

On choosability with separation of planar graphs with lists of different sizes

H. A. Kierstead ^{*} Bernard Lidický [†]

June 25, 2013

Abstract

A (k, d) -list assignment L of a graph G is a mapping that assigns to each vertex v a list $L(v)$ of at least k colors and for any adjacent pair xy , the lists $L(x)$ and $L(y)$ share at most d colors. A graph G is (k, d) -choosable if there exists an L -coloring of G for every (k, d) -list assignment L . This concept is also known as choosability with separation.

It is known that planar graphs are $(4, 1)$ -choosable but it is not known if planar graphs are $(3, 1)$ -choosable. We strengthen the result that planar graphs are $(4, 1)$ -choosable by allowing an independent set of vertices to have lists of size 3 instead of 4.

1 Introduction

Given a graph G , a *list assignment* L is a mapping assigning to each vertex $v \in V(G)$ a list of colors $L(v)$. An *L -coloring* is a vertex coloring φ such that $\varphi(v) \in L(v)$ for each vertex v and $\varphi(x) \neq \varphi(y)$ for each edge xy . A graph G is said to be *k -choosable* if there is an L -coloring for each list assignment L where $|L(v)| \geq k$ for each vertex v . The minimum such k is known as the *choosability* of G , denoted $\chi_\ell(G)$. A graph G is said to be *(k, d) -choosable* if there is an L -coloring for each list assignment L where $|L(v)| \geq k$ for each vertex v and $|L(x) \cap L(y)| \leq d$ for each edge xy .

This concept is called *choosability with separation*, since the second parameter may force the lists of adjacent vertices to be somewhat separated. If G is (k, d) -choosable, then G is also (k', d') -choosable for all $k' \geq k$ and $d' \leq d$. A graph is (k, k) -choosable if and only if it is k -choosable. Clearly, all graphs are $(k, 0)$ -choosable for $k \geq 1$. Thus, for a graph G and each $1 \leq k < \chi_\ell(G)$, there is some threshold $d \in \{0, \dots, k - 1\}$ such that G is (k, d) -choosable but not $(k, d + 1)$ -choosable.

^{*}School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287, USA. E-mail: hal.kierstead@asu.edu. Research of this author is supported in part by NSA grant H98230-12-1-0212.

[†]Department of Mathematics, University of Illinois. E-mail: lidicky@illinois.edu

The concept of choosability with separation was introduced by Kratochvíl, Tuza, and Voigt [4]. They used the following, more general definition. A graph G is (p, q, r) -choosable, if for every list assignment L with $|L(v)| \geq p$ for each $v \in V(G)$ and $|L(u) \cap L(v)| \leq p - r$ whenever u, v are adjacent vertices, G is q -tuple L -colorable. Since we consider only $q = 1$ in this paper, we use a simpler notation. They investigate this concept for both complete graphs and sparse graphs. The study of dense graphs were extended to complete bipartite graphs and multipartite graphs by Füredi, Kostochka, and Kumbhat [2, 3].

Thomassen [5] proved that planar graphs are 5-choosable, and hence they are $(5, d)$ -choosable for all d . Voigt [7] constructed a non-4-choosable planar graph, and there are also examples of non- $(4, 3)$ -choosable planar graphs. Kratochvíl, Tuza, and Voigt [4] showed that all planar graphs are $(4, 1)$ -choosable and asked:

Question 1 ([4]). Are all planar graphs $(4, 2)$ -choosable?

Voigt [6] also constructed a non-3-choosable triangle-free planar graph. Škrekovski [8] observed that there are examples of triangle-free planar graphs that are not $(3, 2)$ -choosable, and posed:

Question 2 ([8]). Are all planar graphs $(3, 1)$ -choosable?

Kratochvíl, Tuza and Voigt [4] proved a partial case of Question 2 by showing that every triangle-free planar graph is $(3, 1)$ -choosable.

Choi et. al [1] proved that every planar graph without 4-cycles is $(3, 1)$ -choosable and that every planar graph without 5-cycles and 6-cycles is $(3, 1)$ -choosable.

In this paper we give a strengthening of the result that every planar graph is $(4, 1)$ -choosable by allowing some vertices to have lists of size three. In a $(4, 1)$ -list assignment L on G , for every $uv \in E(G)$ holds that $|L(u) \cup L(v)| \geq 7$. In a $(3, 1)$ -list assignment L , for every $uv \in E(G)$ holds that $|L(u) \cup L(v)| \geq 5$. An intermediate step is to investigate the case where for every $uv \in E(G)$ holds that $|L(u) \cup L(v)| \geq 6$.

A $(*, 1)$ -list assignment is a list assignment L where $|L(v)| \geq 1$ and $|L(u) \cap L(v)| \leq 1$ for every pair of adjacent vertices u, v .

The main result of this paper is the following theorem.

Theorem 3. *Let G be a planar graph and $I \subseteq V(G)$ be an independent set. If L is a $(*, 1)$ -list assignment such that $|L(v)| \geq 3$ for every $v \in I$ and $|L(v)| \geq 4$ for every $v \in V(G) \setminus I$ then G has an L -coloring.*

The following theorem shows it is not possible to strengthen Theorem 3 by allowing $|L(v)| \geq 2$ for every vertex $v \in V(G)$ and requiring that $|L(u) \cup L(v)| \geq 6$ for every $uv \in E(G)$.

Theorem 4. *For every k there exists a planar graph G and a $(*, 1)$ -list assignment L such that $|L(v)| \geq 2$ for every $v \in V(G)$, $|L(u) \cup L(v)| \geq k$ for every $uv \in E(G)$, and G is not L -colorable.*

We first give some notation. In the next section, we prove Theorem 3 using Thomassen's precoloring extension method. In the last section we show a construction proving Theorem 4.

1.1 Notation

Given a graph G and a cycle $K \subset G$, an edge uv of G is a *chord* of K if $u, v \in V(K)$, but uv is not an edge of K . If G is a plane graph, then let $\text{Int}_K(G)$ be the subgraph of G consisting of the vertices and edges drawn inside the closed disc bounded by K , and let $\text{Ext}_K(G)$ be the subgraph of G obtained by removing all vertices and edges drawn inside the open disc bounded by K . In particular, $K = \text{Int}_K(G) \cap \text{Ext}_K(G)$. Finally, denote the characteristic function of a set S by ι_S . So $\iota_S(x) = 1$ if $x \in S$; else $\iota_S(x) = 0$.

2 Main theorem

In this section, we prove Theorem 3 by proving a slightly stronger theorem that is more amenable to induction. Observe that any list assignment satisfying the assumptions of Theorem 3 also satisfies the conditions of the following theorem.

Theorem 5. *Let G be a plane graph with outer face F and let P be a subpath of F containing at most two vertices. Let $I \subseteq V(G - P)$ be an independent set. If L is a $(*, 1)$ -list assignment satisfying the following conditions:*

- (i) $|L(v)| \geq 4 - \iota_I(v) - \iota_{V(F)}(v) - 2\iota_{V(P)}(v)$ for $v \in V(G)$,
- (ii) P is L -colorable,
- (iii) for every $v \in I$ there is at most one $p \in N(v) \cap V(P)$ with $(L(p) \cap L(v)) \neq \emptyset$,

then G is L -colorable.

Proof. Let $G = (V, E)$ and L be a counterexample where $|V| + |E|$ is as small as possible. Moreover, assume that the sum of the sizes of the lists is also as small as possible subject to the previous condition. Define $L(uv) = L(u) \cap L(v)$ if $uv \in E$; else $L(uv) = \emptyset$. Since G is minimal, we have:

Claim 1. *For all edges $uv, vw, uw \in E \setminus E(P)$*

- (1) $|L(uv)| = 1$;
- (2) $L(u) = \bigcup_{v \in N(u)} L(uv)$; and
- (3) $L(uv) = L(vw)$ implies $L(uv) = L(uw)$ for every triangle uvw .

Proof. For (1), note that $|L(uv)| \leq 1$, and if $L(uv) = \emptyset$ then it suffices to L -color $G - uv$, which is possible by minimality. For (2), the definitions imply $L(u) \supseteq \bigcup_{v \in N(u)} L(uv)$, and if $\gamma \in L(u) \setminus \bigcup_{v \in N(u)} L(uv)$ then L -coloring $G - u$, and then coloring u with γ yields an L -coloring of G . Finally consider (3). By (1), there exists a color γ with $L(uv) = \{\gamma\} = L(vw)$. Thus $\gamma \in L(u) \cap L(w)$, so by definition and (1), $L(uw) = \{\gamma\}$. \square

Claim 2. *G is 2-connected. In particular, F is a cycle.*

Proof. Suppose not. Then there exists $v \in V$ and two induced connected subgraphs G_1 and G_2 of G where $G_1 \cap G_2 = v$ and $G_1 \cup G_2 = G$. Moreover, both G_1 and G_2 have at least two vertices. By symmetry assume that $P \subseteq G_1$. By the minimality of G , there exists an L -coloring φ of G_1 . Let L_2 be a list assignment on $V(G_2)$ such that $L_2(u) = \{\varphi(v)\}$ if $u = v$, and $L_2(u) = L(u)$ otherwise. Since L_2 and G_2 satisfy the assumptions of Theorem 5, there exists an L_2 -coloring ψ of G_2 . Colorings φ and ψ coincide on v ; hence $\varphi \cup \psi$ is an L -coloring of G , a contradiction. \square

Claim 3. (1) $|N(v) \cap V(P)| \leq 1$ for all $v \in I$, and (2) $V(F) \setminus (I \cup V(P)) \neq \emptyset$.

Proof. The minimality of G and (iii) imply (1). Using Claim 2, $F - P$ is a path. Since I is independent, if $V(F) \subseteq I \cup V(P)$ then $|I \cap V(F)| = 1$, contradicting (1). \square

Claim 4. G does not contain a separating triangle with a vertex in I .

Proof. Let $T = xyz$ be a separating triangle in G and let $x \in I$. Assume that $P \subseteq \text{Ext}_T(G)$ and $|V(\text{Int}_T(G))| \geq 4$. By the minimality of G , there exists an L -coloring φ of $\text{Ext}_T(G)$.

Let $G' := \text{Int}_T(G) - z$, $I' := I \setminus V(\text{Ext}_T(G))$ and $P' = xy$. Define a list assignment L' on vertices $u \in V(G')$ in the following way:

$$L'(u) = \begin{cases} \varphi(u) & \text{if } u \in \{x, y\}, \\ L(u) - \varphi(z) & \text{if } uz \in E(G - P'), \\ L(u) & \text{otherwise.} \end{cases}$$

Since $x \in I$, no neighbor of x is in I' . Thus condition (iii) of Theorem 5 is satisfied for G', I', P' and L' . Condition (ii) is witnessed by φ . Since each vertex $u \in N_{G'}(z)$ is on the outer face of G' , but not G , it is straightforward to check that (i) is satisfied. Hence G' has an L' -coloring φ . The coloring $\varphi \cup \psi$ is an L -coloring of G , a contradiction. \square

Claim 5. If xy is a chord of F then neither x nor y is in $V(P)$, and there exists $z \in I \cap V(F)$ such that $|L(z)| = 2 = d(z)$, $L(zx) \neq L(zy)$, and $xzy \subseteq F$.

Proof. Suppose $xy \in E$ is a chord of F . Let G_1 and G_2 be subgraphs of G where $G_1 \cap G_2 = xy$ and $G_1 \cup G_2 = G$. Since xy is a chord, both G_1 and G_2 have at least three vertices. By symmetry assume that $P \subset G_1$.

First suppose G_2 contains exactly three vertices, say x, y, z . Using Claim 1, $2 \leq |L(z)| = |L(zx) \cup L(zy)| \leq d(z) \leq 2$. So $|L(z)| = 2 = d(z)$ and $L(zx) \neq L(zy)$. By condition (i), $|L(z)| = 2$ implies $z \in I \cap V(F)$. Thus $xzy \subseteq F$, since x and y are the only possible neighbors of z . Finally, since $L(zx) \neq L(zy)$, Claim 1.3 implies $L(xy) \not\subseteq L(zx) \cup L(zy)$. Thus $|L(x)|, |L(y)| \geq 2$, and so $x, y \notin P$.

Now suppose for a contradiction that G_2 has at least four vertices. Define G'_1 in the following way. If there exists a vertex $v \in V(G_2) \cap I$ such that v is adjacent to both x and y then G'_1 is obtained from G_1 by adding a new vertex v' adjacent to x and y to the outer face of G . Moreover, let $I' = (I \cap G_1) \cup \{v'\}$ and let L' be an extension of L by defining $L'(v') = L(vx) \cup L(vy)$. Notice that v is unique if it exists, since Claim 4 implies G has no

separating triangles that contain a vertex of I . If no such v exists, let $G'_1 = G_1$, $L' = L$, and $I' = I \cap V(G_1)$. If G'_1 contains v' , neither x nor y is in I . Hence I' is indeed an independent set. Using that $v' \in I'$ is on the outer face, L' satisfies conditions (i,ii,iii). By the minimality of G , there exists an L' -coloring φ of G' which gives an L -coloring of G_1 .

Define a list assignment L_2 on $V(G_2)$ by $L_2(u) = \{\varphi(u)\}$ if $u \in \{x, y\}$, else $L_2(u) = L(u)$. We wish to use xy as P . Conditions (i,ii) of Theorem 5 hold since G satisfies them. For (iii), consider a vertex $w \in I$ with $\{x, y\} \subseteq N(w)$. As remarked above, $w = v$. Since $L'(v) = L(vx) \cup L(vy)$, and φ is an L' -coloring of G' , there exists $u \in \{x, y\}$ with $\varphi(v) \in L(vu)$. Then $\varphi(v) \neq \varphi(u)$ implies $\varphi(u) \notin L(v)$, and (iii) holds. By the minimality of G , there exists an L_2 -coloring ψ of G_2 . Colorings φ restricted to G_1 and ψ coincide on xy ; hence $\phi \cup \psi$ is an L -coloring of G , a contradiction. \square

By the minimality of the sum of the sizes of the lists, we can assume $|V(P)| \geq 1$. Let $F = v_0v_1v_2v_3 \dots v_t$, where $v_0 \in V(P) \subseteq \{v_0, v_1\}$, identifying index i with index $i + t + 1$. Choose $v_i \in V(F) \setminus (I \cup V(P))$ with minimum index i . Such an index exists by Claim 3. Claim 5 implies v_iv_{i-2} is not a chord, and condition (i) implies $L(v_i) - L(v_iv_{i-1}) - L(v_iv_{i+1}) \neq \emptyset$.

Select a set $X \subseteq \{v_i, v_{i+1}, v_{i+2}\}$ and an L -coloring φ of X by the following rules:

- (X1) If v_iv_{i+2} is not a chord then set $X = \{v_i\}$ and pick $\varphi(v_i) \in L(v_i) \setminus (L(v_{i-1}) \cup L(v_{i+1}))$.
- (X2) Else, if there is $c \in L(v_i) \setminus (L(v_{i-1}) \cup L(v_{i+1}) \cup L(v_{i+2}))$, then set $X = \{v_i\}$ and $\varphi(v_i) = c$.
- (X3) Else set $X = \{v_i, v_{i+1}, v_{i+2}\}$. Pick:

- (a) $\varphi(v_{i+2}) \in L(v_{i+2}) \setminus (L(v_{i+3}) \cup L(v_{i+3}v_{i+4}))$;
- (b) $\varphi(v_i) \in L(v_iv_{i+2})$, if $\varphi(v_{i+2}) \notin L(v_iv_{i+2})$; else $\varphi(v_i) \in L(v_iv_{i+1})$;
- (c) $\varphi(v_{i+1}) \in L(v_{i+1}) - \varphi(v_i) - \varphi(v_{i+2})$.

See Figure 1 for an illustration of these rules. Observe that exactly one of (X1), (X2), or (X3) applies and X is well defined. Also, in cases (X2) and (X3), Claim 5 implies $v_{i+1} \in I$, and so $v_{i+2} \notin I$. Thus the sizes of their lists are as claimed in Figure 1. In (X3) either $\varphi(v_i) \in L(v_iv_{i+2})$ or $\varphi(v_{i+2}) \in L(v_iv_{i+2})$. Also, by Claims 1(3) and 5, $L(v_iv_{i+2}) \not\subseteq L(v_{i+1})$. Hence φ is also well defined. Moreover, $d(v_{i+1}) = 2$, and so $N(v_{i+1}) = \{v_i, v_{i+1}\}$.

Let $G' = G - X$, $I' = I \setminus X$, and L' be the list assignment on $V(G')$ defined by

$$L'(v) = L(v) \setminus \{\varphi(x) : x \in N(v) \cap X\}.$$

It suffices to show that G' , L' , I' and P satisfy the assumptions of Theorem 5. Then by the minimality of G , there is an L' -coloring ψ of G' , and by the choice of L' , the function $\psi \cup \varphi$ is an L -coloring of G , a contradiction.

Now we verify that G' , L' , I' and P satisfy the assumptions of Theorem 5. Since I is an independent set, so is I' . Let $M = \{v \in V(G') : L'(v) \neq L(v)\}$. Clearly condition (i) holds for vertices in $V \setminus M$. By Claim 5, all chords have the form v_jv_{j+2} . Thus φ was chosen so that $M \cap V(F) = \emptyset$. Hence the condition (i) is satisfied for $v \in V(F)$. Condition (ii) holds since P did not change. Since $I' \subseteq I$, Claim 3(1) implies condition (iii).

It remains to show that every $v \in M$ satisfies condition (i). Let F' be the outer face of G' . Since each vertex of M has a neighbor in $X \subseteq F$, $M \subseteq F' \setminus F$. Thus it suffices to show that $|L'(v)| \geq |L(v)| - 1$. If $|N(v) \cap X| = 1$ then $|L'(v)| \geq |L(v)| - 1$. Otherwise $|N(v) \cap X| \geq 2$. Then v is handled by rule (X3). So $N(v) \cap X = \{v_i, v_{i+2}\}$ and $L(v_i v_{i+2}) \subset C := \{\varphi(v_i), \varphi(v_{i+2})\}$. If $L(vv_i) \neq L(vv_{i+2})$ then Claim 1(3) implies $L(v_i v_{i+2}) \not\subset L(v)$. Anyway, $|L(v) \cap C| \leq 1$, and we are done. \square

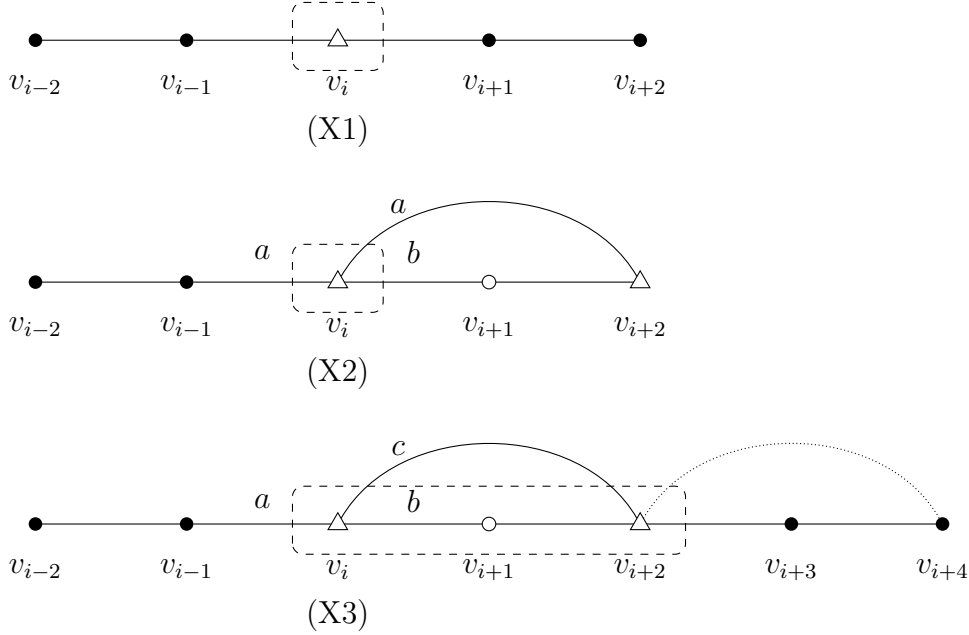


Figure 1: Rules (X1), (X2), and (X3). A black circle is a vertex with arbitrary list size, a white circle is a vertex with list size two, and a triangle is a vertex with list size at least three. The dashed box indicates X and a label on an edge is the common color of lists of its endpoints.

3 Lists of size 3 are necessary

In this section we give a proof of Theorem 4. The construction is analogous to the construction that bipartite graphs are not 2-choosable.

Proof of Theorem 4. Let k be given. Let G be a complete bipartite graph with part X of size $(k-1)^2$ and another part of size 2 formed by vertices a and b . Let L be a list assignment assigning to a a list of colors $\{a_1, \dots, a_{k-1}\}$ and to b a list of colors $\{b_1, \dots, b_{k-1}\}$. To the other vertices, L assigns distinct lists of form $\{a_i, b_j\}$ where $1 \leq i, j \leq k-1$. There are $(k-1)^2$ such lists which is exactly the size of X . Notice that $|L(u) \cup L(v)| = k$ for every edge uv . See Figure 2 for a sketch of G and L .

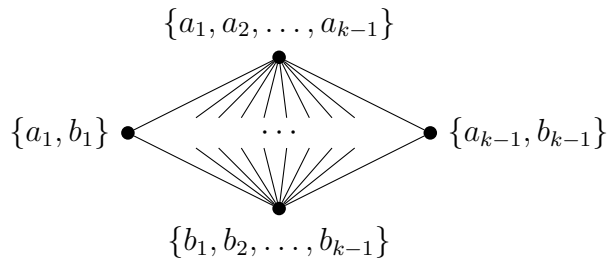


Figure 2: Construction from Theorem 4.

Suppose that there is an L -coloring of G . It assigns colors a_i to a and b_j to b for some $1 \leq i, j \leq k - 1$. However, there is a vertex with list $\{a_i, b_j\}$, a contradiction. Hence G is not L -colorable. \square

References

- [1] I. Choi, B. Lidický, and D. Stolee. On Choosability with Separation of Planar Graphs with Forbidden Cycles, 2013. arXiv:1303.2753.
- [2] Z. Füredi, A. Kostochka, and M. Kumbhat. Choosability with separation of complete graphs. manuscript.
- [3] Z. Füredi, A. Kostochka, and M. Kumbhat. Choosability with separation of complete multipartite graphs and hypergraphs, 2011. arXiv:1109.2969.
- [4] J. Kratochvíl, Z. Tuza, and M. Voigt. Brook’s-type theorems for choosability with separation. *J. Graph Theory*, 27:43–49, 1998.
- [5] C. Thomassen. Every planar graph is 5-choosable. *J. Combin. Theory Ser. B*, 62:180–181, 1994.
- [6] M. Voigt. List colourings of planar graphs. *Discrete Math*, 120:215–219, 1993.
- [7] M. Voigt. A not 3-choosable planar graph without 3-cycles. *Discrete Math*, 146:325–328, 1995.
- [8] R. Škrekovski. A note on choosability with separation for planar graphs. *Ars Combin.*, 58:169–174, 2001.