part III, the term $M_x\pi(X,E(x_{ij}))$ represents maximum profit per acre for an unstratified field with soil fertility uniformly equal to the field mean. With no stratification of variable soil material, it is apparent that this quantity, which is based on preaveraged soil test results, overestimates profit per acre. This fact is recalled from Part III. In terms of Equation 5.3, we see that, with no field subdivision, the matrix $A$ is a null matrix. In this case, the matrix $W$ is equal to $T$, and the term $(1/N)\text{Tr}B_{11}T$ is the correction which is applied to $M_x\pi(X,E(x_{ij}))$ for specifying correct profit $M_xE\pi(X,x_{ij})$ when account is taken of within-field variability. Note that we have used the notation $E$ for $E_sE_x|s$, where $E$ is total expectation. The terms $(1/N)\text{Tr}B_{11}T$ for each field appear in the next-to-last line of Table 1. Note that they range in magnitude from $-1.35$ to $-22.79$ bushels per acre. These may be converted to dollars per acre upon multiplying by $P_y$ dollars per bushel.

When we admit the possibility of strata, the matrix $A$ is no longer a null matrix. For a given type of stratification, we find that the positive quantity $(-\text{Tr}B_{33}A)$ usually increases
as the number of strata increase. This is apparent in Table 1 when it is recognized that the quantity TrB_{11}T is constant for a given field. As a consequence, the term \(\frac{1}{N}(TrB_{11}T - TrB_{33}A)\) typically increases in value as the strata numbers increase for any method of stratification.

For hierarchal stratification, it can be shown that trends noted in the preceding paragraph must follow, i.e., the terms \(-TrB_{33}A\) are nondecreasing with increasing numbers of strata. That this is true may be shown as follows. Suppose a hierarchal field subdivision is at the \(k\)th stage. Then the term \(-TrB_{33}A_k\) is relevant. We observe that \(A_k = T - W_k\), as before. By further partitioning within existing \(k\)-level strata, we may write \(W_k = A_{k+1}^W + W_{k+1}^W\), where the superscript indicates that the "Among" and "Within" arrays are with reference to \(W_k\) as "Total". We may write

\[
-TrB_{33}A_k = -TrB_{33}T + TrB_{33}W_k
= -TrB_{33}T + TrB_{33}A_{k+1}^W + TrB_{33}W_{k+1}^W
\leq -TrB_{33}T + TrB_{33}W_{k+1}^W
\leq -TrB_{33}(T - W_{k+1}^W)
\]  

(5.3a)

where we have used the negative definite property of \(B_{33}\) in establishing the direction of the inequality. But we note that the term on the right of Equation 5.3a is precisely the trace involving the "Among" stratum variance-covariance terms for the \(k+1\)th stage of stratification. Hence we deduce the
inequality $-\text{Tr}B_k^3A_k \leq -\text{Tr}B_{k+1}^3A_{k+1}$. There are several instances of hierarchal stratification exhibited in Table 1. The above inequality is satisfied throughout.

It appears that the quantity $(-\text{Tr}B_k^3A)P_y$ represents the additional profit due to stratification. In light of this observation and the conclusions of a previous paragraph, it is readily seen that Equation 5.3 may be written as the equality

$$E_{x}M_{x}E_{X_s\pi(x,x_{i_j})} = M_{x}\text{EM}(x,x_{i_j}) - \frac{1}{N}\text{Tr}B_{33}^A P_y$$

(5.4)

where the first term on the right represents profit for the field, with no stratification, when account is taken of within-field variability.

We obtain an additional result from Equation 5.3 by considering the upper limit of stratification. In this event, the matrix $A$ becomes the "Total" matrix $T$, and the last term of Equation 5.3 may be written $(1/N)\text{Tr}(B_{11} - B_{33})TP_y$. In terms of symbols introduced in Part III, this modification of Equation 5.3 corresponds to the symbol $EM_{x}\pi(x,x)$. This may be recognized as maximum profit attainable when every small unit of soil receives optimal fertilizer application.

The quantities $(1/N)\text{Tr}(B_{11} - B_{33})T$, in bushels per acre, are given in the last line of Table 1. One is immediately impressed with the large potential, particularly for Field 6. However these quantities appear to be somewhat optimistic. After reexamining the field data and numerical results, the
problem with Field 6 appears to be related to disproportionately large variation in soil phosphorus fertility and extrapolation of the yield function. Although the field mean phosphorus determination was 9.1 pounds per acre, one observation was 100.0, and 10 observations exceeded 20.0. Apparently the yield function is extrapolated far beyond the range of experimental conditions. Data used for estimating the yield function came from 78 experimental sites with a maximum phosphorus test of 11.2 and a mean of 4.1 pounds per acre. In six of the fields involved in the present study, mean phosphorus tests exceeded the experimental mean of 4.1. Similarly, nitrogen means of five fields exceeded the experimental mean.

This serves to illustrate a serious problem associated with practical use of data from fertilizer experiments. As a means of ensuring crop response to fertilizer, agronomists habitually restrict field investigations to areas with deficient soil fertility. It appears that a wider range of experimental soil conditions is necessary for useful prediction equations.

Trends among the quantities given in Table 1 may be followed more readily by reference to the accompanying figures, which are grouped by counties. Ordinates of these graphs are the quantities \((1/N)(\text{TrB}_{11}T - \text{TrB}_{33}A)\) from Table 1, while the abscissae are the number of strata which appear in Tables 5 through 13 of Appendix B. Figures 11, 12, 13, and 14 summa-
Figure 11. Change in yields associated with increasing numbers of strata for three methods of stratifying the fields sampled in Polk County.
Figure 12. Change in yields associated with increasing numbers of strata for three methods of stratifying the fields sampled in Cass County.
Figure 13. Change in yields associated with increasing numbers of strata for three methods of stratifying the fields sampled in Tama County.
Figure 14. Change in yields associated with increasing numbers of strata for three methods of stratifying the fields sampled in Crawford County.
rize information related to the fields sampled in Polk, Cass, Tama, and Crawford Counties, respectively.

The solid dot appearing in each figure for the case of one stratum corresponds to the entry in the next-to-last row of Table 1. This quantity may be interpreted as an index of the total variation of soil nutrients in a field, where each nutrient is weighted according to a particular property of its quantitative contribution to yield. Using this index, it is interesting to note a similar order of variability in the Nicollet-Webster (Figure 11) and the Tama-Muscatine (Figure 13) soils. It is equally interesting to note the rather small indices for the Sharpsburg-Shelby (Figure 12) and Ida-Monona (Figure 14) soils, particularly in light of the fact that slope classes of 11 or greater are common for these soils (see Figures 4, 5, 7, and 9), whereas slope classes do not exceed 4 in the remaining fields (Figures 1, 2, 3, 6, and 8). We conclude that, for the fields studied, variability of soil test quantities may be somewhat larger for gently rolling loess and till soils than for broken, steep topography associated with soils of rather different origin.

Visual observation of trends noted in Figure 11 through 14 prompt us to attempt comparison of alternative bases of field stratification for purposes of soil testing and fertilizer recommendations. We dispose of Fields 4, 5, 7, and 2 by noting little distinction among types of strata; in addition,
increasing the number of strata appears to result in very little additional profit, particularly in Fields 4, 5, and 7.

In terms of additional yields, it appears that natural stratification was superior to strips and squares in Fields 8 and 9. The natural basis for forming strata was evidently inferior to both strips and squares in Field 3. Subdivision into squares appears to be advantageous in Field 1, while Field 6 remains somewhat indeterminate as to superiority of any type of stratification.

For a given basis of stratification, it is apparent that additional yield could be considered as some function of number of strata. Optimum number of strata would be obtained by standard methods reviewed in Part III. However, it appears that cost-price analysis based on sampling alone might prove to be inadequate in this case, particularly when consideration is given to problems and costs of fertilizing small areas. One may visualize some rather complicated cost functions for analyzing such cases.

The following conclusions are inferred from the data presented above. Natural bases for stratifying fields, for purposes of soil tests and fertilizer use, may be inferior to other procedures which ignore soil type, slope, and erosion classes. In some cases, a unit as large as a 40-acre field may properly be sampled and treated as a single unit with no stratification, in spite of extreme conditions of topography.
and differences in soil types.

As a point of further interest, we examined Equation 5.2 empirically for cases in which within-stratum variability is ignored, i.e., when $W$ is assumed to be a null matrix. The term $\text{Tr}B_{11}W$ in Equation 5.2 vanishes, leaving the quantity $\text{Tr}(B_{11} - B_{33})A$. This case corresponds to profit maximization for each stratum, ignoring within-stratum variability but recognizing differences among strata. Since the quantity $\text{Tr}B_{11}T$ was negative for each field, it again follows that maximum profits, averaged over strata, are overestimated when within-stratum variability is ignored. The quantities $(1/N)\text{Tr}(B_{11} - B_{33})A$, for each field and type of stratification, are given in Table 2. Comparing these entries with those of Table 1, we may obtain differences which represent the extent of overestimation caused by assuming no within-stratum variability.

C. Area and Composite Samples

Laboratory results of samples pooled from an area surrounding selected grid points in the field were correlated with analyses of the point samples. Simple correlations between the two types of samples were obtained for soil pH, nitrifiable nitrogen, phosphorus, and potassium. The simple correlation coefficients are given in the first line of Table 3. Although 21 pairs of samples were analyzed, the correla-
Table 2. Changes in yields in bushels per acre expected from increasing numbers of strata in three types of stratification when within-stratum variability is ignored

<table>
<thead>
<tr>
<th>Type of strata</th>
<th>Field number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td><strong>Natural:</strong></td>
<td></td>
</tr>
<tr>
<td>Slope only</td>
<td>0.13</td>
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<tr>
<td>Soil type</td>
<td>2.96</td>
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<tr>
<td>Slope and type</td>
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<tr>
<td>Erosion, slope, and type</td>
<td>3.00</td>
</tr>
<tr>
<td>Mapping unit, erosion, slope, and type</td>
<td>14.52</td>
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<td><strong>Squares:</strong></td>
<td></td>
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<tr>
<td>Quarters</td>
<td>19.97</td>
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<tr>
<td>Sixths</td>
<td>20.48</td>
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<td>24.21</td>
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<td>27.44</td>
</tr>
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<td><strong>Strips:</strong></td>
<td></td>
</tr>
<tr>
<td>Halves</td>
<td>0.28</td>
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<tr>
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<td>Fourths</td>
<td>2.25</td>
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<tr>
<td>Sixths</td>
<td>4.94</td>
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<td><strong>Field:</strong></td>
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Table 2 (Continued).

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<tr>
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<tr>
<td>Sixths</td>
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<tr>
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<td>Twelfths</td>
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<tr>
<td>Sixteenths</td>
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<tr>
<td>Thirds</td>
<td>17.25</td>
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<tr>
<td>Fourths</td>
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<td>19.62</td>
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<tr>
<td><strong>Field:</strong></td>
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<tr>
<td>No subdivision</td>
<td>0.00</td>
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Table 3. Simple correlations between laboratory results of area samples and corresponding point samples and between analyses of composite samples and averages of individual determinations

<table>
<thead>
<tr>
<th>Soil characteristics&lt;sup&gt;a&lt;/sup&gt;</th>
<th>Samples</th>
<th>pH</th>
<th>Nitrifiable nitrogen</th>
<th>Extractable phosphorus</th>
<th>Exchangeable potassium</th>
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</thead>
<tbody>
<tr>
<td>Area vs. point samples</td>
<td>0.96</td>
<td>0.76</td>
<td>0.85</td>
<td>0.80</td>
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<tr>
<td>Average analyses vs. analysis of composite sample</td>
<td>0.98</td>
<td>0.90</td>
<td>0.97</td>
<td>0.98</td>
<td></td>
</tr>
</tbody>
</table>

<sup>a</sup>Soil test results as determined by the Iowa State University Soil Testing Laboratory.

...tions shown for phosphorus and potassium are based on only 20 pairs. The potassium test for one of the samples which exceeded 400 pounds per acre, and hence required recalibration of the flame photometer, was overlooked. Phosphorus results were discarded for one sample pair which appeared to be markedly different from the remaining pairs. Deletion of this one pair of observations resulted in an increase in the sample correlation coefficient from 0.39 to the 0.85 reported in Table 3. Evidence presented appears to indicate that the point sample results may be regarded as representing areas of soil.

Analyses of samples composited from heterogeneous materi-
als were correlated with the arithmetic average of individual determinations for thirty samples. Simple correlation coefficients for soil pH, nitrogen, phosphorus, and potassium are given in the second line of Table 3. The correlation coefficients indicate that the two types of laboratory results may be considered synonymously.

D. Statistical Results

Here we consider previous theoretical results in light of sample estimates rather than population quantities. In the following paragraphs, we obtain unbiased estimates of stratum profits based on a sample of soil test results. We assume an infinite model for within-stratum properties; however, a finite number of well-defined strata of possibly different areal sizes are visualized.

We consider a sample of size $n_i$ drawn from the $i$th stratum. We suppose that observed values of the variable $x_{ij}$ are from a population with mean $\mu_{i}$ and with variance $\sigma_{i}^2$ in the $i$th stratum. On the basis of the sample information, we wish to estimate profit for the stratum. Analyses of Part IV for the finite population lead one immediately to the relevant results, except that here we deal with sample stratum number $n_i$ instead of the population number $N_i$, and sample instead of population moments.

Denoting the correct sample post-average yield function
as \( Y_i = (1/n_1) \sum_j Y_{1j} \), we find, by immediate application of results in Section C of Part IV, the result

\[
Y_i = B_0 + B_1 x_{i1} + x'_i B_{11} x_{i1} + B_2 x + x'_B x + x'_B x + \text{Tr} B_{11} S_1
\]

(5.5)

where we have used the same second-degree polynomial yield function, the sample mean \( x_{i1} \) is defined by \( x_{i1} = (1/n_1) \sum_j x_{1j} \) and where \( S_1 \) represents the q x q matrix of sample moments

\[
S_1 = (1/n_1) \sum_j (x_{1j} - x_{i1})(x_{1j} - x_{i1})'
\]

We note at this point that the term involving \( S_1 \) has arised strictly as a consequence of obtaining the correct sample post-average. As before, we may write Equation 5.5 equivalently as

\[
Y_i = Y(X,x_{i1}) + \text{Tr} B_{11} S_1
\]

(5.6)

Using the relation \( \mathbf{Y}(Y_i) = Y_i P_y - X' P_x \) for estimated profit, and maximizing this with respect to resource application rate given the sample estimate \( Y_i \), we may write the sample estimate of profit for the \( i \)th stratum as the equality

\[
M_Y\hat{\mathbf{Y}}(Y_i) = \left[ B_0 + \frac{1}{2} B_2 B^{-1} \theta + \frac{1}{4} \theta ' B_2^{-1} \theta + (B_1 - \frac{1}{2} B_2 B^{-1}B_{12}) x_{i1}
\right.
\]
\[
+ x'_i B_{44} x_{i1} + \text{Tr} B_{11} S_1\left] P_y - \left[ \frac{1}{2} B_{22} (\theta - B_{12} x_{i1}) \right] P_x \right.
\]

(5.7)

This may be written, in terms of sample quantities, as the equation

\[
M_Y\hat{\mathbf{Y}}(Y_i) = M_Y\hat{\mathbf{Y}}(X,x_{i1}) + \text{Tr} B_{11} S_1 P_y
\]

(5.8)

where the symbol \( B_{44} \) denotes the matrix \((B_1 - \frac{1}{4} B_{12} B_2 B_{12})\).

We obtain the expectation of this maximum profit based on
sample quantities. Taking the expectation of Equation 5.7 with respect to \( x \) for the \( i \)th stratum, and recognizing the relation \( E_x|s = \mu_1 \), we obtain the result

\[
E_x|s \hat{\mu}_x^i (y_i) = \hat{\mu}_x^i (x, \mu_1) + \sigma^2 \left[ (1 - \frac{1}{n_1}) \text{Tr} B_{11} W_1 + \frac{1}{n_1} \text{Tr} B_{44} W_1 \right] \mu_1^i.
\]

(5.9)

We briefly indicate how quantities given in Equation 5.9 were obtained. From Equation 5.8, we note that the term \( E_x|s \text{Tr} B_{11} S_1 \) arises. Taking the conditional expectation of with respect to \( x \), we have the expression

\[
\hat{\mu}_x^i = \hat{\mu}_x^i (x, \mu_1) - \frac{1}{n_1} \sum_j (x_{1j} - \hat{x}_i)(z_{ij} - \hat{z}_i)^2 \quad (5.10)
\]

where we have used the fact that an unbiased sample estimate of the population within-stratum variance-covariance matrix \( \sigma^2 W_1 \) is the expression \( \frac{1}{n_1-1} \sum_j (x_{1j} - \hat{x}_i)(x_{1j} - \hat{x}_i)' \).

We note the quadratic form \( x_i^i B_{44} x_i \) in Equation 5.7. By making use of the identity \( x_i = (x_i - \mu_1) + \mu_1 \), we obtain the quadratic form \( \mu_1^i B_{44} \mu_1 \) in \( \mu_1 \) which becomes a part of \( M_x \mu(X, \mu_1) \) in Equation 5.9. In addition, we obtain a quadratic form involving \( (x_{1j} - \mu_1) \), namely the quantity

\[
E_x|s (x_i - \mu_1)' B_{44} (x_i - \mu_1)
\]

which may be written as \( \text{Tr} B_{44} E_x|s (x_i - \mu_1)(x_i - \mu_1)' \). But we recall that \( x_i \) is a sample mean based on \( n_1 \) observa-
tions, and $\mu_1$ is its expectation. The conditional expectation is the result

$$E_x | s(x_i - \mu_1)(x_i - \mu_1)' = (\sigma^2/n_1)\bar{w}_1.$$  \hfill (5.11)

Combining Equations 5.10 and 5.11 with other terms of Equation 5.7, when the latter are expressed as expectations, we obtain Equation 5.9 previously.

Writing the equivalent form of the matrix $B_{44}$ in Equation 5.9 and simplifying the result, we obtain the equation

$$E_x | sM_x\hat{\pi}(Y_1) = M_x\pi(X, \mu_1)$$

$$+ \left[ \sigma^2 TrB_{11}\bar{w}_1 - (\sigma^2/n_1)TrB_{33}\bar{w}_1 \right] P_y.$$  \hfill (5.12)

Comparing Equation 5.12 for expected sample maximum profit with Equation 3.35 for the corresponding population quantity, we see that the sample estimate of stratum maximum profit is biased by the term $(\sigma^2/n_1)TrB_{33}\bar{w}_1 P_y$. One desirable property of sample estimates is that of unbiasedness. In order to obtain an unbiased estimate from the biased estimator $M_x\hat{\pi}(Y_1)$, we suggest adding the sample term $[n_1/(n_1-1)]TrB_{33}s_1$. It can readily be shown that the revised estimator has the expectation

$$E_x | s \left[ M_x\hat{\pi}(Y_1) + \frac{n_1}{n_1-1}TrB_{33}s_1 \right] = M_x\pi(X, \mu_1) + \sigma^2 TrB_{11}\bar{w}_1 P_y$$  \hfill (5.13)

which is the population quantity. Recalling the population result of Equation 3.35, we observe that application of the operator $E_s$ to Equation 5.13 would lead to unbiased sample
estimates of average stratum profits.

For purposes of the present discussion, we assume a finite number of strata, each with area \( A_i \) such that \( \sum_i A_i = A \), where \( A \) is area of the field in acres. Since the results of this section are expressed in terms of bushels and dollars per acre, we observe that the weighted average of estimated stratum profits may be found directly. For the unbiased estimator we find expected average profit, from Equation 5.13, to be

\[
\frac{1}{A} \sum A_i \pi(x, \mu_1) + TrB_{11} \left( \frac{\sigma^2}{A} \right) \sum A_i w_i p_y.
\]
VI. SUMMARY

Relationships between product output and resource inputs, for given conditions of production, are used in production theory. Some recent applications of theory have utilized quantitative relationships between yield and environmental as well as resource factors, but subject to the implied assumption that environmental conditions are constant throughout the production site. The object of this study was to consider resource use under variable environmental conditions. In particular, fertilizer use was examined under the postulate that soil in situ is variable rather than homogeneous.

Many results of production theory are obtained from operations involving specification of profit and maximization techniques. In the study reported here, analyses were facilitated by introducing a third operation, namely the expectation operator.

Fertilizer recommendations in many states are based on preaveraged laboratory results of soil tests. This procedure arises as a consequence of assumed soil homogeneity. For cases characterized by variable soil conditions, however, few assumptions are required in the conclusion that preaverage analysis may lead to (a) incorrect specification of yields and profits and (b) erroneous profit-maximizing fertilizer application rates. For the special case of a second-degree polyno-
mial yield function, which is assumed throughout the remainder of the discussion, it was concluded that both yields and profits from a given production stratum would be incorrectly specified by use of preaveraged soil tests; in addition, the specification errors were found to be linear functions of within-stratum population variance-covariance terms for the variable soil nutrients.

When a statistical sample is used to provide estimates of yields, profits, and optimum fertilizer rates, the pre- and post-average yields and profits for the sample are related in the same way as those of the population, with the exception that sample instead of population moments are involved. It was found that the sample stratum profit is a biased estimator of the population quantity; however, an unbiased estimator was constructed by adding a linear function of the sample moments.

One remarkable property of the second-degree polynomial yield function is that optimal fertilizer application rates for variable soil conditions are identical to those obtained when homogeneous soil conditions are assumed. There is no specification error associated with use of preaverages instead of postaverages for determining recommended fertilizer rates.

Heterogeneous soil materials were composited and subjected to laboratory tests. When results of the composite analyses were compared with averages of the individual determinations, it was concluded that the two types of analyses
were indistinguishable. In light of discussion given in the preceding paragraph, one may conclude that recommended fertilizer rates may be based on laboratory results of a composite sample when the latter is taken from a particular production area for which the second-degree polynomial yield function holds. This statement does not apply, however, to specification of yields and profits.

Expected maximum stratum profits are properly examined in light of within- and among-strata variability. Expected profit for the variable case differs from that obtained by ignoring variability; the difference is a linear function of population "Among" and "Within" variance-covariance properties. When unbiased sample estimates of maximum stratum profits are used, average values of these estimates are also unbiased.

Intensive field sampling provided a source of empirical evidence regarding methods of stratifying fields for purposes of fertilizer use. Evidence obtained from 9 sampled fields suggests that (a) natural bases for stratification may, in certain cases, be inferior to alternative procedures which ignore soil type, slope, and erosion differences, (b) relatively large areas may be rather homogeneous with respect to soil test results as presently determined in the laboratory, and (c) variability of soil test quantities within a field does not necessarily increase with greater topographical differences exhibited in the field.


VIII. ACKNOWLEDGMENTS

The author extends an expression of appreciation to members of his committee for their inspiring guidance and counsel during the course of study. Special thanks are due the committee co-chairmen, Dr. Herbert T. David and Dr. John T. Pesek.

The author wishes to thank Dr. Lloyd Dumenil, who provided experimental data and suggestions related to one aspect of the study. A final note of thanks is given to the National Science Foundation for major financial support of the research.
IX. APPENDIX A

Material appended here deals with the yield function used in numerical computations. Parameters of this function were estimated as a supplemental study. The required empirical function relates yield to controlled management factors and uncontrolled environmental factors; in particular, soil environment as indicated by laboratory test is considered in this study.

Data were extracted from those given by Dumenil (1958) for corn. Dumenil reported 120 fertilizer rate experiments conducted on Iowa fields during the period 1948 through 1956. Treatment applications, stand levels, soil test results, and treatment means in terms of corn yields were given. Although major soil types of the state were represented, experimental sites were mostly in the northern two-thirds of the state. Dumenil selected data from 93 experiments for which (a) potassium supply was adequate, (b) yields were not thought to be seriously limited by drought or insect damage, and (c) fertilizer treatments applied in the row were excluded. A total of 574 treatment means were involved in his study which examined use of foliar chemical analyses and other information for short-range yield prediction.

In the present study, 15 of the 93 experiments analyzed by Dumenil (1958) were discarded due to unavailable soil test
nitrogen results. In Dumenil's numbering system, these are Experiments 1, 2, 3, 5, 17, 20, 21, 64, 77, 78, 79, 91, 93, 94, and 119. Using data from the remaining 78 experiments, multiple regression techniques were used in conjunction with the 491 observed treatment means. Parameters of a second-degree polynomial model were estimated, where the dependent variable was plot yield in bushels of corn per acre.

Independent variables in the regression, with designating symbols, were soil acidity (pH), soil test nitrogen (n) and phosphorus (p) in pounds of N and P2O5 per acre, fertilizer nitrogen (N) and phosphorus (P) in pounds of N and P2O5 applied per acre, and stand (S) in thousands of plants per acre. Squares and all possible interaction pairs of these independent variables were included in the polynomial model. In addition, a coded variable for preceding crop (C) was introduced. In this code, the numbers 0, 1, 2, and 3 correspond to the crops corn, non-corn row crop, non-legume non-row crop, and leguminous non-row crop, respectively.

Parameters were estimated for two models which were identical except for a constant. An analysis of variance test indicated insignificant reduction in sums of squares due to the constant, so the alternative function was used. The yield function, as estimated without a constant, may be written
\[ \hat{Y} = 27.641430 \, \text{pH} + 3.0600511 \, n - 21.400808 \, p - 5.0872071 \, \text{pH}^2 \\
- 1.0884759 \, \text{pH}n + 1.5020794 \, \text{pH}p - 0.0040878224 \, n^2 \\
- 0.014768080 \, np - 0.11734778 \, p^2 - 0.55155072 \, N \\
- 0.32258109 \, P + 0.075126000 \, S - 0.011414162 \, N^2 \\
+ 0.013019231 \, NP + 0.023855915 \, NS - 0.016056045 \, P^2 \\
+ 0.056025704 \, PS - 8.6748999 \, S^2 + 0.40145179 \, \text{pHN} \\
+ 0.53017849 \, \text{pHP} + 2.5953257 \, \text{pHS} - 0.0040301994 \, nN \\
+ 0.00081080450 \, nP + 0.45134895 \, nS + 0.11275763 \, pN \\
- 0.36635357 \, pP + 1.2348241 \, pS + 5.5124207 \, C. \]

The ratio of "Regression" to "Total" sums of squares, corrected for means, was found to be 0.55. This quantity, commonly called the "coefficient of determination", is usually denoted by the symbol $R^2$.

For use in a particular production problem, we note that the preceding crop C will be known and may be expressed in terms of a code number. In this event, the last term of the above equation may be evaluated prior to other operations. In this light, the term may be interpreted as a crop-dependent parameter in the yield function.
X. APPENDIX B
Table 4. Information pertaining to the nine fields selected for intensive sampling

<table>
<thead>
<tr>
<th>Field No.</th>
<th>County and township</th>
<th>Locationa</th>
<th>Operator and address</th>
<th>Date sampled</th>
<th>Preceding crop</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Polk (Lincoln)</td>
<td>NE40 SW1/4 Sec.4</td>
<td>Richard Alleman, Slater</td>
<td>Aug 59</td>
<td>Meadow</td>
</tr>
<tr>
<td>2</td>
<td>Polk (Lincoln)</td>
<td>SW40 NE1/4 Sec.8</td>
<td>Robert Alleman, Slater</td>
<td>Aug 59</td>
<td>Meadow</td>
</tr>
<tr>
<td>3</td>
<td>Polk (Lincoln)</td>
<td>N1/2 SE40 SE1/4 Sec.2</td>
<td>Vernon Herring, Huxley</td>
<td>Aug 59</td>
<td>West 1/3 - oats East 2/3 - corn</td>
</tr>
<tr>
<td>4</td>
<td>Cass (Grant)</td>
<td>NE40 NW1/4 Sec.35</td>
<td>Walt Glynn, Anita</td>
<td>Apr 61</td>
<td>Corn</td>
</tr>
<tr>
<td>5</td>
<td>Cass (Grant)</td>
<td>N1/2 SE40 SW1/4 Sec.36</td>
<td>Henry Aiff, Anita</td>
<td>Apr 61</td>
<td>Corn</td>
</tr>
<tr>
<td>6</td>
<td>Tama (Crystal)</td>
<td>NE40 SW1/4 Sec.36</td>
<td>Will Podhajsky, Toledo</td>
<td>Apr 61</td>
<td>Corn</td>
</tr>
<tr>
<td>7</td>
<td>Crawford (Charter Oak)</td>
<td>NE40 NW1/4 Sec.5</td>
<td>Louie Andresen, Ute</td>
<td>Apr 61</td>
<td>East 1/2 - Meadow West 1/2 - corn</td>
</tr>
<tr>
<td>8</td>
<td>Tama (Lincoln)</td>
<td>SE40 NW1/4 Sec.21</td>
<td>Alice Costello, Gladbrook</td>
<td>May 61</td>
<td>North 1/2 - meadow South 1/2 - corn</td>
</tr>
<tr>
<td>9</td>
<td>Crawford (Union)</td>
<td>SE40 NW1/4 Sec.15</td>
<td>Charles Green, Dow City</td>
<td>June 61</td>
<td>Corn</td>
</tr>
</tbody>
</table>

aDirections north, south, east, and west are denoted by their traditional symbols. For example, the symbols N1/2 SE40 SW1/4 Sec.36 are read "the north half of the southeast 40 acres in the southwest quarter of Section 36".
Table 5. Among-strata corrected sums of squares and products for alternative stratification methods and total corrected sums of squares and products for Field 1 using soil analyses for acidity (pH), nitrogen (n) and phosphorus (p)

<table>
<thead>
<tr>
<th>Type of strata</th>
<th>Sums of squares and products</th>
<th>Number of strata</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>pH^2</td>
<td>pH x n</td>
</tr>
<tr>
<td>Natural:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Slope only</td>
<td>0.27</td>
<td>-0.31</td>
</tr>
<tr>
<td>Hierarchal:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Soil type</td>
<td>8.04</td>
<td>171.04</td>
</tr>
<tr>
<td>Slope</td>
<td>8.10</td>
<td>169.42</td>
</tr>
<tr>
<td>Erosion</td>
<td>8.10</td>
<td>169.42</td>
</tr>
<tr>
<td>Mapping unit</td>
<td>13.92</td>
<td>178.87</td>
</tr>
<tr>
<td>Geometric:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Squares</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quarters</td>
<td>19.28</td>
<td>0.92</td>
</tr>
<tr>
<td>Sixths</td>
<td>22.50</td>
<td>51.28</td>
</tr>
<tr>
<td>Ninths</td>
<td>23.59</td>
<td>27.11</td>
</tr>
<tr>
<td>Twelfths</td>
<td>24.58</td>
<td>13.44</td>
</tr>
<tr>
<td>Sixteenths</td>
<td>25.82</td>
<td>17.39</td>
</tr>
<tr>
<td>Strips</td>
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<td></td>
</tr>
<tr>
<td>Halves</td>
<td>0.52</td>
<td>43.08</td>
</tr>
<tr>
<td>Thirds</td>
<td>0.84</td>
<td>-6.11</td>
</tr>
<tr>
<td>Fourths</td>
<td>0.63</td>
<td>7.19</td>
</tr>
<tr>
<td>Sixths</td>
<td>1.23</td>
<td>10.81</td>
</tr>
<tr>
<td>Total sums of squares and products</td>
<td>43.90</td>
<td>-29.58</td>
</tr>
</tbody>
</table>
Table 6. Among-strata corrected sums of squares and products for alternative stratification methods and total corrected sums of squares and products for Field 2 using soil analyses for acidity (pH), nitrogen (n) and phosphorus (p)

<table>
<thead>
<tr>
<th>Type of strata</th>
<th>pH²</th>
<th>pH x n</th>
<th>pH x p</th>
<th>n²</th>
<th>n x p</th>
<th>p²</th>
<th>Number of strata</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Slope only</td>
<td>13.37</td>
<td>12.87</td>
<td>21.27</td>
<td>613.05</td>
<td>-58.52</td>
<td>44.24</td>
<td>3</td>
</tr>
<tr>
<td>Hierarchial</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Soil type</td>
<td>13.33</td>
<td>17.84</td>
<td>20.55</td>
<td>23.88</td>
<td>27.51</td>
<td>31.68</td>
<td>2</td>
</tr>
<tr>
<td>Slope</td>
<td>13.33</td>
<td>17.84</td>
<td>20.55</td>
<td>23.88</td>
<td>27.51</td>
<td>31.68</td>
<td>2</td>
</tr>
<tr>
<td>Erosion</td>
<td>13.33</td>
<td>17.84</td>
<td>20.55</td>
<td>23.88</td>
<td>27.51</td>
<td>31.68</td>
<td>2</td>
</tr>
<tr>
<td>Mapping unit</td>
<td>16.83</td>
<td>200.44</td>
<td>31.65</td>
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<td>606.15</td>
<td>66.89</td>
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<td>Geometric:</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Squares</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quarters</td>
<td>12.90</td>
<td>335.60</td>
<td>31.45</td>
<td>20739.21</td>
<td>1413.05</td>
<td>165.20</td>
<td>4</td>
</tr>
<tr>
<td>Sixths</td>
<td>16.27</td>
<td>227.19</td>
<td>-3.79</td>
<td>16381.78</td>
<td>958.84</td>
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<tr>
<td>Ninths</td>
<td>26.15</td>
<td>248.89</td>
<td>-27.52</td>
<td>21395.40</td>
<td>693.08</td>
<td>239.86</td>
<td>9</td>
</tr>
<tr>
<td>Twelfths</td>
<td>29.05</td>
<td>265.93</td>
<td>-31.01</td>
<td>37140.75</td>
<td>1266.08</td>
<td>412.33</td>
<td>12</td>
</tr>
<tr>
<td>Sixteenths</td>
<td>28.81</td>
<td>300.75</td>
<td>-18.15</td>
<td>44741.27</td>
<td>2278.31</td>
<td>562.75</td>
<td>16</td>
</tr>
<tr>
<td>Strips</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Halves</td>
<td>0.87</td>
<td>81.39</td>
<td>9.52</td>
<td>7631.66</td>
<td>893.10</td>
<td>104.51</td>
<td>2</td>
</tr>
<tr>
<td>Thirds</td>
<td>8.83</td>
<td>134.14</td>
<td>4.21</td>
<td>12188.08</td>
<td>1268.21</td>
<td>144.84</td>
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<tr>
<td>Fourths</td>
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<td>77.70</td>
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<td>25546.41</td>
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<td>4</td>
</tr>
<tr>
<td>Sixths</td>
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<td>78.74</td>
<td>0.85</td>
<td>34545.21</td>
<td>2512.06</td>
<td>230.29</td>
<td>6</td>
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<tr>
<td>Total sums of squares and products</td>
<td>43.82</td>
<td>357.15</td>
<td>-39.33</td>
<td>127948.83</td>
<td>10125.96</td>
<td>4617.78</td>
<td>169</td>
</tr>
</tbody>
</table>
Table 7. Among-strata corrected sums of squares and products for alternative stratification methods and total corrected sums of squares and products for Field 3 using soil analyses for acidity (pH), nitrogen (n) and phosphorus (p)

<table>
<thead>
<tr>
<th>Type of strata</th>
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Table 9. Among-strata corrected sums of squares and products for alternative stratification methods and total corrected sums of squares and products for Field 5 using soil analyses for acidity (pH), nitrogen (n) and phosphorus (p)

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Table 10. Among-strata corrected sums of squares and products for alternative stratification methods and total corrected sums of squares and products for Field 6 using soil analyses for acidity (pH), nitrogen (n) and phosphorus (p)

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Table 11. Among-strata corrected sums of squares and products for alternative stratification methods and total corrected sums of squares and products for Field 7 using soil analyses for acidity (pH), nitrogen (n) and phosphorus (p)

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Table 12. Among-strata corrected sums of squares and products for alternative stratification methods and total corrected sums of squares and products for Field 8 using soil analyses for acidity (pH), nitrogen (n) and phosphorus (p)

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Table 13. Among-strata corrected sums of squares and products for alternative stratification methods and total corrected sums of squares and products for Field 9 using soil analyses for acidity (pH), nitrogen (n) and phosphorus (p)

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