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**Second order shadowing approximations for passage through
resonance and capture at resonance**

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Iowa State University, 1991

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**Second order shadowing approximations
for passage through resonance and
capture at resonance**

by

Chao-Pao Ho

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CHAPTER 1. INTRODUCTION

The aim of this paper is to study the phenomenon of “passage through resonance” and “capture at resonance” from the shadowing point of view.

The system to be considered is

$$\begin{aligned} \dot{r} &= \varepsilon f(r, \theta), \\ \dot{\theta} &= r. \end{aligned} \tag{1.1}$$

The phase space is a cylinder; that is, θ is an angle (defined modulo 2π), f is 2π -periodic in θ , r is real (and not restricted to nonnegative real values), and $r = 0$ corresponds to a circle (around the cylinder) and not a point (as in polar coordinates). The circle $r = 0$ then consists of fixed points, and is therefore distinguished from other circles; this circle is the “exact resonance”.

We note that Murdock [2] did this problem using the “leading order inner system” and the “leading order outer system” to describe the “inner region” and the “outer region” except in a certain excluded region. That is, Murdock’s paper only studied the phenomenon of “passage through resonance” in the region $|r| \leq 1$ except the excluded region. In this paper we will use the “second order inner system” and the “second order outer system” to approximate the exact system (1.1) in all regions $|r| \leq 1$. We thus improve the shadowing result obtained by Murdock [2] in two ways: 1) greater accuracy in those regions discussed in [2] and 2) by applying Theorem 3.3 we obtain an estimate in excluded region. The second order inner system is defined by

$$\begin{aligned} \dot{s} &= \varepsilon f(0, \phi) + \varepsilon f_r(0, \phi)s, \\ \dot{\phi} &= s. \end{aligned} \tag{1.2}$$

Here the letters s and ϕ are used in place of r and θ in order to distinguish solutions of (1.2) from solutions of (1.1); however, the points (r, θ) and (s, ϕ) will be considered to belong to the same cylinder. The second order outer system is defined by

$$\begin{aligned}\dot{\rho} &= \varepsilon \bar{f}_1(\rho) + \varepsilon^2 \bar{f}_2(\rho), \\ \dot{\psi} &= \rho + \varepsilon \bar{g}_1(\rho),\end{aligned}\tag{1.3}$$

with

$$\begin{aligned}\bar{f}_1(\rho) &= \frac{1}{2\pi} \int_0^{2\pi} f(\rho, \psi) d\psi, \\ \bar{g}_1(\rho) &= \frac{1}{2\pi} \int_0^{2\pi} u_1(\rho, \psi) d\psi, \\ \bar{f}_2(\rho) &= \frac{1}{2\pi} \int_0^{2\pi} [u_1(\rho, \psi) f_r(\rho, \psi) + v_1(\rho, \psi) f_\theta(\rho, \psi) \\ &\quad - u_{1\theta}(\rho, \psi) \bar{g}_1(\rho) - u_{1r}(\rho, \psi) \bar{f}_1(\rho)] d\psi,\end{aligned}$$

where u_1 and v_1 are 2π -periodic functions of θ which will be defined in Chapter 5. Throughout this paper it will be assumed that

$$\bar{f}_1(\rho) < -k < 0,\tag{1.4}$$

for $|\rho| \leq 1$. This implies that the solution of the outer equations is a slow drift down the cylinder, so that the solutions of (1.1) which cross the resonance are expected to cross it from above. We will define our approximate solution by "patching" solutions of (1.2) and (1.3) together at the distance $\delta(\varepsilon) = \varepsilon^{5/12}$ from resonance. In fact, any height $\gamma = \pm\delta(\varepsilon)$ will do as a patching point, provided $\varepsilon^{1/2} \ll \delta(\varepsilon) \ll \varepsilon^{3/8}$. However, in this paper $\delta(\varepsilon)$ will be chosen $\varepsilon^{5/12}$. (The reason will be given in Chapter 4.)

Before constructing our approximate solutions we want to discuss the qualitative behaviour of the inner solutions. All figures in this paper are drawn for fixed value

of ε . The inner solution can be divided into two different styles. If $f(0, \phi) < 0$, \dot{s} will always be negative and hence all solutions of (1.2) will cross the resonance from above. This is called the “passive resonance case”; see Figure 1.1. On the other hand, if the flow of (1.2) will sometimes be directed upward, and will have rest points on the exact resonance (the “active case”); see Figures 1.2 and 1.3. In this paper there are specific assumptions on the flow of the leading order inner system:

- A1. The leading inner system possesses a homoclinic orbit $q^0(t)$, to a hyperbolic saddle points p_0 .
- A2. Let $\Gamma^0 = \{q^0(t) | t \in \mathbf{R}\} \cup \{p_0\}$. The interior of Γ^0 is filled with a continuous family of periodic orbits $q^\alpha(t)$, $\alpha \in (-1, 0)$.
- A3. $\lim_{\alpha \rightarrow 0} q^\alpha(t) = \Gamma^0$ and $\lim_{\alpha \rightarrow -1} q^\alpha(t) = p_{-1}$ where p_{-1} is a center.
- A4. The only rest points are the saddle p_0 and the center p_{-1} .

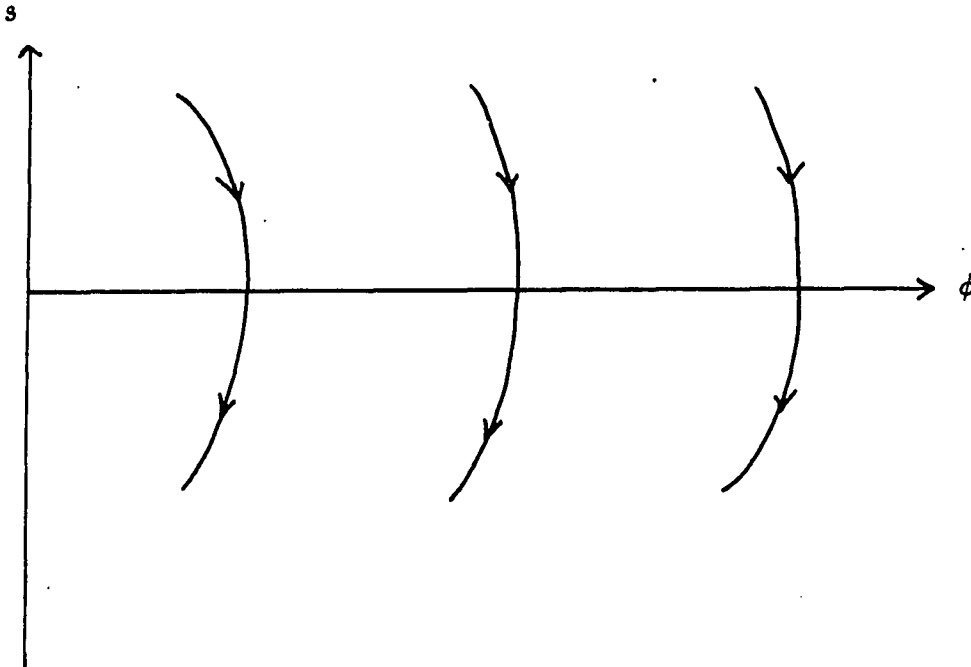


Figure 1.1. Inner approximation in passive case.

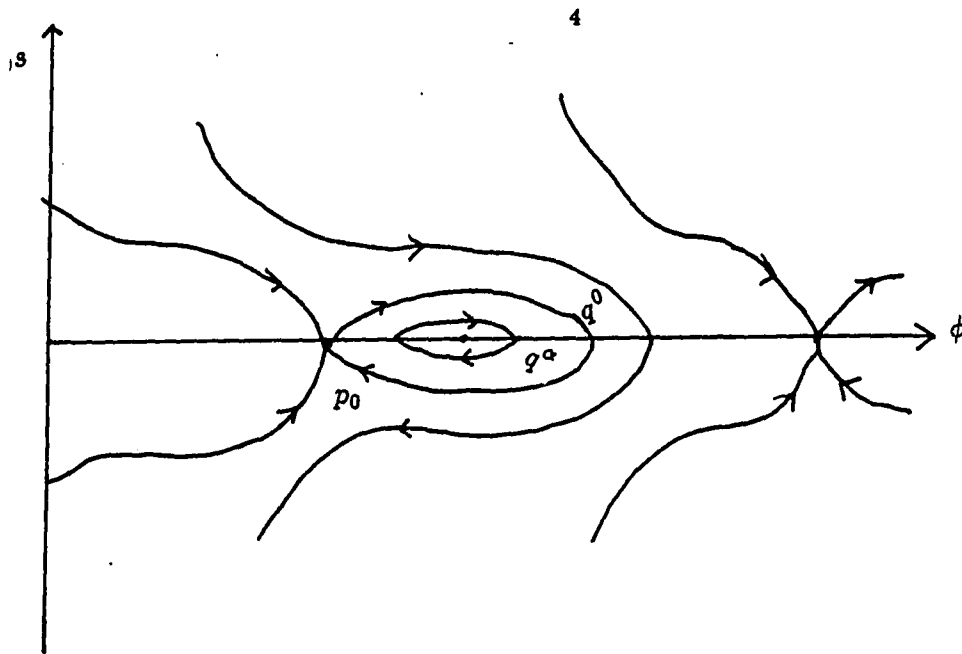


Figure 1.2. Inner approximation (leading order) in active case.

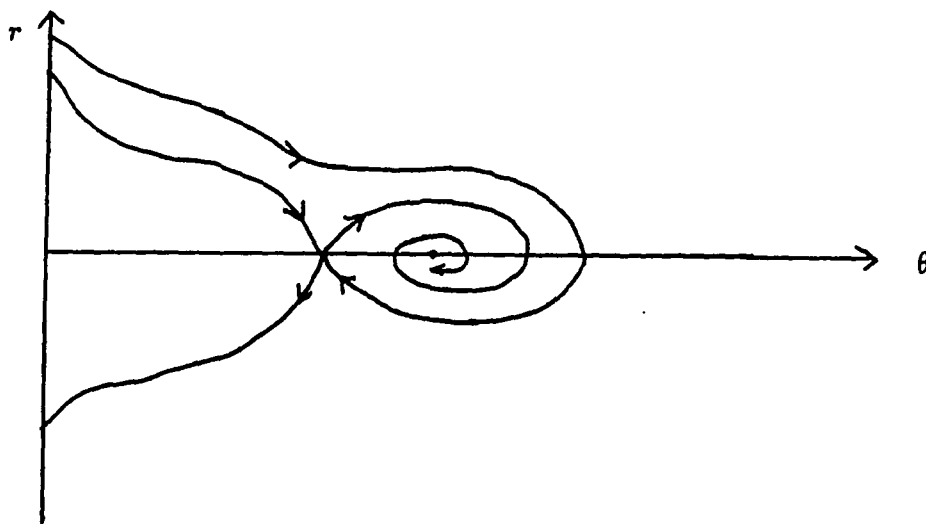


Figure 1.3. Qualitative feature of exact flow in active case.

That is, the flow of leading order inner system has the form shown in Figure 1.2. Since Figure 1.2 contains a homoclinic loop and a nest of periodic solutions, Pexido's theorem implies that the behaviour of Figure 1.2 is not structurally stable. That is, the phase portrait of the exact system (1.1) is not expected to be the same as that of the first order inner system. In fact, a typical phase portrait of the exact system (1.1) near the resonance is shown in Figure 1.3. This figure can be derived from the second order inner system (1.2) and can be shown to be qualitatively correct for (1.1) and (1.2) (very well surveyed in [1, Section 8]). Our first task is to choose a point A on the exact resonance circle, to the right of the stable manifold of the saddle point C , and draw the (second order inner) solution passing through this point; see Figure 1.4. Next, we want to choose a suitable point B near to C such that the solutions crossing the interval AB will pass through the resonance region sufficiently rapidly for standard perturbation methods to be applicable, but solutions crossing BC will not. (Details are given in Chapter 2.) Finally, draw the orbit of the system (1.2) passing through the point B . Thus, the inner region $|r| \leq c\sqrt{\varepsilon}$ can be divided into regions 3, 4, 5 and 6; see Figure 1.4.

Now, we want to define the approximate solutions which we will study. Since our systems are autonomous, we can assume the approximate solutions will cross the exact resonance at time $t = 0$.

First, choose one point ϕ_0 at the exact resonance. Next, let $s(t)$, $\phi(t)$ be the solution of (1.2) such that $s(0) = 0$, $\phi(0) = \phi_0$. For this solution we can define critical times $t_{-2} < t_{-1} < 0 < t_1 < t_2$ such that $s(t_{-2}) = \delta(\varepsilon) = \varepsilon^{5/12}$, $s(t_{-1}) = c\sqrt{\varepsilon}$, $s(t_1) = -c\sqrt{\varepsilon}$ and $s(t_2) = -\delta(\varepsilon) = -\varepsilon^{5/12}$. Further, let the outer solution $\rho_1(t)$, $\psi_1(t)$ of the outer system (1.3) satisfy $\rho_1(t_{-2}) = s(t_{-2})$, $\phi(t_{-2}) = \psi_1(t_{-2})$. Similarly, let $\rho_2(t)$, $\psi_2(t)$ be the outer solution of (1.3) that agrees with the inner solution at time t_2 . Finally, let $\rho_1(t_{-3}) = 1$ and $\rho_2(t_3) = -1$.

Thus, the patched approximate solution can be defined by

$$\begin{aligned}\tilde{r}(t) &= \begin{cases} \rho_1(t), & t_{-3} \leq t \leq t_{-2}, \\ s(t), & t_{-2} \leq t \leq t_2, \\ \rho_2(t), & t_2 \leq t \leq t_3, \end{cases} \\ \tilde{\theta}(t) &= \begin{cases} \psi_1(t), & t_{-3} \leq t \leq t_{-2}, \\ \phi(t), & t_{-2} \leq t \leq t_2, \\ \psi_2(t), & t_2 \leq t \leq t_3. \end{cases}\end{aligned}\tag{1.5}$$

Figure 1.4 is fundamental for the remainder of this paper. This figure shows patched approximate solutions. The solid curves in this figure are solutions of the inner system (1.2) and dotted curves are solutions of the outer system (1.3). These approximate solutions divide the resonance band into eight regions numbered 1 to 8. In this paper, the term “inner region” will refer to regions 3, 4, 5, and 6; “intermediate region” to regions 2 and 7; and “outer region” to 1 and 8. Notice that the inner solution is used in both the inner and intermediate regions. We shall follow solutions from the inner region into regions 7 and 8; the same arguments can be sued backwards in time to handle regions 1 and 2.

As usual we write $f(\varepsilon) = \mathcal{O}(g(\varepsilon))$ if $|f(\varepsilon)| \leq cg(\varepsilon)$ for some $c > 0$ and for ε in some interval $0 \leq \varepsilon \leq \varepsilon_0$. Occasionally we write $f(\varepsilon) \geq \mathcal{O}(g(\varepsilon))$ if $|f(\varepsilon)| \geq cg(\varepsilon)$ for some $c > 0$ and for ε in some interval $0 \leq \varepsilon \leq \varepsilon_0$.

The organization of this paper is as follows:

Chapter 2: The estimates for the regions 3 and 4 are given. We obtain $|r(t) - s(t)| = \mathcal{O}(\varepsilon^{3/2})$, $|\theta(t) - \phi(t)| = \mathcal{O}(\varepsilon)$ in these regions. These results were obtained using the central idea used by Murdock in his paper [2] but with second-order approximation rather than a first order.

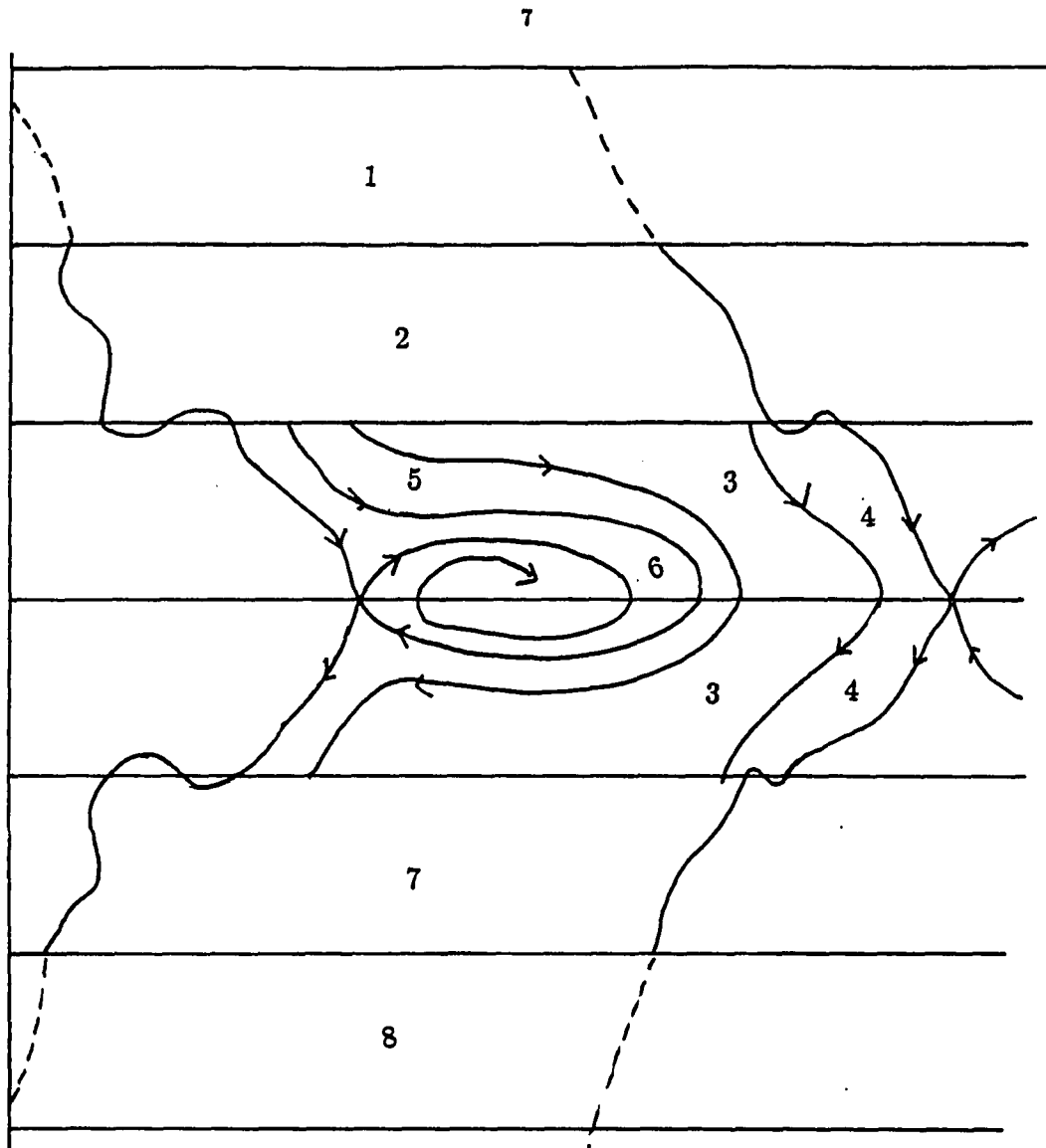


Figure 1.4. The resonance band $-1 \leq r \leq 1$ showing patched approximate solution passing through resonance or capture at resonance. These approximate solutions divide the resonance band into regions 1,2,3,4,5,6,7 and 8.

Chapter 3: The estimates for the regions 5 and 6 are given. The estimates for the region 5 are $|r(t) - s(t)| = \mathcal{O}(\varepsilon^{3/2})$; $|\theta(t) - \phi(t)| = \mathcal{O}(\varepsilon)$. Although we have not obtained a shadowing result in the θ coordinate in all regions 6, we obtain a shadowing result $|r(t) - s(t)| = \mathcal{O}(\varepsilon^{3/2})$, $|\theta(t) - \phi(t)| = \mathcal{O}(\varepsilon)$ in the region 6 except $6''$ and $|r(t) - s(t)| = \mathcal{O}(\varepsilon^{1/2})$ in the region $6''$. These new results are entirely due to the author.

Chapter 4: The estimates for the intermediate region are given. We improve the accuracy of $|r(t) - s(t)|$ and $|\theta(t) - \phi(t)|$ from $\mathcal{O}(\delta^6/\varepsilon^2)$ and $\mathcal{O}(\delta^5/\varepsilon^2)$ (in [2]) to $\mathcal{O}(\delta^{11}/\varepsilon^4)$ and $\mathcal{O}(\delta^{10}/\varepsilon^4)$ in these regions respectively. These results were obtained using the method of Eckhaus extended to the second-order approximation. The method of Eckhaus for the first order is used for Sanders in [4] and [5] and by Murdock in his paper [2].

Chapter 5: The estimates for the outer region are given. We improve the accuracy of $|r(t) - \rho(t)|$ from $\mathcal{O}(\varepsilon^{1/7})$ (in [2]) to $\mathcal{O}(\varepsilon^{7/12})$. Although we have not obtained a shadowing result in the θ coordinate, we improve the accuracy of $|\theta(t) - \psi(t)|$ from $\mathcal{O}(\varepsilon^{-6/7})$ to $\mathcal{O}(\varepsilon^{-5/12})$. The former is the accuracy achieved in reference [2]. These results were obtained using the ideas of Sanders ([4] and [5]) for the outer region, extended to the second-order approximation. Again, these ideas were applied to the first order approximation for our problem by Murdock in his paper [2].

CHAPTER 2.
ESTIMATES IN THE REGIONS 3 AND 4

The first step is to extend the Lemma 2.1 in [2] to $O(\varepsilon^2)$ and prove it. Now consider the following systems:

$$\begin{aligned}\dot{x} &= -\lambda x + p_1(x, y) + \varepsilon p_2(x, y), \\ \dot{y} &= \mu y + q_1(x, y) + \varepsilon q_2(x, y),\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}\dot{\xi} &= -\lambda \xi + p_1(\xi, \eta) + \varepsilon p_2(\xi, \eta) + \varepsilon^2 u(\xi, \eta; \varepsilon), \\ \dot{\eta} &= \mu \eta + q_1(\xi, \eta) + \varepsilon q_2(\xi, \eta) + \varepsilon^2 v(\xi, \eta; \varepsilon),\end{aligned}\tag{2.2}$$

where p_1 and q_1 are of quadratic and higher orders and $u(0, 0; \varepsilon) = 0$, $v(0, 0; \varepsilon) = 0$. This means that the saddle point does not depend on ε . On the other hand, the x -axis and y -axis are the stable and unstable manifolds of the linear system

$$\begin{aligned}\dot{x} &= -\lambda x, \\ \dot{y} &= \mu y.\end{aligned}$$

For small enough $\delta > 0$, and for any $T > 0$, a unique solution of (2.1) will be singled out by requiring that

$$\begin{aligned}x(0) &= \pm\delta, \\ y(T) &= \pm\delta\end{aligned}$$

(with a specific choice of the signs), and similarly for (2.2).

Lemma 2.1. *There exist $\delta > 0$, $\varepsilon_0 > 0$, and $k > 0$ such that for any $T > 0$ the solution of (2.1) and (2.2) satisfying the boundary conditions*

$$\begin{aligned} x(0) &= \xi(0) = \pm\delta, \\ y(T) &= \eta(T) = \pm\delta, \end{aligned} \tag{2.3}$$

satisfy

$$\begin{aligned} |x(t) - \xi(t)| &\leq k\varepsilon^2, \\ |y(t) - \eta(t)| &\leq k\varepsilon^2, \end{aligned} \tag{2.4}$$

for $0 \leq t \leq T$ and $0 \leq \varepsilon \leq \varepsilon_0$.

Proof. Consider the box neighborhood N of the form $|x| < \delta$, $|y| < \delta$; see Figure 2.1. For small enough $\delta > 0$, the joint Lipschitz constant L_1 for p_1 and q_1 in the box neighborhood can be less than any specified quantity, since p_1 and q_1 are of quadratic and higher orders. The Lipschitz constant L_1 is defined by

$$|p_1(x, y) - p_1(x', y')| \leq L_1 \cdot \max\{|x - x'|, |y - y'|\},$$

and

$$|q_1(x, y) - q_1(x', y')| \leq L_1 \cdot \max\{|x - x'|, |y - y'|\},$$

whenever (x, y) and (x', y') are in N . It follows that L_1 can be made arbitrarily small. Further, assume the joint Lipschitz constant L_2 for p_2 and q_2 is defined by

$$|p_2(x, y) - p_2(x', y')| \leq L_2 \cdot \max\{|x - x'|, |y - y'|\},$$

and

$$|q_2(x, y) - q_2(x', y')| \leq L_2 \cdot \max\{|x - x'|, |y - y'|\},$$

whenever (x, y) and (x', y') are in N . Let M be the maximum of $|u|$ and $|v|$ over the box, for ε in some interval $0 \leq \varepsilon \leq \varepsilon_0$. Then

$$\dot{x} - \dot{\xi} = -\lambda(x - \xi) + [p_1(x, y) - p_1(\xi, \eta)] + \varepsilon [p_2(x, y) - p_2(\xi, \eta)] - \varepsilon^2 u(\xi, \eta; \varepsilon)$$

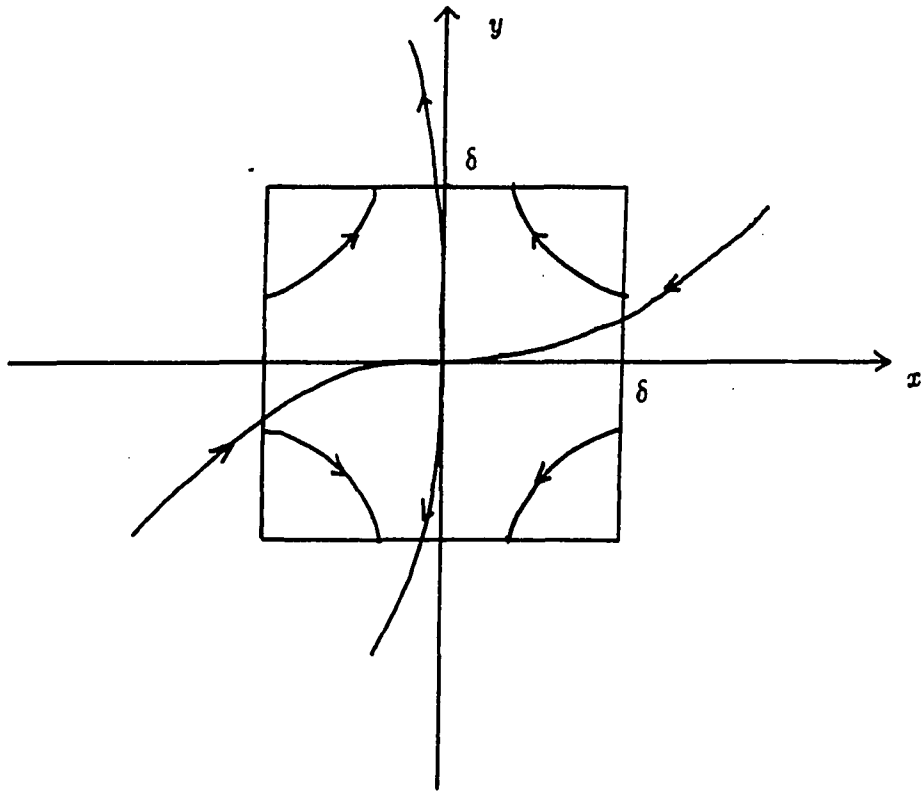


Figure 2.1. Four orbits crossing a box neighborhood of a saddle in time T

which (since $x(0) = \xi(0)$) implies

$$x(t) - \xi(t) = \int_0^t e^{-\lambda(t-\tau)} \{p_1(x(\tau), y(\tau)) - p_1(\xi(\tau), \eta(\tau)) \\ + \varepsilon [p_2(x(\tau), y(\tau)) - p_2(\xi(\tau), \eta(\tau))] - \varepsilon^2 u(\xi(\tau), \eta(\tau); \varepsilon)\} d\tau$$

for all $t \geq 0$. During the interval $0 \leq t \leq T$, while the solutions remain in the box, the quantity in brackets in the integrand can be estimated using L_1 and M . In fact let $K = K(\varepsilon)$ denote the larger of the maximum values of $|x(t) - \xi(t)|$ and $|y(t) - \eta(t)|$ over $0 \leq t \leq T$; then

$$|x(t) - \xi(t)| \leq \int_0^t e^{-\lambda(t-\tau)} [L_1 K + \varepsilon L_2 K + \varepsilon^2 M] d\tau$$

and hence

$$|x(t) - \xi(t)| \leq \frac{1}{\lambda} [L_1 K + \varepsilon L_2 K + \varepsilon^2 M].$$

Similarly, we can obtain

$$|y(t) - \eta(t)| \leq \frac{1}{\mu} [L_1 K + \varepsilon L_2 K + \varepsilon^2 M].$$

Let $C = \max \left\{ \frac{1}{\lambda}, \frac{1}{\mu} \right\}$, then

$$K \leq C [L_1 K + \varepsilon L_2 K + \varepsilon^2 M]$$

or

$$K \leq \frac{CM}{1 - C(L_1 + \varepsilon_0 L_2)} \varepsilon^2$$

provided $1 - C(L_1 + \varepsilon_0 L_2) > 0$ and $0 \leq \varepsilon \leq \varepsilon_0$. The requirement that $1 - C(L_1 + \varepsilon_0 L_2) > 0$ can be met by taking L_1 and ε_0 sufficient small; recall that L_1 can be made small (without changing) by making δ small, as noted at the beginning of the proof. This proves the lemma, with $k = CM/1 - C(L_1 + \varepsilon_0 L_2)$. \square

Remark 1. This lemma is also true when the boundary conditions (2.3) for ξ and η are slightly modified by changing the entering and leaving times by $O(\varepsilon^2)$. Specifically, we may replace $\xi(0) = \pm\delta$ by $\xi(a\varepsilon^2) = \pm\delta$ and $\eta(T) = \pm\delta$ by $\eta(T + b\varepsilon^2) = \pm\delta$. The result (2.4) will remain uniformly valid for $a \in A$, $b \in B$ where A and B are compact sets.

Remark 2. Let $x(t, T, \varepsilon)$, $y(t, T, \varepsilon)$ and $\xi(t, T, \varepsilon)$, $\eta(t, T, \varepsilon)$ be the solutions of (2.1) and (2.2) that cross the box N_δ in time T respectively. Since the constant k in (2.4) is independent on T , we have

$$\begin{aligned} |x(t, T, \varepsilon) - \xi(t, T, \varepsilon)| &\leq k\varepsilon^2, \\ |y(t, T, \varepsilon) - \eta(t, T, \varepsilon)| &\leq k\varepsilon^2, \end{aligned}$$

for $0 \leq t \leq T$ and $0 \leq \varepsilon \leq \varepsilon_0$. So we can insert a function $T(\varepsilon)$ in (2.4). That is,

$$\begin{aligned} |x(t, T(\varepsilon), \varepsilon) - \xi(t, T(\varepsilon), \varepsilon)| &\leq k\varepsilon^2, \\ |y(t, T(\varepsilon), \varepsilon) - \eta(t, T(\varepsilon), \varepsilon)| &\leq k\varepsilon^2 \end{aligned}$$

for $0 \leq t \leq T(\varepsilon)$ and $0 \leq \varepsilon \leq \varepsilon_0$. Thus every ε -dependent family of solutions is shadowed (not only those with fixed crossing time).

The next step is to define the constant c used in drawing the lines $s = \pm c\sqrt{\varepsilon}$ in Figure 1.4. Let $G(\theta)$ be the following antiderivative of $-f(0, \theta)$. For $-f(0, \theta)$ we have

$$-f(0, \theta) = A + \tilde{g}(\theta),$$

where $A = -\frac{1}{2\pi} \int_0^{2\pi} f(0, \theta) d\theta$ and \tilde{g} is a periodic function of mean zero. Then let

$$G(\theta) = A\theta + \tilde{G}(\theta), \tag{2.5}$$

where \tilde{G} is the antiderivative of \tilde{g} having zero mean (and is of course periodic). Let B be the maximum value of $|\tilde{G}|$, and choose

$$c > 2\sqrt{B}. \tag{2.6}$$

Now, we want to introduce the stretched variables R and S and a slow time variable τ into (1.1) and (1.2) respectively. Let

$$\begin{aligned} r &= \sqrt{\varepsilon} R, \\ s &= \sqrt{\varepsilon} S, \\ \tau &= \sqrt{\varepsilon} t, \\ ' &= \frac{d}{d\tau}. \end{aligned} \tag{2.7}$$

Then (1.1) and (1.2) can be written as

$$\begin{aligned} R' &= f(\sqrt{\varepsilon} R, \theta), \\ \theta' &= R \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} S' &= f(0, \phi) + \sqrt{\varepsilon} f_r(0, \phi) S, \\ \phi' &= S, \end{aligned} \tag{2.9}$$

respectively. Let $(0, \theta_0)$ be the saddle point C of the system (1.1). Let $\Theta = \theta - \theta_0$, then

$$\begin{aligned} R' &= f(\sqrt{\varepsilon} R, \Theta + \theta_0) \equiv g(\sqrt{\varepsilon} R, \Theta), \\ \Theta' &= R. \end{aligned}$$

Using the Taylor theorem we can obtain

$$\begin{aligned} g(\sqrt{\varepsilon} R, \Theta) &= g(0, \Theta) + g_r(0, \Theta) \sqrt{\varepsilon} R + \frac{1}{2} g_{rr}(0, \Theta) (\sqrt{\varepsilon} R)^2 \\ &\quad + \frac{1}{6} g_{rrr}(\sqrt{\varepsilon} R^*, \Theta) (\sqrt{\varepsilon} R)^3 \\ &= g(0, 0) + g_\theta(0, 0) \Theta + \frac{1}{2} g_{\theta\theta}(0, \Theta^*) \Theta^2 + g_r(0, \Theta) \sqrt{\varepsilon} R \\ &\quad + \frac{1}{2} g_{rr}(0, \Theta) (\sqrt{\varepsilon} R)^2 + \frac{1}{6} g_{rrr}(\sqrt{\varepsilon} R^*, \Theta) (\sqrt{\varepsilon} R)^3. \end{aligned}$$

Here $\sqrt{\varepsilon} R^* \in (0, \sqrt{\varepsilon} R)$ or $(\sqrt{\varepsilon} R, 0)$ and $\Theta^* \in (0, \Theta)$ or $(\Theta, 0)$. Thus (2.8) can be written as

$$\begin{pmatrix} R' \\ \Theta' \end{pmatrix} = \begin{pmatrix} 0 & g_\theta(0, 0) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} R \\ \Theta \end{pmatrix} + \begin{pmatrix} F(\Theta) \\ 0 \end{pmatrix} + \sqrt{\varepsilon} \begin{pmatrix} G(R, \Theta) \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} H(R, \Theta; \sqrt{\varepsilon}) \\ 0 \end{pmatrix}.$$

Here $g_\theta(0, 0)$ is positive, $F(\Theta) = \frac{1}{2}g_{\theta\theta}(0, \Theta^*)\Theta^2$, $G(R, \Theta) = g_r(0, \Theta)R$ and $H(R, \Theta; \sqrt{\varepsilon}) = \frac{1}{2}g_{rr}(0, \Theta)R^2 + \frac{1}{6}g_{rrr}(\sqrt{\varepsilon} R^*, \Theta)\sqrt{\varepsilon} R^3$. Similarly, (2.9) can be written as

$$\begin{pmatrix} S' \\ \Phi' \end{pmatrix} = \begin{pmatrix} 0 & g_\theta(0, 0) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} S \\ \Phi \end{pmatrix} + \begin{pmatrix} F(\Phi) \\ 0 \end{pmatrix} + \sqrt{\varepsilon} \begin{pmatrix} G(S, \Phi) \\ 0 \end{pmatrix}.$$

Therefore, by diagonalizing the linear part, our systems (2.8) and (2.9) can be written as the forms (2.2) and (2.1) respectively, near the saddle point C , except that ε in (2.2) plays the role of $\sqrt{\varepsilon}$ in (2.8). Thus, applying Remark 2 in Lemma 2.1, we see that there is a neighborhood of C in which every solution of (2.8) is shadowed by a solution of (2.9), the shadowing estimate being

$$\begin{aligned} |R - S| &= \mathcal{O}(\varepsilon), \\ |\theta - \phi| &= \mathcal{O}(\varepsilon). \end{aligned} \tag{2.10}$$

This estimate is expressed in terms of the stretched variables; after finishing the estimates in the inner region (below), it must be restated in the original variables.

Finally, we want to extend (2.10) from the box neighborhood of the saddle to the entire region 4. First choose B close enough to C to lie inside the neighborhood already treated, and draw the orbit of (2.9) through B to divide region 3 from 4. Then draw two horizontal line segments $B'C'$ and $B''C''$ inside the same neighborhood of C , dividing region 4 into three sections, a middle portion (already treated) and the upper and lower portions denoted $4'$ and $4''$; see Figure 2.2. For every orbit of (2.9) in region 3 and 4, the *corresponding orbit* of (2.8) is defined as follows. If the orbit of (2.9) crosses AB , the corresponding orbit is the one with the

same initial conditions at resonance. If the orbit of (2.9) (in region 4) crosses BC , the corresponding orbit of (2.8) is the continuation of the one defined above using Lemma 2.1.

Now, we want to show that (2.10) is true in region 3, 4' and 4''. Using the Taylor theorem, (2.8) and (2.9), we have

$$\begin{aligned} \frac{d}{d\tau} [|R - S| + |\theta - \phi|] &\leq |R' - S'| + |\theta' - \phi'| \\ &\leq |f(0, \theta) - f(0, \phi)| + \sqrt{\varepsilon} |f_r(0, \theta)R - f_r(0, \phi)S| \\ &\quad + \mathcal{O}(\varepsilon) + |R - S| \\ &\leq \ell_1 |\theta - \phi| + \sqrt{\varepsilon_0} \ell_2 [|R - S| + |\theta - \phi|] + |R - S| + \mathcal{O}(\varepsilon) \end{aligned}$$

for suitable Lipschitz constants ℓ_1 and ℓ_2 and $0 \leq \varepsilon \leq \varepsilon_0$. Let $L \equiv \max\{\ell_1 + \sqrt{\varepsilon_0} \ell_2, \sqrt{\varepsilon_0} \ell_2 + 1\}$, then one has

$$\frac{d}{d\tau} [|R - S| + |\theta - \phi|] \leq L [|R - S| + |\theta - \phi|] + \mathcal{O}(\varepsilon).$$

Let E denote $|R - S| + |\theta - \phi|$. Thus, using the Gronwall inequality, there exist constants C_1 , C_2 and C_3 such that

$$E(\tau) \leq E(\tau_0) e^{C_1(\tau - \tau_0)} + \varepsilon C_2 \left[e^{C_3(\tau - \tau_0)} - 1 \right]$$

for $\tau > \tau_0$. Similarly, we can obtain

$$-E(\tau) \geq -E(\tau_0) e^{C_1(\tau_0 - \tau)} - \varepsilon C_2 \left[e^{C_3(\tau_0 - \tau)} - 1 \right]$$

for $\tau_0 > \tau$. Thus, one has

$$E(\tau) \leq E(\tau_0) e^{C_1|\tau - \tau_0|} + \varepsilon C_2 \left[e^{C_3|\tau - \tau_0|} - 1 \right]. \quad (2.11)$$

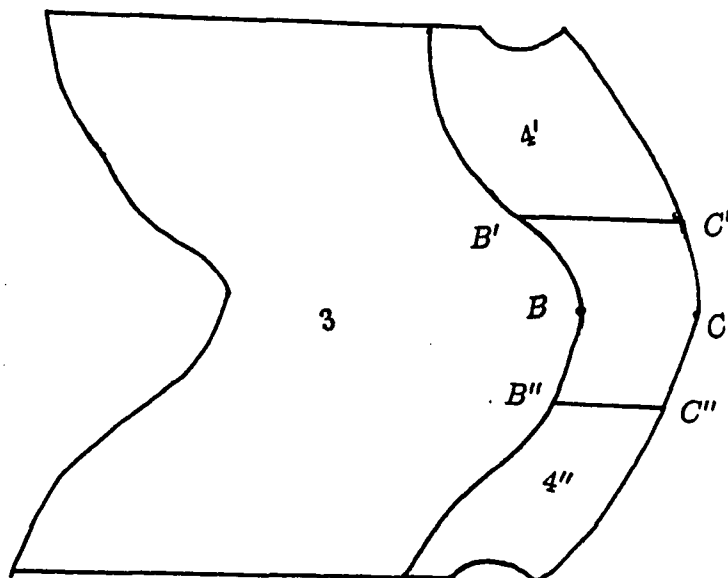


Figure 2.2. Region 3 and the subdivision of Region 4.
The choice of B can be independent of ε ,
but the orbit through B depends on ε .

In region 3, $\tau_0 = 0$ and $E(\tau_0) = 0$, and we have

$$E(\tau) \leq \varepsilon C_2 \left[e^{C_3 |\tau|} - 1 \right]. \quad (2.12)$$

In region 4' and 4'', taking τ_0 to be the time when any particular solution enters these regions (forward or backward from the resonance) we have $E(\tau_0) = \mathcal{O}(\varepsilon)$ from (2.10). Thus

$$E(\tau) = e^{C_1 |\tau - \tau_0|} \mathcal{O}(\varepsilon) + \varepsilon C_2 \left[e^{C_3 |\tau - \tau_0|} - 1 \right]. \quad (2.13)$$

On the other hand, the τ -time taken by solutions to cross these regions is uniformly bounded and hence the exponents in (2.12) and (2.13) are bounded. Therefore,

$$E(\tau) = \mathcal{O}(\varepsilon)$$

in regions 3, 4', and 4''. That is, (2.11) implies that (2.10) holds throughout the inner region. That is,

$$\begin{aligned} |r(t) - s(t)| &= \mathcal{O}(\varepsilon^{3/2}), \\ |\theta(t) - \phi(t)| &= \mathcal{O}(\varepsilon), \end{aligned} \tag{2.14}$$

in the regions 3 and 4.

CHAPTER 3.
ESTIMATES IN THE REGIONS 5 AND 6

The first step in this program is to extend Lemma 2.1 to the case in which the orbit passes through the box neighbourhood N_δ of one saddle point finitely many times. Consider the following two systems of differential equations

$$\dot{y} = f(y) + \varepsilon g(y) \equiv F_\varepsilon(y) \quad y \in \mathbb{R}^2, \quad (3.1)$$

and

$$\dot{z} = f(z) + \varepsilon g(z) + \varepsilon^2 h(z; \varepsilon) \quad z \in \mathbb{R}^2, \quad (3.2)$$

where f , g and h are sufficiently smooth. Assume that the unperturbed system

$$\dot{y} = f(y) \quad (3.3)$$

has two saddle points p_0 and q_0 . Thus there are two corresponding saddle points $p(\varepsilon)$ and $q(\varepsilon)$ of (3.1) with $p(\varepsilon) = p_0 + \mathcal{O}(\varepsilon)$ and $q(\varepsilon) = q_0 + \mathcal{O}(\varepsilon)$. Further, our specific assumptions are

- A1. Assume $h(p(\varepsilon); \varepsilon) = 0 = h(q(\varepsilon); \varepsilon)$. That is, $p(\varepsilon)$ and $q(\varepsilon)$ are also saddle points of (3.2).
- A2. (3.3) possesses a trajectory joining the saddle points p_0 and q_0 ; see Figure 3.1(a).
- A3. For $\varepsilon \neq 0$, the stable manifold $W^s(q(\varepsilon))$ and the unstable manifold $W^u(p(\varepsilon))$ of the system (3.1) and (3.2) are separated in the same way, either as in Figure 3.1(b) or (c). It follows that solutions passing near both p and q fall into three types: $(+, +)$, $(+, -)$, $(-, -)$ (see Figure 3.1(b)) or $(+, +)$, $(-, +)$, $(-, -)$, (see Figure 3.1(c)).

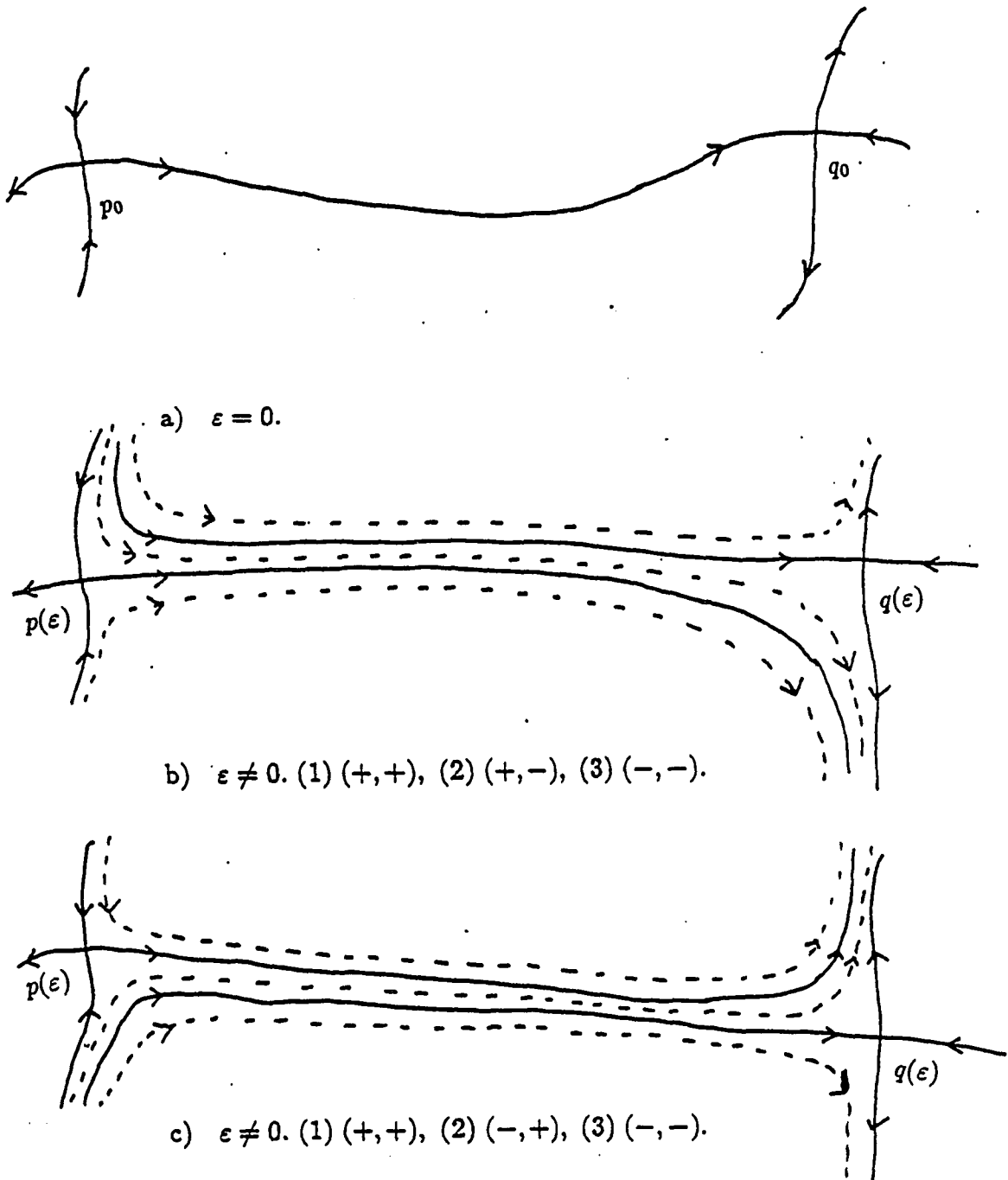


Figure 3.1. The flows of (3.10) and its perturbed systems

In our derivation of the main result (Theorem 3.3) of this chapter, we will need the following definition and theorems.

Definition. Let F be a c^r vector field defined on a manifold M . A tubular flow for F is a pair (\mathcal{F}, Φ) where \mathcal{F} is an open set in M and Φ is a c^r diffeomorphism of \mathcal{F} onto the cube $I^m = I \times I^{m-1} = \{(u, v) \in \mathbb{R} \times \mathbb{R}^{m-1}; |u| < 1 \text{ and } |v_i| < 1, i = 1, 2, \dots, m-1\}$ which takes the trajectories of F in \mathcal{F} to the straight lines $I \times \{v\} \subset I \times I^{m-1}$. The open set \mathcal{F} is called a flow box for the field F .

The next theorem describes the local behaviour of the orbits in a neighbourhood of a regular point.

Theorem 3.1. (Tubular Flow Theorem) *Let F be a c^r vector field defined on a manifold M and let $p \in M$ be a regular point of F . Let C be the vector field on I^m defined by $C(u, v) = (1, 0, \dots, 0)$. Then there exists a c^r diffeomorphism $\phi: V_p \rightarrow C$, for some neighbourhood V_p of p in M , taking trajectories of F to trajectories of C . Thus, if $p \in M$ is a regular point of F then there exists a flow box containing p .*

Proof. A proof of this theorem can be found in [3]. \square

The next theorem describes the global behaviour of the orbits in a neighbourhood of an arc which is compact (but not a closed curve).

Theorem 3.2. (Long Tubular Flow Theorem) *Let $\Gamma \subset M$ be an arc of a trajectory of F that is compact but not a closed curve. Then there exist a tubular flow (\mathcal{F}, Φ) of F such that $\Gamma \subset \mathcal{F}$.*

Proof. A good proof of this theorem can be found in [3]. Here, we only give a partial proof. Let $\alpha: [-\varepsilon, a + \varepsilon] \rightarrow M$ be an integral curve of F such that $\alpha([0, a]) = \Gamma$ and $\alpha(t) \neq \alpha(t')$ for $t \neq t'$. Let us consider the compact set $\tilde{\Gamma} = \alpha([-\varepsilon, a + \varepsilon])$. Thus, using the Theorem 3.1, there exists a finite cover $\{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n\}$ of $\tilde{\Gamma}$ by flow boxes. Without loss of generality we can assume each \mathcal{F}_i intersects only \mathcal{F}_{i-1} and \mathcal{F}_{i+1} ; see Figure 3.2. Let $-\varepsilon = t_1 < t_2 < \dots < t_n = a + \varepsilon$ be such that

$p_i = \alpha(t_i) \in \mathcal{F}_i \cap \tilde{\Gamma}$ and let us write I_d for $\{(0, v) \in I \times I; |v| < d\}$. Let (\mathcal{F}_i, Φ_i) be the tubular flows corresponding to the flow boxes above. It is clear that $\Sigma_1 = \Phi_1^{-1}(I_d)$ is a section transversal to F because Φ_1 is a local diffeomorphism and $p_1 \in \Sigma_1$. Define $\Sigma_p = F_t(\Sigma_1)$ where F_t is the flow of F at times t and $\alpha(t) = p \in \tilde{\Gamma}$. Thus $\mathcal{F} = \bigcup_{p \in \tilde{\Gamma}} \Sigma_p$ is a neighbourhood of Γ ; see Figure 3.3.

Given $z \in \mathcal{F}$ then there is $p \in \tilde{\Gamma}$ such that $z \in \Sigma_p$. The projection $\Pi_1: \mathcal{F} \rightarrow \tilde{\Gamma}$ defined by $\Pi_1(z) = p$ is a c^r map. On the other hand, since $p = F_t(p_1)$, $F_{-t}(z) \in \Sigma_1$. The projection $\Pi_2: \mathcal{F} \rightarrow \Sigma_1$ defined by $\Pi_2(z) = F_{-t}(z)$ is also a c^r map. Let $g_1: \tilde{\Gamma} \rightarrow I$ and $g_2: \Sigma_1 \rightarrow I$ be two diffeomorphisms. Thus $\Phi: \mathcal{F} \rightarrow I \times I$ defined by

$$\Phi(z) = (g_1 \Pi_1(z), g_2 \Pi_2(z))$$

is a diffeomorphism and hence (\mathcal{F}, Φ) is a tubular flow which contains Γ . \square

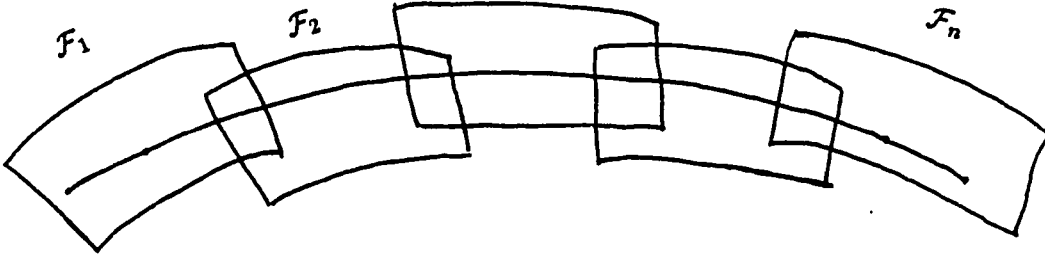


Figure 3.2. A cover of $\tilde{\Gamma}$ by flow boxes

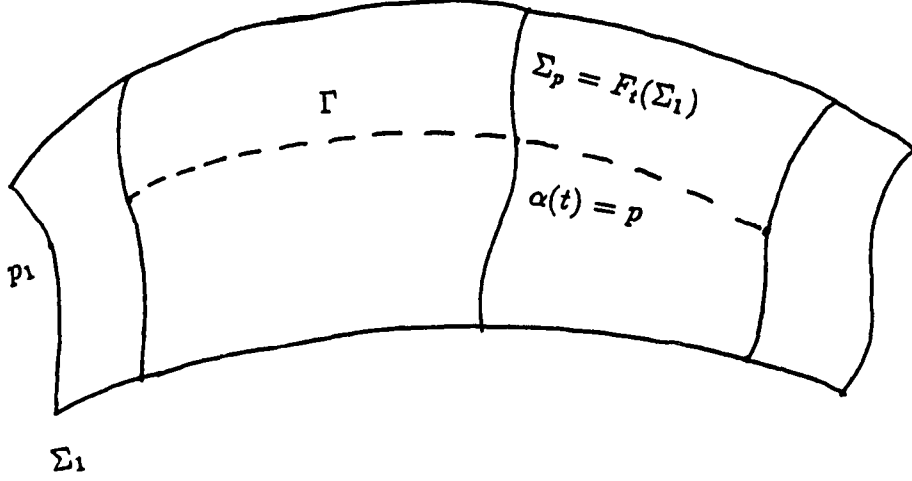


Figure 3.3. A neighbourhood \mathcal{F} of Γ

Let ϕ_ε be the transformation defined on $\tilde{N}_\delta = \{(\tilde{y}_1, \tilde{y}_2) \mid |\tilde{y}_1| < \delta, |\tilde{y}_2| < \delta\}$ such that the transformation ϕ_ε^{-1} defined on the neighbourhood $U_\delta \equiv U_\delta(p(\varepsilon))$ of $p(\varepsilon)$ carries the systems (3.1) and (3.2) (in the region U_δ) into

$$\begin{aligned}\dot{\tilde{y}}_1 &= +\tilde{\lambda}\tilde{y}_1 + \tilde{p}_1(\tilde{y}_1, \tilde{y}_2) + \varepsilon\tilde{p}_2(\tilde{y}_1, \tilde{y}_2), \\ \dot{\tilde{y}}_2 &= -\tilde{\mu}\tilde{y}_2 + \tilde{q}_1(\tilde{y}_1, \tilde{y}_2) + \varepsilon\tilde{q}_2(\tilde{y}_1, \tilde{y}_2),\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\dot{\tilde{z}}_1 &= +\tilde{\lambda}\tilde{z}_1 + \tilde{p}_1(\tilde{z}_1, \tilde{z}_2) + \varepsilon\tilde{p}_2(\tilde{z}_1, \tilde{z}_2) + \varepsilon^2\tilde{u}(\tilde{z}_1, \tilde{z}_2; \varepsilon), \\ \dot{\tilde{z}}_2 &= -\tilde{\mu}\tilde{z}_2 + \tilde{q}_1(\tilde{z}_1, \tilde{z}_2) + \varepsilon\tilde{q}_2(\tilde{z}_1, \tilde{z}_2) + \varepsilon^2\tilde{v}(\tilde{z}_1, \tilde{z}_2; \varepsilon).\end{aligned}\tag{3.5}$$

On the other hand, let ψ_ε be the transformation defined on $\hat{N}_\delta = \{(\hat{y}_1, \hat{y}_2) \mid |\hat{y}_1| < \delta, |\hat{y}_2| < \delta\}$ such that the transformation ψ_ε^{-1} defined on the neighbourhood $V_\delta \equiv$

$V_\delta(q(\varepsilon))$ of $q(\varepsilon)$ carries the systems (3.1) and (3.2) (in the region V_δ) into

$$\begin{aligned}\dot{\hat{y}}_1 &= -\hat{\lambda}\hat{y}_1 + \hat{p}_1(\hat{y}_1, \hat{y}_2) + \varepsilon\hat{p}_2(\hat{y}_1, \hat{y}_2), \\ \dot{\hat{y}}_2 &= +\hat{\mu}\hat{y}_2 + \hat{q}_1(\hat{y}_1, \hat{y}_2) + \varepsilon\hat{q}_2(\hat{y}_1, \hat{y}_2),\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}\dot{\hat{z}}_1 &= -\hat{\lambda}\hat{z}_1 + \hat{p}_1(\hat{z}_1, \hat{z}_2) + \varepsilon\hat{p}_2(\hat{z}_1, \hat{z}_2) + \varepsilon^2\hat{u}(\hat{z}_1, \hat{z}_2; \varepsilon), \\ \dot{\hat{z}}_2 &= +\hat{\mu}\hat{z}_2 + \hat{q}_1(\hat{z}_1, \hat{z}_2) + \varepsilon\hat{q}_2(\hat{z}_1, \hat{z}_2) + \varepsilon^2\hat{v}(\hat{z}_1, \hat{z}_2; \varepsilon).\end{aligned}\tag{3.7}$$

Here \tilde{p}_1 , \tilde{q}_1 , \hat{p}_1 and \hat{q}_1 are of quadratic and higher orders. Further, $\tilde{u}(0, 0; \varepsilon) = 0 = \tilde{v}(0, 0; \varepsilon)$ and $\hat{u}(0, 0; \varepsilon) = 0 = \hat{v}(0, 0; \varepsilon)$. The assumption A3 tells us there exists an orbit on the $W^u(p(\varepsilon))$ such that this orbit connects U_δ and V_δ . Thus, by Theorem 3.3, there is a tubular flow (\mathcal{F}, Φ) of F_ε such that \mathcal{F} is a (tubular) neighbourhood of this orbit and hence the transformation Φ_ε carries the system (3.1) (in the region \mathcal{F}) into

$$\begin{aligned}\dot{u} &= 1, \\ \dot{v} &= 0,\end{aligned}\tag{3.8}$$

and it follows that the same Φ_ε carries (3.2) (in the region \mathcal{F}) into

$$\begin{aligned}\dot{R} &= 1 + \varepsilon^2 K_1(R, S; \varepsilon), \\ \dot{S} &= 0 + \varepsilon^2 K_2(R, S; \varepsilon).\end{aligned}\tag{3.9}$$

Theorem 3.3. *There exist $\delta > 0$, $\varepsilon_0 > 0$ and $k > 0$ such that for any ε in $0 \leq \varepsilon \leq \varepsilon_0$ and for any solution family $y(t, \varepsilon)$ of (3.1) completely passing through U_δ , completely passing through V_δ , such that $y(t, \varepsilon)$ is of the same type $(+, +)$, $(+, -)$, $(-, -)$ or $(+, +)$, $(-, +)$, $(-, -)$ for each small ε , there exists a corresponding solution family $z(t, \varepsilon)$ of (3.2) which satisfies*

$$\|y(t, \varepsilon) - z(t, \varepsilon)\| \leq k\varepsilon^2$$

for all t such that $y(t, \varepsilon) \in U_\delta \cup \mathcal{F} \cup V_\delta$. That is, each solution family $y(t, \varepsilon)$ of (3.1) is ε^2 -shadowed by a solution family $z(t, \varepsilon)$ of (3.2) in the regions U_δ , \mathcal{F} and V_δ .

Before proving this theorem we need the following lemmas. Let $u(t, \alpha_0), v(t, \alpha_0)$ be the solution of (3.8) satisfying the initial condition

$$\begin{aligned} u(0, \alpha_0) &= 0, \\ v(0, \alpha_0) &= \alpha_0. \end{aligned} \tag{3.10}$$

Thus, $u(t, \alpha_0) = t$ and $v(t, \alpha_0) = \alpha_0$. Let $R(t, \alpha; \varepsilon), S(t, \alpha; \varepsilon)$ denote the solution of (3.9) satisfying the initial condition

$$\begin{aligned} R(0, \alpha; \varepsilon) &= 0, \\ S(0, \alpha; \varepsilon) &= \alpha. \end{aligned} \tag{3.11}$$

From the constructions \mathcal{F} and U_δ the cross section $\Sigma_1 = \Phi_1^{-1}(I_d)$ is the intersection of \bar{U}_δ and \mathcal{F} . Thus, the vertical line I_d is the “initial edge” of the systems (3.8) and (3.9). On the other hand, the “terminal edges” of the systems (3.8) and (3.9) are not vertical lines, but satisfy the following relations: there is a function $g(\cdot, \cdot)$ such that the terminal edge of (3.8) is given by

$$u = g(v, \varepsilon), \tag{3.12}$$

and the terminal edge of (3.9) for each ε in $0 \leq \varepsilon \leq \varepsilon_0$ is given by

$$R = g(S, \varepsilon). \tag{3.13}$$

(see Figure 3.4). The function g is given by $g = \Phi_{\varepsilon_1} (\Phi_{\varepsilon_2} | \bar{V}_\delta \cap \mathcal{F})^{-1}$ where Φ_{ε_i} is the i th coordinate of Φ_ε . Notice that the solution $u(t, \alpha_0), v(t, \alpha_0)$ of (3.8) reaches its terminal

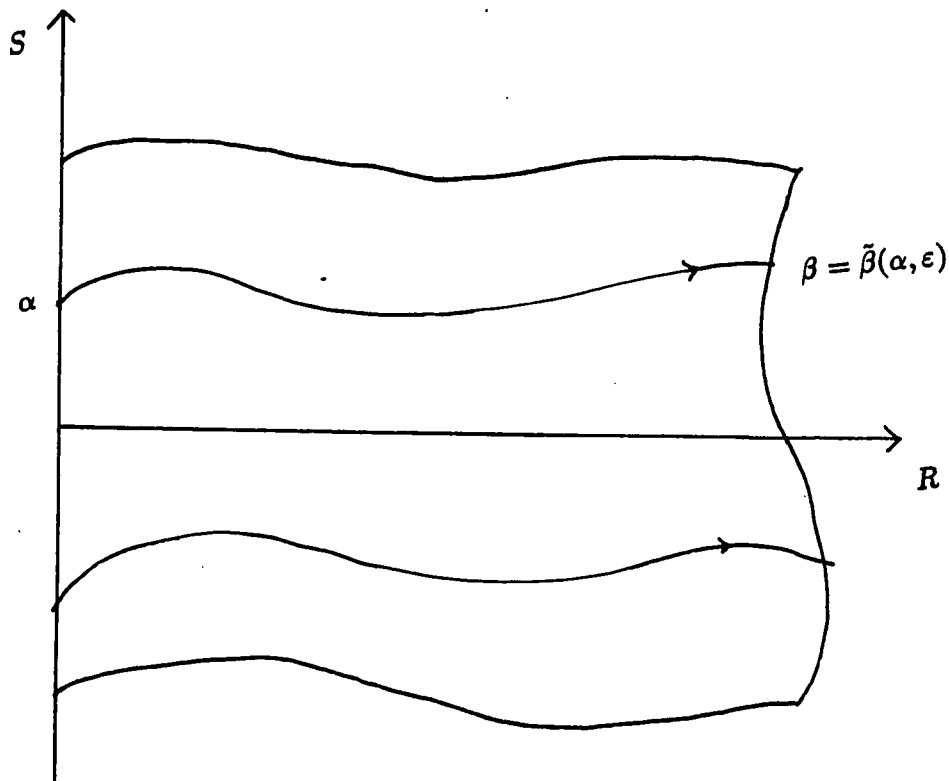


Figure 3.4. The terminal and initial edges of (3.8) and (3.9)

edge at time (see (3.18))

$$\hat{T} \equiv g(\alpha_0, \epsilon). \quad (3.14)$$

For the purposes of the next lemma, we assume that K_1 and K_2 in (3.9) are extended smoothly to the entire plane so that solutions can be continued beyond the terminal edge.

Lemma 3.4. *For any $\hat{T} > 0$, there exists $\epsilon_0 > 0$ such that for $|\epsilon| \leq \epsilon_0$, the solution $R(t, \alpha; \epsilon)$, $S(t, \alpha; \epsilon)$ exists for $|t| \leq \hat{T} + 1$.*

Proof. This lemma is a corollary of general results of this type in perturbation

theorem. It can be proved by using the implicit function theorem in a Banach space of functions, or by contraction mapping arguments. \square

Equations (3.9) and (3.11) imply that the solutions $R(t, \alpha; \varepsilon)$, $S(t, \alpha; \varepsilon)$ of (3.9) satisfies the following integral equations

$$\begin{aligned} R(t, \alpha; \varepsilon) &= t + \varepsilon^2 \int_0^t K_1(R(\tau), S(\tau); \varepsilon) d\tau, \\ S(t, \alpha; \varepsilon) &= \alpha + \varepsilon^2 \int_0^t K_2(R(\tau), S(\tau); \varepsilon) d\tau. \end{aligned}$$

Let $R(t, \alpha; \varepsilon)$, $S(t, \alpha; \varepsilon)$ cross the terminal edge at the “ β -point” $(g(\beta, \varepsilon), \beta)$ at time T (β and T will be determined later; see (3.16)). Thus,

$$\begin{aligned} g(\beta, \varepsilon) &= T + \varepsilon^2 \int_0^T K_1(R(\tau), S(\tau); \varepsilon) d\tau, \\ \beta &= \alpha + \varepsilon^2 \int_0^T K_2(R(\tau), S(\tau); \varepsilon) d\tau. \end{aligned} \tag{3.15}$$

Let

$$\begin{aligned} \Phi_1(\alpha, \beta, T, \varepsilon) &= g(\beta, \varepsilon) - T - \varepsilon^2 \int_0^T K_1(R(\tau), S(\tau); \varepsilon) d\tau, \\ \Phi_2(\alpha, \beta, T, \varepsilon) &= \beta - \alpha - \varepsilon^2 \int_0^T K_2(R(\tau), S(\tau); \varepsilon) d\tau, \end{aligned}$$

and

$$\Phi(\alpha, \beta, T, \varepsilon) = (\Phi_1(\alpha, \beta, T, \varepsilon), \Phi_2(\alpha, \beta, T, \varepsilon)).$$

Since

$$\left. \frac{\partial(\Phi_1, \Phi_2)}{\partial(\beta, T)} \right|_{\varepsilon=0} = \begin{vmatrix} g_\beta & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0,$$

there are

$$\begin{aligned}\beta &= \tilde{\beta}(\alpha, \varepsilon) \\ T &= \tilde{T}(\alpha, \varepsilon)\end{aligned}$$

so that the solution $R(t, \alpha; \varepsilon)$, $S(t, \alpha; \varepsilon)$ of (3.9) beginning at $(0, \alpha)$ crosses the “terminal edge” at the “ β -point” $(g(\beta, \varepsilon), \beta)$ at time T . Since the solution $u(t, \alpha_0)$, $v(t, \alpha_0)$ of (3.8) reach its terminal edge when $\varepsilon = 0$ at time $T_0 = g(\alpha_0, 0)$, it is easy to see that if $\alpha(\varepsilon) = \alpha_0 + \mathcal{O}(\varepsilon^2)$ then

$$\begin{aligned}\tilde{\beta}(\alpha(\varepsilon), \varepsilon) &= \alpha_0 + \mathcal{O}(\varepsilon^2), \\ \tilde{T}(\alpha(\varepsilon), \varepsilon) &= T_0(\alpha_0) + \mathcal{O}(\varepsilon).\end{aligned}$$

So if $\alpha = \alpha_0 + \alpha_2 \varepsilon^2 + \mathcal{O}(\varepsilon^3)$ then

$$\begin{aligned}\tilde{\beta}(\alpha(\varepsilon), \varepsilon) &= \alpha_0 + \beta_2(\alpha_0, \alpha_2) \varepsilon^2 + \mathcal{O}(\varepsilon^3), \\ \tilde{T}(\alpha(\varepsilon), \varepsilon) &= T_0(\alpha_0) + \varepsilon T_1(\alpha_0) \\ &\quad + \varepsilon^2 T_2(\alpha_0, \alpha_2) + \mathcal{O}(\varepsilon^3),\end{aligned}$$

for suitable β_2 and T_2 (they are computable by solving for the implicit functions by series techniques; see Lemma 3.5).

Lemma 3.5. *For any α_0 , α_2 , there exist $\beta_2 = \beta_2(\alpha_0, \alpha_2)$, $T_1 = T_1(\alpha_0)$, $T_2 = T_2(\alpha_0, \alpha_2)$, and $\varepsilon_0 > 0$ such that*

$$\begin{aligned}\tilde{\beta}(\alpha(\varepsilon), \varepsilon) &= \alpha_0 + \beta_2(\alpha_0, \alpha_2) \varepsilon^2 + \mathcal{O}(\varepsilon^3), \\ \tilde{T}(\alpha(\varepsilon), \varepsilon) &= T_0(\alpha_0) + \varepsilon T_1(\alpha_0) + \varepsilon^2 T_2(\alpha_0, \alpha_2) + \mathcal{O}(\varepsilon^3),\end{aligned}\tag{3.16}$$

and the solution $R(t, \alpha; \varepsilon)$, $S(t, \alpha; \varepsilon)$ of (3.9) also satisfies

$$\begin{aligned}|R(t, \alpha; \varepsilon) - u(t, \alpha_0)| &= \mathcal{O}(\varepsilon^2), \\ |S(t, \alpha; \varepsilon) - v(t, \alpha_0)| &= \mathcal{O}(\varepsilon^2),\end{aligned}\tag{3.17}$$

for $0 \leq t \leq \tilde{T}$ and $0 \leq \varepsilon \leq \varepsilon_0$.

Proof. Lemma 3.4 tells us that for any α there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| \leq \varepsilon_0$, the system (3.9) has a solution $R(t, \alpha; \varepsilon)$, $S(t, \alpha; \varepsilon)$ in the time interval $0 \leq t \leq T_0 + 1$.

The first line of (3.15) tells us

$$\begin{aligned} & g(\alpha_0, 0) + \varepsilon g_\varepsilon(\alpha_0, 0) + \varepsilon^2 \left[g_\beta(\alpha_0, 0)\beta_2 + \frac{1}{2}g_{\varepsilon\varepsilon}(\alpha_0, 0) \right] + \mathcal{O}(\varepsilon^3) \\ &= T_0(\alpha_0) + \varepsilon T_1(\alpha_0) + \left[T_2(\alpha_0, \alpha_2) + \int_0^{T_0(\alpha_0)} K_1(\tau, \alpha_0; 0) d\tau \right] \varepsilon^2 + \mathcal{O}(\varepsilon^3), \end{aligned}$$

by using the Taylor theorem. Thus we have

$$\begin{aligned} T_0(\alpha_0) &= g(\alpha_0, 0), \\ g_\varepsilon(\alpha_0, 0) &= T_1(\alpha_0), \\ T_2(\alpha_0, \alpha_2) &= g_\beta(\alpha_0, 0)\beta_2 + \frac{1}{2}g_{\varepsilon\varepsilon}(\alpha_0, 0) - \int_0^{T_0(\alpha_0)} K_1(\tau, \alpha_0; 0) d\tau. \end{aligned} \tag{3.18}$$

The second line of (3.15) tells us

$$\alpha_0 + \beta_2(\alpha_0, \alpha_2)\varepsilon^2 + \mathcal{O}(\varepsilon^3) = \alpha_0 + \left[\alpha_2 + \int_0^{T_0(\alpha_0)} K_2(\tau, \alpha_0; 0) d\tau \right] \varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

by using the Taylor theorem. Thus

$$\beta_2(\alpha_0, \alpha_2) = \alpha_2 + \int_0^{T_0(\alpha_0)} K_2(\tau, \alpha_0, 0) d\tau. \tag{3.19}$$

From (3.18) and (3.19) we obtain

$$\begin{aligned} T_2(\alpha_0, \alpha_2) &= g_\beta(\alpha_0, 0) \left[\alpha_2 + \int_0^{T_0(\alpha_0)} K_2(\tau, \alpha_0; 0) d\tau \right] + \frac{1}{2}g_{\varepsilon\varepsilon}(\alpha_0, 0) \\ &\quad - \int_0^{T_0(\alpha_0)} K_1(\tau, \alpha_0; 0) d\tau. \end{aligned}$$

Finally, we want to show that the solution $R(t, \alpha; \varepsilon)$, $S(t, \alpha; \varepsilon)$ satisfies (3.17).

Let $T_0 + 1 \geq \max_{0 \leq \varepsilon \leq \varepsilon_0} |T(\varepsilon)|$, $M_1 = \max_{\substack{0 \leq \varepsilon \leq \varepsilon_0 \\ 0 \leq t \leq T_0 + 1}} |K_1(R(\tau), S(\tau); \varepsilon)|$,

$M_2 = \max_{\substack{0 \leq \varepsilon \leq \varepsilon_0 \\ 0 \leq t \leq T_0 + 1}} |K_2(R(\tau), S(\tau); \varepsilon)|$ and

$$\begin{aligned} |R(t, \alpha; \varepsilon) - u(t, \alpha_0)| &= |t + \varepsilon^2 \int_0^t K_1(R(\tau), S(\tau); \varepsilon) d\tau - t| \\ &= |\varepsilon^2 \int_0^t K_1(R(\tau), S(\tau); \varepsilon) d\tau| \\ &\leq \varepsilon^2 \int_0^t |K_1(R(\tau), S(\tau); \varepsilon)| d\tau \\ &\leq \varepsilon^2 \int_0^{T_0 + 1} M_1 \\ &= \varepsilon^2 M_1 (T_0 + 1). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |S(t, \alpha; \varepsilon) - v(t, \alpha_0)| &= |\alpha + \varepsilon^2 \int_0^t K_2(R(\tau), S(\tau); \varepsilon) d\tau - \alpha_0| \\ &\leq |\alpha - \alpha_0| + \varepsilon^2 \int_0^t |K_2(R(\tau), S(\tau); \varepsilon)| d\tau \\ &\leq \alpha_2 \varepsilon^2 + \mathcal{O}(\varepsilon^3) + M_2 (T_0 + 1) \varepsilon^2 \\ &\leq [\alpha_2 + M_2 (T_0 + 1)] \varepsilon^2 + \mathcal{O}(\varepsilon^3). \quad \square \end{aligned}$$

Now, we want to prove the Theorem 3.3. First, we examine the neighbourhood U_δ of $p(\varepsilon)$. In the region U_δ , the transformation ϕ_ε^{-1} carries the systems (3.1) and (3.2) into (3.4) and (3.5) respectively. Let the solution $\tilde{y}(t, \varepsilon) = (\tilde{y}_1(t, \varepsilon), \tilde{y}_2(t, \varepsilon))$

of (3.4) takes time $T_p \equiv T_p(\varepsilon)$ to cross the box \tilde{N}_δ . That is, the solution $\tilde{y}(t, \varepsilon)$ of (3.4) satisfies the boundary conditions

$$\begin{aligned}\tilde{y}_1(T_p, \varepsilon) &= \delta, \\ \tilde{y}_2(0, \varepsilon) &= \delta.\end{aligned}\tag{2.30}$$

Thus, using the Remarks 1 and 2 in the Lemma 2.1, there is a compact set A such that the solution $\tilde{z}(t, \varepsilon; a) = (\tilde{z}_1(t, \varepsilon; a), \tilde{z}_2(t, \varepsilon; a))$ of (3.5) satisfying

$$\begin{aligned}\tilde{z}_1(T_p + a\varepsilon^2 + \mathcal{O}(\varepsilon^3), \varepsilon; a) &= \delta, \\ \tilde{z}_2(0, \varepsilon; a) &= \delta,\end{aligned}\tag{3.21}$$

satisfies

$$\|\tilde{z}(t, \varepsilon; a) - \tilde{y}(t, \varepsilon)\| = \mathcal{O}(\varepsilon^2),$$

for $0 \leq t \leq T_p$, $0 \leq \varepsilon \leq \varepsilon_0$ and $a \in A$. That is, $\tilde{y}(t, \varepsilon)$ is ε^2 -shadowed by $\tilde{z}(t, \varepsilon; a)$ in the region \tilde{N}_δ for all $a \in A$. Let $z(t, \varepsilon; a)$ denote the solution of (3.2) corresponding to $\tilde{z}(t, \varepsilon; a)$ under the mapping ϕ_ε ; we understand $z(t, \varepsilon; a)$ to be continued, for all t , outside of the neighborhood U_δ . Thus, $\tilde{y}(t, \varepsilon)$ is ε^2 -shadowed by $z(t, \varepsilon; a)$ in the region U_δ for all $a \in A$. Similarly, there is a compact set B such that $\tilde{y}(t, \varepsilon)$ is ε^2 -shadowed by a solution $z(t, \varepsilon; b)$ in the region V_δ for all $b \in B$.

Next, we want to show that there exist $a \in A$, $b \in B$ such that $z(t, \varepsilon; a)$ and $z(t, \varepsilon; b)$ can match together and retain their accuracy in the region \mathcal{F} . For each $a \in A$ there exists a corresponding ‘‘exit point’’ on the ‘‘initial edge’’ I_d of the system (3.9). We will denote this point by $\hat{\alpha}^+(a, \varepsilon)$ when it is above the unstable manifold, and $\hat{\alpha}^-(a, \varepsilon)$ when it is below. On the other hand, for each $b \in B$ there exists a corresponding ‘‘entering point’’ $\hat{\beta}^\pm(b, \varepsilon)$ on the ‘‘terminal edge’’ of the system (3.9); see Figure 3.4. Now, we want to consider the case which the exit point and the entering point are $\hat{\alpha}^+(a, \varepsilon)$ and $\hat{\beta}^+(b, \varepsilon)$ respectively. (We can use the same technique to discuss the other types: $(+, -)$, $(-, -)$, $(-, +)$ etc.) Thus, using the Lemma 3.5, the solution $R(t, \alpha; \varepsilon)$, $S(t, \alpha; \varepsilon)$ of (3.9) with the initial condition

$R(0, \alpha; \varepsilon) = 0$, $S(0, \alpha; \varepsilon) = \alpha \equiv \hat{\alpha}^+(a, \varepsilon)$ (without loss of generality we can assume the solution satisfies the initial condition at time zero, since it is an autonomous system) will reach the terminal edge at

$$\begin{aligned}\beta &= \tilde{\beta}(\hat{\alpha}^+(a, \varepsilon), \varepsilon), \\ T &= \tilde{T}(\hat{\alpha}^+(a, \varepsilon), \varepsilon).\end{aligned}$$

Our aim is to choose a and b so that

$$\tilde{\beta}(\hat{\alpha}^+(a, \varepsilon), \varepsilon) = \hat{\beta}^+(b, \varepsilon) \tag{3.22}$$

which guarantees that the solutions connect. The function $\hat{\alpha}^+(a, \varepsilon)$ is decreasing in a for fixed ε because increasing a increases the time in the box \tilde{N}_δ and this means lowering the exit point $\hat{\alpha}^+(a, \varepsilon)$. So $\hat{\alpha}^+$ is invertible (for fixed ε). On the other hand, $\tilde{\beta}(\alpha, \varepsilon)$ is increasing in α for fixed ε (by following solutions across the “tube”; see Figure 3.4). So $\tilde{\beta}$ is invertible (for fixed ε). Thus, (3.22) is solvable for a unique a for each b . That is, there exist $a \in A$ and $b \in B$ such that $z(t, \varepsilon; a)$ and $z(t, \varepsilon; b)$ can match together and in fact coincide. Let $z(t, \varepsilon) \equiv z(t, \varepsilon; a) = z(t, \varepsilon; b)$ be this solution. On the other hand, the Lemma 3.5 tells us that this solution retains its accuracy in the region \mathcal{F} . Then the solution $y(t, \varepsilon)$ is ε^2 -shadowed by the solution $z(t, \varepsilon)$ in the regions U_δ , \mathcal{F} and V_δ . \square

Remarks.

- a) We can use the same idea to extend this theorem to the case which the orbit passes through the neighbourhood N_δ of one saddle point finitely many times.
- b) We also can improve this theorem to the case which has finitely many saddle points.

We can use the technique we treated in the region 4 and the Theorem 3.3 to

obtain the follow results

$$\begin{aligned} |r(t) - s(t)| &= \mathcal{O}(\varepsilon^{3/2}), \\ |\theta(t) - \phi(t)| &= \mathcal{O}(\varepsilon), \end{aligned} \tag{3.23}$$

in the region 5. Finally, we want to treat region 6. First draw two line segments $D'E'$ and $D''E''$ inside the δ -neighbourhood N_δ of C (see Figure 3.5). Thus, the region 6 is divided into three sections, a middle portion and the upper and lower portions denoted $6'$ and $6''$. The middle portion can be treated by using the Lemma 2.1. (The technique is the same as we did in the region 4.) Thus, we can obtain

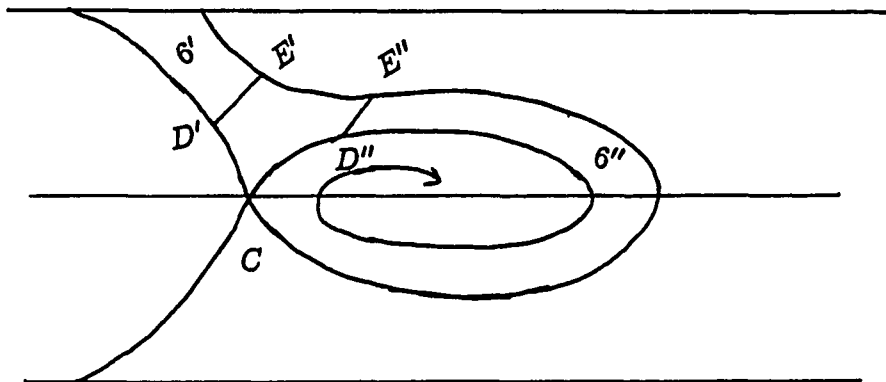


Figure 3.5. The subdivision of region 6.

$$\begin{aligned} |R - S| &= \mathcal{O}(\varepsilon), \\ |\theta - \phi| &= \mathcal{O}(\varepsilon), \end{aligned} \tag{3.24}$$

in the middle portion. For every orbit of (2.8) (an “exact” orbit) in region 6, the corresponding orbit of (2.9) (or “approximate” orbit) is defined as follows. Since the orbit passes through the neighbourhood N_δ , the corresponding orbit is the

continuation of the one treated in the middle portion. In the region $6'$ we can use the idea we did in the region $4'$ to obtain

$$\begin{aligned} |R - S| &= \mathcal{O}(\varepsilon), \\ |\theta - \phi| &= \mathcal{O}(\varepsilon), \end{aligned} \tag{3.25}$$

in the upper portion $6'$. In the region $6''$ we only can obtain a shadowing result in the r coordinate. Since the portion $6''$ is in the region $|r| \leq c\sqrt{\varepsilon}$, we have

$$|r(t) - s(t)| \leq 2c\sqrt{\varepsilon},$$

in the capture region $6''$. Thus, one has

$$\begin{aligned} |r(t) - s(t)| &= \mathcal{O}(\varepsilon^{3/2}), \\ |\theta(t) - \phi(t)| &= \mathcal{O}(\varepsilon), \end{aligned} \tag{3.26}$$

in the region 6 except $6''$ and

$$|r(t) - s(t)| = \mathcal{O}(\varepsilon^{1/2}), \tag{3.27}$$

in the region $6''$.

Remarks.

- a) We are trying to extend the Theorem 3.3 to the case which the orbit passes the neighbourhood N_δ of one saddle point infinitely many times as $\varepsilon \rightarrow 0$. Further, we are trying to use one Lyapunov's function to treat the fixed neighbourhood of one sink. Then we could perhaps improve the accuracy of $|r(t) - s(t)|$ and obtain a shadowing result in the θ coordinate in the capture region $6''$.
- b) Murdock suggested that I try to transform the given system (r, θ) in the capture region $6''$ to the action-angle variables (J, Φ) of the first order inner system (which is conservative and has a periodic regime instead of a capture

region), then average the action-angle system and obtain the new averaged system. Then we could construct an improved approximate solution by patching the inner system (s, ϕ) and the new averaged system at the line segment $D''E''$. Thus, we could perhaps improve the accuracy of $|r(t) - s(t)|$ and obtain a shadowing result in the θ coordinate in the capture region. Even if this is possible it is not very important, because all captured solutions tend toward the sink and in applications the fact of capture is more important than a detailed approximation of the transients.

CHAPTER 4
ESTIMATES IN THE
INTERMEDIATE REGION

As mentioned above, we will treat region 7, since 2 proceeds symmetrically. From (2.14), (3.23) and (3.26) we have the initial error upon entering 7. That is,

$$\begin{aligned} |r(t_1) - s(t_1)| &= \mathcal{O}(\varepsilon^{3/2}), \\ |\theta(t_1) - \phi(t_1)| &= \mathcal{O}(\varepsilon). \end{aligned} \tag{4.1}$$

Since θ (on the exact solution) and ϕ are monotone decreasing functions of t during the period under consideration, there exists a smooth monotone function $t^*(t) \equiv \phi^{-1}(\theta(t))$ defined for $t_1 \leq t \leq t_2$ such that

$$\theta(t) = \phi(t^*(t)). \tag{4.2}$$

Let ϕ_1 and ϕ_2 be the values of ϕ at times t_1 and t_2 , as shown in Figure 4.1. Our first task is to show that

$$|\phi_2 - \phi_1| = \mathcal{O}\left(\frac{\delta^2}{\varepsilon}\right), \tag{4.3}$$

$$t_2 - t_1 = \mathcal{O}\left(\frac{\delta}{\varepsilon}\right), \tag{4.4}$$

$$t^*(t_1) = t_1 + \mathcal{O}(\sqrt{\varepsilon}), \tag{4.5}$$

$$\frac{dt^*}{dt} = 1 + \mathcal{O}\left(\frac{\delta^8}{\varepsilon^3}\right), \tag{4.6}$$

provided $\varepsilon^{1/2} \ll \delta \ll \varepsilon^{3/8}$. Except for (4.5) and (4.6), these equations are the same as in [2], but the proofs are slightly different (in technical details) because

our equations contain additional terms which must be estimated. First, we want to check (4.3). Equation (1.2) tells us

$$[s(t)]^2 - [s(t_1)]^2 = 2\varepsilon [G(\phi_1) - G(\phi)] + 2\varepsilon \int_{\phi_1}^{\phi} s(\hat{\phi}) f_r(0, \hat{\phi}) d\hat{\phi}. \quad (4.7)$$

or, using (2.5),

$$[s(t)]^2 - [s(t_1)]^2 = 2\varepsilon \left\{ A(\phi_1 - \phi) + \tilde{G}(\phi_1) - \tilde{G}(\phi) + \int_{\phi_1}^{\phi} s(\hat{\phi}) f_r(0, \hat{\phi}) d\hat{\phi} \right\}. \quad (4.8)$$

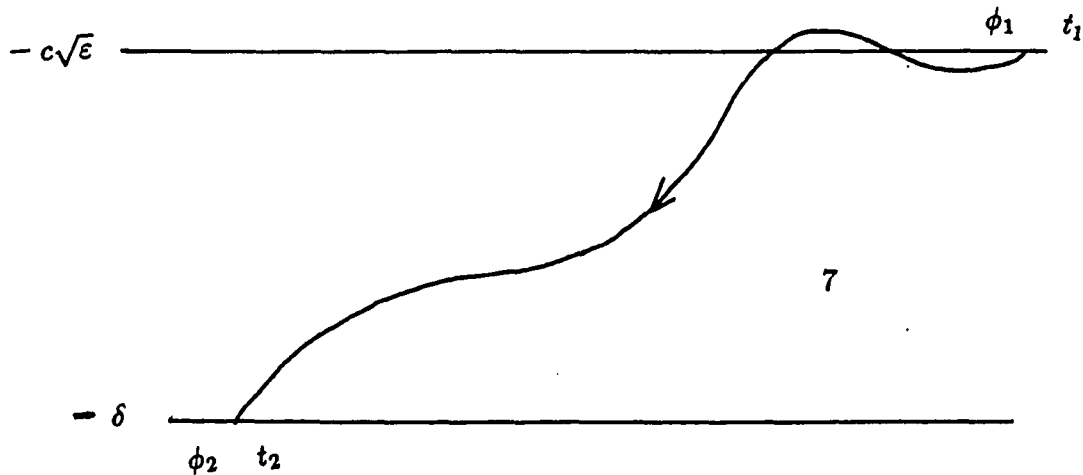


Figure 4.1. A typical orbit crossing region 7.

For the special case, $t = t_2$ one has

$$\delta^2 - c\varepsilon^2 = 2\varepsilon \left\{ A(\phi_1 - \phi_2) + \tilde{G}(\phi_1) - \tilde{G}(\phi_2) + \int_{\phi_1}^{\phi_2} s(\hat{\phi}) f_r(0, \hat{\phi}) d\hat{\phi} \right\}.$$

Since \tilde{G} is bounded, $\tilde{G}(\phi_1) - \tilde{G}(\phi_2) = \mathcal{O}(1)$. Thus, using $\int_{\phi_1}^{\phi_2} s(\hat{\phi}) f_r(0, \hat{\phi}) d\hat{\phi} = \mathcal{O}(\delta)|\phi_1 - \phi_2|$ and $\varepsilon^{1/2} \ll \delta \ll 1$, we have

$$|\phi_1 - \phi_2| = \mathcal{O}(1) + \mathcal{O}\left(\frac{\delta^2}{\varepsilon}\right) = \mathcal{O}\left(\frac{\delta^2}{\varepsilon}\right).$$

Next, we want to check (4.4). Although our real concern is with the active case, it is helpful to note that the proof is trivial in the passive case. In the passive case the term $f(0, \phi)$ is negative. So $f(0, \phi) + s f_r(0, \phi) < 0$ for small s . (This is true in the inner and intermediate regions whenever ε is small enough.) That is, $f(0, \phi) + s f_r(0, \phi)$ is bounded away from zero. Since the crossing time is the distance across the intermediate band (which is $\mathcal{O}(\delta)$) divided by a lower bound for the speed (which is $\mathcal{O}(\varepsilon)$), this implies (4.4) is true for the passive case. Now, we want to show that (4.4) is true for the active case. Equation (4.7), together with $s(t) = \frac{d\phi}{dt}$ and $s(t_1) = c\sqrt{\varepsilon}$, implies

$$\frac{dt}{d\phi} = -\frac{1}{\sqrt{\varepsilon}} \frac{1}{\sqrt{c^2 + 2[G(\phi_1) - G(\phi)] + 2 \int_{\phi_1}^{\phi} s(\hat{\phi}) f_r(0, \hat{\phi}) d\hat{\phi}}}.$$

Thus,

$$t_2 - t_1 = \frac{1}{\sqrt{\varepsilon}} \int_{\phi_2}^{\phi_1} \frac{d\phi}{\sqrt{c^2 + 2[G(\phi_1) - G(\phi)] + 2 \int_{\phi_1}^{\phi} s(\hat{\phi}) f_r(0, \hat{\phi}) d\hat{\phi}}}. \quad (4.9)$$

Let $M = \max_{0 \leq \hat{\phi} \leq 2\pi} |f_r(0, \hat{\phi})|$. There exists $\phi^* \in [\phi, \phi_1]$ such that

$$\begin{aligned} c^2 + 2[G(\phi_1) - G(\phi)] + 2 \int_{\phi_1}^{\phi} s(\hat{\phi}) f_r(0, \hat{\phi}) d\hat{\phi} \\ = c^2 + 2 \left[A(\phi_1 - \phi) + \tilde{G}(\phi_1) - \tilde{G}(\phi) \right] - 2s(\phi^*) f_r(0, \phi^*) (\phi_1 - \phi). \end{aligned}$$

It is easy to see (recalling $B = \max |\tilde{G}'|$) that

$$\begin{aligned} c^2 - 4B + 2(A - M\delta)(\phi_1 - \phi) &\leq c^2 + 2 \left[A(\phi_1 - \phi) + \tilde{G}(\phi_1) - \tilde{G}(\phi) \right] \\ &\quad - 2s(\phi^*) f_r(0, \phi^*) (\phi_1 - \phi) \quad (4.10) \\ &\leq c^2 + 4B + 2(A + M\delta)(\phi_1 - \phi). \end{aligned}$$

Therefore, the integrand in (4.9) is bounded below (above) by $1/\sqrt{c^2 + 4B + 2(A + M\delta)(\phi_1 - \phi)}$ ($1/\sqrt{c^2 - 4B + 2(A - M\delta)(\phi_1 - \phi)}$), which also has a nonvanishing denominator whenever ε (and hence δ) is small enough, in view of $c > 2\sqrt{B}$ (see (2.6)). So the integral here is proper and can be found to be $\mathcal{O}(\sqrt{\phi_1 - \phi_2})$. Thus, using (4.3), one has

$$t_2 - t_1 = \mathcal{O}\left(\frac{\delta}{\varepsilon}\right).$$

The proof of (4.5) begins with $\phi(t^*(t_1)) = \theta(t_1) = \phi(t_1) + \mathcal{O}(\varepsilon)$, from (4.2) and (4.1). Therefore ϕ travels a distance of $\mathcal{O}(\varepsilon)$ between the time t_1 and $t^*(t_1)$ (although we do not know which of these times is earlier). Equation (4.8) tells us

$$\left[\dot{\phi}(t) \right]^2 = c^2 \varepsilon + 2\varepsilon \left[A(\phi_1 - \phi) + \tilde{G}(\phi_1) - \tilde{G}(\phi) + \int_{\phi_1}^{\phi} s(\hat{\phi}) f_r(0, \hat{\phi}) d\hat{\phi} \right] \quad (4.11)$$

Since $A - M\delta$ is positive whenever ε is small enough, (4.10) and (4.11) imply $\frac{d\phi}{dt} \geq \mathcal{O}(\sqrt{\varepsilon})$. Since an upper bound for the time from t_1 to $t^*(t_1)$ is the distance

(which is $\mathcal{O}(\varepsilon)$) divided by a lower bound for the speed (which is $\mathcal{O}(\sqrt{\varepsilon})$), this implies

$$t^*(t_1) = t_1 + \mathcal{O}(\sqrt{\varepsilon}).$$

Next, we want to prove (4.6). Equation (1.1) can be written as

$$\dot{r} = \varepsilon f(0, \theta) + \varepsilon r F(r, \theta),$$

$$\dot{\theta} = r.$$

or

$$\frac{1}{2} [r(t)]^2 - \frac{1}{2} [r(t_1)]^2 = \varepsilon [G(\theta_1) - G(\theta)] + \varepsilon \int_{\theta_1}^{\theta} r(\hat{\theta}) F(r(\hat{\theta}), \hat{\theta}) d\hat{\theta}. \quad (4.12)$$

Equation (4.2) implies

$$\dot{\theta}(t) = \dot{\phi}(t^*(t)) \cdot \frac{dt^*}{dt},$$

or

$$\frac{dt^*}{dt} = \frac{\dot{\theta}(t)}{\dot{\phi}(t^*(t))}. \quad (4.13)$$

Equation (1.1) implies

$$\dot{r} = \mathcal{O}(\varepsilon),$$

and hence

$$dr = \mathcal{O}(\varepsilon) dt.$$

Thus, for $t_1 \leq t \leq t_2$, one has

$$r(t) = r(t_1) + \mathcal{O}(\varepsilon)(t - t_1).$$

Using (4.1), (4.4) and $\varepsilon^2 \ll \varepsilon^{1/2} \ll \delta$ we have

$$r(t) = \mathcal{O}(\delta) \quad (4.14)$$

Therefore, using (4.14), $\dot{\theta} = r$ and (4.4), one has

$$\begin{aligned}\theta - \theta_1 &= \theta(t) - \theta(t_1) \\ &= \dot{\theta}(t - t_1) \\ &= \mathcal{O}(\delta)\mathcal{O}\left(\frac{\delta}{\varepsilon}\right) \\ &= \mathcal{O}\left(\frac{\delta^2}{\varepsilon}\right).\end{aligned}\tag{4.15}$$

From (4.11) and (4.2), the denominator $\dot{\phi}(t^*(t))$ can be written as

$$\dot{\phi}(t^*(t)) = \sqrt{\varepsilon} \sqrt{c^2 + 2[G(\phi_1) - G(\theta)] + 2 \int_{\phi_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta}}, \tag{4.16}$$

or, using $\int_{\phi_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} = \mathcal{O}(\delta)(|\theta - \phi_1|) = \mathcal{O}(\delta)(|\theta - \theta_1| + \mathcal{O}(\varepsilon)) = \mathcal{O}(\delta^3/\varepsilon)$,

$$\dot{\phi}(t^*(t)) = \sqrt{\varepsilon} \sqrt{c^2 + 2[G(\phi_1) - G(\theta)] + \mathcal{O}(\delta^3/\varepsilon)}. \tag{4.17}$$

Similarly, using (4.12), we have

$$\dot{\theta} = \sqrt{[r(t_1)]^2 + 2\varepsilon[G(\theta_1) - G(\theta)] + 2\varepsilon \int_{\theta_1}^{\theta} r(\hat{\theta}) F(r(\hat{\theta}), \hat{\theta}) d\hat{\theta}}. \tag{4.18}$$

From (4.14) and (4.15) we have

$$\begin{aligned}\int_{\theta_1}^{\theta} r(\hat{\theta}) F(r(\hat{\theta}), \hat{\theta}) d\hat{\theta} &= \mathcal{O}(\delta)(|\theta - \theta_1|) \\ &= \mathcal{O}(\delta)\mathcal{O}\left(\frac{\delta^2}{\varepsilon}\right) \\ &= \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right)\end{aligned}\tag{4.19}$$

Using (4.1) and $s(t_1) = c\sqrt{\varepsilon}$ it is easy to see that

$$[r(t_1)]^2 = c^2\varepsilon + \mathcal{O}(\varepsilon^2) \tag{4.20}$$

Thus, using (4.19) and (4.20), (4.18) can be written as

$$\begin{aligned}\dot{\theta}(t) &= \sqrt{\varepsilon} \sqrt{c^2 + \mathcal{O}(\varepsilon) + 2[G(\theta_1) - G(\theta)] + \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right)} \\ &= \sqrt{\varepsilon} \sqrt{c^2 + 2[G(\theta_1) - G(\theta)] + \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right)}.\end{aligned}\tag{4.21}$$

Equations (4.1) and (4.17) imply that

$$\begin{aligned}\dot{\phi}(t^*(t)) &= \sqrt{\varepsilon} \sqrt{c^2 + 2[G(\theta_1 + \mathcal{O}(\varepsilon)) - G(\theta)] + \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right)} \\ &= \sqrt{\varepsilon} \sqrt{c^2 + 2[G(\theta_1) - G(\theta)] + \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right) + \mathcal{O}(\varepsilon)} \\ &= \sqrt{\varepsilon} \sqrt{c^2 + 2[G(\theta_1) - G(\theta)] + \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right)}.\end{aligned}\tag{4.22}$$

Therefore, (4.13), (4.21) and (4.22) imply that

$$\begin{aligned}\frac{dt^*}{dt} &= \sqrt{\frac{c^2 + 2[G(\theta_1) - G(\theta)] + \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right)}{c^2 + 2[G(\theta_1) - G(\theta)] + \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right)}} \\ &= \sqrt{1 + \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right)} \\ &= 1 + \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right),\end{aligned}\tag{4.23}$$

provided $\varepsilon^{1/2} \ll \delta \ll \varepsilon^{1/3}$. Of course, this does not yet quite prove the equation (4.6), but it gives a preliminary estimate of the same quantity. We will use this preliminary estimate to show the following results:

$$|\theta(t) - \phi(t)| = \mathcal{O}\left(\frac{\delta^5}{\varepsilon^2}\right),\tag{4.24}$$

$$|r(t) - s(t)| = \mathcal{O}\left(\frac{\delta^6}{\varepsilon^2}\right),\tag{4.25}$$

for $t_1 \leq t \leq t_2$. (These results will be used to prove (4.6) which will in turn be used to improve these results.) From (4.23), assuming $t_1 \leq t \leq t_2$, we have

$$t^*(t) - t^*(t_1) = t - t_1 + \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right)(t - t_1).$$

Equations (4.4) and (4.5) imply that

$$\begin{aligned} t^*(t) - t &= t^*(t_1) - t_1 + \mathcal{O}\left(\frac{\delta^3}{\varepsilon}\right)\mathcal{O}\left(\frac{\delta}{\varepsilon}\right) \\ &= \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}\left(\frac{\delta^4}{\varepsilon^2}\right) \\ &= \mathcal{O}\left(\frac{\delta^4}{\varepsilon^2}\right). \end{aligned} \tag{4.26}$$

Thus using $\dot{\phi} = s = \mathcal{O}(\delta)$ and (4.26), one has

$$\begin{aligned} |\theta(t) - \phi(t)| &= |\phi(t^*(t)) - \phi(t)| \\ &\leq \left(\max |\dot{\phi}|\right) |t^*(t) - t| \\ &= \mathcal{O}(\delta) \cdot \mathcal{O}\left(\frac{\delta^4}{\varepsilon^2}\right) \\ &= \mathcal{O}\left(\frac{\delta^5}{\varepsilon^2}\right), \end{aligned}$$

provided $\varepsilon^{1/2} \ll \delta \ll \varepsilon^{2/5}$. (A stronger upper bound will be imposed on δ shortly.) Equations (4.14) and (1.1) imply that

$$\dot{r} = \varepsilon f(0, \theta) + \mathcal{O}(\varepsilon\delta) \tag{4.27}$$

Thus (1.2), (4.27) and (4.24) imply that

$$\frac{d}{dt}|r(t) - s(t)| = \mathcal{O}\left(\frac{\delta^5}{\varepsilon}\right) + \mathcal{O}(\varepsilon\delta). \tag{4.28}$$

Integrating (4.28) from t_1 to t with $t \leq t_2$, using $|r(t_1) - s(t_1)| = \mathcal{O}(\varepsilon^{3/2})$ and $t_2 - t_1 = \mathcal{O}(\delta/\varepsilon)$, one has

$$|r(t) - s(t)| = \mathcal{O}(\varepsilon^{3/2}) + \mathcal{O}\left(\frac{\delta^6}{\varepsilon^2}\right) + \mathcal{O}(\delta^2) = \mathcal{O}\left(\frac{\delta^6}{\varepsilon^2}\right).$$

Now, we want to check (4.6). It follows from (1.1) that one has

$$\dot{r} = \varepsilon f(0, \theta) + \varepsilon f_r(0, \theta)r + \varepsilon r^2 H(r, \theta),$$

and hence

$$\dot{\theta} = \sqrt{[r(t_1)]^2 + 2\varepsilon [G(\theta_1) - G(\theta)] + 2\varepsilon \int_{\theta_1}^{\theta} r(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + 2\varepsilon \int_{\theta_1}^{\theta} r^2(\hat{\theta}) H(r(\hat{\theta}), \hat{\theta}) d\hat{\theta}} \quad (4.29)$$

Thus, using (4.25) and (4.15), one has

$$\begin{aligned} \int_{\theta_1}^{\theta} r(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} &= \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}\left(\frac{\delta^6}{\varepsilon^2}\right) \int_{\theta_1}^{\theta} f_r(0, \hat{\theta}) d\hat{\theta} \\ &= \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}\left(\frac{\delta^6}{\varepsilon^2}\right) \mathcal{O}(|\theta - \theta_1|) \\ &= \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}\left(\frac{\delta^6}{\varepsilon^2}\right) \mathcal{O}\left(\frac{\delta^2}{\varepsilon}\right) \\ &= \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}\left(\frac{\delta^8}{\varepsilon^3}\right). \end{aligned} \quad (4.30)$$

Equations (4.14) and (4.15) imply that

$$\int_{\theta_1}^{\theta} r^2(\hat{\theta}) H(r(\hat{\theta}), \hat{\theta}) d\hat{\theta} = \mathcal{O}(\delta^2)|\theta - \theta_1| = \mathcal{O}\left(\frac{\delta^4}{\varepsilon}\right). \quad (4.31)$$

Thus, using (4.20), (4.30) and (4.31), (4.29) can be written as

$$\begin{aligned} \dot{\theta} &= \sqrt{\varepsilon \left[c^2 + 2[G(\theta_1) - G(\theta)] + 2 \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}\left(\frac{\delta^8}{\varepsilon^3}\right) + \mathcal{O}\left(\frac{\delta^4}{\varepsilon}\right) + \mathcal{O}(\varepsilon) \right]} \\ &= \sqrt{\varepsilon \left[c^2 + 2[G(\theta_1) - G(\theta)] + 2 \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}\left(\frac{\delta^8}{\varepsilon^3}\right) \right]}. \end{aligned} \quad (4.32)$$

Since $\phi_1 = \theta_1 + \mathcal{O}(\varepsilon)$, one has

$$\begin{aligned}
\int_{\phi_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} &= \int_{\theta_1 + \mathcal{O}(\varepsilon)}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} \\
&= \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \int_{\theta_1 + \mathcal{O}(\varepsilon)}^{\theta_1} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} \\
&= \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}(\varepsilon\delta).
\end{aligned} \tag{4.33}$$

Thus, using (4.16) and (4.33), one has

$$\begin{aligned}
\dot{\phi}(t^*(t)) &= \sqrt{\varepsilon} \sqrt{c^2 + 2[G(\phi_1) - G(\theta)] + 2 \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}(\varepsilon\delta)} \\
&= \sqrt{\varepsilon} \sqrt{c^2 + 2[G(\theta_1) - G(\theta)] + 2 \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}(\varepsilon\delta) + \mathcal{O}(\varepsilon)} \\
&= \sqrt{\varepsilon} \sqrt{c^2 + 2[G(\theta_1) - G(\theta)] + 2 \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}(\varepsilon)}.
\end{aligned} \tag{4.34}$$

Therefore, using (4.13), (4.32) and (4.34), one has

$$\begin{aligned}
\frac{dt^*}{dt} &= \sqrt{\frac{c^2 + 2[G(\theta_1) - G(\theta)] + 2 \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}\left(\frac{\delta^8}{\varepsilon^3}\right)}{c^2 + 2[G(\theta_1) - G(\theta)] + 2 \int_{\theta_1}^{\theta} s(\hat{\theta}) f_r(0, \hat{\theta}) d\hat{\theta} + \mathcal{O}(\varepsilon)}} \\
&= \sqrt{1 + \mathcal{O}\left(\frac{\delta^8}{\varepsilon^3}\right)} \\
&= 1 + \mathcal{O}\left(\frac{\delta^8}{\varepsilon^3}\right),
\end{aligned}$$

provided $\varepsilon^{1/2} \ll \varepsilon \ll \varepsilon^{3/8}$. This completes the proof of (4.6). Finally, we can repeat the argument from (4.22) to (4.27) to obtain the following improvement of (4.24) and (4.25):

$$\begin{aligned} |\theta(t) - \phi(t)| &= \mathcal{O}\left(\frac{\delta^{10}}{\varepsilon^4}\right) = o(1), \\ |r(t) - s(t)| &= \mathcal{O}\left(\frac{\delta^{11}}{\varepsilon^4}\right) = o(1), \end{aligned} \tag{4.35}$$

for $t_1 \leq t \leq t_2$.

Remark. Since $\varepsilon^{1/2} \ll \delta \ll \varepsilon^{3/8}$, we can choose $\delta = \varepsilon^{5/12}$. An additional reason for this choice will appear in equation (5.19).

CHAPTER 5
ESTIMATES IN THE OUTER REGION

The first step is to consider a near-identity transformation

$$\begin{aligned} r &= x + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y), \\ \theta &= y + \varepsilon v_1(x, y), \end{aligned} \tag{5.1}$$

where u_i and v_i are 2π -periodic in y . This transformation carries (1.1) into

$$\begin{aligned} \dot{x} &= \varepsilon \bar{f}_1(x) + \varepsilon^2 \bar{f}_2(x) + \varepsilon^3 R_1(x, y; \varepsilon), \\ \dot{y} &= x + \varepsilon \bar{g}_1(x) + \varepsilon^2 R_2(x, y; \varepsilon), \end{aligned} \tag{5.2}$$

in the outer region. Here

$$\begin{aligned} \bar{f}_1(x) &= \frac{1}{2\pi} \int_0^{2\pi} f(x, y) dy, \\ \bar{g}_1(x) &= \frac{1}{2\pi} \int_0^{2\pi} u_1(x, y) dy, \\ \bar{f}_2(x) &= \frac{1}{2\pi} \int_0^{2\pi} [u_1(x, y) f_x(x, y) + v_1(x, y) f_y(x, y) \\ &\quad - u_{1y}(x, y) \bar{g}_1(x) - u_{1x}(x, y) \bar{f}_1(x)] dy. \end{aligned}$$

We have the following estimates for R_1 and R_2 :

$$\begin{aligned} R_1 &= \mathcal{O}\left(\frac{1}{|x|^4}\right), \\ R_2 &= \mathcal{O}\left(\frac{1}{|x|^3}\right). \end{aligned} \tag{5.3}$$

as $|x| \rightarrow 0$, uniformly for all y and for $0 \leq \varepsilon \leq \varepsilon_0$.

Our first task is to establish (5.3). From there we show that the error estimate for our patched approximate solution in the outer region is

$$\begin{aligned} |r - \rho| &= \mathcal{O}(\varepsilon^{7/12}) = o(1), \\ |\theta - \psi| &\neq o(1). \end{aligned} \quad (5.4)$$

Let u_i in (5.1) be for the moment an arbitrary smooth function. Differentiating (5.1) with respect to time and using (1.1), it is easy to see that

$$\begin{aligned} \dot{x} &= \frac{\varepsilon f(x + \varepsilon u_1 + \varepsilon^2 u_2, y + \varepsilon v_1) - \varepsilon(u_{1y} + \varepsilon u_{2y})\dot{y}}{1 + \varepsilon u_{1x} + \varepsilon^2 u_{2x}} \\ &= \varepsilon f_1(x, y) + \varepsilon^2 f_2(x, y) + \varepsilon \frac{h_1(x, y; \varepsilon)}{1 + \varepsilon u_{1x} + \varepsilon^2 u_{2x}}, \end{aligned} \quad (5.5)$$

with

$$\begin{aligned} f_1(x, y) &= f(x, y) - x u_{1y}, \\ f_2(x, y) &= u_1 f_r + v_1 f_\theta - x u_{2y} - \bar{g}_1 u_{1y} - u_{1x} \bar{f}_1, \\ h_1(x, y; \varepsilon) &= f(x + \varepsilon u_1 + \varepsilon^2 u_2, y + \varepsilon v_1) - f(x, y) - \varepsilon [u_1 f_r + v_1 f_\theta] \\ &\quad - \varepsilon^2 [u_{2x} \bar{f}_1 + u_{1x} \bar{f}_2 + u_{2y} \bar{g}_1 + u_{1y} R_2] - \varepsilon^3 u_{2y} R_2. \end{aligned}$$

In order to make the first two terms of the second line of (5.5) equal to $\varepsilon \bar{f}_1(x)$ and $\varepsilon^2 \bar{f}_2(x)$ respectively, we should take

$$u_{1y} = \frac{f(x, y) - \bar{f}_1(x)}{x}, \quad (5.6)$$

and

$$u_{2y} = \frac{u_1(x, y) f_r(x, y) + v_1(x, y) f_\theta(x, y) - \bar{g}_1(x) u_{1y}(x, y) - u_{1x}(x, y) \bar{f}_1(x) - \bar{f}_2(x)}{x}. \quad (5.7)$$

Similarly, we can obtain

$$\begin{aligned} \dot{y} &= \frac{x + \varepsilon u_1 + \varepsilon^2 u_2 - \varepsilon v_{1x} \dot{x}}{1 + \varepsilon v_{1y}} \\ &= x + \varepsilon g_1(x, y) + \varepsilon \frac{h_2(x, y; \varepsilon)}{1 + \varepsilon v_{1y}}. \end{aligned} \quad (5.8)$$

Here

$$\begin{aligned} g_1(x, y) &= u_1(x, y) - xv_{1y}(x, y), \\ h_2(x, y; \varepsilon) &= \varepsilon [u_2 - v_{1x}\bar{f}_1 - g_1v_{1y}] - \varepsilon^2 v_{1x}\bar{f}_2 - \varepsilon^3 v_{1x}R_1, \end{aligned}$$

and

$$v_{1y} = \frac{u_1(x, y) - \bar{g}_1(x)}{x}. \quad (5.9)$$

Since the right-hand side of (5.6) has zero mean, its integral (with respect to y) is periodic, hence bounded in y for fixed x , so that $u_1(x, y)$ exists and can be written as

$$u_1(x, y) = \tilde{u}_1(x, y)/x,$$

where \tilde{u}_1 is bounded. From this and from the quotient rule we find

$$\begin{aligned} u_1(x, y) &= \mathcal{O}\left(\frac{1}{|x|}\right) = \mathcal{O}\left(\frac{1}{\delta}\right), \\ u_{1y}(x, y) &= \mathcal{O}\left(\frac{1}{|x|}\right) = \mathcal{O}\left(\frac{1}{\delta}\right), \\ u_{1x}(x, y) &= \mathcal{O}\left(\frac{1}{|x|^2}\right) = \mathcal{O}\left(\frac{1}{\delta^2}\right). \end{aligned} \quad (5.10)$$

Here the big-oh symbols involving $|x|$ hold uniformly in y as $x \rightarrow 0$, and the big-oh symbols involving δ hold uniformly for x and y in any ε -dependent region of the form $|x| \geq k\delta(\varepsilon)$, as $\varepsilon \rightarrow 0$. (A specific value of k will be chosen below.)

Similarly, we integrate (5.9), using the definition of \bar{g}_1 , to obtain v_1 , then integrate (5.7), using all of the previous definitions, to obtain u_2 . From this we obtain the estimates

$$\begin{aligned} u_2(x, y) &= \mathcal{O}\left(\frac{1}{|x|^3}\right) = \mathcal{O}\left(\frac{1}{\delta^3}\right), \\ u_{2y}(x, y) &= \mathcal{O}\left(\frac{1}{|x|^3}\right) = \mathcal{O}\left(\frac{1}{\delta^3}\right), \\ u_{2x}(x, y) &= \mathcal{O}\left(\frac{1}{|x|^4}\right) = \mathcal{O}\left(\frac{1}{\delta^4}\right), \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} v_1(x, y) &= \mathcal{O}\left(\frac{1}{|x|^2}\right) = \mathcal{O}\left(\frac{1}{\delta^2}\right), \\ v_{1y}(x, y) &= \mathcal{O}\left(\frac{1}{|x|^2}\right) = \mathcal{O}\left(\frac{1}{\delta^2}\right), \\ v_{1x}(x, y) &= \mathcal{O}\left(\frac{1}{|x|^3}\right) = \mathcal{O}\left(\frac{1}{\delta^3}\right), \end{aligned} \tag{5.12}$$

in a region $|x| \geq k\delta$. Equations (5.10) and (5.11) imply that $1 + \varepsilon u_{1x} + \varepsilon^2 u_{2x}$ is bounded away from zero in $|x| \geq k\delta$, and it follows that (5.5) is well-defined and also the asymptotic order of the last term in (5.5) is the same as that of εh_1 . Since the order of εh_1 is $\mathcal{O}(\varepsilon^3/x^4)$, this proves that the first line of (5.3). Similarly, we can show the second line of (5.3).

Our aim now is to compare solutions of the following systems during the time interval $t_2 \leq t \leq t_3$:

$$\begin{aligned} \dot{r} &= \varepsilon f(r, \theta), \\ \dot{\theta} &= r, \end{aligned} \tag{5.13}$$

$$\begin{aligned} \dot{x} &= \varepsilon \bar{f}_1(x) + \varepsilon^2 \bar{f}_2(x) + \varepsilon^3 R_1(x, y; \varepsilon), \\ \dot{y} &= x + \varepsilon \bar{g}_1(x) + \varepsilon^2 R_2(x, y; \varepsilon), \end{aligned} \tag{5.14}$$

$$\begin{aligned} \dot{\rho} &= \varepsilon \bar{f}_1(\rho) + \varepsilon^2 \bar{f}_2(\rho), \\ \dot{\psi} &= \rho + \varepsilon \bar{g}_1(\rho). \end{aligned} \tag{5.15}$$

The initial condition at time t_2 for (5.15) lies on $\rho = -\delta$ (see the definition of our patched approximate solutions, (1.5)), and the initial condition for (5.13) is related to it by (4.35). That is,

$$|r(t_2) - \rho(t_2)| = \mathcal{O}\left(\frac{\delta^{11}}{\varepsilon^4}\right). \tag{5.16}$$

(throughout this discussion, r and ρ are solution families depending on ε although this dependence is omitted in the notation.) The initial condition for (5.14) is obtained from that for (5.13) via (5.1). That is,

$$r(t_2) = x(t_2) + \mathcal{O}\left(\frac{\varepsilon}{\delta}\right). \quad (5.17)$$

Thus, one has

$$x(t_2) = \rho(t_2) + \mathcal{O}\left(\frac{\delta^{11}}{\varepsilon^4}\right) + \mathcal{O}\left(\frac{\varepsilon}{\delta}\right) = \mathcal{O}(\delta). \quad (5.18)$$

That is, there is a constant k such that $x(t_2) \leq -k\delta$ for all solutions. This is the value of k to be used in the previous argument establishing (5.10), (5.11) and (5.12). Furthermore (1.4) and (5.14) imply that for small enough ε , \dot{x} is negative. Therefore, $x(t) \leq -k\delta$ for $t \geq t_2$. Finally, we want to show (5.4). Equation (5.18) implies

$$|x(t_2) - \rho(t_2)| = \mathcal{O}\left(\frac{\delta^{11}}{\varepsilon^4}\right) + \mathcal{O}\left(\frac{\varepsilon}{\delta}\right).$$

Thus, choosing $\delta = \varepsilon^{5/12}$, one has

$$|x(t_2) - \rho(t_2)| = \mathcal{O}\left(\varepsilon^{7/12}\right). \quad (5.19)$$

On the other hand, using (1.1), one has

$$t_3 - t_2 = \mathcal{O}\left(\frac{1}{\varepsilon}\right). \quad (5.20)$$

(1.4) and (5.15) imply that

$$\dot{\rho} = \varepsilon \bar{f}_1(\rho) + \varepsilon^2 \bar{f}_2(\rho) \leq -\varepsilon K, \quad (5.21)$$

for some positive constant K and $0 \leq \varepsilon \leq \varepsilon_0$. Integrating (5.21) from t_2 to t with $t \leq t_3$ using $\rho(t_2) = -\delta$ one has

$$\rho(t) \leq -\delta - \varepsilon K(t - t_2).$$

Thus, for $n \geq 2$, one has

$$\begin{aligned}
 \int_{t_2}^{t_3} \frac{\varepsilon dt}{|\rho(t)|^n} &\leq \int_{t_2}^{t_3} \frac{\varepsilon dt}{[\delta + \varepsilon K(t - t_2)]^n} \\
 &= \frac{1}{K(-n+1)} [\delta + \varepsilon K(t - t_2)]^{-n+1} \Big|_{t_2}^{t_3} \\
 &= \frac{1}{K(n-1)} \left\{ \delta^{-n+1} - [\delta + \varepsilon K(t_3 - t_2)]^{-n+1} \right\} \\
 &= \mathcal{O}\left(\frac{1}{\delta^{n-1}}\right).
 \end{aligned} \tag{5.22}$$

(Note that $\delta + \varepsilon K(t_3 - t_2) = \mathcal{O}(1)$ since $t_3 - t_2 = \mathcal{O}(1/\varepsilon)$.)

Equation (5.19) implies

$$x(t_2) = \rho(t_2) + o(1) \text{ as } \varepsilon \downarrow 0$$

and hence

$$\frac{1}{|x(t_2)|} < \frac{2}{|\rho(t_2)|}. \tag{5.23}$$

for sufficiently small ε . Thus, there is a nonempty interval $[t_2, t_2 + \tau^*)$, for some $\tau^* \in (0, \frac{T}{\varepsilon}]$, such that

$$\frac{1}{|x(t)|} < \frac{2}{|\rho(t)|}, \quad t \in [t_2, t_2 + \tau^*) \tag{5.24}$$

in view of the continuity of x and ρ and (5.23). Thus, for any ζ between x and ρ , we have

$$\frac{1}{|\zeta(t)|} < \frac{2}{|\rho(t)|}, \quad t \in [t_2, t_2 + \tau^*). \tag{5.25}$$

Using the differential equations (5.14), (5.15) and the mean-value theorem, we

obtain, for any $t \in [t_2, t_2 + \tau^*)$,

$$\begin{aligned}
|x(t) - \rho(t)| &\leq |x(t_2) - \rho(t_2)| + \varepsilon \int_{t_2}^t |\bar{f}_1(x(\tau)) - \bar{f}_1(\rho(\tau))| d\tau \\
&\quad + \varepsilon^2 \int_{t_2}^t |\bar{f}_2(x(\tau)) - \bar{f}_2(\rho(\tau))| d\tau \\
&\quad + \varepsilon^3 \int_{t_2}^t |R_1(x(\tau), y(\tau); \varepsilon)| d\tau \\
&\leq |x(t_2) - \rho(t_2)| + \varepsilon \int_{t_2}^t |\bar{f}'_1(\zeta_1(\tau))| |x(\tau) - \rho(\tau)| d\tau \\
&\quad + \varepsilon^2 \int_{t_2}^t |\bar{f}'_2(\zeta_2(\tau))| |x(\tau) - \rho(\tau)| d\tau \\
&\quad + \varepsilon^3 \int_{t_2}^t |R_1(\rho(\tau), y(\tau); \varepsilon)| d\tau \\
&\quad + \varepsilon^3 \int_{t_2}^t |R_{1x}(\zeta_3(\tau), y(\tau); \varepsilon)| |x(\tau) - \rho(\tau)| d\tau.
\end{aligned}$$

Here ζ_1 , ζ_2 and ζ_3 are between x and ρ .

Differentiating \bar{f}_1 , \bar{f}_2 and R_1 with respect to x and using (5.10), (5.11) and (5.12), we obtain

$$\begin{aligned}
|\bar{f}'_1(x)| &= \mathcal{O}(1), \\
|\bar{f}'_2(x)| &= \mathcal{O}\left(\frac{1}{|x|^3}\right), \\
|R_{1x}(x)| &= \mathcal{O}\left(\frac{1}{|x|^5}\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|x(t) - \rho(t)| &\leq |x(t_2) - \rho(t_2)| + \varepsilon \int_{t_2}^t L \left\{ 1 + \frac{\varepsilon}{|\zeta_2^3(\tau)|} + \frac{\varepsilon^2}{|\zeta_3^5(\tau)|} \right\} |x(\tau) - \rho(\tau)| d\tau \\
&\quad + \varepsilon^2 \int_{t_2}^t \frac{L\varepsilon}{|\rho(\tau)|^4} d\tau,
\end{aligned} \tag{5.26}$$

for some suitable constant L . Applying (5.25) to (5.26) we see that

$$\begin{aligned}
|x(t) - \rho(t)| &\leq |x(t_2) - \rho(t_2)| + \varepsilon \int_{t_2}^t L \left\{ 1 + \frac{8\varepsilon}{|\rho^3(\tau)|} + \frac{32\varepsilon^2}{|\rho^5(\tau)|} \right\} |x(\tau) - \rho(\tau)| d\tau \\
&\quad + \varepsilon^2 \int_{t_2}^t \frac{L\varepsilon}{|\rho(\tau)|^4} d\tau, \quad t \in [t_2, t_2 + \tau^*].
\end{aligned}$$

Thus, using the Gronwall lemma, one has

$$\begin{aligned}
|x(t) - \rho(t)| &\leq \left[|x(t_2) - \rho(t_2)| + \varepsilon^2 \int_{t_2}^t \frac{L\varepsilon}{|\rho(\tau)|^4} d\tau \right] \\
&\quad \times e^{\varepsilon \int_{t_2}^t L \left\{ 1 + \frac{8\varepsilon}{|\rho^3(\tau)|} + \frac{32\varepsilon^2}{|\rho^5(\tau)|} \right\} d\tau}
\end{aligned} \tag{5.27}$$

Applying (5.22) to (5.27) we obtain, $t \in [t_2, t_2 + \tau^*]$,

$$\begin{aligned}
|x(t) - \rho(t)| &\leq \left[|x(t_2) - \rho(t_2)| + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^3}\right) \right] e^{\mathcal{O}(1) + \mathcal{O}\left(\frac{\varepsilon}{\delta^2}\right) + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^4}\right)} \\
&\leq |x(t_2) - \rho(t_2)| + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^3}\right),
\end{aligned} \tag{5.28}$$

in view of $\mathcal{O}(1) + \mathcal{O}(\varepsilon/\delta^2) + \mathcal{O}(\varepsilon^2/\delta^4) = \mathcal{O}(1)$. Suppose there is some ζ_0 between x and ρ , some ε in $0 \leq \varepsilon \leq \varepsilon_0$ and some τ^{**} in $[\tau^*, \frac{T}{\varepsilon}]$ such that

$$\frac{1}{|\zeta_0(t_2 + \tau^{**})|} = \frac{2}{|\rho(t_2 + \tau^{**})|}. \tag{5.29}$$

Thus (5.29) can be written as

$$\frac{2}{|\rho(t_2 + \tau^{**})|} = \frac{1}{|\rho(t_2 + \tau^{**})|} + \frac{1}{|\zeta_0(t_2 + \tau^{**})|} - \frac{1}{|\rho(t_2 + \tau^{**})|}.$$

Thus, using the triangle inequality,

$$\begin{aligned} \frac{2}{|\rho(t_2 + \tau^{**})|} &\leq \frac{1}{|\rho(t_2 + \tau^{**})|} + \left| \frac{1}{\zeta_0(t_2 + \tau^{**})} - \frac{1}{\rho(t_2 + \tau^{**})} \right| \\ &\leq \frac{1}{|\rho(t_2 + \tau^{**})|} + \frac{|\rho(t_2 + \tau^{**}) - \zeta_0(t_2 + \tau^{**})|}{|\zeta_0(t_2 + \tau^{**})| |\rho(t_2 + \tau^{**})|}. \end{aligned} \quad (5.30)$$

Equation (5.29) implies

$$\zeta_0(t_2 + \tau^{**}) = \frac{1}{2}\rho(t_2 + \tau^{**}). \quad (5.31)$$

Applying (5.31) to (5.30) we see that

$$\begin{aligned} \frac{2}{|\rho(t_2 + \tau^{**})|} &\leq \frac{1}{|\rho(t_2 + \tau^{**})|} + \frac{|\rho(t_2 + \tau^{**}) - \zeta_0(t_2 + \tau^{**})|}{\frac{1}{2}|\rho(t_2 + \tau^{**})|^2} \\ &\leq \frac{1}{|\rho(t_2 + \tau^{**})|} + 2 \frac{|\rho(t_2 + \tau^{**}) - x(t_2 + \tau^{**})|}{|\rho(t_2 + \tau^{**})|^2}, \end{aligned}$$

in view of $\zeta_0 \in [\rho, x]$ or $[x, \rho]$. Thus using (5.28),

$$\frac{2}{|\rho(t_2 + \tau^{**})|} \leq \frac{1}{|\rho(t_2 + \tau^{**})|} \left(1 + 2 \frac{|x(t_2) - \rho(t_2)| + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^3}\right)}{|\rho(t_2 + \tau^{**})|} \right).$$

Since $|\rho(t_2 + \tau^{**})| \geq \delta$, $|x(t_2) - \rho(t_2)| = \mathcal{O}(\varepsilon^{7/12})$, and $\delta = \varepsilon^{5/12}$, we obtain

$$\begin{aligned} \frac{2}{|\rho(t_2 + \tau^{**})|} &\leq \frac{1}{|\rho(t_2 + \tau^{**})|} \left(1 + \frac{\mathcal{O}(\varepsilon^{7/12})}{\varepsilon^{5/12}} \right) \\ &= \frac{1}{|\rho(t_2 + \tau^{**})|} \left(1 + \mathcal{O}(\varepsilon^{1/6}) \right) \\ &= \frac{1}{|\rho(t_2 + \tau^{**})|} (1 + o(1)) \quad \text{as } \varepsilon \downarrow 0, \end{aligned}$$

which contradicts the existence of ζ_0 and τ^{**} . That is, for any ζ between ρ and x we have

$$\frac{1}{|\zeta(t)|} < \frac{2}{|\rho(t)|},$$

whenever $t \in [t_2, t_2 + \tau^{**}]$ with $0 < \varepsilon\tau^{**} \leq L$. Thus (5.28) is valid in time interval $[t_2, t_3]$. That is, for $t \in [t_2, t_3]$,

$$|x(t) - \rho(t)| = |x(t_2) - \rho(t_2)| + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^3}\right) = \mathcal{O}(\varepsilon^{7/12}), \quad (5.32)$$

in view of (5.19). Using (5.10), (5.11) and (5.1) one has

$$|r(t) - x(t)| = \mathcal{O}\left(\frac{\varepsilon}{\delta}\right) + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^3}\right) = \mathcal{O}(\varepsilon^{7/12}). \quad (5.33)$$

Using the triangle inequality, (5.32) and (5.33) we have

$$|r(t) - \rho(t)| = \mathcal{O}(\varepsilon^{7/12}) = o(1). \quad (5.34)$$

in the outer region, proving the first line of (5.4). The remainder of the argument is very standard. One has

$$\frac{d}{dt}|\theta - \psi| \leq |r - \rho| + \varepsilon|\bar{g}_1(\rho)| = \mathcal{O}(\varepsilon^{7/12}) + \mathcal{O}\left(\frac{\varepsilon}{\delta}\right) = \mathcal{O}(\varepsilon^{7/12}), \quad (5.35)$$

in view of $|\bar{g}_1(\rho)| = \mathcal{O}(1/\delta)$. Integrating this from t_2 to $t \leq t_3$ one has, using (4.35),

$$\begin{aligned} |\theta(t) - \psi(t)| &\leq |\theta(t_2) - \psi(t_2)| + \mathcal{O}(\varepsilon^{7/12})(t - t_2) \\ &= \mathcal{O}\left(\frac{\delta^{10}}{\varepsilon^4}\right) + \mathcal{O}(\varepsilon^{-5/12}) \\ &= \mathcal{O}(\varepsilon^{-5/12}) \\ &\neq o(1). \end{aligned}$$

Remark.

- a) We can also use (5.22) together with the same idea as above, applied to the first order system to improve the accuracy of $|r(t) - \rho(t)|$ from $\mathcal{O}(\varepsilon^{1/7})$ to $\mathcal{O}(\varepsilon^{4/7})$. The former is the accuracy achieved in reference [2].

- b) Of course, we have not obtained a shadowing result in the θ coordinate. However, we improve the accuracy of $|\theta(t) - \psi(t)|$ from $\mathcal{O}(\varepsilon^{-6/7})$ (in [2]) to $\mathcal{O}(\varepsilon^{-5/12})$.
- c) In general, one would expect to use $\tilde{r} = \rho + \varepsilon u_1(\rho, \psi) + \varepsilon^2 u_2(\rho, \psi)$ as an approximate solution of the exact system (5.13). However, $|r(t) - \tilde{r}(t)| = \mathcal{O}(\varepsilon^{7/12}) = |r(t) - \rho(t)|$, that is, using \tilde{r} in place of ρ does not increase the accuracy of the approximation in our case. This is the reason for choosing $\rho(t)$ as an approximation of the exact system (5.13) in this paper.
- d) To obtain (5.34) it is not necessary to include the v_1 term in (5.1). This was included in an attempt to obtain a shadowing estimate for θ . We used v_1 in this presentation in order to show that it improves the θ estimate slightly but not enough to obtain $o(1)$.

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