MINIMUM RANK AND MAXIMUM EIGENVALUE MULTIPLICITY OF SYMMETRIC TREE SIGN PATTERNS

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Abstract. The set of real matrices described by a sign pattern (a square matrix whose entries are elements of $\{+,-,0\}$) has been studied extensively. A simple graph has been associated with the set of symmetric matrices having a zero-nonzero pattern of off-diagonal entries described by the graph. In this paper, we present a unified approach to the study of the set of symmetric matrices described by a sign pattern and the set of matrices associated with a graph allowing loops, with the presence or absence of loops describing the zero-nonzero pattern of the diagonal. We call any family of matrices having a common graph a cohort. For a cohort whose graph is a tree, we provide an algorithm for the calculation of the maximum of the multiplicities of eigenvalues of any matrix in the cohort. For a symmetric tree sign pattern or tree that allows loops, this algorithm allows exact computation of maximum multiplicity and minimum rank, and can be used to obtain a symmetric integer matrix realizing minimum rank.

Key words. Sign pattern matrix, symmetric tree sign pattern, minimum rank, maximum multiplicity, tree, graph, cohort.

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1. Introduction. Much of the material we present is inspired by previous work in two somewhat different but related areas: sign patterns of matrices, and graphs of matrices.

Sign pattern matrices have many important applications; in fact, the study of sign patterns arose more than fifty years ago in economics. Brualdi and Shader [4] provide a thorough mathematical treatment of sign patterns through 1995. For a current survey with extensive bibliography, see Hall and Li [8].

Recently there has been substantial interest in minimum rank and the related question of the maximal multiplicity of an eigenvalue for sign patterns, e.g., [5], [7]. In addition, many other papers concerning related parameters of sign patterns, such as inertia [9], rank [10], diagonalizability [17], etc. have appeared. In the last ten years there have been numerous papers on minimum rank and multiplicities of eigenvalues for symmetric matrices associated with a graph, e.g., [13], [3], [1], [2]. There are similarities in techniques and results in the study of sign patterns and matrices of graphs, but also important differences, caused by the issue of what set of matrices is associated with a graph or a sign pattern.

We unify these two approaches and apply results from graphs to sign patterns. In Section 1, we introduce this new approach and terminology, and in Section 2, we discuss the use of permutation digraphs to determine singularity. The remaining sections deal exclusively with the case in which the graph is a tree or forest. In Section 3, we specialize and extend the results of the previous sections. The Parter-Wiener Theorem and parameters related to the maximum multiplicity of an eigenvalue are discussed in Section 4. In Section 5, we present an algorithm that allows computation of minimum rank and maximum multiplicity of an eigenvalue for symmetric tree sign patterns, and in Section 7, we show how to use the algorithm to obtain a symmetric integer matrix of minimum

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rank. Section 6 contains some technical results about paths. The main results are summarized in Section 8.

We begin by introducing some terminology needed for our unified approach. Let $\mathcal{N} = \{1, \ldots, n\}$. An $n \times n$ matrix $B = [b_{ij}],\ i, j \in \mathcal{N}$ can be described in a natural way as being indexed by $\mathcal{N}$. Every matrix discussed in this paper is real and square. Because we will be extracting submatrices of submatrices, and because we will be associating principal submatrices with induced subgraphs, we will need to retain information about the original row and column indices. Thus we explicitly attach the index set to the matrix.

An *index set* is a finite set of positive integers. We require every matrix B to have an index set, denoted $\iota(B)$ and denote the entries of B by these indices. If B has index set $\iota(B)$, then B is an $|\iota(B)| \times |\iota(B)|$ matrix, $B = [b_{ij}]$ with $i, j \in \iota(B)$ and B is written as a square array using the natural order of the indices. The standard index set for an $n \times n$ matrix is $\mathcal{N}$, and this is used for an ordinary matrix (that does not arise from a graph or as a principal submatrix).

Matrix functions, such as the rank and the spectrum of B are computed ignoring the index set (here the spectrum $\sigma(B)$ is the multiset of roots of the characteristic polynomial). We will use the definition of the determinant in terms of permutations, with the permutations acting on the index set; this results in the same value of the determinant as obtained by ignoring the index set and evaluating as usual.

If B is a matrix and $R \subseteq \iota(B)$, define the *principal submatrix* $B[R]$ to be the submatrix of B lying in rows and columns that have indices in R, together with the index set R. This definition has the desirable feature that if $R \subseteq Q \subseteq \iota(B)$, $B[Q][R] = B[R]$; note that this is not true in the traditional definition (where the index set is ignored), as a principal submatrix lying in rows and columns 2 and 3 is implicitly reindexed as the 2×2 matrix with rows and columns 1 and 2, so $B[\{2,3\}][\{2\}]$ would yield the 3,3-entry of B, rather than the 2,2-entry. We also define $B(R)$ to be the principal submatrix obtained from B by deleting from B all rows and columns with indices in R, with $\iota(B(R)) = \overline{R}$, where $\overline{R} = \iota(B) - R$. Equivalently, $B(R) = B[\,\overline{R}\,]$. If R and Q are disjoint subsets of $\iota(B)$, then $B(R)(Q) = B(R \cup Q)$. When $\{k\}$ is a singleton set, we use $B(k)$ to denote $B(\{k\})$.

A *sign pattern matrix* (sign pattern for short) is a square matrix $Z = [z_{ij}]$ whose entries z_{ij} are elements of $\{+, -, 0\}$, with index set $\iota(Z)$. For Z a sign pattern and $R \subseteq \iota(Z)$, define the *principal subpattern* $Z[R]$ to be the subpattern of Z lying in rows and columns that have indices in R, together with the index set R. Define $Z(R) = Z[\,\overline{R}\,]$; when $\{k\}$ is a singleton set, $Z(\{k\})$ is denoted $Z(k)$.

For a real number b, the *sign* of b, denoted $\operatorname{sgn}(b)$, is $+, -, 0$ according as $b > 0, b < 0, b = 0$. For B a matrix, define $\mathcal{Z}^\ell(B)$ to be the sign pattern matrix with $(\mathcal{Z}^\ell(B))_{ij} = \operatorname{sgn}(b_{ij})$ and $\iota(\mathcal{Z}^\ell(B)) = \iota(B)$. The *qualitative class* of sign pattern Z is

$$\mathcal{Q}^\ell(Z) = \{B : \mathcal{Z}^\ell(B) = Z\}.$$

Note that the traditional notation for the qualitative class of Z is $\mathcal{Q}(Z)$. We have included the superscript because we will be considering both the situation in which the diagonal is restricted and that in which it is free. The "ℓ" comes from loop, as the graphs involved have loops, and indicates the diagonal is restricted.

It is traditional in the study of sign patterns to say that a sign pattern Z *requires* property P if every matrix in $\mathcal{Q}^\ell(Z)$ has property P and to say that Z *allows* property P if there exists a matrix in $\mathcal{Q}^\ell(Z)$ that has property P. In our study of minimum rank, we are interested in sign patterns that allow singularity, or equivalently, that do not require nonsingularity. A sign pattern Z is *sign nonsingular* (SNS) if Z requires nonsingularity, i.e., if every matrix $B \in \mathcal{Q}^\ell(Z)$ is nonsingular.

As usual, an (indexed) matrix is nonsingular if and only if its determinant is nonzero. A sign pattern Z has *signed determinant* if the sign of the determinant of B is the same for every matrix $B \in \mathcal{Q}^\ell(Z)$. Saying that Z has signed determinant 0 is the same as saying Z requires singularity. Many results about sign patterns are known, including the following.

THEOREM 1.1. *(SNS Theorem) [4, pp. 7-8]:*

1. *A sign pattern has signed determinant 0 if and only if the standard determinant expansion has no nonzero terms.*
2. *A sign pattern has signed determinant + if and only if there is a nonzero term in the determinant expansion and every nonzero term is signed + (the sign of the term is the product of the signs of the entries from the pattern and the sign of the permutation).*
3. *A sign pattern has signed determinant – if and only if there is a nonzero term in the determinant expansion and every nonzero term is signed –.*
4. *A sign pattern is* SNS *if and only if there is a nonzero term in the determinant expansion and every nonzero term has the same sign.*
5. *A sign pattern has signed determinant if and only if in the standard determinant expansion either every nonzero term has the same sign or there are no nonzero terms.*

In contrast to the study of sign patterns, the study of matrices associated with a graph has traditionally ignored the diagonal and required the matrices to be symmetric. We will explore the effect of requiring the matrices associated with a sign pattern to be symmetric, and of requiring each diagonal entry of a matrix associated with a graph to be zero or nonzero according to whether the graph has a loop or not.

For our purposes, a *graph* allows loops but does not allow multiple edges. A *simple graph* is a graph that does not have loops. The set of vertices $V(G)$ of G is a finite set of positive integers. An edge of G is an unordered multiset of two vertices of G, denoted vw or vv, and the set of edges of G is denoted $E(G)$. If G is a graph, the *simple graph associated with* G, $\widehat{G}$, is obtained from G by suppressing all loops. We will also use $\widehat{G}$ to denote an arbitrary simple graph. If $R \subseteq V(G)$, $G - R$ is the graph obtained from G be deleting all vertices in R and all edges incident with a vertex in R. An *induced subgraph* of G is a graph of the form $G - R$, and is also denoted $\langle \overline{R} \rangle$ (where $\overline{R} = V(G) - R$). A *component* of a graph G is a maximal connected induced subgraph of G.

A sign pattern Z is *symmetric* if for all $i, j \in \iota(Z)$, $z_{ij} = z_{ji}$. A matrix or sign pattern is *combinatorially symmetric* if for all i, j in the index set, either the i, j and j, i entries are both nonzero, or they are both 0. Let B be a combinatorially symmetric matrix and let Z be a combinatorially symmetric sign pattern. Then, we define

- $\mathcal{G}^\ell(B)$ to be the graph with vertices $\iota(B)$ such that ij is an edge of $\mathcal{G}^\ell(B)$ if and only if $b_{ij} \neq 0$.
- $\mathcal{G}(B)$ to be the simple graph with vertices $\iota(B)$ such that ij is an edge of $\mathcal{G}(B)$ if and only if $i \neq j$ and $b_{ij} \neq 0$. Note the diagonal is ignored.
- $\mathcal{G}^\ell(Z)$ to be the graph with vertices $\iota(Z)$ such that ij is an edge of $\mathcal{G}^\ell(Z)$ if and only if $z_{ij} \neq 0$.
- $\mathcal{G}(Z)$ to be the simple graph with vertices $\iota(Z)$ such that ij is an edge of $\mathcal{G}(Z)$ if and only if $i \neq j$ and $z_{ij} \neq 0$. Note the diagonal is ignored.

EXAMPLE 1.2. Let $A = \begin{bmatrix} 0 & 1 & -1 & 3 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 3 & 0 & 0 & -6 \end{bmatrix}$ with $\iota(A) = \{1, 2, 3, 4\}$. Then $\mathcal{Z}^\ell(A) =$

$\begin{bmatrix} 0 & + & - & + \\ + & + & 0 & 0 \\ - & 0 & + & 0 \\ + & 0 & 0 & - \end{bmatrix}$. The graph of A, $\mathcal{G}^\ell(A)$, is shown in Figure 1.1. We will call this graph G_1; we refer to it later.

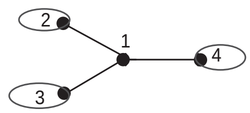

FIG. 1.1. $G_1 = \mathcal{G}^\ell(A)$

Let Z be a symmetric sign pattern, G a graph, and $\widehat{G}$ a simple graph. Then we define

- $\mathcal{Q}^\ell(G) = \{B : B$ is a combinatorially symmetric matrix and $\mathcal{G}^\ell(B) = G\}$.
- $\mathcal{S}^\ell(G) = \{A : A$ is a symmetric matrix and $\mathcal{G}^\ell(A) = G\}$.
- $\mathcal{S}^\ell(Z) = \{A : A$ is a symmetric matrix and $\mathcal{Z}^\ell(A) = Z\}$.
- $\mathcal{S}(\widehat{G}) = \{A : A$ is a symmetric matrix and $\mathcal{G}(A) = \widehat{G}\}$.

Recall that $\mathcal{Q}^\ell(Z)$ has already been defined. $\mathcal{S}(\widehat{G})$ is the traditional class of symmetric matrices associated with a simple graph. We could also define $\mathcal{Q}(\widehat{G})$ analogously, but will not have occasion to use this set ($\mathcal{S}(\widehat{G})$ is defined here primarily to discuss its connection with the literature).

OBSERVATION 1.3. *Let Z be a symmetric sign pattern and A a symmetric matrix. The following statements are clear.*

1. $\mathcal{G}^\ell(\mathcal{Z}^\ell(A)) = \mathcal{G}^\ell(A)$.
2. $\mathcal{S}^\ell(Z) \subseteq \mathcal{S}^\ell(\mathcal{G}^\ell(Z))$.
3. $\mathcal{S}^\ell(Z) \subseteq \mathcal{Q}^\ell(Z)$.
4. $\mathcal{S}^\ell(G) \subseteq \mathcal{Q}^\ell(G)$.

A symmetric sign pattern *allows symmetric singularity* if there is a matrix $A \in \mathcal{S}^\ell(Z)$ that is singular.

OBSERVATION 1.4. *Let Z be a symmetric sign pattern. If Z allows symmetric singularity, then Z has signed determinant 0 or has terms of opposite sign.*

The distinction between $\mathcal{S}^\ell(Z)$ and $Q^\ell(Z)$ can have a significant effect on minimum rank and sign nonsingularity, as noted in [9], and illustrated in the example below.

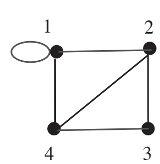

FIG. 1.2. $G_2 = \mathcal{G}^\ell(Z)$

EXAMPLE 1.5.
Let $Z = \begin{bmatrix} + & + & 0 & + \\ + & 0 & + & + \\ 0 & + & 0 & + \\ + & + & + & 0 \end{bmatrix}$ with $\iota(Z) = \{1, 2, 3, 4\}$. Let $A = \begin{bmatrix} d_1 & a_{12} & 0 & a_{14} \\ a_{12} & 0 & a_{23} & a_{24} \\ 0 & a_{23} & 0 & a_{34} \\ a_{14} & a_{24} & a_{34} & 0 \end{bmatrix} \in \mathcal{S}^\ell(Z)$,

where $a_{ij}, d_1 > 0$. The graph of Z is shown in Figure 1.2. We call this graph G_2; we refer to it later. Then $\det A = a_{14}^2 a_{23}^2 - 2a_{12}a_{23}a_{34}a_{14} + a_{12}^2 a_{34}^2 + 2d_1 a_{23}a_{34}a_{24} = (a_{14}a_{23} - a_{12}a_{34})^2 + 2d_1 a_{23}a_{34}a_{24} > 0$, so every symmetric matrix having sign pattern Z has positive determinant, and Z requires symmetric sign nonsingularity, but the determinant expansion has both positive and negative terms. Note that if the sign pattern Z_- is obtained from Z by replacing the $1,1$-entry by $-$, then $\mathcal{G}^\ell(Z_-) = G_2$, but Z_- allows symmetric singularity.

In order to prove results for families of matrices associated with both sign patterns and graphs, we note that in both situations we are studying a set of matrices, and all the matrices in the set under examination have the same simple graph (in fact, both have the same graph, but it is the simple graph that we will use in Section 5). The sets $\mathcal{Q}^\ell(Z)$, $\mathcal{Q}^\ell(G)$, $\mathcal{S}^\ell(Z)$ and $\mathcal{S}^\ell(G)$ have the additional property that we can assemble disjoint principal submatrices arising from distinct matrices into one matrix in the set. We introduce some terminology to allow us to simultaneously discuss these families of matrices.

Let V be a finite set of positive integers and let $\mathcal{Q}_V$ denote the set of all real combinatorially symmetric matrices having index set V. A *semicohort* K is a nonempty subset of $\mathcal{Q}_V$ such that for all $B_1, B_2 \in K$, $\mathcal{G}(B_1) = \mathcal{G}(B_2)$. The *index set* of K is V; this is denoted by $\iota(K) = V$. Let K be a semicohort. Since every matrix in K has the same simple graph, we can define this graph to be the *simple graph of* K, $\mathcal{G}(K) = \mathcal{G}(B)$ for $B \in K$. Let $R \subseteq \iota(K)$, and $i \in \iota(K)$. Define

$$K[R] = \{B[R] : B \in K\}, \quad K(R) = \{B(R) : B \in K\}, \quad \text{and } K(i) = K(\{i\}).$$

A *cohort* (also called a V*-cohort* if we wish to emphasize V) is a semicohort K satisfying the additional property that for any disjoint subsets $R_1, \ldots, R_h \subseteq \iota(K)$ and matrices $B_1, \ldots, B_h \in K$, there must exist a matrix $B \in K$ such that for all $i = 1, \ldots, h$, $B[R_i] = B_i[R_i]$.

Observation 1.6.

1. *If Z is a symmetric sign pattern, then $\mathcal{Q}^\ell(Z)$ and $\mathcal{S}^\ell(Z)$ are both $\iota(Z)$-cohorts.*
2. *If G is a graph, then $\mathcal{Q}^\ell(G)$ and $\mathcal{S}^\ell(G)$ are both $V(G)$-cohorts and $\mathcal{G}(\mathcal{Q}^\ell(G)) = \mathcal{G}(\mathcal{S}^\ell(G)) = \widehat{G}$.*
3. *If $\widehat{G}$ is a simple graph, then $\mathcal{S}(\widehat{G})$ is a $V(\widehat{G})$-cohort and $\mathcal{G}(\mathcal{S}(\widehat{G})) = \widehat{G}$.*

If K is a cohort and $\langle R \rangle$ is a component of $\mathcal{G}(K)$, then $K[R]$ is called a *component of* K. A component of K is a family of principal submatrices of the matrices in K. A cohort S is a *symmetric cohort* if every matrix in S is symmetric. Note that a symmetric sign pattern gives rise to both symmetric and nonsymmetric cohorts, $\mathcal{S}^\ell(Z)$ and $\mathcal{Q}^\ell(Z)$, and both of these sets of matrices have been studied.

One of the parameters of primary interest in this work is the minimum rank of a set of matrices. Let Z be a symmetric sign pattern. The minimum rank and symmetric minimum rank have been defined, e.g., [5], as $\operatorname{mr}(Z) = \min\{\operatorname{rank} B : B \in \mathcal{Q}^\ell(Z)\}$ and $\operatorname{smr}(Z) = \min\{\operatorname{rank} A : A \in \mathcal{S}^\ell(Z)\}$, respectively. For a simple graph $\widehat{G}$, the (symmetric) minimum rank has been defined, e.g., [3], as $\operatorname{mr}(\widehat{G}) = \min\{\operatorname{rank} A : A \in \mathcal{S}(\widehat{G})\}$.

We will define the minimum rank of a cohort and apply that definition to specific cohorts. If K is a cohort, define the *minimum rank of* K to be

$$\operatorname{mr}(K) = \min\{\operatorname{rank} B : B \in K\}.$$

For a sign pattern Z, the *minimum rank of* Z is

$$\operatorname{mr}^\ell(Z) = \operatorname{mr}(\mathcal{Q}^\ell(Z)),$$

and if Z is symmetric, the *symmetric minimum rank of Z* is

$$\operatorname{smr}^\ell(Z) = \operatorname{mr}(\mathcal{S}^\ell(Z)).$$

For a sign pattern Z, our terms $\operatorname{smr}^\ell(Z)$ and $\operatorname{mr}^\ell(Z)$ mean the same thing as the terms $\operatorname{smr}(Z)$ and $\operatorname{mr}(Z)$ in [5], and as noted there, obviously $\operatorname{smr}^\ell(Z) \geq \operatorname{mr}^\ell(Z)$. For a graph G, the *symmetric minimum rank of G* is

$$\operatorname{smr}^\ell(G) = \operatorname{mr}(\mathcal{S}^\ell(G)).$$

For a simple graph $\widehat{G}$, where no restriction is placed on the diagonal of associated matrices, the *symmetric minimum rank of $\widehat{G}$* is

$$\operatorname{smr}(\widehat{G}) = \operatorname{mr}(\mathcal{S}(\widehat{G})).$$

When discussing multiplicity of an eigenvalue of a real matrix, it is generally necessary to distinguish between algebraic and geometric multiplicity (see for example the discussion in [7]). For symmetric matrices this is unnecessary; the multiplicity of eigenvalue λ for the symmetric matrix A will be denoted by $m_A(\lambda)$. For a symmetric cohort S, define the *maximum multiplicity of λ* to be the maximum multiplicity of eigenvalue λ allowed by S,

$$M_\lambda(S) = \max\{\mathrm{m}_A(\lambda) : A \in S\}.$$

If S is a symmetric cohort and there exists a matrix $A \in S$ such that $\lambda \in \sigma(A)$, then we say S *allows eigenvalue* λ. So, S allows eigenvalue λ if and only if $M_\lambda(S) \geq 1$. If S allows eigenvalue zero then S *allows singularity*. For a symmetric sign pattern Z and real number λ, the *maximum multiplicity of λ for Z* is

$$M_\lambda^\ell(Z) = M_\lambda(\mathcal{S}^\ell(Z)).$$

For a graph G, the the *maximum multiplicity of λ for G* is

$$M_\lambda^\ell(G) = M_\lambda(\mathcal{S}^\ell(G)).$$

For a simple graph $\widehat{G}$, where no restriction is placed on the diagonal of associated matrices, the *maximum multiplicity of λ for $\widehat{G}$* is

$$M_\lambda(\widehat{G}) = M_\lambda(\mathcal{S}(\widehat{G})).$$

Observation 1.7. *If S is a symmetric cohort, then $M_0(S) + \operatorname{mr}(S) = |\iota(S)|$.*

Lemma 1.8. *Let G be a graph, $\widehat{G}$ a simple graph, and Z a symmetric sign pattern. Then,*
1. *$M_\lambda(\widehat{G}) = M_\mu(\widehat{G})$ for any λ, μ,*
2. *$M_\lambda^\ell(G) = M_\mu^\ell(G)$ for any $\lambda \neq 0$ and $\mu \neq 0$,*
3. *$M_\lambda^\ell(Z) = M_\mu^\ell(Z)$ if $\operatorname{sgn}(\lambda) = \operatorname{sgn}(\mu)$.*

Proof. To establish the first statement, note that $A' = A + (\mu - \lambda)I$ has the same simple graph as A, and $m_{A'}(\mu) = m_A(\lambda)$. To establish the third statement, note that if $\lambda \neq 0$, and $\operatorname{sgn}(\lambda) = \operatorname{sgn}(\mu)$, then $A' = \frac{\mu}{\lambda}A$ has the same sign pattern as A, and $m_{A'}(\mu) = m_A(\lambda)$. For the second statement, it is not necessary to assume $\operatorname{sgn}(\lambda) = \operatorname{sgn}(\mu)$. □

In this paper, $M_\lambda^\ell(Z)$ is the maximum multiplicity of eigenvalue λ that is allowed in the symmetric matrices with symmetric sign pattern Z. In [7], algebraic multiplicities of eigenvalues of not necessarily symmetric matrices associated with a sign pattern are studied, and it is established that the only eigenvalue that can require repetition is 0.

LEMMA 1.9. *Let K be a cohort and λ a real number. Let $K[R_i], i = 1, \ldots, k$ be the components of K.*

1. *If K is symmetric, then $M_\lambda(K) = \sum_{i=1}^k M_\lambda(K[R_i])$.*
2. $\mathrm{mr}(K) = \sum_{i=1}^k \mathrm{mr}(K[R_i])$.

Proof. This follows from the fact that $A = \oplus_{i=1}^k A[R_i]$ for $A \in K$, and the additivity of rank and multiplicity of direct summands. □

We now state a well known and powerful tool for understanding eigenvalue multiplicity, the *Interlacing Theorem*, which applies to all real symmetric matrices.

THEOREM 1.10. *(Interlacing Theorem) [12]. If the eigenvalues of a symmetric matrix A are $\lambda_1 \le \lambda_2 \cdots \le \lambda_n$, $k \in \iota(A)$, and the eigenvalues of $A(k)$ are $\mu_1 \le \mu_2 \le \cdots \le \mu_{n-1}$, then $\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \cdots \le \lambda_{n-1} \le \mu_{n-1} \le \lambda_n$.*

We can apply this theorem to cohorts.

COROLLARY 1.11. *(Interlacing Corollary for Cohorts) If S is a symmetric cohort, $R \subseteq \iota(S)$ then*

$$M_\lambda(S) - |R| \le M_\lambda(S(R)) \le M_\lambda(S) + |R|.$$

Proof. We prove that for $k \in \iota(S)$, $M_\lambda(S) - 1 \le M_\lambda(S(k)) \le M_\lambda(S) + 1$, and the more general result follows by repeated application.

Choose $A \in S$ such that $m_A(\lambda) = M_\lambda(S)$. Then $M_\lambda(S(k)) \ge m_{A(k)}(\lambda) \ge \mathrm{m}_A(\lambda) - 1 = M_\lambda(S) - 1$. Choose $A' \in S$ such that $m_{A'(k)}(\lambda) = M_\lambda(S(k))$. Then $M_\lambda(S(k)) = m_{A'(k)}(\lambda) \le \mathrm{m}_{A'}(\lambda) + 1 \le M_\lambda(S) + 1$. □

The following lemma will be used in the study of maximum multiplicity of nonzero eigenvalues in Section 5.

LEMMA 1.12. *Let G be a graph and let Z be a symmetric sign pattern. Let S be the symmetric cohort $\mathcal{S}^\ell(G)$ or $\mathcal{S}^\ell(Z)$.*

1. *If $\mathcal{G}(S)$ has an edge, then S allows any nonzero eigenvalue.*
2. *If G has a loop, then $\mathcal{S}^\ell(G)$ allows any nonzero eigenvalue.*
3. *If Z has a positive (negative) diagonal entry, then $\mathcal{S}^\ell(Z)$ allows any positive (negative) eigenvalue.*

Proof. Suppose $\mathcal{G}(S)$ has edge kj with $k \ne j$. Choose $A \in S$ with $a_{kj} = a_{jk} = 1$ (or $a_{kj} = a_{jk} = -1$) and $a_{kk}, a_{jj} \in \{0, 0.1, -0.1\}$, depending on whether the loop is present (or $z_{kk}, z_{jj} \in \{0, +, -\}$). Then $\det(A[\{k, j\}]) \le -0.99$, so $A[\{k, j\}]$ must have both a positive and a negative eigenvalue. Then, by the Interlacing Theorem, A has both a positive and a negative eigenvalue.

For the second and third statements, apply the Interlacing Theorem to the 1×1 matrix associated with the loop or the correctly signed diagonal entry. □

2. Singularity and Permutation Digraphs. For the study of minimum rank (and maximum multiplicity of the eigenvalue zero), we will need to determine whether a cohort allows singularity. We will study both sign patterns and graphs by means of permutation digraphs. A *digraph* is a directed graph; a digraph allows loops but does not allow multiple edges. A directed edge is called an *arc* and denoted as an ordered pair, (v, w) or (v, v). An induced subdigraph is defined analogously to an induced subgraph. If $v \neq w$, a digraph is permitted to have both of the arcs (v, w) and (w, v), and this pair of arcs is a 2*-cycle*. More generally, the k*-cycle* or *cycle* $(v_1, v_2, \ldots, v_k)$ is the sequence of arcs $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)$ with $v_1, v_2, \ldots, v_{k-1}, v_k$ distinct.

If G is a graph, the *digraph of* G is the digraph $\mathcal{D}(G)$ with $V(\mathcal{D}(G)) = V(G)$ and $E(\mathcal{D}(G)) = \{(i,j),(j,i) : i \neq j \text{ and } ij \in E(G)\} \cup \{(i,i) : ii \in E(G)\}$. If D is a digraph, the *underlying graph of* D is the graph with vertex set $V(D)$ and edge set $\{ij : \text{at least one of } (i,j),(j,i) \in E(D)\}$. The graph G is *acyclic* if $\mathcal{D}(G)$ has no cycles of length three or more.

Let D be a digraph. A digraph P is a *permutation digraph of* D if $V(P) = V(D)$, every arc of P is an arc of D and P is a union of disjoint cycles. A permutation digraph is also called a "composite cycle" [7], but this term is sometimes used to denote the associated product of entries [9]. If G is a graph, a *permutation digraph of* G is a permutation digraph of $\mathcal{D}(G)$. Let P be a permutation digraph of G that is the disjoint union of the cycles $(a_1, \ldots, a_{k_a}), (b_1, \ldots, b_{k_b}), \ldots, (c_1, \ldots, c_{k_c})$. Let π be the permutation of the vertices of G that is the product of these cycles, i.e., $\pi = (a_1, \ldots, a_{k_a})(b_1, \ldots, b_{k_b}) \ldots (c_1, \ldots, c_{k_c})$. Then P is denoted P_π and π is the associated permutation of P_π. Let $\mathrm{Perm}(G)$ denote the set of all permutations π such that P_π is a permutation digraph of G.

Example 2.1. For G_1 shown in Figure 1.1, $\mathrm{Perm}(G_1) = \{(2)(3)(14), (3)(4)(12), (4)(2)(13)\}$.

Observation 2.2. *Let A be a symmetric matrix with index set $\iota(A) = \{i_1, \ldots, i_n\}$. Then the following is obvious:*

$$\mathrm{Det} A = \sum_{\pi \in \mathrm{Perm}(\mathcal{G}^\ell(A))} \mathrm{sgn}(\pi) a_{i_1 \pi(i_1)} \cdots a_{i_n \pi(i_n)},$$

where the sum over the empty set is zero.

Let $Sym(V)$ denote the symmetric group on V, i.e., the group of permutations of V. Let $\pi, \tau \in Sym(V)$. We say π *is equivalent to* τ, denoted $\pi \sim \tau$, if the (disjoint) cycles of π can be placed in one to one correspondence with the (disjoint) cycles of τ such that each cycle is matched to itself or its inverse. Two permutation digraphs are *equivalent* if their associated permutations are equivalent.

Observation 2.3.
1. *$\sim$ is an equivalence relation on $Sym(V)$.*
2. *If $\pi \sim \tau$, then* $\mathrm{sgn}(\pi) = \mathrm{sgn}(\tau)$

For $i \leq j$, let x_{ij} be independent indeterminates. For a symmetric sign pattern Z, let Z_x be the symmetric matrix (with $\iota(Z_x) = \iota(Z)$) such that for $i \leq j$, both the i,j- and j,i-entries of Z_x are equal to $z_{ij} x_{ij}$.

Observation 2.4. *Let Z be a symmetric sign pattern. Then,*

1. *Equivalent permutation digraphs of $\mathcal{G}^\ell(Z)$ contribute identical signed products of x_{ij} to* $\det Z_x$*,*
2. *Nonequivalent permutation digraphs of $\mathcal{G}^\ell(Z)$ contribute distinct products of x_{ij} to* $\det Z_x$*.*

We treat $\det Z_x$ as a polynomial in the x_{ij}'s. A *term* is the integer multiple of a product of x_{ij}'s that results from grouping all identical products of x_{ij}'s together.

EXAMPLE 2.5. The graph G_2 shown in Figure 1.2 has six permutation digraphs: a pair of equivalent 4-cycles, a pair of equivalent permutation digraphs consisting of a 1-cycle and a 3-cycle, and 2 nonequivalent permutation digraphs consisting of two 2-cycles each. So for the sign pattern Z in Example 1.5, $\det Z_x$ is the sum of of the 4 terms $x_{14}^2 x_{23}^2$, $-2x_{12}x_{23}x_{34}x_{14}$, $x_{12}^2 x_{34}^2$, and $2x_{11}x_{23}x_{34}x_{24}$.

OBSERVATION 2.6. *Let G be a graph. The following statements follow from Observations 2.2, 2.3, 2.4, and the SNS Theorem 1.1.*

1. *G has no permutation digraphs if and only if for every sign pattern Z such that $\mathcal{G}^\ell(Z) = G$, all the terms in the standard expansion of the determinant of Z are* 0 *if and only if every such sign pattern Z has signed determinant 0.*
2. *G has, up to equivalence, exactly one permutation digraph if and only if for every symmetric sign pattern Z, such that $\mathcal{G}^\ell(Z) = G$, there is exactly one nonzero term in* $\det Z_x$*. In this case, every such symmetric sign pattern is* SNS*.*
3. *G has at least two nonequivalent permutation digraphs if and only if for every symmetric sign pattern Z, such that $\mathcal{G}^\ell(Z) = G$, there are at least two distinct nonzero terms in* $\det Z_x$*.*

We say the graph G *requires singularity* if B is singular for every $B \in \mathcal{Q}^\ell(G)$ and G *requires symmetric singularity* if A is singular for every $A \in \mathcal{S}^\ell(G)$. The graph G *requires nonsingularity* if B is nonsingular for every $B \in \mathcal{Q}^\ell(G)$. The graph G *requires symmetric nonsingularity* if A is nonsingular for every $A \in \mathcal{S}^\ell(G)$. The graph G is *ambiguous* if there exist $A_1, A_2 \in \mathcal{S}^\ell(G)$ such that A_1 is singular and A_2 is nonsingular. The graph G allows *symmetric singularity* if G requires singularity or is ambiguous.

OBSERVATION 2.7. *Let G be a graph. The following statements follow from Observations 2.2 and 2.3.*

1. *If G has no permutation digraphs, then G requires singularity.*
2. *If G has, up to equivalence, exactly one permutation digraph, then G requires symmetric nonsingularity.*
3. *If G is ambiguous, then G has at least two nonequivalent permutation digraphs.*
4. *If G allows symmetric singularity, then either G has at least two nonequivalent permutation digraphs or G has no permutation digraphs.*

EXAMPLE 2.8. The graph G_1 shown in Figure 1.1 is ambiguous (the matrix A in Example 1.2 is singular, but if any one of the three nonzero diagonal entries is perturbed it will no longer be singular). The graph G_3 shown in Figure 2.1 requires nonsingularity and G_4 requires singularity. These latter statements can be verified by examining permutation digraphs and applying Observation 2.7.

THEOREM 2.9. *Let G be a graph. The following are equivalent.*

1. *G has no permutation digraphs.*
2. *For every symmetric sign pattern Z such that $\mathcal{G}^\ell(Z) = G$, all the terms in the standard expansion of the determinant of Z are* 0*.*

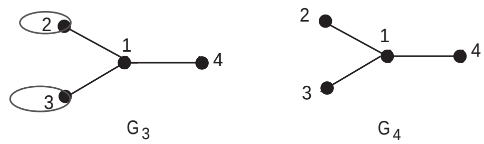

Fig. 2.1. *Graph G_3 that requires nonsingularity and graph G_4 that requires singularity*

3. *Every symmetric sign pattern Z such that $\mathcal{G}^\ell(Z) = G$ has signed determinant 0.*
4. *G requires singularity.*
5. *G requires symmetric singularity.*
6. *For X a symmetric matrix of independent indeterminates such that $\mathcal{G}^\ell(X) = G$,* $\det X = 0$.

Proof. The equivalence of 1, 2, 3 is Observation 2.6.1. That item 3 implies item 4 implies item 5 is obvious. The negation of item 6 implies the negation of item 5, because $\det X$ is a polynomial in the entries of X and a formally nonzero polynomial has a nonzero value as a function.

We show that the negation of item 1 implies the negation of item 6: Let G be a graph with at least one permutation digraph. Let X be a symmetric matrix of independent indeterminates such that $\mathcal{G}^\ell(X) = G$. The determinant of X is a sum of products of entries of X associated with permutation digraphs. Two such terms contain exactly the same entries of X if and only if their associated permutation digraphs D_π and D_τ satisfy $\pi \sim \tau$. Then π, τ have the same sign, so the terms do not cancel, and $\det X$ is nonzero. □

We will show in the next section that the converses of the other statements in Observation 2.7 are true for trees and forests.

Question 2.10. *Are the converses of the second and third statements in Observation 2.7 true for all graphs?*

The following lemma will be needed in the next section.

Lemma 2.11. *If graph G is ambiguous, then there is a symmetric sign pattern Z with $\mathcal{G}^\ell(Z) = G$ that does not have signed determinant.*

Proof. Let G be ambiguous. Then there exists $A \in \mathcal{S}^\ell(G)$ having $\det A = 0$. Then $\mathcal{G}^\ell(\mathcal{Z}^\ell(A)) = G$ and $\mathcal{Z}^\ell(A)$ is not SNS. Since G does not require singularity, G must contain permutation digraphs. Thus no sign pattern having graph equal to G, including $\mathcal{Z}^\ell(A)$, has signed determinant 0. So Z does not have signed determinant. □

3. General results for Trees and Tree Sign Patterns. Many standard terms such as tree, path, star, etc., are defined for simple graphs. To distinguish between graphs and simple graphs, we will preface these terms with the word "simple" when referring to a simple graph. We extend these terms to graphs by ignoring loops. Thus, a graph T is a *tree* if its associated simple graph $\widehat{T}$ is a simple tree (equivalently, T is connected and acyclic) and is a *forest* if $\widehat{T}$ is a simple forest (i.e., T is acyclic). The graphs G_1, G_3, G_4 in Figures 1.1 and 2.1 are trees. If K is a cohort and $\mathcal{G}(K)$ is a simple tree (simple forest), then K is called a *tree cohort* (*forest cohort*).

A combinatorially symmetric sign pattern Z is a *tree sign pattern* (*forest sign pattern*) if $\mathcal{G}^\ell(Z)$ is a tree (forest); equivalently, Z is a tree sign pattern (forest sign pattern) if $\mathcal{G}(Z)$ is a simple tree (simple forest). The sign pattern Z in Example 1.2 is a symmetric tree sign pattern.

Note that for a tree or forest, since there are no cycles of length greater than 2, any two distinct permutation digraphs are nonequivalent. The results in Lemmas 3.1 - 3.6 below are generally known.

Lemma 3.1. [5] *Let Z be a symmetric forest sign pattern and $B \in \mathcal{Q}^\ell(Z)$. Then there exists a positive diagonal matrix D such that $A = DBD^{-1}$ is symmetric and has the same sign pattern as B, i.e., $A \in \mathcal{S}^\ell(Z)$. Thus* $\operatorname{smr}(Z) = \operatorname{mr}(Z)$.

Corollary 3.2. *Let Z be a symmetric forest sign pattern.*

1. *The following are equivalent:*
   a) *Z requires symmetric singularity.*
   b) *Z requires singularity.*
   c) *Z has signed determinant 0.*
2. *The following are equivalent:*
   a) *Z requires symmetric nonsingularity.*
   b) *Z requires nonsingularity, i.e., Z is* SNS.
3. *The following are equivalent:*
   a) *Z allows symmetric singularity.*
   b) *Z allows singularity.*
   c) *Z has signed determinant 0 or* $\det Z$ *has both positive and negative terms.*

Lemma 3.3. *Let Z be a symmetric forest sign pattern. There exists a nonsingular diagonal sign pattern D such that all nonzero off-diagonal entries of DZD^{-1} are $+$.*

Proof. Without loss of generality, we can assume Z is a tree. Therefore, there exists exactly one path between any two vertices of Z. Let D be the diagonal matrix with index set $\iota(Z)$ defined by $D = \operatorname{diag}(d_{\iota(1)}, d_{\iota(2)}, \ldots, d_{\iota(n)})$. Set $d_{\iota(1)} = 1$. For any vertex v at distance k from $\iota(1)$, let $P(v_0 = \iota(1), v_1, \ldots, v_{k-1}, v_k = v)$ be the path from $\iota(1)$ to v with vertex set $V = \{v_0, v_2, \ldots, v_k\}$ and edge set $E = \{e_{v_1}, e_{v_2}, \ldots, e_{v_k}\}$ where $e_{v_l} = \{v_{l-1}, v_l\}$. Set $d_v = g_{v_1} g_{v_2} \ldots g_{v_k}$ where

$$g_{v_l} = \begin{cases} 1 & \text{if } z_{v_{l-1},v_l} = +, \\ -1 & \text{if } z_{v_{l-1},v_l} = -, \end{cases} \qquad \text{for } l = 1, 2, \ldots, k.$$

□

Observation 3.4. *It follows from Lemmas 3.1 and 3.3 that, when studying a property of a symmetric forest sign pattern Z that is preserved by diagonal similarity, such as rank, spectrum, multiplicity of an eigenvalue, nonsingularity, etc. we may assume all nonzero off-diagonal entries of Z are positive and may restrict our attention to symmetric matrices.*

Lemma 3.5. [5] *If Z is a symmetric forest sign pattern such that all nonzero off-diagonal entries of Z are $+$ and $B \in \mathcal{Q}^\ell(Z)$, then there exist positive diagonal matrices D_1, D_2 such that all the nonzero off-diagonal entries of $A = D_1 B D_2$ are one, and $A \in \mathcal{S}^\ell(Z)$. Thus,* $\operatorname{mr}(Z)$ *can be achieved by a matrix all of whose nonzero off-diagonal entries are one.*

Lemma 3.6. [5] *Let Z be a forest sign pattern. There exists a nonsingular diagonal sign pattern D and symmetric forest sign pattern Z_1 such that $Z = Z_1 D$.*

Observation 3.7. *It follows from the preceding lemma that the minimum rank of a forest sign pattern may be achieved by a matrix all of whose off-diagonal entries are in* $\{0, 1, -1\}$.

The reductions to a symmetric tree sign pattern, to having all nonzero off-diagonal entries of the sign pattern be $+$, and to having all nonzero off-diagonal entries of a matrix be one are not valid for the study of eigenvalue multiplicity. An example is given in [6] of an $n \times n$ sign pattern Z such that the graph of Z is a path, but there is a nilpotent matrix in $\mathcal{Q}^\ell(Z)$. Since the graph is a path, $\operatorname{mr}(Z) = n - 1$.

When working with graphs of sign patterns, the following lemma can expedite the determination that a symmetric tree sign pattern allows singularity, and will be used in Section 5.

LEMMA 3.8. *Let Z be a symmetric forest sign pattern. If there is a subset $R \subseteq \iota(Z)$ such that $\det Z[R]$ has both positive and negative terms and $\mathcal{G}^\ell(Z[\overline{R}])$ has a permutation digraph, then Z allows symmetric singularity.*

Proof. Since $\mathcal{G}^\ell(Z[\overline{R}])$ has a permutation digraph, $\det Z[\overline{R}]$ has a nonzero term. The product of a nonzero term in $\det Z[\overline{R}]$ with terms of opposite signs in $\det Z[R]$ produces terms of opposite signs in $\det Z$. The result then follows from Corollary 3.2. □

THEOREM 3.9. *Let T be a forest.*

1. *T requires symmetric singularity if and only if for any symmetric sign pattern Z, $\mathcal{G}^\ell(Z) = T$ implies Z has signed determinant 0.*
2. *T requires symmetric nonsingularity if and only if for any symmetric sign pattern Z, $\mathcal{G}^\ell(Z) = T$ implies Z is* SNS .
3. *T is ambiguous if and only if there exists a symmetric sign pattern Z with $\mathcal{G}^\ell(Z) = T$ that does not have signed determinant.*

Proof. Statement 1 is true for all graphs (cf. Theorem 2.9). It is sufficient to prove the third statement, since each of the mutually exclusive possibilities {requires symmetric singularity, requires symmetric nonsingularity, ambiguous} can come from only one of {every symmetric sign pattern has signed determinant 0, every symmetric sign pattern is SNS, there exists a symmetric sign pattern that does not have signed determinant} and vice versa.

Suppose there is a symmetric sign pattern Z such that $\mathcal{G}^\ell(Z) = T$ and Z does not have signed determinant. Then by the SNS Theorem, Z is not SNS and does not have signed determinant 0. Then there are two matrices $B_1, B_2 \in \mathcal{Q}^\ell(Z)$ such that $\det B_1 = 0$ and $\det B_2 \neq 0$. By Lemma 3.1, there are positive diagonal matrices D_1, D_2 such that $A_i = D_i B_i D_i^{-1}$ are symmetric. Then $A_i \in \mathcal{S}^\ell(T)$, $\det A_1 = 0$, and $\det A_2 \neq 0$, so T is ambiguous. The converse is true for all graphs by Lemma 2.11. □

LEMMA 3.10. *Let T be a simple forest. If the order of T is odd, then T has no permutation digraphs. If the order of T is even, then T has at most one permutation digraph.*

Proof. Since T has no loops, the only cycles are 2-cycles. Thus any permutation digraph is a union of disjoint 2-cycles, so if $|T| = n$ is odd, then there are no permutation digraphs of T. Suppose the order $n = 2k$ of T is even. We show by induction on k that there is at most one permutation digraph of T. The result is clear for $k = 1$, i.e., $n = 2$. Assume true for k. Let the order of T be $2(k+1) = 2k+2$. If T has an isolated vertex, T has no permutation digraphs; otherwise, let v be a vertex of degree 1, and let u be the unique neighbor of v. In any permutation digraph of T, the 2-cycle (edge) vu must appear since there is no other way to cover v. So delete u and v from T to obtain simple forest $T - \{u, v\}$ of order $2k$, which by the induction hypothesis has at most one permutation digraph. □

LEMMA 3.11. *Let T be a forest that has at least two permutation digraphs. Then T has a loop ii such that there is a permutation digraph of T that includes ii and another permutation digraph of T that does not include ii.*

Proof. Let P_1 and P_2 be distinct permutation digraphs of forest T. If P_1 and P_2 do not have identical loops, then one has a loop that is not in the other. If they have identical loops, then let L be the set of vertices at which P_1 and P_2 have loops. Removing all loops from P_1 and P_2 gives two distinct permutation digraphs of the simple graph $\widehat{T-L}$, contradicting Lemma 3.10. □

THEOREM 3.12.

1. *A forest T requires symmetric singularity if and only if T has no permutation digraphs.*
2. *A forest T requires symmetric nonsingularity if and only if T has a unique permutation digraph.*
3. *A forest T is ambiguous if and only if T has at least two permutation digraphs.*

Proof. Statement 1 was established for all graphs in Theorem 2.9. We will show that for any forest T, if T has at least two permutation digraphs, then T is ambiguous. This result, in conjunction with Observation 2.7.3, establishes statement 3, and statement 2 follows from statements 1 and 3.

Let T be a forest that has at least two permutation digraphs. We show there is a symmetric sign pattern Z with $\mathcal{G}^\ell(Z) = T$ that does not have signed determinant. Then by Theorem 3.9, T is ambiguous.

Choose any symmetric sign pattern Z such that $G^\ell(Z) = T$. Compute the standard determinant expansion of Z, which has at least two nonzero terms. If there are terms of opposite sign, then Z does not have signed determinant. Now suppose all nonzero terms have the same sign. By Lemma 3.11, there is a loop ii of T and a permutation digraph that includes ii and another permutation digraph that does not include ii. Reverse the sign of diagonal element i in Z to obtain a new sign pattern Z_1. The determinant of Z_1 is obtained from the determinant of Z by reversing the signs of exactly those terms associated with permutation digraphs containing loop ii. Thus at least one term changes sign and at least one does not. Thus Z_1 does not have signed determinant. □

Corollary 3.13. *Let T be a forest. Then T allows symmetric singularity if and only if T has no permutation digraphs or at least two permutation digraphs.*

4. The Parter-Wiener Theorem for Cohorts. A *path cover* of a simple graph $\widehat{G}$ is a disjoint union of simple paths covering all vertices such that each simple path occurs as an induced subgraph of $\widehat{G}$. The *path cover number* is the minimum number of paths in a path cover. A *minimum path cover* is a path cover that achieves the path cover number. Definitions equivalent to the following were given in [13] for simple graphs (here we apply these in the obvious way to our definition of matrix, which has an index set, and use our notation "smr", etc.).

- $M(\widehat{T}) = \max\{\mathrm{m}_A(\lambda) : A \in \mathcal{S}(\widehat{T})\}$.
- $\mathrm{smr}(\widehat{T}) = \min\{\mathrm{rank} A : A \in \mathcal{S}(\widehat{T})\}$.
- $P(\widehat{T}) =$ path cover number.
- $\Delta(\widehat{T}) = \max\{p_Q - |Q| : Q \subseteq V(\widehat{T})$ and $\widehat{T} - Q$ consists of p_Q disjoint paths$\}$.

By Lemma 1.8, $M(\widehat{G}) = M_\lambda(\widehat{G})$ for any λ. One of the main results of [13] is that for $\widehat{T}$ a simple tree, $\Delta(\widehat{T}) = P(\widehat{T}) = M(\widehat{T}) = n - \mathrm{smr}(\widehat{T})$ (technically, this was proved for matrices with index set $\mathcal{N}$, but the same proof works for an arbitrary index set). Note that for all simple graphs, $M(\widehat{G}) = n - \mathrm{smr}(\widehat{G})$ (cf. Observation 1.7). These two related parameters, maximum multiplicity and (symmetric) minimum rank, are the ones of most interest. The power of the result $\Delta(\widehat{T}) = P(\widehat{T}) = M(\widehat{T})$ for simple trees lies in the algorithms for computation of Δ and P [16], [11], which render the otherwise challenging computation of M straightforward.

We first show that for graphs (with loops) neither the parameter P nor an obvious revision of this parameter is useful, restricting our attention for the moment to eigenvalue 0, which relates to minimum rank. For a graph G, we define $P_0^\ell(G) = \max\{$the number of paths in a minimum path cover of G that allow symmetric singularity$\}$.

The next example shows we need not have equality between $M_0^\ell(T)$ and $P_0^\ell(T)$.

Example 4.1. Let T_{DP} be the double path shown in Figure 4.1. Then T_{DP} is covered by the two paths $(3,1,4)$ and $(5,2,6)$, which form a minimum path cover since $P(\widehat{T}_{DP}) = 2$. Since these paths have at least two permutation digraphs (e.g., $(13)(4), (3)(14)$), they allow symmetric singularity by Corollary 3.13, and so $P_0^\ell(T_{DP}) = 2$. It is not possible to delete a vertex from T_{DP} and obtain three components that allow symmetric singularity, so it will follow from the Parter-Wiener Theorem 4.3 below that $M_0^\ell(T_{DP}) \leq 1$. Thus, $M_0^\ell(T_{DP}) < P_0^\ell(T_{DP})$. (By examination of permutation digraphs we see T_{DP} allows singularity, so $M_0^\ell(T_{DP}) = 1$.)

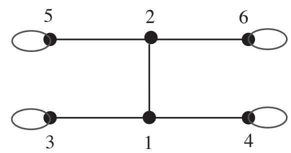

Fig. 4.1. *The double path T_{DP}*

Thus we see that it is not possible to use the path cover number to compute the minimum rank of a looped tree. Furthermore, Δ cannot be used directly because the connection between a path and a desired eigenvalue has been lost. In the next section we introduce a new parameter, but first we need to state the theorem that makes the new parameter useful.

The *Parter-Wiener Theorem*, which applies to matrices whose simple graph is a simple tree, is a powerful theorem for dealing with eigenvalue multiplicity. Let A be a symmetric matrix. Index $k \in \iota(A)$ is a *Parter-Wiener vertex of A for eigenvalue* λ if $\mathrm{m}_{A(k)}(\lambda) = \mathrm{m}_A(\lambda) + 1$. Furthermore, k is a *strong Parter-Wiener vertex of A for* λ if λ is an eigenvalue of at least three of the direct summands of A corresponding to components of $\mathcal{G}(A) - k$ and k is a Parter-Wiener vertex of A for λ.

Theorem 4.2. *(Parter-Wiener Theorem) [18], [19], [15] If A is a symmetric matrix, $\mathcal{G}(A)$ is a simple tree, and $\mathrm{m}_A(\lambda) \geq 2$, then there is a strong Parter-Wiener vertex of A for λ.*

We adapt the Parter-Wiener Theorem to cohorts.

Theorem 4.3. *(Parter-Wiener Theorem for Cohorts) Let S be a symmetric tree cohort. If $M_\lambda(S) \geq 2$, then there exists $k \in \iota(S)$ such that $M_\lambda(S(k)) = M_\lambda(S)+1$ and $S(k)$ has at least three components that allow eigenvalue λ.*

Proof. If $M_\lambda(S) \geq 2$, then there exists $A \in S$ such that $m_A(\lambda) = M_\lambda(S) \geq 2$. So by the Parter-Wiener Theorem, there exists $k \in \iota(A) = \iota(S)$ such that k is a Parter-Wiener vertex of A for λ. That is, λ is an eigenvalue of the principal submatrices $A[R_i]$ of A corresponding to at least three of the components $\langle R_i \rangle$ of $\mathcal{G}(A(k)) = \mathcal{G}(S(k))$ and $m_{A(k)}(\lambda) = m_A(\lambda)+1 = M_\lambda(S)+1$. Thus, $S(k)$ must have at least three components that allow eigenvalue λ and $M_\lambda(S(k)) \geq M_\lambda(S) + 1$. But $M_\lambda(S(k)) \leq M_\lambda(S) + 1$ by the Interlacing Corollary. □

An index k with the properties in Theorem 4.3 is called a *strong Parter-Wiener vertex* for S. A *high degree vertex* in a forest T is a vertex whose degree is at least three in the associated simple forest $\widehat{T}$. Note that only a high degree vertex of $\mathcal{G}(S)$ can be a strong PW-vertex of S.

5. Algorithm for Determination of Minimum Rank and Maximum Multiplicity for Trees and Tree Sign Patterns. Chen, Hall, Li and Wei [5] give a variety of lower bounds for the minimum rank of a tree sign pattern. Specifically, both the diameter and half the number of loops of $\mathcal{G}(Z)$ are lower bounds for the minimum rank of tree sign pattern Z. The authors also provide a means of computing the exact value of minimum rank for certain sign patterns having "star-like" graphs. In [5], a tree is called star-like if it has exactly one high degree vertex. Such a graph is also called a generalized star (e.g., [14]), although the latter term has been applied to simple graphs rather than graphs that allow loops, and sometimes does not require the existence of a high degree vertex (i.e., a path is considered a generalized star). In this section, we introduce a parameter $\mathcal{C}_\lambda$ and give an algorithm for its computation that allows explicit calculation of the minimum rank of a symmetric tree sign pattern.

For a symmetric cohort S and $R \subseteq \iota(S)$, define $c_\lambda(R)$ to be the number of components of $S(R)$ that allow eigenvalue λ. Then our readily computable new parameter is

$$\mathcal{C}_\lambda(S) = \max\{c_\lambda(R) - |R| \ : R \subseteq \iota(S)\}.$$

THEOREM 5.1. *For any symmetric tree cohort S, $\mathcal{C}_\lambda(S) = M_\lambda(S)$.*

Proof. Let S be a symmetric tree cohort. Let Q be a subset of vertices such that $c_\lambda(Q) = \mathcal{C}_\lambda(S) + |Q|$. Let $S[R_1], \ldots, S[R_{c_\lambda(Q)}]$ be the components of $S(Q)$ that allow eigenvalue λ. Since $S[R_i]$ allows eigenvalue λ, there must be a matrix $A_i \in S$ such that $\lambda \in \sigma(A_i[R_i])$. By the definition of cohort, there is a matrix $A \in S$ such that $A[R_i] = A_i[R_i]$ for $i = 1, \ldots, c_\lambda(Q)$, so $\lambda \in \sigma(A[R_i])$. Thus $m_{A(Q)}(\lambda) = c_\lambda(Q)$ and $M_\lambda(S(Q)) \geq c_\lambda(Q)$. Then by the Interlacing Corollary, $M_\lambda(S) \geq c_\lambda(Q) - |Q| = \mathcal{C}_\lambda(S)$.

We show by induction on the order of S that $\mathcal{C}_\lambda(S) = M_\lambda(S)$. Note first that for any S such that $M_\lambda(S) = 1$, $\mathcal{C}_\lambda(S) \geq M_\lambda(S)$ by choosing $R = \emptyset$. Now assume the theorem is true for every symmetric tree cohort S' having $|\iota(S')| < |\iota(S)|$. If $M_\lambda(S) = 1$, then $\mathcal{C}_\lambda(S) \geq M_\lambda(S)$. If $M_\lambda(S) > 1$, then by Theorem 4.3, there exists an index k such that $M_\lambda(S(k)) = M_\lambda(S) + 1$. Each component $S[R_i]$ of $S(k)$ is a symmetric tree cohort and $|\iota(S[R_i])| < |\iota(S)|$, so by the induction hypothesis, $\mathcal{C}_\lambda(S[R_i]) = M_\lambda(S[R_i])$. Thus there exists a subset $Q_i \subseteq R_i$ such that there are $M_\lambda(S[R_i]) + |Q_i|$ components of $S[R_i]$ that allow eigenvalue λ. Let $Q = (\cup Q_i) \cup \{k\}$. Then by Lemma 1.9, $S(Q)$ has $\sum M_\lambda(S_i) + \sum |Q_i| = M_\lambda(S(k)) + \sum |Q_i| = M_\lambda(S) + 1 + |Q| - 1$ components that allow eigenvalue λ, so $\mathcal{C}_\lambda(S) \geq M_\lambda(S)$. □

Graphs and sign patterns are our objects of interest, so we specialize the definition of $\mathcal{C}$ for these cohorts. For G a graph and Z a symmetric sign pattern, define

$$\mathcal{C}^\ell_\lambda(G) = \mathcal{C}_\lambda(\mathcal{S}^\ell(G)) \quad \text{and} \quad \mathcal{C}^\ell_\lambda(Z) = \mathcal{C}_\lambda(\mathcal{S}^\ell(Z)).$$

OBSERVATION 5.2. *Let G be a graph. When computing $\mathcal{C}^\ell_0(G)$, by Corollary 3.13, $c_0(R)$ is the number of components of $G - R$ that have either no permutation digraphs or at least two permutation digraphs. For $\lambda \neq 0$, by Lemma 1.12, $c_\lambda(R)$ is the number of components of $G - R$ that have an edge (with a loop considered to be an edge).*

OBSERVATION 5.3. *Let Z be a symmetric sign pattern. For $\lambda \neq 0$, $c_\lambda(R)$ is the number of components of $Z(R)$ that have a nonzero off-diagonal entry or a diagonal entry whose sign matches the sign of λ. If Z is a forest sign pattern, we can apply Lemma 3.8 to a component to show it allows symmetric singularity. To show a component does not allow symmetric singularity we can show its graph has a unique permutation digraph.*

For K a cohort (or $\widehat{T}$ a simple tree), we say K (or $\widehat{T}$) is *R-free* if $R \cap \iota(K) = \emptyset$ ($R \cap V(\widehat{T}) = \emptyset$). We are now ready to present the algorithm to determine a set $Q \subseteq \iota(S)$ that can be used to compute of $\mathcal{C}_\lambda(S)$.

ALGORITHM 5.4. *Let S be a symmetric tree cohort and let $\widehat{T} = \mathcal{G}(S)$.*
Initialize: $Q = \emptyset, i = 1$ and H is the set of all high degree vertices of $\widehat{T}$.
While $H \neq \emptyset$:

1. *Set $\widehat{T}_i$ = the unique component of $\widehat{T} - Q$ that contains an H-vertex.*
2. *Set $S_i = S[V(\widehat{T}_i)]$, the associated component of $S(Q)$.*
3. *Set $Q_i = \emptyset$.*
4. *Set $W_i = \{w \in H$: all but possibly one component of $\widehat{T}_i - w$ is H-free$\}$.*
5. *For each vertex $w \in W_i$,*
   if there are at least two H-free components of $S_i(w)$ that allow eigenvalue λ, then $Q_i = Q_i \cup \{w\}$.
6. *$Q = Q \cup Q_i$.*
7. *Remove all the vertices of W_i from H.*
8. *For each $v \in H$,*
   if $\deg_{\widehat{T}-Q} v \leq 2$, remove v from H.
9. *$i = i + 1$.*

In Theorem 5.8 below we will show that for the set Q produced by Algorithm 5.4,

$$c_\lambda(Q) - |Q| = \mathcal{C}_\lambda(S).$$

Before doing so, we illustrate how the algorithm is used in several examples. As noted in Observation 5.3, it is easy to determine whether a component allows a positive or allows a negative eigenvalue for a sign pattern or a graph (cf. Example 5.6 below). However, the case of $\lambda = 0$ is of more interest, because of the connection to minimum rank, so we begin with that example, even though it is more difficult.

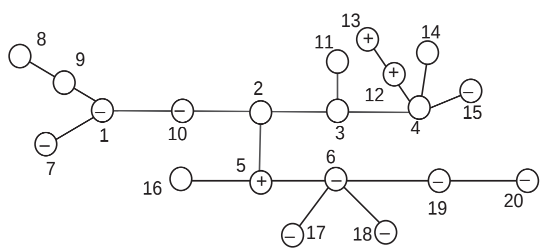

FIG. 5.1. *The signed tree $\mathcal{G}(Z)$*

EXAMPLE 5.5. We compute the minimum rank of the tree sign pattern Z, shown in Figure 5.1, by computing $M_0^\ell(Z)$. The sign of each diagonal entry is shown on the vertex, with the absence of a sign indicating 0; the signs of the nonzero off-diagonal entries can be assumed to be + by Observation 3.4. Initially, $Q = \emptyset,\ i = 1$, and $H = \{1, 2, 3, 4, 5, 6\}$ is the set of high degree vertices.

For the first iteration of Algorithm 5.4, $\widehat{T}_1 = \mathcal{G}(Z)$, and $W_1 = \{1, 4, 6\}$.

Deletion of vertex 1 leaves two H-free components of $\mathcal{S}^\ell(Z)$, but neither allows singularity. Thus $1 \notin Q_1$.

Deletion of vertex 4 leaves three H-free components, two of which allow symmetric singularity (note $Z_{14,14} = 0$ and $\det Z[\{12, 13\}] = (z_{12,12}z_{13,13}) + (-z_{12,13}z_{13,12}) = (+)\ +\ (-)$, which allows symmetric singularity). Thus $4 \in Q_1$.

Deletion of vertex 6 leaves three H-free components, but only one allows symmetric singularity. Thus $6 \notin Q_1$.

Vertex 3 is no longer high degree, and so is removed from H also.

Now $Q = Q_1 = \{4\}$, $H = \{2, 5\}$, and the signed forest $\mathcal{G}(Z) - Q_1$ is shown in Figure 5.2 (the only labels now shown are for vertices currently in H).

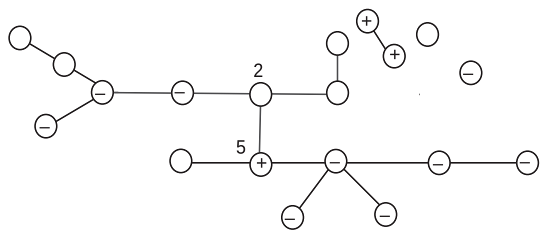

FIG. 5.2. *The signed forest $\mathcal{G}(Z) - Q_1$ resulting from the first iteration of Algorithm 5.4*

For the second iteration of Algorithm 5.4, $\widehat{T}_2$ is the component that contains 2 and 5, and $W_2 = \{2, 5\}$.

$\widehat{T}_2 - 2$ has two H-free components. Vertex 2 is not an element of Q_2 because $Z[\{3, 11\}]$ has a unique permutation digraph (use Figure 5.1 to see the vertex numbers), and so does not allow symmetric singularity. It is unnecessary to verify that $Z[\{1, 7, 8, 9, 10\}]$ allows symmetric singularity.

$\widehat{T}_2 - 5$ has two H-free components. The component $Z[\{16\}]$ requires singularity because $Z_{16,16} = 0$. The fact that the component $Z[\{6, 17, 18, 19, 20\}]$ allows symmetric singularity can be established by Lemma 3.8 with $R = \{19, 20\}$. Thus $5 \in Q_2$.

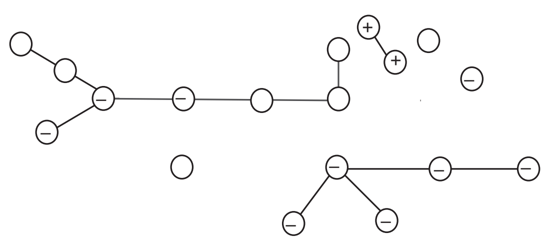

FIG. 5.3. *The signed forest $\mathcal{G}(Z) - Q$*

Thus $Q = \{4, 5\}$. The signed forest $\mathcal{G}(Z) - Q$ is shown in Figure 5.3. It is clear from this figure and Lemma 3.8 that $Z(Q)$ has five components that allow symmetric singularity (use $R = \{1, 7\}$ for $Z[\{1, 2, 3, 7, 8, 9, 10, 11\}]$). Since $|Q| = 2$, by Theorems 5.1 and 5.8, $M_0^\ell(Z) = \mathcal{C}_0^\ell(Z) = 5 - 2 = 3$. Thus $\operatorname{mr}^\ell(Z) = \operatorname{smr}^\ell(Z) = 20 - 3 = 17$. Note that the lower bound for minimum rank given by Corollary 2.9 of [5] is 6, since $\mathcal{G}^\ell(Z)$ has 12 loops, and the lower bound given by the diameter (Corollary 2.3 of [5]) is 8. A specific symmetric integer matrix $A \in \mathcal{S}^\ell(Z)$ of rank 17 is constructed in Example 7.3 below.

As a comparison, note that if the graph in Figure 5.1 is viewed as a simple tree $\widehat{T}$ (the signs are ignored and the diagonal is unrestricted) and the algorithm is applied to $\widehat{T}$, $Q = \{1, 2, 4, 6\}$ and $\widehat{T} - Q$ consists of 11 paths, so $M(\widehat{T}) = 11 - 4 = 7$ and $\operatorname{smr}(\widehat{T}) = 20 - 7 = 13$, but only 2 of the 11 paths allow symmetric singularity when the diagonal entries are restricted as shown in Figure 5.1.

EXAMPLE 5.6. Let Z be the symmetric tree sign pattern shown in Figure 5.1. We compute

$M_{-1}^{\ell}(Z)$. Initially, $Q = \emptyset,\ i = 1$ and $H = \{1,2,3,4,5,6\}$ is the set of high degree vertices.

For the first iteration of Algorithm 5.4, $\widehat{T_1} = \mathcal{G}(Z)$, and $W_1 = \{1,4,6\}$.

Deletion of vertex 1 leaves two H-free components of $\mathcal{S}^{\ell}(Z)$ that allow a negative eigenvalue, since z_{89} is nonzero and $z_{77} = -$. Thus $1 \in Q_1$.

Deletion of vertex 4 leaves three H-free components, two of which allow a negative eigenvalue. Thus $4 \in Q_1$.

Deletion of vertex 6 leaves three H-free components that allow a negative eigenvalue. Thus $6 \in Q_1$.

Vertices 3 and 5 are no longer high degree, and so are removed from H also.

Now $Q = Q_1 = \{1,4,6\}$, $H = \{2\}$, and the signed forest $\mathcal{G}(Z) - Q_1$ is shown in Figure 5.4 (the only labels now shown are for vertices currently in H).

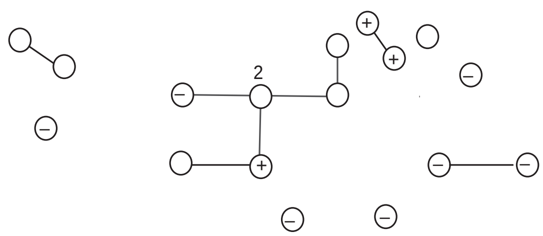

FIG. 5.4. *The signed forest* $\mathcal{G}(Z) - Q_1$

For the second iteration of Algorithm 5.4, $\widehat{T_2}$ is the component that contains 2, and $W_2 = \{2\}$. $\widehat{T_2} - 2$ has three H-free components. The components $Z[\{10\}]$, $Z[\{3,11\}]$, and $Z[\{5,16\}]$ each allow a negative eigenvalue, so $2 \in Q_2$.

Thus $Q = \{1,2,4,6\}$ and the forest $\mathcal{G}(Z) - Q$ (with signs of diagonal entries) is shown in Figure 5.5. It is clear from this figure and Lemma 1.12 that $Z(Q)$ has ten components that allow a negative eigenvalue. Since $|Q| = 4$, by Theorems 5.1 and 5.8, $M_{-1}^{\ell}(Z) = \mathcal{C}_{-1}^{\ell}(Z) = 10 - 4 = 6$. Construction of a specific symmetric integer matrix $A \in \mathcal{S}^{\ell}(Z)$ with $m_A(-1) = 6$ is discussed in Example 7.10.

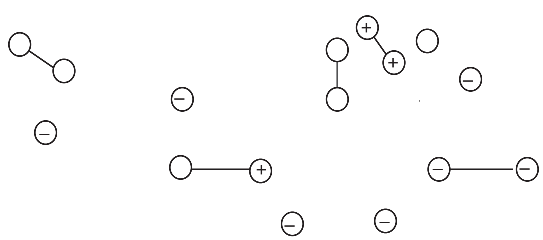

FIG. 5.5. *The signed forest* $\mathcal{G}(Z) - Q$

EXAMPLE 5.7. We apply Algorithm 5.4 to compute the symmetric minimum rank of the tree T shown in Figure 5.6 by computing $M_0^{\ell}(T)$. Here $S = \mathcal{S}^{\ell}(T)$ and the simple tree in Algorithm 5.4 is actually $\widehat{T}$, but the components in question must be examined in T itself, so we refer to the components of T rather than the components of $\mathcal{S}^{\ell}(T)$. Initially, $Q = \emptyset,\ i = 1$ and $H = \{1,2,3,4,5,6,7,8\}$ is the set of high degree vertices.

For the first iteration of Algorithm 5.4, $T_1 = T$, and $W_1 = \{1,3,6,7\}$.

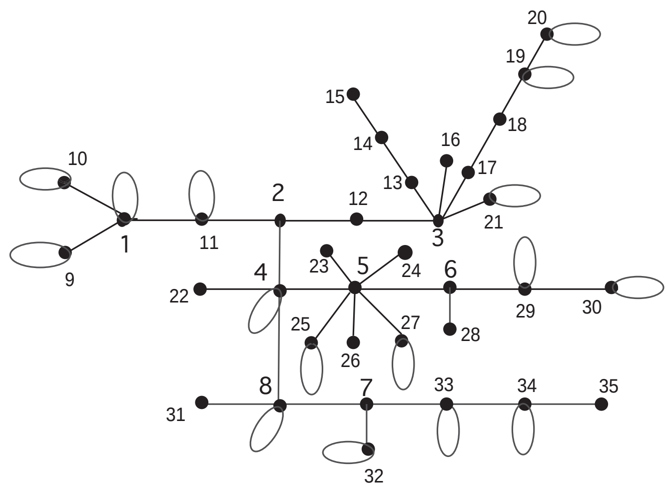

FIG. 5.6. *The tree* $T = T_1$

Deletion of vertex 1 leaves two H-free components both of which require symmetric nonsingularity. Thus $1 \notin Q_1$.

Deletion of vertex 3 leaves four H-free components, three of which allow symmetric singularity. This can be seen by considering permutation digraphs, or applying Lemma 6.1 below. Thus $3 \in Q_1$.

Deletion of vertex 6 leaves two H-free components, both of which allow symmetric singularity. Thus $6 \in Q_1$.

Deletion of vertex 7 leaves two H-free components, both which require symmetric nonsingularity. Thus $7 \notin Q_1$.

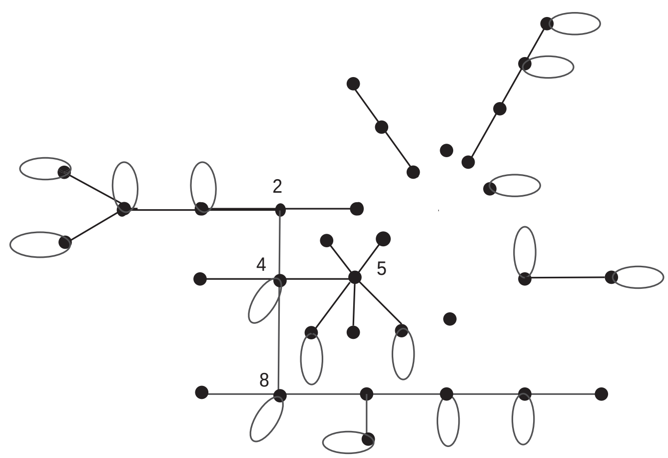

FIG. 5.7. *The forest* $T - Q_1$ *resulting from the first iteration of Algorithm 5.4*

Now $Q = Q_1 = \{3, 6\}$, $H = \{2, 4, 5, 8\}$ and the forest $T - Q_1$ is shown in Figure 5.7 (the only labels shown are for vertices currently in H).

For the second iteration of Algorithm 5.4, T_2 is the component that contains $2, 4, 5, 8$, and

$W_2 = \{2, 5, 8\}$:

$T_2 - 2$ has two H-free components, both of which allow symmetric singularity. The fact that the component that contains vertex 1 (look at Figure 5.6 in order to see that label) allows singularity follows from Theorem 3.12. Thus $2 \in Q_2$.

$T_2 - 5$ has five H-free components, three of which allow symmetric singularity. Thus $5 \in Q_2$.

$T_2 - 8$ has two H-free components, both of which allow symmetric singularity. Thus $8 \in Q_2$.

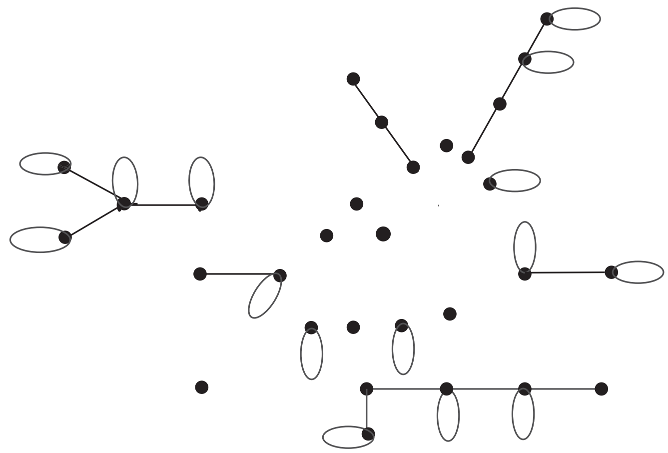

FIG. 5.8. *The forest $T - Q$*

Thus $Q = \{2, 3, 5, 6, 8\}$ and $T - Q$ is shown in Figure 5.8. There is no third iteration since the only vertex remaining in H after the removal of W_2, i.e. 4, no longer has high degree, and so is removed from H also.

Since $T - Q$ has twelve components which allow symmetric singularity, by Theorems 5.1 and 5.8 below, $M_0^\ell(T) = \mathcal{C}_0^\ell(T) = 12 - 5 = 7$. Thus $\mathrm{smr}^\ell(T) = 35 - 7 = 28$. Construction of a specific symmetric integer matrix $A \in \mathcal{S}^\ell(T)$ of rank 28 is discussed in Example 7.6.

We now prove that the set Q produced by Algorithm 5.4 realizes $\mathcal{C}_\lambda(S)$.

THEOREM 5.8. *For any cohort S whose simple graph is a simple tree, $\mathcal{C}_\lambda(S) = c_\lambda(Q) - |Q|$ for Q the set of vertices determined by Algorithm 5.4.*

Proof. Let S be a cohort such that $\widehat{T} = \mathcal{G}(S)$ is a simple tree. Perform Algorithm 5.4, recording the number r of iterations performed and the sets Q_i and W_i produced in iteration i. Let $W = \cup_{j=1}^r W_j$ and let $W_0 = U_0 = Q_0 = \emptyset$. For $i = 1, \ldots, r$, $\widehat{T_i}$ is the tree used in the ith iteration of the algorithm, and we let $\widehat{T}_{r+1} = \emptyset$.

Now we partition the set $U = \iota(S) - W$ into subsets U_i. Note first that $\widehat{T} - W$ is a disjoint union of paths, because if a vertex v has high degree in $\widehat{T} - W$, then the algorithm would not have terminated after r steps (in fact, W is a set that realizes $\Delta(\widehat{T})$, but that is not relevant here). Since $\widehat{T}$ is connected, each path P of $\widehat{T} - W$ has one or more vertices having neighbor(s) in W. Define $\omega(P)$ to be the maximum of the indices i such that a vertex of P has a neighbor in W_i. Then define U_i to be the set of all vertices in all paths P such that $\omega(P) = i$. Note $U = \cup_{j=1}^r U_j$ and $\widehat{T} = W \cup U$.

Let X be a set of vertices of $\widehat{T}$. We say

- X has property α at level i if $(\cup_{j=1}^i U_j) \cap X = \emptyset$

- X has property β at level i if $(\cup_{j=1}^{i}(W_j - Q_j)) \cap X = \emptyset$
- X has property γ at level i if $\cup_{j=1}^{i} Q_j \subseteq X$

If X has property φ at level i, then X has property φ at level j for $j < i$ ($\varphi \in \{\alpha, \beta, \gamma\}$). For $v \in X$, define $X(v)$ be the set obtained from X by removing v from X. If X has property φ at level i and $v \notin Q$, then clearly $X(v)$ also has property φ at level i.

Let $v \in W_{i+1} \cup U_{i+1}$. By construction, $v \in \widehat{T}_{i+1}$. If X has property γ at level i, the component C of $\widehat{T} - X(v)$ (or the component C of $\widehat{T} - X$ if $v \notin X$) that contains v is contained in $\widehat{T}_{i+1}$, because $\widehat{T}_{i+1}$ is a connected component of $\widehat{T} - \cup_{j=1}^{i} Q_j$ and $\cup_{j=1}^{i} Q_j \subseteq X$.

Note that any set X has properties α, β and γ at level 0, because $U_0 = W_0 = Q_0 = \emptyset$. Assume that X has properties α, β and γ at level $i < r$. We show that we can find a set X_γ of vertices of $\widehat{T}$ such that X_γ has properties α, β and γ at level $i+1$ and $c_X - |X| \le c_{X_\gamma} - |X_\gamma|$. Note that if Y has properties α, β and γ at level r, then $Y = Q$, so repeated application of this step shows $c_X - |X| \le c_Q - |Q|$, i.e., $\mathcal{C}_\lambda(S) = c_Q - |Q|$.

Suppose that X has properties α, β, γ at level i, but does not have property α at level $i+1$. Then there is a vertex u in U_{i+1} that is in X. By the algorithm, u has degree 2 or less in $\widehat{T}_{i+1}$. Since the component C of $\widehat{T} - X(u)$ that contains u is contained in $\widehat{T}_{i+1}$, $\deg_C u \le 2$, so removing u from C creates at most one additional component. Thus $c_X - |X| \le c_{X(u)} + 1 - (|X(u)| + 1) = c_{X(u)} - |X(u)|$. So if X_α is obtained from X by removing every vertex of U_{i+1} that is in X, then X_α has property α at level $i+1$ and properties β and γ at level i, and $c_X - |X| \le c_{X_\alpha} - |X_\alpha|$.

Suppose that X_α does not have property β at level $i+1$. Then there is a vertex $w \in W_{i+1} - Q_{i+1}$ that is in X_α. Let C be the component of $\widehat{T} - X_\alpha(w)$ that contains w. Since X_α has properties β and γ at level i and property α at level $i+1$, any component of $C - w$ that is not in $\widehat{T}_{i+2}$ is a component of $\widehat{T}_{i+1} - w$. Since $w \notin Q_{i+1}$, at most one such component of S allows eigenvalue λ, i.e., $S[V(C-w)]$ has at most one component not in $\widehat{T}_{i+2}$ that allows eigenvalue λ, and so at most two components that allow eigenvalue λ. Then $c_{X_\alpha} - |X_\alpha| \le c_{X_\alpha(w)} + 1 - (|X_\alpha(w)| + 1) = c_{X_\alpha(w)} - |X_\alpha(w)|$. So if X_β is obtained from X_α by removing every vertex of $W_{i+1} - Q_{i+1}$ that is in X_α, then X_β has properties α and β at level $i+1$ and property γ at level i, and $c_{X_\alpha} - |X_\alpha| \le c_{X_\beta} - |X_\beta|$.

Suppose that X_β does not have property γ at level $i+1$. Then there is a vertex $q \in Q_{i+1}$ that is not in X_β. Let C be the component of $\widehat{T} - X_\beta$ that contains q. Since X_β has properties α and β at level $i+1$ and γ at level i, any component of $C - q$ that is not in $\widehat{T}_{i+2}$ is a component of $\widehat{T}_{i+1} - q$. So $S[V(C-q)]$ has at least two components that allow eigenvalue λ. Then $c_{X_\beta \cup \{q\}} - |X_\beta \cup \{q\}| \ge c_{X_\beta} + 1 - (|X_\beta| + 1) = c_{X_\beta} - |X_\beta|$. So if X_γ is obtained from X_β by adding every vertex of Q_{i+1} that is not in X_β, then X_γ satisfies properties α, β and γ at level $i+1$, and $c_{X_\beta} - |X_\beta| \le c_{X_\gamma} - |X_\gamma|$. □

6. Singularity of Paths. By the nature of Algorithm 5.4, many of the components we examine are paths. Thus it is helpful to have information about singular paths.

If we begin at one end of a path and number the vertices consecutively (starting with 1), we say a vertex is *odd* or *even* depending on the parity of its number. A loop inherits its parity from its vertex. For an odd order path, it is irrelevant to this labeling which end is chosen for the start; for an even path what matters for singularity/nonsingularity is odd before even, and this is true for one starting end if and only if it is true for the other. To precisely describe a path, we can denote it by $P(d_1, \ldots, d_n)$, where $d_i \in \{0, l\}$ and $d_i = l$ if and only if vertex i has a loop.

Proposition 6.1.
1. *An odd order path requires symmetric singularity if and only if it has no odd loops.*
2. *An odd order path allows symmetric singularity if and only if it has no loops or at least two odd loops.*

3. *No even order path requires symmetric singularity.*
4. *An even order path allows symmetric singularity if and only if it has an odd loop preceding an even loop.*

Proof. By Theorem 3.12 and Corollary 3.13, a tree requires symmetric singularity if and only if it has no permutation digraphs, and allows symmetric singularity if and only if it has no permutation digraphs or at least two permutation digraphs.

For (1) and (2), any permutation digraph of an odd order path must contain an odd loop, and any odd loop can be combined with 2-cycles to produce a permutation digraph.

An even order path always has at least one permutation digraph consisting of disjoint 2 cycles; this establishes (3).

For (4), assume the vertices of P are $\{1, \ldots, 2k\}$, for some positive integer k. Since P has at least one permutation digraph, it allows singularity if and only if it has at least two permutation digraphs, i.e. if and only if there is a permutation digraph that contains loops. If the path has loops at $2s+1$ and $2t$, for t and s positive integers with $2s+1 < 2t$, then P has the following additional permutation digraph:
$(1\ 2)\ldots(2s-1\ 2s)(2s+1)(2s+2\ 2s+3)\ldots(2t-2\ 2t-1)(2t)(2t+1\ 2t+2)\ldots(2-1\ 2k)$. In any permutation digraph that contains loops, the first loop to appear must appear at an odd vertex, and there must be at least one subsequent loop at an even vertex. □

7. Finding a Symmetric Integer Matrix Realizing Minimum Rank for Trees and Tree Sign Patterns. In this section, we show how to use Algorithm 5.4 to obtain an integer matrix realizing the minimum rank of a tree sign pattern or a tree that allows loops. This algorithm can be applied to a forest or forest sign pattern by executing it on each component separately.

Before performing Algorithm 7.1 below, a tree sign pattern Z should be preprocessed by applying Lemma 3.6 to determine a nonsingular diagonal sign pattern D_1 and a symmetric tree sign pattern Z_1 such that $Z = Z_1 D_1$. When an integer matrix $A_1 \in \mathcal{S}^\ell(Z_1)$ with $\operatorname{rank} A_1 = \operatorname{mr}^\ell(Z_1)$ is obtained, then $A = A_1 D_1^1$ is a matrix having the desired properties, where D_1^1 is the matrix obtained from sign pattern D_1 by replacing $+$ by 1 and $-$ by -1.

Algorithm 7.1. *Let $S = \mathcal{S}^\ell(Z)$ or $\mathcal{S}^\ell(T)$, where T is a tree and Z is a symmetric tree sign pattern. To construct an integer matrix $A \in S$ having* $\operatorname{rank} A = \operatorname{mr}(S)$*:*

1. *Apply Algorithm 5.4 to S to find the subset Q of indices to be deleted. Let the indices of the components of $S(Q)$ be denoted by $R_i, i = 1, \ldots, h$.*
2. *For each i, construct a rational symmetric singular matrix $A_i \in S[R_i]$.*
3. *Construct a matrix A such that $A[R_i] = A_i$ and $A \in S$, using $0, 1,$ or -1 for any as yet unspecified entry.*
4. *If necessary, multiply by a scalar to obtain an integer matrix.*

It is clear how to perform each of the steps in Algorithm 7.1 except step 2. Algorithm 7.2 (respectively, 7.4) below gives a procedure for finding a rational singular matrix in $\mathcal{S}^\ell(Z)$ (respectively, $\mathcal{S}^\ell(T)$) that is usually simple to use in practice. We prove that the algorithm for trees (7.4) does produce a rational singular matrix. We prove (in Lemma 7.7 below) that it is always theoretically possible to find a rational singular matrix having a given symmetric tree sign pattern that allows singularity.

Algorithm 7.2. *Let Z be a symmetric tree sign pattern that allows but does not require singularity. To construct a rational singular matrix A having $\mathcal{Z}^\ell(A) = Z$:*

1. *Apply the method given in the proof of Lemma 3.3 to compute a nonsingular diagonal sign pattern D such that $Z_1 = DZD^{-1}$ has all nonzero off-diagonal entries equal to $+$.*

2. *Construct rational* $A_1 \in \mathcal{S}^\ell(Z_1)$ *as follows:*
   a) *Set all nonzero off-diagonal entries of* A_1 *equal to* 1.
   b) *For* $j = 1, \dots, r$*, where* $r = |\iota(Z_1)|$*, set the* j*th diagonal entry to the* j*th diagonal entry of* Z_1 *times* x_j*, where the* x_j *are independent indeterminates.*
   c) *Compute* $\det A_1 = p(x_1, \dots, x_r)$.
   d) *Select a variable* x_s *that appears in one of a pair of terms of opposite sign and not in the other.*
   e) *Express* p *as* $p(x_1, \dots, x_r) = \pm(x_s q_1(x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_r) - q_2(x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_r))$, *where* q_1 *and* q_2 *each contain at least one positive term.*
   f) *If possible, choose rational values of* $x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_r$ *to make both* q_1 *and* q_2 *positive; otherwise the algorithm does not produce the desired matrix.*
   g) *With the chosen values of the* x_j*, set* $x_s = \frac{q_2(x_1,\dots,x_{s-1},x_{s+1},\dots,x_r)}{q_1(x_1,\dots,x_{s-1},x_{s+1},\dots,x_r)}$
3. $A = D^{-1} A_1 D$.

We illustrate Algorithms 7.1 and 7.2 in the next example. Algorithm 7.2 calls for setting all nonzero off-diagonal elements to one. The *adjacency matrix* $\mathcal{A}(\widehat{G})$ of a simple graph $\widehat{G}$ is a 0,1-matrix that has 1's in exactly the off-diagonal entries corresponding to the edges of the graph. Thus it is convenient to describe each matrix constructed by giving only its diagonal, since the matrix is the sum of the adjacency matrix for $\mathcal{G}(Z[R])$ or $\mathcal{G}(Z)$ and the diagonal matrix.

Example 7.3. Let Z be the symmetric tree sign pattern shown in Figure 5.1 (assuming the nonzero off-diagonal entries of Z are already $+$). Algorithm 5.4 has been applied to this sign pattern in Example 5.5. For each of the components $Z[\{1,2,3,7,8,9,10,11\}]$, $Z[\{6,17,18,19,20\}]$, and $Z[\{12,13\}]$, we will produce a rational singular matrix $A \in \mathcal{S}^\ell(Z)$ that is the sum of the $\mathcal{A}(\mathcal{G}(Z[R]))$ and a rational diagonal matrix. Let d_i denote the ith diagonal element of the matrix A. Note that choices are involved and many other matrices could be obtained from the algorithm.

We illustrate steps 2(a) to 2(g) of Algorithm 7.2 on $Z[\{6,17,18,19,20\}]$. The matrix produced by steps 2(a) and 2(b) is $Z_x = \begin{bmatrix} -x_6 & 1 & 1 & 1 & 0 \\ 1 & -x_{17} & 0 & 0 & 0 \\ 1 & 0 & -x_{18} & 0 & 0 \\ 1 & 0 & 0 & -x_{19} & 1 \\ 0 & 0 & 0 & 1 & -x_{20} \end{bmatrix}$. Step 2(c) yields

$$\det Z_x = -x_{17} - x_{18} + x_{17}x_{18}x_{20} + x_{17}x_{19}x_{20} + x_{18}x_{19}x_{20} + x_{17}x_{18}x_6 - x_{17}x_{18}x_{19}x_{20}x_6.$$

We select x_{17} as our chosen variable in step 2(d), and step 2(e) yields

$$\det Z_x = -(x_{17}(1 - x_{18}x_{20} - x_{19}x_{20} - x_{18}x_6 + x_{18}x_{19}x_{20}x_6) - (-x_{18} + x_{18}x_{19}x_{20})),$$

$$q_1(x_6, x_{18}, x_{19}, x_{20}) = 1 - x_{18}x_{20} - x_{19}x_{20} - x_{18}x_6 + x_{18}x_{19}x_{20}x_6,$$
$$q_2(x_6, x_{18}, x_{19}, x_{20}) = -x_{18} + x_{18}x_{19}x_{20}.$$

In step 2(f), we choose $x_6 = 2, x_{18} = 1, x_{19} = 2, x_{20} = 2$, so $\det Z_x = 3 - x_{17}$. In step 2(g), $x_{17} = 3$, and thus $d_6 = -2, d_{17} = -3, d_{18} = -1, d_{19} = -2, d_{20} = -2$.

For $Z[\{12,13\}]$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is singular, so let $d_{12} = d_{13} = 1$.

For $Z[\{1,2,3,7,8,9,10,11\}]$, $\det Z_x = 1 - x_1x_7$, so we choose $x_1 = 1, x_7 = 1, x_{10} = 1$. The resulting diagonal entries are $d_1 = -1, d_7 = -1, d_{10} = -1$ (the latter value is irrelevant to the determinant, but must have the correct sign).

The only remaining undetermined diagonal entry is d_5, since vertex 5 was deleted. We choose $d_5 = 1$. Then the matrix we have constructed is

$$A = \mathcal{A}(\mathcal{G}(Z)) + \operatorname{diag}(-1, 0, 0, 0, 1, -2, -1, 0, 0, -1, 0, 1, 1, 0, -1, 0, -3, -1, -2, -2),$$

and $\operatorname{rank} A = 17$.

The algorithm for trees is simpler.

ALGORITHM 7.4. *Let T be a tree that allows but does not require singularity. To construct a rational singular matrix A having $\mathcal{G}^\ell(A) = T$:*

a) Set all nonzero off-diagonal entries of A equal to 1.

b) For $j = 1, \ldots, r$, where $r = |V(T)|$, if T has a loop at vertex j, set the jth diagonal entry to x_j; otherwise set the jth diagonal entry to zero.

c) Compute $\det A = p(x_1, \ldots, x_r)$. *Since T allows but does not require singularity, there are at least two nonzero terms.*

d) Select a variable x_s that appears in one of the terms and not in the other.

e) Express p as
$p(x_1, \ldots, x_r) = x_s q_1(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r) - q_2(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r)$, *where both q_1 and q_2 contain at least one nonzero term.*

f) Choose rational values of $x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r$ to make both q_1 and q_2 nonzero.

g) With the chosen values of the x_j, set $x_s = \frac{q_2(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r)}{q_1(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r)}$

LEMMA 7.5. *Let T be a tree that allows symmetric singularity. If T allows but does not require symmetric singularity, then Algorithm 7.4 will produce a singular symmetric rational matrix $A \in \mathcal{S}^\ell(T)$. If T requires symmetric singularity, then any symmetric rational matrix with graph T is a singular matrix.*

Proof. Let T_x be the symmetric matrix such that all nonzero off-diagonal entries are equal to one, and having x_i as its ith diagonal entry (if it is nonzero), where the x_i are independent indeterminates. Since T allows but does not require singularity, T has at least two permutation digraphs. By Lemma 3.11, there is a loop ss that is in one permutation digraph that is not in another permutation digraph. So we can write $\det T_x = x_s q_1(x_i) - q_2(x_i)$, where both q_1 and q_2 are nonzero polynomials in the variables $x_i, i \neq s$. We can choose rational values a_{ii} for the variables $x_i, i \neq s$ that make $q_1(a_{ii}) \neq 0$ and $q_2(a_{ii}) \neq 0$. Let $a_{ss} = \frac{q_2(a_{ii})}{q_1(a_{ii})}$. Then the matrix A having nonzero diagonal entries a_{ii} is a rational symmetric singular matrix with $\mathcal{G}^\ell(A) = T$. □

EXAMPLE 7.6. In Example 5.7, Algorithm 5.4 was applied to the tree in Figure 5.6. The components are shown in Figure 5.8. It is not difficult to apply Algorithm 7.4 to each component to choose integer values for the diagonal that when added to the adjacency matrix produce a singular matrix. One particular set of choices to produce such singular matrices yields $A = \mathcal{A}(\mathcal{G}(Z)) + \operatorname{diag}(3, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1, -1, 1, 0)$, and $\operatorname{rank} A = 28$. There are many other possible choices that achieve this rank.

We now prove it is always theoretically possible to find a singular symmetric rational matrix having a given tree sign pattern that allows singularity.

LEMMA 7.7. *If Z is a symmetric tree sign pattern that allows singularity, then there exists a singular symmetric rational matrix $A \in \mathcal{S}^\ell(Z)$.*

Proof. If Z requires singularity then any symmetric rational matrix with sign pattern Z may be chosen, so assume Z does not require singularity. Note $|\iota(Z)| \geq 2$.

We say a tree sign pattern Z is *minimally singular* if for every index $s \in \iota(Z)$ such that $z_{ss} \neq 0$, $Z(s)$ is nonsingular. Any nondiagonal sign pattern of size two that allows singularity is minimally singular. We first show that it is possible to find the desired singular rational matrix if Z is minimally singular.

Let Z_x be a matrix having all nonzero off-diagonal entries equal to one and having $z_{ii}x_i$ as the ith diagonal entry, where the x_i are independent indeterminates. Since Z allows but does not require singularity, as in the proof of Lemma 7.5 there is a variable x_s that appears in one term and does not appear in another term. Then $\det Z_x = x_s q_1(x_i) - q_2(x_i)$, where both $q_1(x_i)$ and $q_2(x_i)$ are nonzero polynomials in the variables $x_i, i \neq s$. By Lemma 3.5, there is a singular matrix $\widetilde{A} = [\tilde{a}_{ij}]$ in $\mathcal{S}^\ell(Z)$ all of whose nonzero off-diagonal entries are one, so there are values $\tilde{a}_i = |\tilde{a}_{ii}|$ that make $\tilde{a}_s q_1(\tilde{a}_i) - q_2(\tilde{a}_i) = \det \widetilde{A} = 0$. Note that $\det \widetilde{A}(s) = \pm q_1(\tilde{a}_i)$ and $\widetilde{A}(s) \in \mathcal{S}^\ell(Z(s))$, so by the hypothesis that Z is minimally singular, $q_1(\tilde{a}_i) \neq 0$. Since $\tilde{a}_s > 0$, $\operatorname{sgn}(q_2(\tilde{a}_i)) = \operatorname{sgn}(q_1(\tilde{a}_i))$. Thus we can perturb the $\tilde{a}_i, i \neq s$, slightly to rational values a_i so that $\operatorname{sgn}(q_j(a_i)) = \operatorname{sgn}(q_j(\tilde{a}_i)), j = 1, 2$. Let $a_s = \frac{q_2(a_i)}{q_1(a_i)}$. Then the matrix A with diagonal defined by $a_{ii} = z_{ii}a_i$ and having all nonzero off-diagonal entries equal to one is the desired singular rational matrix.

Now we consider the case where Z is not assumed minimally singular. Identify all the permutation digraphs of $\mathcal{G}^\ell(Z)$. If an edge $vw, v \neq w$, is not in any of these permutation digraphs, then remove it, obtaining graph G', in which every edge $vw, v \neq w$, appears in at least one permutation digraph of G'. Let Z' be the symmetric forest sign pattern obtained from Z by changing to zero any off-diagonal entry whose corresponding edge was removed. An edge is called *isolated* if the component of G' that contains the edge has only two vertices. If $vw, v \neq w$, is in every permutation digraph, then no 2-cycle corresponding to another edge incident with v or w can appear in a permutation digraph of G', so vw is isolated.

Choose a minimally singular principal subpattern $Z'[R]$ of Z'. Carry out the procedure described above to find index s, polynomials $q_j, j = 1, 2$, and a symmetric singular rational matrix $A[R] \in Z'[R]$ such that if $a_i = |a_{ii}|$, then $\operatorname{sgn}(q_1(a_i)) = \operatorname{sgn}(q_2(a_i)) \neq 0$ and $a_s = \frac{q_2(a_i)}{q_1(a_i)}$. Even if $\mathcal{G}^\ell(Z'[R])$ is not a component of $\mathcal{G}^\ell(Z')$, there must exist a permutation digraph in $\mathcal{G}^\ell(Z'[\overline{R}])$, since any edge that is not isolated is not required to appear in a permutation digraph. So $Z'[\overline{R}]$ does not require singularity, and we can choose a matrix $A[\overline{R}] \in Z'[\overline{R}]$ such that $0 \neq \det A[\overline{R}] = f(a_j), j \in \overline{R}, a_j = |a_{jj}|$. Now all diagonal elements of A have been determined. For an edge of $\mathcal{G}^\ell(Z)$ between two vertices in R or an edge between two vertices in $\overline{R}$, set the corresponding entry of A to be one; the values of these entries are irrelevant in computing the determinant of A, as these edges were removed to obtain G'. Assign all remaining nonzero off-diagonal entries to be ϵ. Then there exists a polynomial $g(x_i, x_j)$ with $i \in R, j \in \overline{R}$, such that $\det A = f(a_j)(a_s q_1(a_i) - q_2(a_i)) + \epsilon^2 g(a_i, a_j) = a_s f(a_j) q_1(a_i) - (f(a_j) q_2(a_i) - \epsilon^2 g(a_i, a_j))$. Choose ϵ rational and sufficiently small so that $\operatorname{sgn}(f(a_j)q_2(a_i) - \epsilon^2 g(a_i, a_j)) = \operatorname{sgn}(f(a_j)q_2(a_i))$. ☐

Although it works well in practice, we have not proved that step 2(f) of Algorithm 7.2 will always produce values for $x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r$ that make q_1, q_2 both positive; in fact, for some choice of x_s that may be impossible, as is demonstrated in the next example.

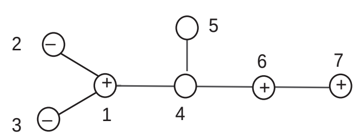

FIG. 7.1. *The tree* $\mathcal{G}(Z)$

EXAMPLE 7.8. Let Z be the tree sign pattern shown in Figure 7.1, with all nonzero off-diagonal positions being $+$. Then $\det Z_x = (1 - x_6 x_7)(x_1 x_2 x_3 + x_2 + x_3)$, so it is not possible to use any of x_1, x_2, x_3 as x_s in Algorithm 7.2, even though each of these variables appears in both a positive and a negative term. In this example, if either x_6 or x_7 is chosen as x_s, the algorithm will produce the desired matrix.

We now turn our attention to constructing a rational matrix having maximum multiplicity for a nonzero rational eigenvalue.

ALGORITHM 7.9. *Let $S = \mathcal{S}^\ell(Z)$ or $\mathcal{S}^\ell(G)$, where G is a tree and Z is a symmetric tree sign pattern. Given a rational number λ, to construct a symmetric rational matrix $A \in S$ having $m_A(\lambda) = M^\ell(S)$:*

1. *Apply Algorithm 5.4 to S to find the subset Q of indices to be deleted. Let the indices of the components of $S(Q)$ be denoted by $R_i, i = 1, \ldots, h$.*
2. *For each i, construct a rational symmetric matrix $A_i \in S[R_i]$ having eigenvalue λ.*
3. *Construct a matrix A such that $A[R_i] = A_i$ and $A \in S$, using $0, 1,$ or -1 for any as yet unspecified entry.*

Again, it is clear how to perform each of the steps in Algorithm 7.9 except step 2. Although we do not present formal algorithms for step 2 for the nonzero case, it is usually not hard to construct a rational matrix having the desired rational eigenvalue, as illustrated in the next example.

EXAMPLE 7.10. Let Z be the symmetric tree sign pattern shown in Figure 5.1 (assuming the nonzero off-diagonal entries of Z are already $+$). Algorithm 5.4 has been applied to this sign pattern for eigenvalue -1 in Example 5.6 (see Figure 5.4). Table 7.1 lists matrices having eigenvalue -1 and components for which they should be used to assemble a matrix $A \in \mathcal{S}^\ell(Z)$ having $m_A(-1) = 6$. For nonzero eigenvalues, it is not always possible to have all the nonzero off-diagonal entries be one, so we are no longer using the sum of the adjacency matrix and a diagonal matrix. Instead, one embeds the matrices shown in Table 7.1 in the appropriate places.

TABLE 7.1

Matrix	R
[-1]	{7},{10},{15},{17},{18}
$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$	{12,13}
$\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$	{5,16}
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	{3,11},{8,9}
$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$	{19,20}

8. Conclusions. In this section, we restate our main results explicitly for tree sign patterns and trees.

THEOREM 8.1. *For any symmetric tree sign pattern Z, $\mathcal{C}^\ell_\lambda(Z) = M^\ell_\lambda(Z)$. The following parameters can be computed by using Algorithm 5.4 to compute $\mathcal{C}^\ell_\lambda(Z)$: the maximum multiplicity of any positive eigenvalue, which is equal to $M^\ell_1(Z)$; the maximum multiplicity of any negative* *eigenvalue, which is equal to* $M^{\ell}_{-1}(Z)$*; and the maximum multiplicity of eigenvalue zero,* $M^{\ell}_{0}(Z)$*. The minimum rank of any tree sign pattern can be computed and there is an integer matrix realizing the minimum rank in* $\mathcal{S}^{\ell}(Z)$.

THEOREM 8.2. *For any tree* T, $\mathcal{C}^{\ell}_{\lambda}(T) = M^{\ell}_{\lambda}(T)$. *The following parameters can be computed by using Algorithm 5.4 to compute* $\mathcal{C}^{\ell}_{\lambda}(T)$*: the maximum multiplicity of any nonzero eigenvalue, which is equal to* $M^{\ell}_{1}(T)$*; and the maximum multiplicity of eigenvalue zero,* $M^{\ell}_{0}(Z)$*. There exists a matrix* $A \in \mathcal{S}^{\ell}(T)$ *such that every off-diagonal element of* A *is* 0 *or* 1*, the diagonal of* A *is rational, and* $\operatorname{rank} A = \operatorname{smr}^{\ell}(T)$.

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