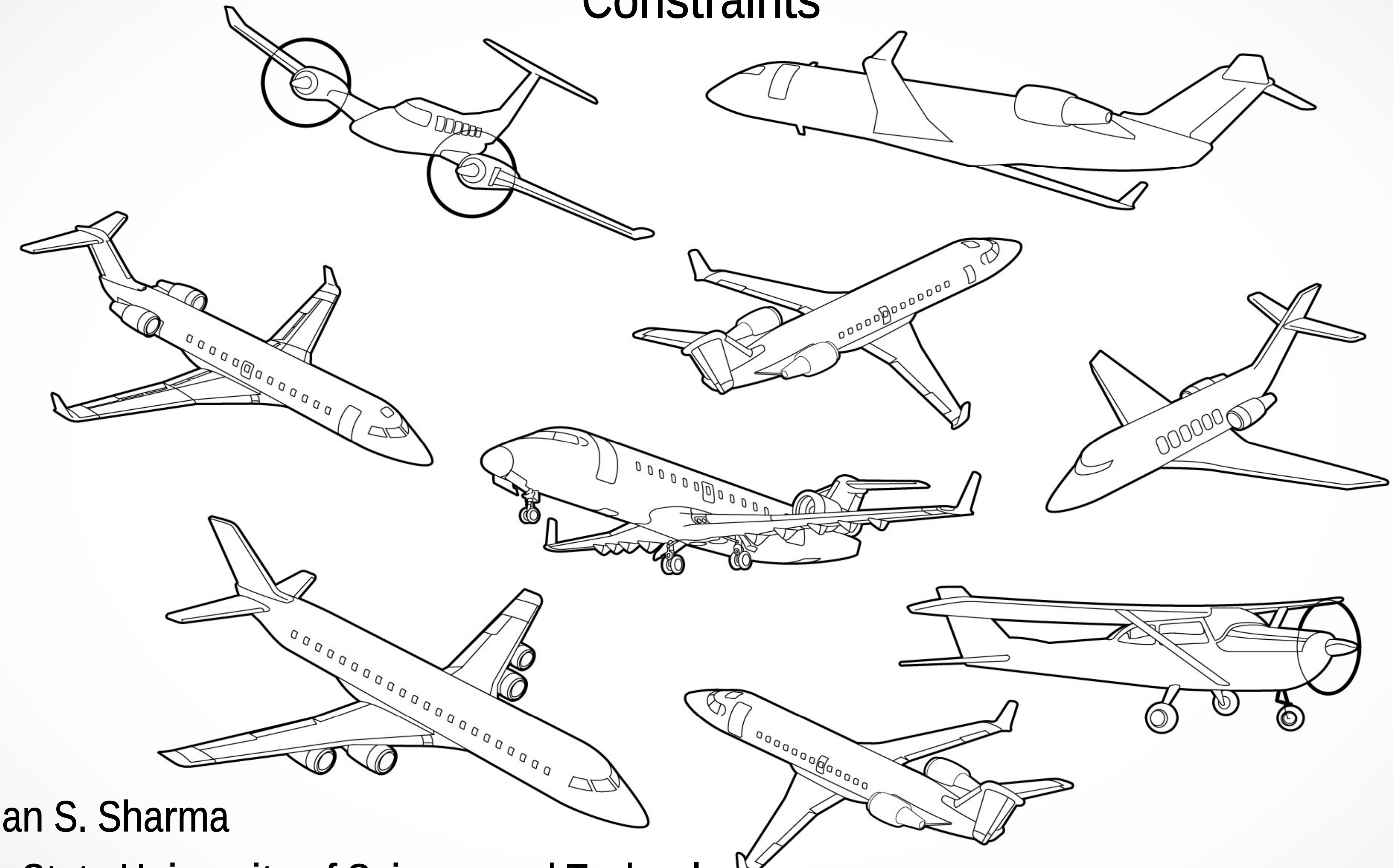
Computational Approach to Function Minimization and Optimization with Constraints



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SYMPOSIUM ON UNDERGRADUATE RESEARCH & CREATIVE EXPRESSION

Presentation Overview

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3. MATLAB Optimization Tools
4. Tested Functions
5. $f(x) = -5x_1 - 4x_2 - 6x_3$
6. $f(x) = \frac{1}{2}{x_1}^2 + {x_2}^2 - x_1x_2 - 2x_1 - 6x_2$
7. Ackley's Function
8. Cross-in-tray Function
9. Conclusion
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11. Acknowledgements

Introduction

- Various methods of optimization one can employ in MATLAB.
- Multiple functions of various types are selected and each optimization process is rigorously tested.
- To determine the best optimizer to be used for a set type of function.
  - MATLAB Environment
  - Processes tested for accuracy and computation time
- These processes are then used to test the function minimization process of typical optimization testing functions.

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Testing Functions

Common Benchmarks:

1. Unimodal, convex, multidimensional, and continuous.
   - Can cause slow or poor convergence to a single global extremum.
2. Multimodal, two-dimensional with a small number of local extremes, and continuous.
   - Functions of this type are employed to test the quality of an optimization tool or process.
   - Typically have few local extremes with a single global extreme.
3. Multimodal, two-dimensional with a huge number of local extremes, and continuous.
   - Higher number of local extremes.
   - Further test the quality of the tool used.
4. Multimodal, multi-dimensional with a large number of local extremes, and continuous.
   - Typical functions to appear in actual practice.

MATLAB Optimization Tools

- quadprog – quadratic programming
- lsqcurvefit – solved nonlinear curve-fitting (data-fitting) problems in least-squares sense
- lsqnonlin – Solve nonlinear least-squares (nonlinear data-fitting) problems
- fminsearch – Find minimum of unconstrained multivariable function using derivative-free method
- fminunc – Find minimum of unconstrained multivariable function
- linprog – Solve linear programming problems
- lsqlin – Solve constrained linear least-squares problems
- lsqnonlin – Solve nonlinear least-squares (nonlinear data-fitting) problems
- lsqnonneg – Solve nonnegative least-squares constraint problem
- fminbnd – Find minimum of single-variable function on fixed interval
- fmincon – Find minimum of constrained nonlinear multivariable function
- fseminf – Find minimum of semi-infinitely constrained multivariable nonlinear function
- bintprog – Solve binary integer programming problems

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Tested Functions

1. $f(x) = -5x_1 - 4x_2 - 6x_3$
2. $f(x) = \frac{1}{2}{x_1}^2 + {x_2}^2 - x_1x_2 - 2x_1 - 6x_2$
3. $\sum_{k=1}^{10}(2 + 2k - e^{kx_1} - e^{kx_2})^2$ starting at the point x = (0.3, 0.4)
4. $f(x) = 3{x_1}^2 + 2x_1x_2 + {x_2}^2$
5. $f(x) = -x_1x_2x_3$ starting at x = (10,10,10)
6. Ackley's Function
7. Bukin Function No. 6
8. Three-hump camel function
9. Easom function
10. Holder table function
11. Cross-in-tray function

$f(x) = -5x_1 - 4x_2 - 6x_3$

Subject to the following:

- $x_1 - x_2 + x_3 \leq 20$
- $3x_1 + 2x_2 \leq 30$
- $0 \leq x_2$

$3x_1 + 2x_2 + 4x_3 \leq 42$

$0 \leq x_1$

$0 \leq x_3$

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$f(x) = -5x_1 - 4x_2 - 6x_3$

Utilization of linprog:

```
Optimization terminated.
Elapsed time is 0.014165 seconds.

x =

    0.0000
   15.0000
    3.0000


ans =

    0.0000
    1.5000
    0.5000


ans =

    1.0000
    0.0000
    0.0000

fx >>
```

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$f(x) = -5x_1 - 4x_2 - 6x_3$

Utilization of lsqnonlin:

```
>> leastsquarenonlin
Warning: Trust-region-reflective algorithm requires at least as many
equations as variables; using Levenberg-Marquardt algorithm instead.
> In lsqncommon at 56
  In lsqnonlin at 237
  In leastsquarenonlin at 4

Local minimum found.

Optimization completed because the size of the gradient is less than
the default value of the function tolerance.

<stopping criteria details>

Elapsed time is 0.008820 seconds.
>> x

x =

   -5.0649   10.9481   -3.0779

fx >>
```

$f(x) = -5x_1 - 4x_2 - 6x_3$

Utilization of fminsearch:

```
>> minsearch

Exiting: Maximum number of function evaluations has been exceeded
         - increase MaxFunEvals option.
         Current function value: -95997968077443712000000000000000

x =

   1.0e+54 *

   -0.0006    2.3277    0.0486

Elapsed time is 0.020214 seconds.
>> 
```

$f(x) = -5x_1 - 4x_2 - 6x_3$

Utilization of fminunc:

```
>> minunc
Warning: Gradient must be provided for trust-region algorithm;
  using line-search algorithm instead.
> In fminunc at 383
  In minunc at 3

Problem appears unbounded.

fminunc stopped because the objective function value is less than
or equal to the default value of the objective function limit.

<stopping criteria details>

Elapsed time is 0.024999 seconds.
>> x

x =

   1.0e+19 *

    1.1398    0.9118    1.3677

>>
```

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$f(x) = -5x_1 - 4x_2 - 6x_3$

Utilization of fmincon:

```
>> mincon
Warning: The default trust-region-reflective algorithm does not solve
problems with the constraints you have specified. FMINCON will use the
active-set algorithm instead. For information on applicable algorithms, see
Choosing the Algorithm in the documentation.
> In fmincon at 501
  In mincon at 5
Warning: Your current settings will run a different algorithm
(interior-point) in a future release.
> In fmincon at 506
  In mincon at 5

Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the default value of the function tolerance,
and constraints are satisfied to within the default value of the constraint to

<stopping criteria details>

Active inequalities (to within options.TolCon = 1e-06):
  lower      upper     ineqlin   ineqnonlin
                          2
                          3
                          4
Elapsed time is 0.021024 seconds.
>> x

x =

    0.0000   15.0000    3.0000

>>
```

$f(x) = -5x_1 - 4x_2 - 6x_3$

Evaluation:

- Linear equation
- linprog optimization tool yielded the accurate result in the least time possible:

$$x_1 = 0 \quad x_2 = 15 \quad x_3 = 3$$

Time Elapsed = 0.014165 seconds

$$f(x) = \frac{1}{2}{x_1}^2 + {x_2}^2 - x_1 x_2 - 2x_1 - 6x_2$$

Subject to the following:

- $x_1 + x_2 \leq 2$
- $-x_1 + 2x_2 \leq 2$
- $2x_1 + x_2 \leq 3$
- $0 \leq x_1$
- $0 \leq x_2$

- Polynomial function with quadratic roots.

$$f(x) = \frac{1}{2}{x_1}^2 + {x_2}^2 - x_1x_2 - 2x_1 - 6x_2$$

Method 1 solutio … o optimize.

- The algorithm … imize the function. 'active-set' ca… creases the computationa… lgorithm is effective on some probl… traints. It is not a large-scale alg…

```
Elapsed time is 0.005229 seconds.

x =

    0.6667
    1.3333


fval =

   -8.2222


exitflag =

     1


ans =

    0.3820
    2.6180

>>
```

$$f(x) = \frac{1}{2}{x_1}^2 + {x_2}^2 - x_1x_2 - 2x_1 - 6x_2$$

Method 2 soluti … ize.

- . The algorith … to compare its operation an … used algorithm 'active-set'. 'i … , sparse, complex prob … ms. The algorithm sat … n recover from NaN or Inf re

```
                                                    First-orde
 Iter          f(x)          Feasibility     optimalit
    0     -2.000000e+00      1.000e+01      4.500e+0
    1     -2.630486e+01      0.000e+00      9.465e-0
    2     -2.639877e+01      0.000e+00      3.914e-0
    3     -2.639881e+01      0.000e+00      3.069e-1

Minimum found that satisfies the constraints.

Optimization completed because the objective f
feasible directions, to within the default val
and constraints are satisfied to within the de

<stopping criteria details>

Elapsed time is 0.010300 seconds.

fval =

  -26.3988


eflag =

     1

fx >>
```

$$f(x) = \frac{1}{2}{x_1}^2 + {x_2}^2 - x_1 x_2 - 2x_1 - 6x_2$$

Utilization of lsqnonlin to optimize:

```
>> leastsquarenonlin
Warning: Trust-region-reflective
algorithm requires at least as many
equations as variables; using
Levenberg-Marquardt algorithm
instead.
> In lsqncommon at 56
  In lsqnonlin at 237
  In leastsquarenonlin at 4

Local minimum found.

Optimization completed because the size of the gradient is less than
the default value of the function tolerance.

<stopping criteria details>

Elapsed time is 0.024857 seconds.

x =

   -0.0662    0.0228

>>
```

$$f(x) = \frac{1}{2}{x_1}^2 + {x_2}^2 - x_1x_2 - 2x_1 - 6x_2$$

Utilization of fminsearch to optimize:

```
>> minsearch

x =

   10.0000    8.0000

Elapsed time is 0.004693 seconds.
>> 
```

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$$f(x) = \frac{1}{2}{x_1}^2 + {x_2}^2 - x_1 x_2 - 2x_1 - 6x_2$$

Utilization of fminunc to optimize:

```
>> minunc
Warning: Gradient must be provided
for trust-region algorithm;
  using line-search algorithm
  instead.
> In fminunc at 383
  In minunc at 3

Local minimum found.

Optimization completed because the size of the gradient is less than
the default value of the function tolerance.

<stopping criteria details>


x =

   10.0000    8.0000

Elapsed time is 0.018594 seconds.
>>
```

$$f(x) = \frac{1}{2}{x_1}^2 + {x_2}^2 - x_1 x_2 - 2x_1 - 6x_2$$

Utilization of fmincon to optimize:

```
constraints you have specified.
FMINCON will use the active-set
algorithm instead. For information
on applicable algorithms, see
Choosing the Algorithm in the
documentation.
> In fmincon at 501
  In mincon at 5
Warning: Your current settings will
run a different algorithm
(interior-point) in a future
release.
> In fmincon at 506
  In mincon at 5

Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the default value of the function tolerance,
and constraints are satisfied to within the default value of the constraint

<stopping criteria details>

Active inequalities (to within options.TolCon = 1e-06):
  lower      upper     ineqlin   ineqnonlin
                          1
                          2

x =

    0.6667    1.3333         0

Elapsed time is 0.087329 seconds.
>> |
```

$$f(x) = \frac{1}{2}{x_1}^2 + \ {x_2}^2 - \ x_1x_2 - 2x_1 - 6x_2$$

Evaluation:

- Polynomial function with quadratic roots.
- quadprog optimization toolset yielded the most accurate result in the shortest time possible.

- $x_1 = 0.6667 \quad x_2 = 1.333$
- $Eigenvalues\ of\ matrix\ H = 0.3820 \ \ \& \ \ 2.6180$
- Time Elapsed = 0.005229 seconds

Ackley's Function

N-dimensional multimodal function which has a large number of local minima but only has one global minimum.

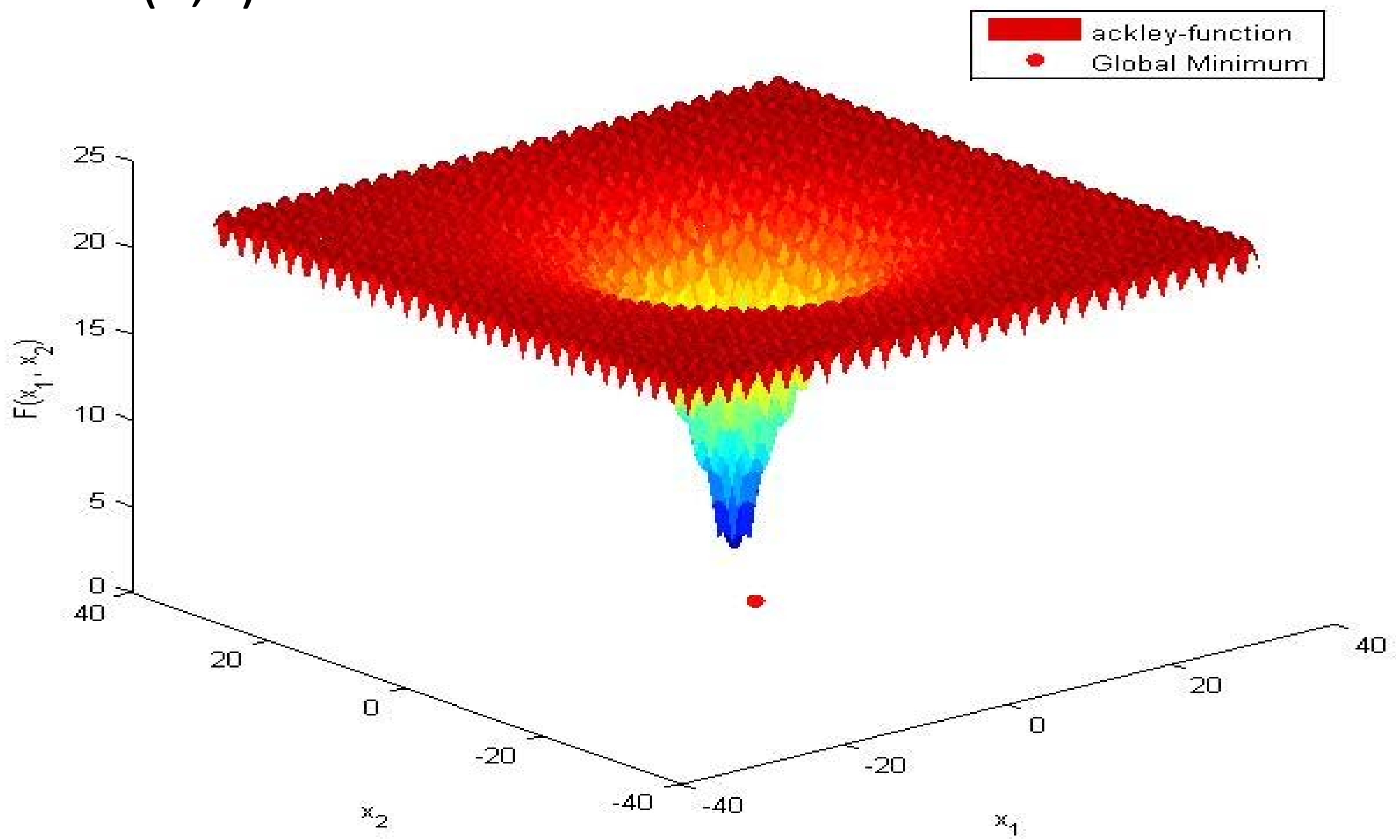
1. Set a relevant domain
2. Converging on the minimum value

$$f(x,y) = -20\exp\left(-0.2\sqrt{0.5(x^2 + y^2)}\right) - \exp\left(0.5\left(Cos(2\pi x) + Cos(2\pi y)\right)\right) + 20 + e$$

- Subject to the following: $-5 \leq x, y \leq 5$

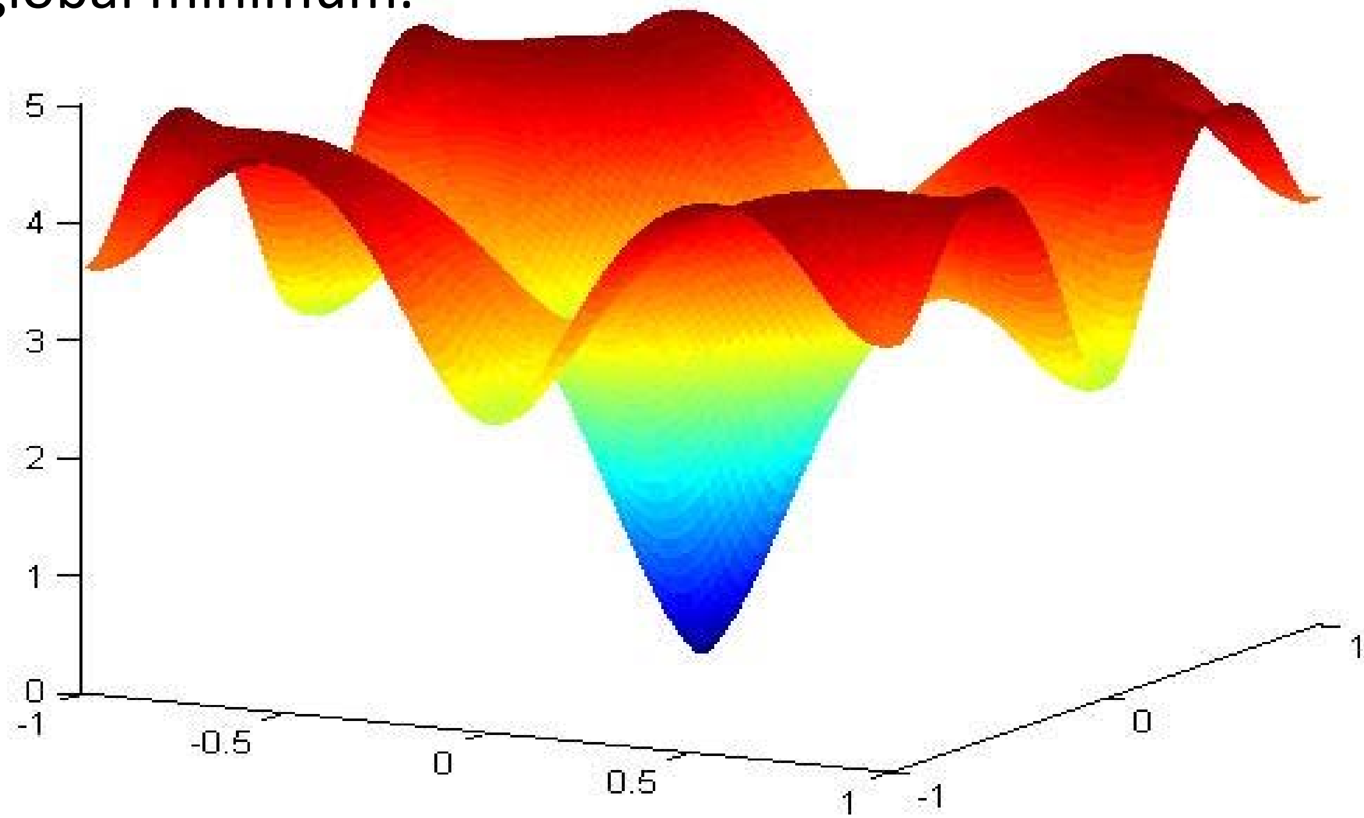
Ackley's Function

Solution (0,0)



Ackley's Function

Changing domain to [-1,1] we see the following convergence on the global minimum.



Cross-in-tray Function

$$f(x,y) = -0.0001\left(\left|\sin(x)\sin(y)\exp\left(\left|100-\frac{\sqrt{x^2+y^2}}{\pi}\right|\right)\right|+1\right)^{0.1}.$$

- Has multiple global minima but four distinct global minima.
- Function is typically evaluated on the square x – [-10, 10] and y – [-10, 10].
- The global minima are as follows:
  1. (1.3491, -1.3491)
  2. (1.3491, 1.3491)
  3. (-1.3491, 1.3491)
  4. (-1.3491, -1.3491)

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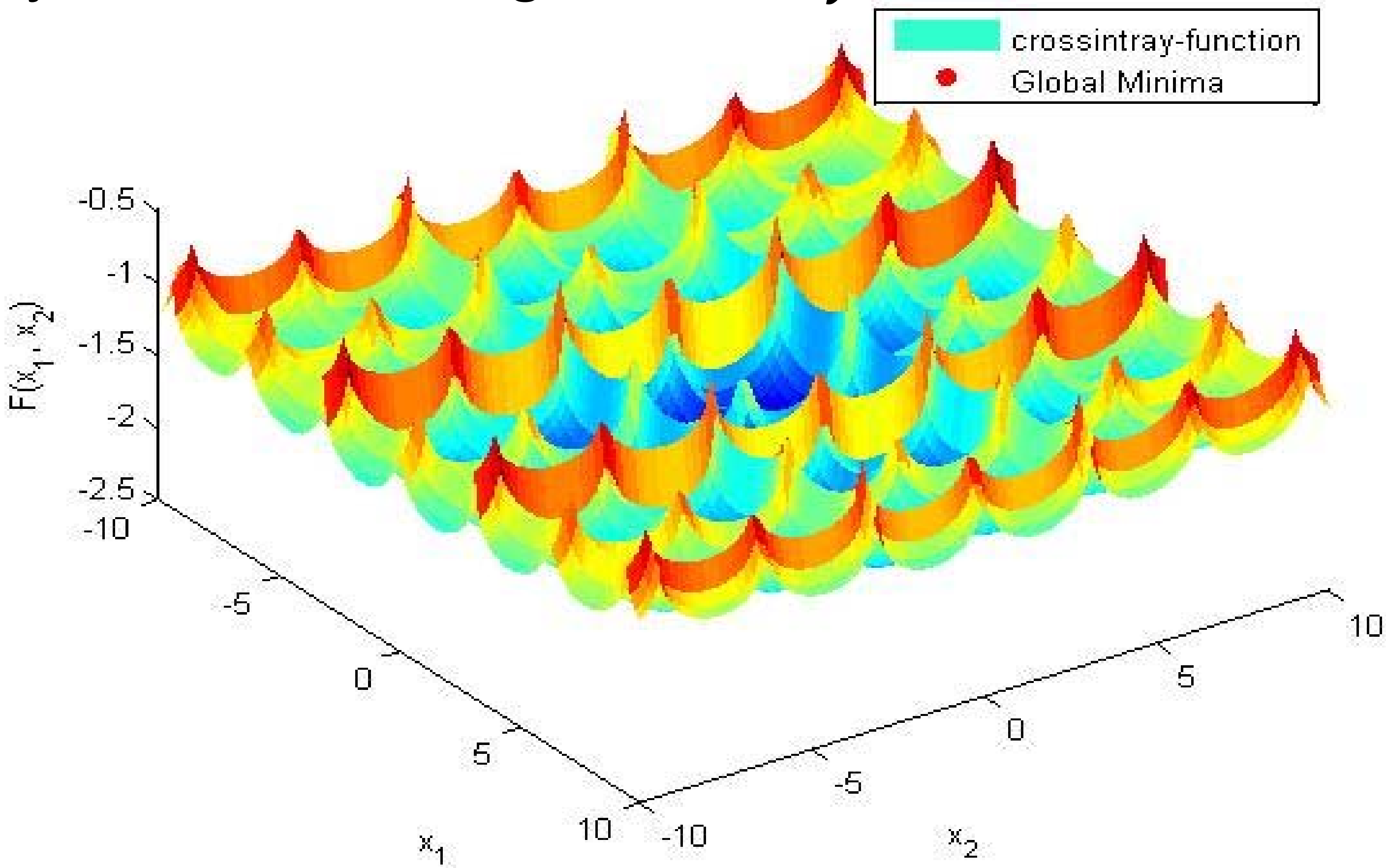
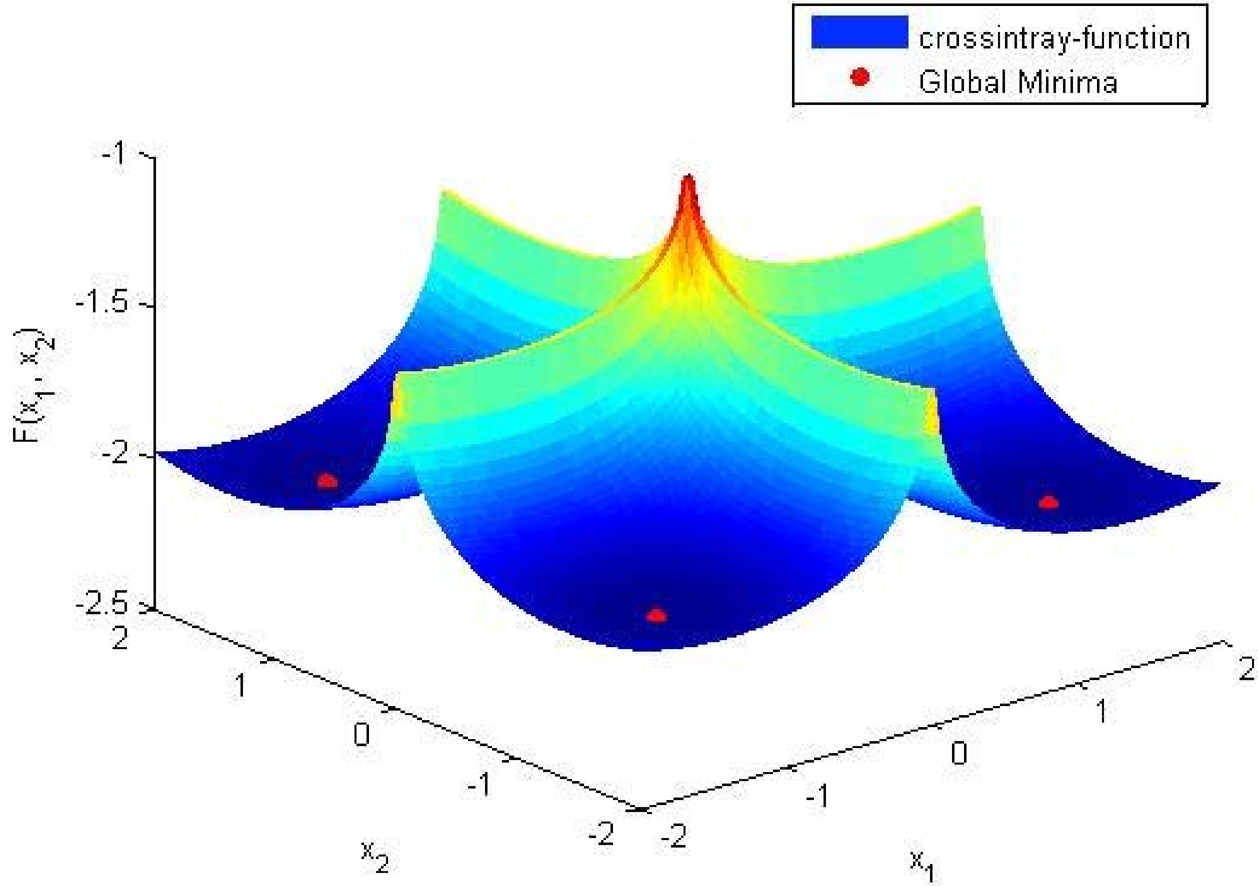
Cross-in-tray Function

Subject to the following $-10 \leq x, y \leq 10$



Cross-in-tray Function

Subject to the following $-2 \leq x, y \leq 2$



Conclusion

Judging from the computation time and the accuracy of each optimization algorithm a table has been prepared depicting the type of optimization to be employed given a particular type of constraint and function.

Constraint Type	Objective Type				
	Linear	Quadratic	Least Squares	Smooth nonlinear	Non-smooth
None	N/A	quadprog	lsqcurvefit, lsqnonlin	fminsearch, fminunc	fminsearch
Bound	linprog	quadprog	lsqcurvefit, lsqlin, lsqnonlin, lsqnoneg	fminbnd, fmincon, fseminf	fminbnd
Linear	linprog	quadprog	lsqlin	fmincon, fseminf	
General Smooth	fmincon	fmincon	fmincon	fmincon, fseminf	
Discrete	bintprog	N/A	N/A	N/A	N/A

Future Plans

- Research submitted to the Journal of Optimization
- Continue research modifying developed optimization toolsets
- Currently working on a nozzle optimization tool.
- Develop a Wing Planform Exploration Tool (PET)
  - Aerodynamic Analysis
  - Structural Analysis
  - Cost
  - Ease of manufacturing

Acknowledgements

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