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The theoretical input-output system with flexible technological coefficients based on the two-stage level CES-type production function

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Iowa State University, 1990
The theoretical input-output system
with flexible technological coefficients
based on the two-stage level CES-type production function

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1. INTRODUCTION

In an economic system, many (variable) factors exert on and variate the economic phenomena through their causal interactions within resource restraints. The impacts of such factors lead to the price and the quantity variations of economic resources. Under these changing circumstances, economic units such as consumers (or purchasers) and producers (or sellers) pursue to make rational choices. The respective choice makings are usually assumed to be performed through the consumers' satisfaction-maximization rule and the producers' gain-maximization rule (or the producers' cost-minimization rule), subject to the money and/or resource constraints. Such choice makings of economic units have influences on demand for and supply of resources and, in turn, on the price system in the market mechanism. In other words, the (variations of) production-consumption activities and supply-demand of resources initiate (or are affected by) (the changes of) the economic phenomena in an economy through the price system.

Like this, the economic phenomena vary in response to the various economic behaviors (and vise versa). They are actually interdependent in an economy. Under such an interdependent economic system, a method of systematically quantifying the interrelationships among the economic behaviors is the input-output analysis, which was begun with W. W. Leontief's paper of 1936. The ideas of input-output analysis
have been applied to a number of areas (for instance, theoretical regional analysis proposed by Walter Isard (1951)) by many economists and analysts.

Specially, Metzler (1951b) and Atsumi (1981) analyzed the combined effects of the tax-and-subsidy on price movements, that is, on the competitive supply prices of output, in the Leontief input-output model. From such economic analyses of the interindustrial price effects of the tax-and-subsidy, both of them reached to the same conclusion, though the assumptions each of them used are different, as follows: In the new position of equilibrium, after all prices have been adjusted to the tax-and-subsidy, the price of the taxed commodity rises and the price of the subsidized commodity falls; the primary effect of the tax or subsidy exceeds the secondary effect by changing the cost of production in the taxed or subsidized industry. On the other hand, for the commodities of the other industries, price movements are ambiguous; that is, they may either rise or fall. However, they were concerned with the effects of the tax-and-subsidy only on price movements rather than movements of the productive resources.

On the other hand, to my knowledge, Theil and Tilanus (1964) among many economists utilized the input-output analysis were the first who simply and comprehensively introduced price sensitivities on the output in the input-output relation.

^Metzler (1951b) demonstrated the combined effects of the specific tax-and-specific subsidy on price movements under the restrictive assumptions that (1) the production coefficients are held constant; (2) the outputs or the final demands are also constant; (3) the primary resources are also remaining constant: this implies that there is no explicit role for the primary factors and the imported foreign factors, so that possibilities of substitution between such factors and the intermediate-good factors have to be ignored. On the other hand, Atsumi (1981) stated the combined effects of the *ad valorem* tax-and *ad valorem* subsidy as well as the specific tax-and-specific subsidy on price movements after he relaxed the assumptions (2) and (3) Metzler (1951b) used.
They however discussed the case in which all producing sectors are characterized by Uzawa's (1962) type (or modified Mukerji's (1963) type) CES production function which has the linear homogeneity and the same elasticity of substitution among the factors of production. They dealt only with the factors of production (i.e., intermediate factors) which are distributed by $N$ origin-sectors in the input-output table.

The limited framework they used has the following problems: First, they excluded the productive factors such as the primary factors and the imported factors as arguments in the production function. Second, the same elasticity of substitution among the intermediate factors does not at least describe the interactions among 'all' factors of production in the production process. Thus, their limited framework with the problems mentioned above cannot catch the proper impacts of price sensitivities on the economy.

Most economists and analysts show tendencies that they prefer to work with models designed for a specific purpose and stressing a particular set of structural relations rather than to create ever larger general-purpose models. To my knowledge, most of studies analyzed the input-output theory under the conventional framework with the Leontief-type rigid production function.

1.1 The Objectives of the Study

The objectives of this study are as follows:

First, this study is to develop the theoretical input-output system with flexible technological coefficients based on the two-stage level CES-type production function within the frameworks of the traditional input-output theory and the neo-classical theory of production and cost.
Second, this study is to analyze the economic effects of the exogenous economic element such as the world market price of the imported material (for instance, oil) on the prices of outputs, the equilibrium domestic outputs, and the domestic resource allocations in a small importing country, under the theoretical input-output system with flexible technological coefficients.

1.2 The Composition of the Study

This study is developed as follows.

Chapter 2 and Chapter 3 give a review of the characteristics of the Leontief input-output system which is needed for our economic impact analysis. Chapter 2 is concerned with the historical background of appearance of the input-output system and Chapter 3 deals with both the fundamental and the extended frameworks of the input-output relations. In the following Chapter 4, the general production function is defined and the assumptions or regularity conditions for the neo-classical theory of production and costs are provided and described.

Chapter 5 is concerned with the concepts of substitutability and/or complementarity relationships between the factors of production utilized in the production process. Such relationships are expounded in relation to the price and the output elasticities and the elasticity of substitution. In Chapter 6, we examine the price structures containing taxes, tariffs, the world price of the imported foreign factor, and the rate of foreign exchange. Also, the impacts of taxes, tariffs, and the foreign exchange rate on economic welfare are explained briefly.

In Chapter 7 and Chapter 8, on the basis of the general production function discussed in Chapter 4, the specific production function - the two-stage level CES-
type production function is introduced for our purpose. The two-stage level CES-type production function can properly reflect the factor substitutabilities to the production of output. Chapter 7 expounds the two-stage level CES-type production function in association with (strong) functional separability. On the other hand, in Chapter 8, that production function is described with relevance to the input-output framework. The following Chapter 9 shows the two-stage optimization methodology concerning how to obtain the optimal factor-resource demand functions and how to calculate the production parameters in each industry in an economy; the elasticities of substitution of inputs both within the same factor category (group) and in the different factor categories (groups).

In Chapter 10, we derive the general system for getting the endogenous prices of the intermediate factors (commodities). The solutions of that system depend on the technological indicator, the elasticities of factor substitution, and the exogenous prices of the non-intermediate factors. On the basis of such solutions of the general price system, we finally derive the general open input-output system with the flexible input-output coefficients which reflects technological improvements, factor substitutabilities, and relative prices in producing outputs and in the allocation of the production factors. And we briefly examine the special cases such as the Cobb-Douglas case and the Marx-Leontief case under the special assumptions.

In Chapter 11, with the $3 \times 3$ analytical model based on the general system discussed in Chapter 9 through Chapter 10, we algebraically and numerically analyze the economic effects of the change in the world market price of the imported foreign factor (e.g., oil) on the prices of outputs, the equilibrium domestic outputs, and the domestic resource allocations in a small importing country.
Chapter 12 and Chapter 13 are Summary and Conclusions, and Appendix, respectively.
2. INPUT-OUTPUT SYSTEM AND GENERAL EQUILIBRIUM SYSTEM

2.1 Literature Review

For the background devoted to the birth of Leontief's input-output system, we need to investigate a specific historical origin of the system.

Quesnay, physiocrat, in 1769 first introduced the schema of general interdependence among sectors through a drastic simplification of the economic system into three interacting sectors - farmers, artisan, landowner - in his Tableau Économique. In other words, Quesnay's Tableau Économique illustrated the phenomenon of mutual interdependence among industries by a zigzag graphic diagram. Out of this emerged a conception of the closed stationary state as a circular flow which in each period repeats itself. Quesnay's idea of general interdependence among sectors in the economic system provided the foundation of economic analysis for production, consumption, and distribution. In other words, it prepared the ground for an analysis of the interrelations of the whole economic system. This Quesnay's establishment became one of origins of the Leontief's input-output (closed) model.¹ Quesnay's basic idea, too, seems to be the basis of general equilibrium system.

In the multi-market mechanism in an economy, commodity and factor prices

¹See Blaug (1987, pp. 24-28) and Leontief (1951, p. 9).
are determined simultaneously. This proposition cannot be expounded by partial equilibrium analysis framework. In other words, the existence of $N$ partial equilibria does not guarantee general equilibrium for the whole economy consisted of $N$ markets.

Cournot (1897) had realized that "for a complete and precise solution of the partial problems of the economic system, it is inevitable that one must consider the system as a whole." Cournot, however, thought that the problem of general equilibrium was beyond the resources of mathematical analysis.

Walras (1874/1954) suggested the construction of equations relating input and output though it does not quite add up to the input-output analysis. Walras seized the partial problems of the economic system, and established the Walrasian system of general market equilibrium made up of a set of simultaneous equations. Specifically, he first formulated the state of the economic system at any point of time as the solution of a system of simultaneous equations consisted of the consumers' demand for commodities, the supply of commodities by producers, and the equilibrium conditions (or market-clearing conditions) that supply equals demand on every market - the factor market and the commodity market. This expresses the interdependence of all prices and quantities. The problem of determining the existence of general market equilibrium amounts to the problem of finding a unique (complete and consistent) solution for a set of simultaneous equations. The usefulness of the general equilibrium system depends upon whether or not the system has a unique solution.

Walras showed that the system is capable of being solved, at least in principle. He thought that if the number of equations to be solved equals the number of unknowns (or variables) to be determined, a complete and consistent solution

---

\(^2\text{Cournot, too, was the first writer to define and draw a demand function.}\)
of the Walrasian system exists. However, equality of the number of equations and the number of unknowns is neither necessary nor sufficient for the existence of a unique solution to a system of equations.\(^3\) Even though a unique solution exists, the weakness of the Walrasian system follows that a unique solution of general market equilibrium may involve nonpositive prices or quantities of commodities and factors of production. This fact means that the Walrasian general equilibrium system allows for negative (or zero) prices and negative (or zero) quantities of commodities and productive factors which are economically meaningless. This also implies that the Walrasian system allows for free goods and nuisance goods. Hence, Walras' demonstration of the existence of a general equilibrium is not only mathematically clumsy but unsatisfactory. There is, however, an architectonic quality to the whole performance as the achievement of theoretical analysis. Here we should note in relation to the input-output system that Walras introduced the fixed technical coefficients of production in his general market equilibrium system.\(^4\) Since L. Walras did not give conclusive arguments that the simultaneous equations of the general market equilibrium have a unique solution, many economists and mathematicians tried to generalize the Walrasian equation systems of mathematical economics.

Cassel (1924), the popularizer of the Walrasian system, presented his formulation of the Walrasian system with four basic principles: (1) demand for each final commodity is a function of the prices of all final commodities; (2) zero profits for all producers; (3) fixed technical coefficients relating the utilization of primary factors of production

\(^3\)For more details see Dorfman, Samuelson, and Solow (1958, p. 350).

\(^4\)Walras states that he takes the fixed technological coefficients of production as given only for convenience, that of course technical substitution is possible, that the technical coefficients depend on the prices of the factors of production. See Dorfman, Samuelson, and Solow (1958, p. 348).
to output of final commodities; (4) equality of supply and demand on each market.

Cassel's system may be written as follows:

\[ X_i = f_i(P_1, P_2, ..., P_m) \]  
\[ \sum_j \alpha_{ij} W_j = P_i \text{ for all } i, \]  
\[ \sum_i \alpha_{ij} X_i = r_j \text{ for all } j, \]

where \( P_j \) = the price of final commodity \( j \),
\( X_i \) = the demand for final commodity \( i \),
\( \alpha_{ij} \) = the amount of primary factor \( j \) utilized in production of one unit of commodity \( i \); this is technological coefficients of the Walras-Leontief type,
\( W_j \) = the price of factor \( j \),
\( r_j \) = the amount of factor \( j \) available initially.

Neisser (1932) and Stackelberg (1933) raised questions of existence and uniqueness of a solution to the Cassel's formulation of the Walrasian system, in relation to the requirement that prices and quantities be represented by nonnegative numbers. Neisser (1932, pp. 424-425) mentioned that the Cassel's formulation of the Walrasian system might have negative prices or quantities as solutions, which are economically meaningless. He also observed that even some variability in the technical coefficients might not be sufficient to remove the inconsistency. Stackelberg (1933) indicated that equation (2.3) would have, in general, no solution if the number of factors \( r_j \) is larger than that of commodities \( X_i \). He noted that the economic meaning of this inconsistency was that some of the equations in (2.3) would become inequalities, with the corresponding resources (commodities) becoming free goods. He argued that this implied the loss of a certain number of equations and hence the indeterminacy of the
rest of the system. For this reason, he held that the assumption of fixed technical coefficients could not be maintained and the possibility of substitution in production must be admitted.\(^5\)

Zeuthen (1933) and Schlesinger (1934) made a suggestion that economic theory should expound not only the nonnegative prices and quantities (produced) of scarce commodities but also which goods are scarce and which are free (i.e., have a zero prices).\(^6\) Zeuthen (1933, pp. 2-3, 6) argued that the resources in the Casselian system were properly only the scarce resources; however, it could not be regarded as known a priori which resources (commodities) are free and which are not. Hence, equations (2.3) should be rewritten as follows:

\[ \sum_i \alpha_{ij} X_j \leq r_j, \]

with the additional statement that the price \( W_j = 0 \) if the strict inequality holds for any \( j \). Schlesinger (1934) took up Zeuthen's modification and made a suggestion that it might resolve difficulties found by Neisser and Stackelberg.\(^7\)

Wald (1935, 1951) proved the existence and uniqueness of solution to the system expressing the above problems (i.e., the Casselian system). He concerned and presented a static model of production in which each commodity in demand can be produced in one way. In the productive system, Wald assumed that all fixed proportions among the factors of production and the single output of every process

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\(^5\)See Arrow and Debreu (1954, pp. 94-95).

\(^6\)Zeuthen and Neisser pointed out that the market determines which goods (resources) shall be free and which scarce. That is, there is no external symbol of intrinsic scarcity or abundance. It depends on the structure of demand, on the availability of complementary factors, on production relations. Thus, the list of resources appearing in the Casselian system must be all-inclusive.

\(^7\)See Arrow and Debreu (1954, p. 95).
are nonnegative and at least one of fixed proportions in the production process of a commodity is positive, that is, every commodity requires at least one input in its production. He also assumed that the total availabilities of primary factors are given by nature and that there is a given static structure of demand. On the demand side, he makes assumptions concerning the demand functions (satisfying a monotonicity condition) instead of deriving them from the utility-maximization. Under these assumptions, he reached the conclusion that the general equilibrium system (i.e., the Casselian system) possesses an economically meaningful solution in the variables - quantities and prices of commodities and the factors of production.

Von Neumann (1945) generalized Wald’s model of production. He also adopted the fixed technological coefficients in producing a commodity, as L. Walras and A. Wald did. He introduced the concept of intermediate commodities (factors); that is, a commodity appears simultaneously as a factor of production of one production activity and as output of another activity but there are no primary factors. Since this circularity idea was extended to commodities demanded by consumers, Von Neumann’s model is a closed productive system, a pure production model, with no inflow of primary factors from outside or outflow of final products out of the system considered. He assumed that all productive activities are performed under unchanging technology. He treated prices as determined in competitive markets so as to satisfy a zero-profit condition on all activities engaged in.®

Arrow and Debreu (1954) proved the existence of a general equilibrium for a competitive economy with an integrated model of production, exchange, and consumption. They provided two theorems stating general conditions under which a

®See Koopmans (1951, pp. 1-2).
competitive equilibrium will exist. The first theorem asserts that if every individual has initially some positive quantity of every commodity available for sale, then a competitive equilibrium will exist. The second theorem asserts the existence of competitive equilibrium if there are some types of primary factor (e.g., labor) with the following two properties: (1) each individual can supply some positive amount of at least one such type of primary factor (e.g., labor), (2) each type of primary factor (e.g., labor) has a positive usefulness in the production of desired commodities.\textsuperscript{9}

Blaug (1987) says that the Walrasian system has an unique, economically meaningful solution, provided that (1) returns to scale are constant or diminishing, (2) there are no joint products or external effects either in production or in consumption, and (3) all goods are gross substitutes for each other, in the sense that a rise in the price of one good will always produce positive excess demand for at least one other.\textsuperscript{10}

With this background in economic theory, Leontief (1936) introduced the fundamentals of input-output theory. However, Leontief (1941), in his The Structure of the American Economy, 1919-1929, first presented the closed input-output system for an analysis of the structure of the economy through the input-output relations in detail. This was the first attempt to apply the general equilibrium theory to the analysis of an economic reality. Leontief's attempt was a considerable simplification of Walras' general market equilibrium system up to the point where the equations involved in it could be estimated statistically. In other words, the Leontief input-output system was an alternative theoretical formulation of Walras' general equilibrium system, which

\textsuperscript{9}See Arrow and Debreu (1954, p. 69).
\textsuperscript{10}See Blaug (1987, p. 577).
showed the structural interdependence in an economy in stationary equilibrium. The procedure of simplification was as follows: First, he aggregated (grouped) the countless individual commodities and numerous independent economic activity units such as production units (producers) and consumption units (consumers) which entered Walras' system into comparatively few outputs and industries. Secondly, he dropped the supply equations for primary factors of production and the demand equations for final consumptions. Thirdly, he adopted the particular rigid type of production function in linear form.

The system showed the interrelationships among the different parts (industries, sectors) of a national economy. In other words, the Leontief input-output system expresses mainly the typical productive and distributive interrelations in the structure of the national economy. Specifically, the system expounds the intersectoral, physical flows - inflows or outflows in the physical units; costs (outlays, expenditures) or revenues (sales, receipts) in the monetary unit - of commodities from one industry to the other industries which are all the productive industries. Here we should note that this system - the closed input-output system - does not include the autonomous sector\(^{11}\) because in this system consumers are also treated as a productive industry (sector) which utilizes the outputs of the other industries in direct proportion to its outputs. In other words, the autonomous sector is explained within the system. This means that it plays a role of an endogenous variable in the system. Consequently, the system does not involve the components of final demand such as personal (household) consumption, investment (capital formulation including

\(^{11}\)The autonomous sector has no output, represents final demand one and is unexplained within the model. In other words, the autonomous sector plays a role of an exogenous variable in the system.
inventories), government spending, export. In sum, all sectors are endogenous in a “closed” input-output system. This system represents a comprehensive view of the structure of the economy as a whole.

On the other hand, Leontief (1944) introduced first the open system of input-output relations and described a method of estimating the quantitative relationship which exists between the primary demand for the products of all the various sectors (industries) of the national economy, on the one hand, and the total output and employment of primary factor in each of them in the open system, on the other. Specifically, he explained the method of computation of the total outputs and corresponding employment of primary factor (i.e., labor) of all the different industries from the given final demands available for consumption (and for new investment), based on the assumption that the production of a given quantity of any particular types of commodity requires a definite technically determined amount of direct labor combined with certain also technically determined amount of products of other industries. That is, he adopted the labor input coefficient showing the relationship between the primary factor (i.e., labor) and the total output of the separate industries.

The theoretical concepts underlying the Leontief input-output systems have been adapted to the objective of investigating quantitative policy from an analysis of observable variables of a more or less aggregative type. That is, the system provides an empirical basis for an analysis of the effects, on the activity levels in individual industries of given changes in the composition of final demand by industries supplying final commodities.

The Leontief system assumes the following: The production processes of an industry are regarded as one activity (process). The fixed technological coefficients
are assumed. The way of measurement of technological coefficients (or input-output coefficients) is the observation of the money value of all commodities flowing from one industry (sector) to each other industry in a census year. The Leontief system precludes the separate measurement of alternative processes to produce the same commodity (although there are many processes in the given technology.) In other words, the system assumes that each industry (sector) produces a single, homogeneous commodity. This means that each industry has one production activity in a process among many processes under the given technology. Also, constant returns to scale is assumed. The input-output system does not allow joint production.

In the Leontief system, substitution possibilities between factors of production have not been explicitly introduced. This implies that the Leontief-type production function does not show substitutability between factors of production. To avoid this limitation, we need another conditions such as engineering information: production function.
3. THE FUNDAMENTAL FRAMEWORK OF THE INPUT-OUTPUT RELATION

3.1 Introduction

The modern economy is no isolated island. Production activities are highly specialized, and industries (sectors) in an economy are interdependent. The variations in a single industry in the economy will thus generate a series of repercussions throughout the entire economy. The changes in any industry (sector) will, of course, be small, but they are transmitted from one industry to the other over the entire economy so that their cumulative total effect may be considerable. The extent to which repercussions generated by an industry have influences on the other industries depends on the degree of interdependence among various industries in the economy.

It is crucial, therefore, that we can evaluate the total effects, both direct and indirect, of a change in the economic behavior. How can we catch the total impacts of some change in a systematic way? Input-output analysis provides such a method. Input-output analysis is a method of systematically quantifying the mutual interrelationships among the various industries of a complex economic system. In the input-output system, the economy is divided into many industries, and the flow of commodities and productive factors shows the systematic relationships among them. These systematical interrelationships are input-output relations. In other words, an
input-output relation depicts the intersectoral flows of commodities between all the individual industries (sectors) of an economy (over a stated period of time).

Since input-output analysis is a study of the mutual interdependence of the different sections of the economy and one of the analytical tools often used to analyze an economy in a general equilibrium framework, we examine the fundamental system for input-output analysis.

3.2 The Framework of the Leontief Input-Output System

We examine two systems of the Leontief input-output relation: the open system and the closed system.

3.2.1 The Leontief Open Input-Output System

An economic system consists of a large number of producing and distributing industries. We can call a distributing industry (sector) and a producing industry (sector) as an origin-sector and a destination-sector\(^1\) in terms of the input-output system, respectively. Now, consider a national economy composed of a finite number of industries (sectors). Let the national economy be subdivided into a finite number \(N\) of industries (sectors) and an autonomous sector, i.e., final demand sector. In other words, there is an excess in production to satisfy final demand. The Leontief

\(^1\)In an economic system, there are usually two types of economic units: consumption units (i.e., consumers) and production units (i.e., producers: firms or industries) or buyers and sellers. From the viewpoint of the input-output system, a distributing sector or an origin-sector amounts to a producer or a seller of intermediate goods utilized by other industries and of final goods. On the contrary, a producing sector or a destination-sector amounts to a consumer or a buyer of intermediate goods utilized by itself and of the other factors of production such as the primary factors.
system adopts the following assumptions:  

1. Each industry (sector) in an economy produces only a single homogeneous commodity. This means that each commodity (or group of commodities) is supplied by a single industry or sector of production.

2. Constant proportionality between the productive factors utilized and the output produced by each industry exists. In other words, the productive factors purchased by each destination-sector from an origin-sector are a function of only the level of output of that destination-sector. This assumption implies that there are (i) fixed technological coefficients, (ii) the linear homogeneity (i.e., constant returns to scale) in production, and that the technology is such that no substitution is possible.

3. The total effect of performing different production processes is the sum total of the separate effects. This additivity assumption rules out positive and negative externalities - external economies and diseconomies.

We use the following notations for explaining the Leontief input-output system.

- $Q_i$ = the total physical output of an origin-sector $i$, where $i = [1, ..., N]$, $Q_i \in (0, \infty)$;
- $Q_{ij}$ = the physical amount of the output of an origin-sector $i$ utilized - as its factor of production: an intermediate factor - by a destination-sector $j$, where $i, j \in [1, N]$;

---


3A good or commodity is a material thing or service that has the capacity of directly satisfying human wants or needs, or can be used to produce something that capacity or both. Final goods are passed immediately into the hands of individuals.
$F_i$ = the quantity of the product of an origin-sector $i$ delivered to the autonomous sector, where $i = [1, ..., N]$. That is, $F_i$ is the final demand, which is not utilized in the process of production. The autonomous sector is composed of personal (household) consumption, investment (including inventories), government spending, and exports. $F_i$ therefore enters into the net national product of the economy.

Mathematically, we define the technological (input-output) coefficients under the assumption 2 mentioned above as follows:

$$\phi_{ij} = \frac{Q_{ij}}{Q_j}, \quad (3.1)$$

Or equivalently,

$$Q_{ij} = \phi_{ij} Q_j, \quad (3.2)$$

where $i, j \in [1, N]$ and $\phi_{ij} \in [0, \infty]$.

The expression (3.1) means that the technological (input-output) coefficient is the quantity of the product of an origin-sector $i$, $Q_{ij}$, utilized by a destination-sector $j$ per unit of its total output, $Q_j$.

(Consumers) to satisfy wants. Intermediate goods are currently produced goods also utilized in the productive process. Bhatia (1982, p. 319) describes intermediate goods as follows: In an economy some goods are utilized only for producing other commodities and others serve both as intermediate inputs and final goods. Following Bhatia’s viewpoint, we will call the former ‘pure’ intermediate goods and the latter, ‘dual’ intermediate goods.

$^4$The expression (3.1) is the type of relationship between the factors of production and the physical output originally utilized by L. Walras in his first formulation of the general market equilibrium theory.

$^5$W. W. Leontief, a pioneer of the input-output analysis, usually assumes that $\phi_{ii} = 0$, that is, a producing industry does not utilize any of its own commodity as a factor of production in producing itself. However, we assume that the technological coefficient $\phi_{ii}$ is not necessary zero because we consider the possibility that the industry requires some of its own commodity as a necessary input in its production process.
The two identities below form the fundamentals of the Leontief open system analysis. First, the total output of each origin-sector is divided into intermediate-good factors, $Q_{ij}$'s, and final products, $F_i$'s. This means that the total product of each distributing sector is the sum total of the commodity $i$ utilized by a destination-sector $j$ for producing $Q_j$, and final demands. This can be expressed mathematically in the following way:

$$Q_i = \sum_{j=1}^{N} Q_{ij} + F_i, \quad i \in [1, N].$$

(3.3)

The expression (3.3) is the basic material balance equation of the Leontief open input-output system. In other words, it represents the allocation of the total output of the origin-industry $i$. This is the external relationship representing the mutual interdependence of different industries of the economy. The right-hand side of the expression (3.3) represents the supply of output of industry $i$ and the left-hand side of it, the demand for output $Q_i$.

Second, the output level of each producing sector $j$ is a function of the intermediate factors utilized by a corresponding sector. This statement can be expressed by the general production function as below:

$$Q_j = Q_j \left[ Q_{1j}, Q_{2j}, \ldots, Q_{Nj}, \hat{X}_j \right]$$

$$= \min \left[ \frac{Q_{1j}}{\phi_{1j}}, \frac{Q_{2j}}{\phi_{2j}}, \ldots, \frac{Q_{Nj}}{\phi_{Nj}}, \frac{\hat{X}_j}{\phi_j} \right],$$

(3.4)

where $j \in [1, N]$, $\hat{X}_j$ = the total utilization of primary factors in a producing sector $j$.

The expression (3.4) states the internal input (cost)-output structure of the individual industries. Note that the Leontief input-output system assumes that the production
function has the linear homogeneity (i.e., constant returns to scale) in the productive
factors.

Now, making use of the expressions (3.2) and (3.3), we can get the Leontief
open input-output system for the 'correct' equilibrium solution. Substitution of the
expression (3.2) into (3.3) leads to the following reduced form:

\[ Q_i = \sum_{j=1}^{N} \phi_{ij}Q_j + F_i, \quad i \in [1,N]. \] (3.5)

When written out, (3.5) reads:

\[
\begin{align*}
Q_1 &= \phi_{11}Q_1 + \phi_{12}Q_2 + \ldots + \phi_{1N}Q_N + F_1 \\
Q_2 &= \phi_{21}Q_1 + \phi_{22}Q_2 + \ldots + \phi_{2N}Q_N + F_2 \\
&\vdots \\
Q_N &= \phi_{N1}Q_1 + \phi_{N2}Q_2 + \ldots + \phi_{NN}Q_N + F_N.
\end{align*}
\] (3.6)

Or equivalently,

\[
\begin{align*}
(1 - \phi_{11})Q_1 - \phi_{12}Q_2 - \ldots - \phi_{1N}Q_N &= F_1 \\
-\phi_{21}Q_1 + (1 - \phi_{22})Q_2 - \ldots - \phi_{2N}Q_N &= F_2 \\
&\vdots \\
-\phi_{N1}Q_1 - \phi_{N2}Q_2 - \ldots + (1 - \phi_{NN})Q_N &= F_N.
\end{align*}
\] (3.7)

This system shows that the interdependence among producing and distributing
industries of the given economy is described by a set of linear equations expressing
the balances between the total factors of production and the aggregate output of each
commodity produced and utilized in the course of the given period of time.
The technological structure of the entire system can be expressed by the matrix of technological coefficients of all its sectors. In matrix notation, the system (3.6) or (3.7) can be rewritten as

$$
\begin{bmatrix}
1 - \phi_{11} & -\phi_{12} & \cdots & -\phi_{1N} \\
-\phi_{21} & 1 - \phi_{22} & \cdots & -\phi_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
-\phi_{N1} & -\phi_{N2} & \cdots & 1 - \phi_{NN}
\end{bmatrix}
\begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_N
\end{bmatrix}
=
\begin{bmatrix}
F_1 \\
F_2 \\
\vdots \\
F_N
\end{bmatrix}
$$

Thus, the system (3.8) can also be written as below:

$$
Q = F
$$

where $\Psi^0 = [\phi_{ij}]$; the $(N \times N)$ structure matrix of the technological coefficients in the Leontief open system of the economy,

$$
Q = [Q_1, Q_2, ..., Q_N]^T; \text{ an } (N \times 1) \text{ column vector of the total outputs of producing sectors,}
$$

$$
F = [F_1, F_2, ..., F_N]^T; \text{ an } (N \times 1) \text{ column vector of final demands.}
$$

The problem for the Leontief open system is to find the necessary and sufficient condition for the existence of a vector $Q \geq 0$ such that $[I - \Psi^0]Q = F$ for any $F \geq 0$.

If the Leontief (technological) matrix $[I - \Psi^0]$ is nonsingular, that is, $\text{det}[I - \Psi^0] \neq 0$,

The expression (3.9) determines the relations between levels of production $Q$ and final demands $F$. From the expression (3.9), we can get $Q = \Psi^0Q + F$ (1). Using the iterative procedure, we obtain $Q = \sum_{k=0}^{\infty}(\Psi^0)^kF$ (2). The expression (2) shows that each round in the iteration consists of raising the matrix to a successively higher power, which gives the method the name of expansion in powers. We can rewrite the expression (2) as $Q = (I + \Psi^0)F + \sum_{k=2}^{\infty}(\Psi^0)^kF$ (3). From the expression (3), we can know the total effect of a given final demand on the equilibrium output level. The first term of (3) represents the direct effect and the second term a series of indirect effect.  

---

6The expression (3.9) determines the relations between levels of production $Q$ and final demands $F$. From the expression (3.9), we can get $Q = \Psi^0Q + F$ (1). Using the iterative procedure, we obtain $Q = \sum_{k=0}^{\infty}(\Psi^0)^kF$ (2). The expression (2) shows that each round in the iteration consists of raising the matrix to a successively higher power, which gives the method the name of expansion in powers. We can rewrite the expression (2) as $Q = (I + \Psi^0)F + \sum_{k=2}^{\infty}(\Psi^0)^kF$ (3). From the expression (3), we can know the total effect of a given final demand on the equilibrium output level. The first term of (3) represents the direct effect and the second term a series of indirect effect.
then the inverse Leontief matrix \([I - \Psi^o]^{-1}\) exists. If so, under the condition of \(\det[I - \Psi^o] \neq 0\), the system (3.9) will have the unique equilibrium solution \(Q^o = [Q^o_1, Q^o_2, \ldots, Q^o_N] \geq 0\) through:

\[
Q = \left[I - \Psi^o\right]^{-1}P. \tag{3.10}
\]

Here we require that:

\[
\left[I - \Psi^o\right]^{-1} \geq 0.
\]

For a necessary and sufficient condition for the existence of a non-negative inverse Leontief (technological) matrix \([I - \Psi^o]^{-1}\), we need the following theorem.

**3.2.1.1 Theorem 3.1**: Let \(H = [h_{ij}]\) be an \((N \times N)\) matrix with \(h_{ii} > 0\) and \(h_{ij} \leq 0\) for \(i \neq j\); then the following conditions are equivalent.

(i) There exists a \(Q \geq 0\) such that \(HQ > 0\).

(ii) (Hawkins-Simon condition)

All the successive principal minors of the matrix \(H = [h_{ij}]\) are positive. That is,

\[
det(H_k) > 0,
\]

where

\[
H_k = \begin{bmatrix}
    h_{11} & h_{12} & \ldots & h_{1k} \\
    h_{21} & h_{22} & \ldots & h_{2k} \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{k1} & h_{k2} & \ldots & h_{kk}
\end{bmatrix}
\]

(iii) $H^{-1} \geq 0$.

In the end, when final demand $F = [F_1, F_2, ..., F_N]^T$ is exogenously given and the structure matrix $\Psi = [\phi_{ij}]$ is predetermined, we can derive the unique equilibrium output levels - production requirements necessary to satisfy final demand - in each producing sector, $Q^\circ = [Q_1^\circ, Q_2^\circ, ..., Q_N^\circ]^T \geq 0$ from the expression (3.10).

3.2.2 The Leontief Closed Input-Output System

The Leontief closed system also considers an economic system consisting of a finite number $N$ of industries (sectors) under the same assumptions as for the Leontief open system. When the autonomous sector (i.e., final demand sector) of the Leontief open input-output system is included in the system as just another sector, the system becomes the closed input-output system. In the Leontief closed input-output system, the autonomous sector, i.e., final demand and the primary factors do not appear; in their position will be the input requirements and the output of the newly conceived sector. All outputs (commodities) now are intermediate factors in nature, since every output produced is utilized only for satisfying the input requirements of the $(N + 1)$ sectors in the system. In other words, each industry produces an output which is just sufficient to meet the demands of all industries.

The following identities, instead of the expressions (3.3) and (3.4) of the Leontief open input-output system, form the fundamentals of the Leontief closed input-output system. First, the total output of any sector is allocated only among intermediate factors. Mathematically, the above fact can be expressed as below:

$$ Q_i = \sum_{j=1}^{N+1} Q_{ij}, \quad i \in [1, (N+1)]. $$

(3.11)
Second, the output level of each producing sector \( j \) is a function of only intermediate factors utilized by that producing sector. Therefore, the general production function (3.4) is transformed as follows:

\[
Q_j = Q_j \left[ Q_{1j}, Q_{2j}, \ldots, Q_{Nj}, Q_{(N+1)j} \right],
\]

(3.12)

where \( j \in [1,(N+1)] \). This means that \( X_j \) which is the total utilization of primary factors in a producing sector \( j \) in the Leontief open system has disappeared in the Leontief closed system, and is replaced by an endogenous sector \( Q_{(N+1)j} \). The production function (3.12) is also assumed to be homogeneous degree of one in the productive factors.

With the same way used in the case of the Leontief open system we can obtain the Leontief closed system. Substitution of the expression (3.2) into (3.11) yields the following reduced form:

\[
Q_i = \sum_{j=1}^{N+1} \phi_{ij} Q_j, \quad i \in [1,(N+1)].
\]

(3.13)

The specific system of the expression (3.13) is as follows:

\[
\begin{align*}
Q_1 &= \phi_{11} Q_1 + \phi_{12} Q_2 + \ldots + \phi_{1N} Q_N + \phi_{1,N+1} Q_{N+1} \\
Q_2 &= \phi_{21} Q_1 + \phi_{22} Q_2 + \ldots + \phi_{2N} Q_N + \phi_{2,N+1} Q_{N+1} \\
&\quad \vdots \\
Q_N &= \phi_{N1} Q_1 + \phi_{N2} Q_2 + \ldots + \phi_{NN} Q_N + \phi_{N,N+1} Q_{N+1} \\
Q_{N+1} &= \phi_{N+1,1} Q_1 + \phi_{N+1,2} Q_2 + \ldots + \phi_{N+1,N} Q_N + \phi_{N+1,N+1} Q_{N+1}.
\end{align*}
\]

Or equivalently,

\[
0 = (1 - \phi_{11}) Q_1 - \phi_{12} Q_2 - \cdots - \phi_{1N} Q_N - \phi_{1,N+1} Q_{N+1}
\]

(3.15)
\[ 0 = -\phi_{21}Q_1 + (1 - \phi_{22})Q_2 - \cdots - \phi_{2N}Q_N - \phi_{2,N+1}Q_{N+1} \]

\[ 0 = -\phi_{N1}Q_1 - \phi_{N2}Q_2 - \cdots + (1 - \phi_{NN})Q_N - \phi_{N,N+1}Q_{N+1} \]

\[ 0 = -\phi_{N+1,1}Q_1 - \phi_{N+1,2}Q_2 - \cdots - \phi_{N+1,N}Q_N + (1 - \phi_{N+1,N+1})Q_{N+1}. \]

In matrix notation, a homogeneous equation system (3.15) can be rewritten as below:

\[
\begin{bmatrix}
(1 - \phi_{11}) & -\phi_{12} & \cdots & -\phi_{1N} & -\phi_{1,s} \\
-\phi_{21} & (1 - \phi_{22}) & \cdots & -\phi_{2N} & -\phi_{2,s} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\phi_{N1} & -\phi_{N2} & \cdots & (1 - \phi_{NN}) & -\phi_{N,s} \\
-\phi_{s,1} & -\phi_{s,2} & \cdots & -\phi_{s,N} & (1 - \phi_{s,s})
\end{bmatrix}
\begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_N \\
Q_s
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
\]

(3.16)

Note that \( s = N + 1 \).

The above matrix system (3.16) can be described in the concise form as follows:

\[
\begin{bmatrix}
I - \Psi^C
\end{bmatrix}
Q = 0,
\]

(3.17)

where \( \Psi^C = \left[ \phi_{ij} \right] \); an \(((N + 1) \times (N + 1))\) structure matrix of the technological coefficients of the Leontief closed system,

\[ Q = [Q_1, Q_2, \ldots, Q_{N+1}]^T; \] an \(((N + 1) \times 1)\) column vector of the total outputs of producing sectors, \( \mathbf{0} = \) the null vector.

Since the matrix system (3.17) is a homogeneous equation system, this system can have a nontrivial solution if and only if the \((N + 1) \times (N + 1)\) Leontief matrix \([I - \Psi^C]\) is singular, i.e., \( \text{det}[I - \Psi^C] = 0 \). In the Leontief closed system, there are no primary factors; hence if flows are in money term, each column sum in the structure
matrix \( \Psi^c \) must be exactly equal to 1; that is,
\[
\sum_{i=1}^{N+1} \phi_{ij} = 1, \ j \in [1, (N+1)].
\] (3.18)

In every column of the Leontief matrix \([I - \Psi^c]\), we can obtain the following result:
\[
\phi_{1j} = 1 - \sum_{i=2}^{N+1} \phi_{ij}, \ j \in [1, N+1].
\] (3.19)

The expression (3.19) implies that the \((N+1)\) rows of the Leontief matrix \([I - \Psi^c]\) are linearly dependent. As a result, \(det[I - \Psi^c] = 0\). Therefore, the system (3.15) or (3.16) does possess nontrivial solutions; in fact, it has an infinite number of them. This implies that no unique equilibrium output level exists in the Leontief closed system. Thus the equilibrium output levels \(Q^o = [Q^o_1, Q^o_2, \ldots, Q^o_{N+1}]^T\) are determined in proportion to one another, however cannot fix their absolute levels unless the additional restrictions are imposed on the input-output system.

### 3.3 The Extended Input-Output Quantity-Flow Relation

In the preceding section we showed and discussed an analytical tool (i.e., mathematical form) of an economic system. Now, in this section we introduce the input-output quantity-flow table which represents a descriptive device of an economic system.

The quantitative input-output Table 3.1 depicts and provides a framework for measuring the flows of the factors of production and outputs between the origin-industries and the destination-industries of the (national) economy during the given period of time. In other words, the input-output table is the input-output accounting
system showing the interrelations arising from production of outputs and distribution (or allocation) of the factors of production such as the intermediate-good factors, the primary factors unproduced in the current system, and the imported foreign factors. We can analyze and trace the interrelations of the economic phenomena existing among the various industries by making use of the input-output accounting system under the appropriate assumptions about economic behavior and definitions of the variables appeared in the input-output analysis (for instance, see Section 3.2.1).

The main characteristics of the quantitative input-output Table 3.1 can be described in the following way. First, each industry plays roles as the destination-industry producing output, i.e., a producer of a commodity, or a purchaser and an utilizer of the factors of production, on the one hand, and as the origin-industry distributing (or supplying) the intermediate factors of production, that is, a supplier of the intermediate-good factor, on the other.

Second, all the entries in the input-output table are the flows measured in the physical units or the monetary value units in the given time period. Here we should note the accounting units of the entries employed in the rows and the columns of the input-output system. Specifically, in terms of the physical (accounting) units, the entries in any row are all measured in the same (or homogeneous) physical units. So, we can add across the rows for calculating the total physical quantity of output and/or the factor of production distributed or allocated to the destination industries. However, items in the same column are not measured in the same units because each item has the proper, heterogeneous physical unit such as ton, cubic contents (m³), and the numbers of commodities. As a result, we cannot add down the columns in the input-output table. On the contrary, there are no problems in following the
additivity rule both in any row and in any column if the monetary value units are used.\(^8\)

Third, each column in the input-output table shows the internal input (or cost) structure of the corresponding industry. In other words, a column represents one point on the production function of the corresponding industry. Specifically, items in the same column express the factors of production such as the intermediate factors, the primary factors, and the imported foreign factors in the same production function. In the monetary value terms, therefore, the sum of each column gives the total cost of producing the industry's output. The total cost is composed of the intermediate-factor cost \(c_{1j}\) (see explanation of Block I), the primary-factor cost \(c_{2j}\) (see explanation of Block III) and the imported-factor cost \(c_{3j}\) (see explanation of Block I). On the other hand, each row indicates a state of the distribution quantities of outputs or the factors of production to producing industries and final demand. If the output produced or distributed is measured in the monetary value terms, total output in terms of the physical units amounts to total revenue of each origin-industry in the monetary value terms.

Fourth, in an accounting sense, the direct payment for primary factors such as labor, land, capital (including depreciation allowances), taxes, and so forth, comprises the value added in the sector. In other words, all the value added is measured by primary factor costs and imported factor costs. In relation to the national income accounting, the value added which is equal to the values of final demand has the

\(^8\)W. W. Leontief used originally the monetary value units in his input-output system and most of the input-output tables are recorded in the monetary terms. In relation to this point, Dorfman, Samuelson, and Solow (1958) say as follows: The whole subject [of the Leontief model] appears to be more a branch of money national-income accounting than the structure of physical production.
same meaning as the gross national product which represents a measure of the performance of the economy (in the case that imports are treated as a deduction from final demand).

On the bases of the mentioned above, we can describe the quantitative input-output Table 3.1 more in detail. The fundamental structure of the input-output Table 3.1 (to be utilized in our analysis) is derived from the division of demand (or utilization) of both output and the factors of production into two categories - intermediate and final demands - and the corresponding division of the factors of production into three categories - the intermediate-good, the primary, and the imported foreign factors. In other words, the separation between intermediate and final demand of output and among the intermediate-good, the primary, and the imported foreign factors leads to six types of transactions, which are shown in the six blocks of Table 3.1.

**Block I** comprises the main portion of the inter-industry transactions. In other words, **Block I** indicates intermediate utilizations of the factors of production in the production process. Each item $Q_{gj}^O$ in **Block I** shows the physical quantity of output or the intermediate-good factor $g$ utilized by the destination-industry $j$ during the given time period. The sum total of outputs in a row of **Block I** indicates the total output distributed (or supplied) by the origin-industry $g$ to the destination-industries during the given period of time. On the other hand, the sum total (in the monetary value terms) of outputs in a column of **Block I** expresses the total cost of the intermediate factors utilized by the destination-sector $j$ during the given time period: $c_{1j} = \sum_{g=1}^{G} W_g Q_{gj}^O$. In relation to the national income (or GNP) accounting, **Block I** is a double counting part.
Block II shows the outputs consumed by final (or autonomous) demand sector which is composed of personal (household) consumption (C), investment (I), government spending (G), and exports (or export demand) (E). Block II comprises the main part of the gross national product. For example, \( C_g \) and \( G_g \) are the physical quantities of output \( Q_g \) consumed by households and government, respectively, during the given period of time. \( I_g \) is the physical amount of output \( Q_g \) invested (and not consumed) in the given time period. \( E_g \) is the physical portion of output \( Q_g \) exported to the foreign countries during the given period. Converting the physical quantities of the mentioned above into the monetary-value amounts, they can be expressed as follows: \( W_g C_g \) (total values consumed by households), \( W_g I_g \) (total investment values including inventory values), \( W_g G_g \) (total values absorbed by government), \( W_g E_g \) (total export values).

Block III indicates the amounts of the primary factors such as labor, natural resources (e.g., land), capital, taxes, savings and depreciation allowances, and so on, utilized by producing industries. Each item \( \dot{X}^o_{mj} \) in Block III shows the physical quantity of the primary factor \( m \) utilized by the destination-industry \( j \) during the given time period. The sum total of the monetary values of the primary factor in any row of Block III represents factor-owners' incomes transferred (or paid) from the destination-industries. The sum total of factor owners' incomes is a part of total value-added in the national income accounting system. On the other hand, the sum total of the values of the primary factors in any column of Block III denotes the primary-factor cost of the corresponding producing industry: 

\[
\sigma_{2j} = \sum_{m=1}^{M} \dot{W}_m \dot{X}^o_{mj}.
\]

Block IV shows the amounts of the primary factors consumed by final demand sector. As in Block II, \( \dot{C}_m \) and \( \dot{G}_m \) are the physical quantities of the primary
factors \( X_m \) consumed and employed by households and government, respectively, during the given period of time. \( I_g \) is the physical amount of the primary factors \( X_m \) not utilized and consumed in the given time period. \( \tilde{E}_m \) is the physical portions of the primary factors \( X_m \) exported to the foreign countries during the given period. In terms of the monetary units, the mentioned above can be written as follows: \( \bar{W}_m \tilde{C}_m, \bar{W}_m \hat{I}_m, \bar{W}_m \hat{G}_m, \bar{W}_m \hat{E}_m \). For example, **Block IV** includes (i) intra-household transactions, (ii) household savings and expenditures for depreciation of consumer durables or capital goods, (iii) factor payments by government (e.g., wages and salaries of government employees and interest payment by government), (iv) tax payments by government, (v) depreciation allowances for public facilities and government deficits, (vi) sales and excise taxes, and so on.

**Block V** represents the amounts of the imported foreign factors such as rubber, uranium, and crude oil utilized by producing industries. Each item \( X^*_{zj} \) in **Block V** shows the physical quantity of the imported foreign factor \( z \) utilized by the destination-industry \( j \) during the given time period. The sum total of the values of the imported foreign factors in any column of **Block V** gives the total imported-factor cost of the corresponding producing industry: \( c^*_3 j = \sum_{z=1}^{Z} W^D_z X^*_{zj} \). The imported-factor cost varies with the rate of foreign exchange, even though the international (or world) price of the imported factor is unchanged.

**Block VI** expresses the amounts of the imported factors consumed by final demand sector. For instance, **Block VI** is composed of (i) household purchase of imports (e.g., \( C^*_z \)), (ii) government expenditures by utilizing the foreign factors, (iii) balance of trade, etc.. In the quantitative input-output Table 3.1, \( E^*_z \geq 0 (z \in [1, Z]) \). This states that there are possibilities of re-exporting (or re-selling) the imported
foreign factors to the rest-of-the-world. We however assume that $E^*_z = 0 (z \in [1, Z])$.

In the microeconomic principle, producers (firms, industries) obtain the competitive zero-profit in a long-run competitive equilibrium. We therefore can put total revenue (TR) or value of total output (VTO) equal to total cost (TC) or value of total input (VTI) in the monetary terms. This also amounts to put price ($W_j^o, W_j$) of output equal to unit cost or average cost. Specifically, we can expound this in relation to the quantitative input-output Table 3.1. TR (or VTO) is the sum total of the monetary values of output in any row passing left and right across Block I and Block II. That is, $VQ_g (g \in [1, G])$ in the total sale (TS) column denotes TR (or VTO) of the origin-industry $g$ in the given time period. TC (or VTI) indicates the sum total of the monetary values of inputs in any row passing up and down through Block I, Block III, and Block V. $C_j^o (j \in [1, G])$ in the total expenditure (TE) row represents TC (or VTI) of a producing industry $j$.

For example,

$$VQ_G = C_G^o,$$

where

$$VQ_G = W_G \left[ \sum_{j=1}^G Q_j G_j + \left( HCG + I_G + G + E_G \right) \right].$$

$$C_G^o = \sum_{g=1}^G W_g Q_g G + \sum_{m=1}^M \hat{W}_m \hat{x}_m G + \sum_{z=1}^Z W_z^D \hat{x}_{zG}^*.$$

This example explains that total revenue (TR) of industry $G$ is equal to total cost (TC) of that industry.

In sum, under the competitive zero-profit condition, the sum total of any column within Block I, Block III, and Block V would, by definition, be the same as the sum
total of the corresponding row within Block I, Block II, in Table 3.1. Furthermore, adding across the TE row and down the TS column, we can obtain gross total (GT) of TE and TS (in the monetary terms).

\[
GT = \sum_{j=1}^{G} C^G_j + \left( VHC_G + VIG + VG_G + VEG \right). \tag{3.23}
\]

\[
GT = \sum_{g=1}^{G} VQ_g + \sum_{m=1}^{M} V\hat{X}_m + \sum_{z=1}^{Z} VX^*_z. \tag{3.24}
\]

Since \( \sum_{j=1}^{G} C^G_j = \sum_{g=1}^{G} VQ_g \) as mentioned above, we get

\[
\left( VHC_G + VIG + VG_G + VEG \right) = PFC + FFC, \tag{3.25}
\]

where \( PFC = \sum_{m=1}^{M} V\hat{X}_m \) and \( FFC = \sum_{z=1}^{Z} VX^*_z \). This states that the value of total final demand is equal to total primary-factor cost (PFC) plus total imported-foreign-factor cost (FFC).
Table 3.1: Quantitative Input-Output Relation

<table>
<thead>
<tr>
<th>Origin Sectors</th>
<th>Destination sectors</th>
<th>Final Demand</th>
<th>TS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_1 )</td>
<td>( Q_1 )</td>
<td>( H^C_1 )</td>
<td>( I_1 )</td>
</tr>
<tr>
<td>( W_2 )</td>
<td>( Q_{11} )</td>
<td>( H^C_2 )</td>
<td>( I_2 )</td>
</tr>
<tr>
<td>:</td>
<td>( Q_{12} )</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>( W_g )</td>
<td>( Q_{g1} )</td>
<td>( H^C_g )</td>
<td>( I_g )</td>
</tr>
<tr>
<td>:</td>
<td>( Q_{g2} )</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>( W_G )</td>
<td>( Q_{G1} )</td>
<td>( H^C_G )</td>
<td>( I_G )</td>
</tr>
<tr>
<td>:</td>
<td>( Q_{G2} )</td>
<td>:</td>
<td>:</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Primary Factors</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_1 )</td>
<td>( X_1 )</td>
</tr>
<tr>
<td>( W_2 )</td>
<td>( X_{11} )</td>
</tr>
<tr>
<td>:</td>
<td>( X_{12} )</td>
</tr>
<tr>
<td>( W_m )</td>
<td>( X_{m1} )</td>
</tr>
<tr>
<td>:</td>
<td>( X_{m2} )</td>
</tr>
<tr>
<td>( W^M )</td>
<td>( X_M )</td>
</tr>
<tr>
<td>:</td>
<td>( X_{M1} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Imported Factors</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( W^D_1 )</td>
<td>( X^*_1 )</td>
</tr>
<tr>
<td>( W^D_2 )</td>
<td>( X^*_2 )</td>
</tr>
<tr>
<td>:</td>
<td>( X^*_{j1} )</td>
</tr>
<tr>
<td>( W^D_z )</td>
<td>( X^*_z )</td>
</tr>
<tr>
<td>:</td>
<td>( X^*_{z1} )</td>
</tr>
</tbody>
</table>

| TC (TE) | \( C^o_1 \) | \( V^H \) | \( VI \) | \( VG \) | \( VE \) | \( GT \) |
4. THE NEO-CLASSICAL THEORY OF PRODUCTION AND COSTS

4.1 Introduction

An economy\(^1\) is a complex organizational system: a system for organizing economic decisions (activities) such as production of commodities - goods and services - and distribution of commodities among consumers (or consumption units) and producers (or production units) who are rational units of economic decision. In other words, an economic system is the set of arrangements through which the economy's resources are utilized so as to produce the commodities that satisfy the wants and needs of the members of the economy. Possibly the simplest and most versatile arrangement that has been devised is the market mechanism. In essence, the market mechanism involves consumers (buyers) and producers (sellers) in voluntary exchange. As an economic system the market mechanism performs the fundamental functions of allocating the economy's scarce resources among a variety of utilizations and distributes the output of production among consumers. Under the market mechanism, production is performed by rational producers - production sectors: a firm, industry - in an

\(^1\)In a modern society, economic decisions (activities) of the various unisolated units influence each other; they are interdependent. The totality of interdependent units of economic decision is called an economy or an economic system. See Lange (1946, p. 25).
economic system. In production processes, a producer utilizes the knowledge (information) of input and output prices and the technology related to the combination of factors of production to yield goods and services. In other words, production processes are subject to the availability of factors of production - natural resources (or raw materials), capital, human resources (e.g., labor), and semifinished products (or intermediate outputs) - and to the technical characteristics of the production process.

What is the notion of production? Conventionally, production is defined as the creation of utility (or satisfaction, happiness) which means the ability of a good or service to gratify human wants. Gould and Ferguson (1980, p. 121) depict production normally requiring various types of the factors of production as the creation of goods or services people will purchase. Heathfield and Wibe (1987, p. 1-2) expounds the concept of production as follows: Production may be regarded as a transformation from one state of the world to another. But not all such transformations are acts of production because consumption, too, may be regarded as a transformation from one state of the world to another. We can define an act of production as any act which transforms the world from a less to a more preferred state. There are four ways in which the state of the world may be so changed: (1) The quantity of a good may be changed; (2) the quality of the good may be changed; (3) the geographical location of a good can be changed: e.g., delivery of a good; (4) the time location of a good can be changed: e.g., storage of a good to deliver in the future.

Finally, Heathfield and Wibe (1987) conclude that production may be defined as any activity the net result of which is to increase the degree of compliance between the quantity, quality and distribution (spatial and temporal) of commodities and a given preference pattern. Debreu (1959a, pp. 37-38) says simply that production is
a specification of the quantities of all inputs and all outputs of a producer.

In sum, production can be defined as transformation (or conversion) of factors of production into more beneficial commodities completely specified physically, spatially, temporally to satisfy human wants.

### 4.2 Production Sets: Technology

A rational producer yields outputs by the technology\(^2\) which is all the technical information about the combination of various factors of production. We need a convenient way to describe the production possibilities of the rational producer, i.e., which combinations of factors of production and outputs are feasible.

We assume that the factors of production and outputs are measured in terms of quantities - stocks or flows. We also assume that the factors of production are distinguished by the time \(t\) - the discrete time or the continuous time - and the location \(s\) in which they are available. Furthermore, it is assumed that a rational producer has \(n \in (0, \infty)\) commodities to serve as the factors of production and/or outputs. Under these assumptions we represent a specific production plan (process, set). For a producer, a production plan is a specification of the quantities (stocks or flows) of all his/her factors of production and all his/her outputs. A technically possible production (plan) is represented by a vector \(Q_{ts} = (Q_{1ts}, Q_{2ts}, \ldots, Q_{nts})\) in \(R^n\), the \(n\)-dimensional Euclidean space, where \(Q_{mts}\) \((m = 1, 2, \ldots, n)\) is negative.

\(^2\)Technology is society’s pool of knowledge or information regarding the industry arts. It consists of knowledge used by industry regarding the principles of physical and social phenomena, knowledge regarding the application of these principles to production, and knowledge regarding the day-to-day operations of production. For more details, see Mansfield (1971, pp. 9-38).
(i.e., $Q_{mts} < 0$) when the $m$-th component is utilized as a net input at the location $s$ at the time (period) $t$ and positive ($Q_{mts} > 0$) when the $m$-th component serves as a net output at the location $s$ and at the time (period) $t$. Such a vector is called a net output vector (or, netput vector). The set of all feasible production plans, i.e., netput vectors, is called the producer's production (possibilities) set and denoted by $\hat{Q}_{ts}$, a subset of $\mathbb{R}^n$. In other words,

$$Q_{ts} \in \hat{Q}_{ts},$$

$$\hat{Q}_{ts} \subset \mathbb{R}^n.$$  \hfill (4.1)  

The production set $\hat{Q}_{ts}$ depicts all patterns of the factors of production and outputs that are feasible. Hence, it gives us a complete description of the technological possibilities facing a producer.

In sum, the set $\hat{Q}_{ts}$ is the set of all the technically possible production processes in a given economy. We assume $\hat{Q}_{ts} \subset \mathbb{R}^n$, and $Q_{ts} \in \hat{Q}_{ts}$ denotes a production process of a producer. The quantity $Q_{mts}$ indicates the amount of the $m$-th component involved in a production process $Q$.

### 4.3 Input Requirement Set

Let us consider a rational producer that yields only one output. In this case, we can write the net bundle as $(Q, -X)$ where $X$ is a vector of factors of production that can yield the output $Q$. We can then define the input requirement set or the production input set as:

$$V(Q) = \{X \in \mathbb{R}_+^n \mid (Q, -X) \in \hat{Q}\},$$  \hfill (4.3)

where $\mathbb{R}_+^n$ is the nonnegative orthant of $\mathbb{R}^n$, the $n$-dimensional Euclidean space,
\( \bar{Q} \) is the production set. (For notational convenience, we omit the subscripts for location and time.)

This implies that the input requirement set (or the production set) gives all input bundles that produce exactly \( Q \). We can also define an isoquant as follows:

\[
I(Q) = \{ X \in \mathbb{R}^n_+ \mid X \in V(Q), X \notin V(Q') \text{ for } Q' > Q \}. \tag{4.4}
\]

This means that the isoquant gives all input bundles that yields exactly \( Q \).

### 4.4 Production Function

#### 4.4.1 Introduction

The production function is the core concept in the economic theory of production and costs. Such a production function is quite an old item in the economists' analytical paraphernalia. It was introduced by Wicksteed (1894, p. 4) with one simple remark: "...the product being a function of the factors of production we have \( P = f(a, b, c, ...) \)." However, the concept of a production function did not come into popular utilization until at least the first decade of this century. Chambers (1988, pp. 6-7) explains the development of the production function as follows: "...well into the 1930s, significant contributions were being made to the general understanding of the production function. ...one can say that the classical economists, although likely grasping the rudiments of such a formulation, did not clearly define the production function. But the concept was firmly rooted in the professional jargon of economists

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4See Schumpeter (1953) and Stigler (1965) for the genealogy of the production function.
by the 1930s. Therefore, it is perhaps best to attribute origination of the concept to the entire school of early marginalists and neo-classical economists. After all, the marginal revolution in economics clearly encouraged the development and refinement of analytical concepts like the production function that facilitate exact reasoning in economic analysis."

In any case, the production function plays an crucial role in the theory of production and costs because production functions imply particular cost functions, often the self-dual of the production function.

4.4.2 The Definition and General Form

We discussed in the previous section that production is transformation of certain factors of production into outputs under a given technology. The technology existing at a given point in time sets limits on how much output can be produced with a given quantity of factors of production. Now, it is crucial to know how much output can be produced with certain combinations of factors of production and what, if any, alternatives there are to producing particular outputs in particular ways. Given the level of technology, there is generally a wide range of possible methods of producing a particular commodity. Each possible method requires certain factors of production to produce the commodity in question. The production function is an attempt at defining these alternatives (methods). In other words, the production function is an attempt at mathematically specifying the range of technical possibilities - choice of a production process - open to producers. It has a functional form for the production process. Thus, the production function can be defined as the mathematical expression of the relationship between the quantities of factors of production and the quantities
of outputs given producer's current state of technological knowledge.

In short, the production function relates the quantities of factors of production in a production process to the output of that process. For any quantitative combination of inputs, the production function defines the maximum amount of output to be realized. In other words, the production function shows the maximum quantities of output technologically available from given combination of quantities of the factors of production. In addition, the production function can show the possibilities of substitution between the factors of production to yield a given output, assumed to be a single one.

Let $Q \in R_+$ denote the quantity of output, where $R_+ = \text{the set of the non-negative real numbers}$, $X = (X_1, X_2, \ldots, X_n)$: a vector of the factors of production. Then, the input requirement set of a technology is defined as:

$$V(Q) = \{X \in R^*_+| (Q,-X) \in \tilde{Q}, \; \tilde{Q} \subset R^{n+1}\}.$$ (4.5)

The production function is defined as follows:

$$F(X) = \max\{Q| \; (Q,-X) \in \tilde{Q}\}.$$ (4.6)

The expression implies that the production function represents the maximal output obtainable from the vector of factors of production $X$. This maximum would actually exist in the production set. Thus, the production function can be written as follows;

$$Q = f(X_1, X_2, \ldots, X_n).$$ (4.7)

In sum, the production function (4.7) shows implicitly that physical and chemical laws govern the relationships between output and the factors of production.
4.4.3 Regularity Conditions: Assumptions

The neo-classical theory of production and cost essentially consider the optimal allocation of the factors of production that minimizes the total cost of production for each given level of output, and the characteristic of the cost functions derived from production functions under neo-classical properties. Thus, we need the following regularity conditions throughout our study.\(^5\)

[H.1] (Perfect competition and No free good):\(^6\) The price of output and the prices of the factors of production are strictly positive, continuously variable, and given regardless of the quantities of the factors of production.

[H.2] Quantities of factors of production can be any nonnegative, real numbers.

[H.3] The production function \(f(X)\) is finite, positive, and single-valued; \(f(X) \in [0, \infty)\) (by virtue of [H.2]). This states that every finite factors generates a unique, finite output.

[H.4]\(^7\) The production set

\[ V_1(Q) = \{ X \mid X \in \mathbb{R}^n_+, Q \leq f(X) \} \]

\(^5\)Uzawa's (1964) and Diewert's (1971) assumptions are similar to these conditions, but less restrictive.

\(^6\)Throughout the whole of classical and neo-classical economic literatures we learned that perfect competition in some sense achieves efficiency in the maximization of satisfactions of individual consumers and producers. So, we assume perfect competition in the factor market and the commodity market. Since free goods such as air, water do not give rise to any restrictions on allocation decisions, and negative prices reflect the positive cost of disposing of certain free goods, we take the assumption of no free good.

\(^7\)Regularity condition [H.4] is a generalization of the neo-classical condition that the production function \(f(X)\) is a concave function, which, in turn, is a generalization of the classical condition that the production function exhibits diminishing returns with respect to any single factor of production.
is a convex set for every $Q \geq 0$. This also implies that the production function $f(X)$ defined over a convex set $V_1(Q)$ is strictly quasi-concave if

$$f(X) > \min[f(X^1), f(X^2)],$$

where $X = \theta X^1 + (1 - \theta)X^2$, $\theta \in (0,1)$, and $X^1, X^2 \in V_1(Q)$ with $X^1 \neq X^2$.

4.4.3.1 Definition 4.1: A set $V_1(Q)$ is said to be convex if for all $X^1 \in V_1(Q), X^2 \in V_1(Q), X = \theta X^1 + (1 - \theta)X^2 \in V_1(Q)$ for all $\theta \in [0,1]$.

4.4.3.2 Lemma: The concave production function $f(X)$ defined over a convex set $V_1(Q)$ is also quasi-concave.

[H.5] The production function $f(X)$ is everywhere continuous and everywhere at least twice-continuously differentiable, and the Hessian matrix is non-singular.

4.4.3.3 Definition 4.2: (i) The function $f(X)$ is continuous at $X = X_0$ if for every $\varepsilon > 0$, there exists a $\delta > 0 \exists$ if $|X - X_0| < \delta, |f(X) - f(X_0)| < \varepsilon$; equivalently, (ii) the function $f(X)$ is continuous at $X_0$ if $\lim_{x \to x_0} f(X) = f(X_0)$.

[H.6] The set $V_1(Q)$ is a closed and non-empty set consisting of nonnegative factors for all $Q > 0$.

4.4.3.4 Definition 4.3: Given a fixed set $\Omega$, let $\tau$ be a collection of subsets of $\Omega$. Then $\tau$ is a topology over $\Omega$ if and only if (i) $\Omega \in \tau$, the empty set $\phi \in \tau$; (ii) $G_i \in \tau (i = 1, 2, ..., n)$ implies $\bigcap_{i=1}^{n} G_i \in \tau$; (iii) $G_\alpha \in \tau$ for all $\alpha \in \tau$ means $\bigcup_{\alpha \in \tau} G_\alpha \in \tau$. $(\Omega, \tau)$ is a topological space and a member of $\tau$ is called an open set.

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8For more details, see Diewert (1971, pp. 485-486).
A subset $V_1(Q)$ of $\Omega$ is called a closed set if and only if its complement is an open set, that is, $[V_1(Q)]^c \in \tau$, where $[V_1(Q)]^c = \Omega \setminus V_1(Q)$.9

[H.7] All first-order derivatives are positive, that is, all marginal physical productivities are positive and finite for all positive factor bundles: $f_m > 0 \ (m = 1, 2, \ldots, n)$. This characterizes the economic region of the production function $f(X)$. Hence, if $X' \geq X$, then $f(X') \geq f(X)$.

[H.8] All factors of production are essential10, that is,

$$\lim_{x_m \to 0} f(X) = 0, (m = 1, 2, \ldots, n). \tag{4.8}$$

(i) $f(\bar{0}) = 0$, where $\bar{0} =$ the null vector (weak essentiality);

(ii) $f(X_1, X_2, \ldots, X_{m-1}, 0, X_{m+1}, \ldots, X_n) = 0$ for all $X_m$ (strong essentiality).

[H.9] (Ray property)

$$\lim_{\pi \to \infty} f(\pi X) = \infty,$$

where $\pi =$ a scalar, $X > 0$.

[H.10] The production function $f(X)$ is a homothetic function.11

### 4.5 Uniqueness of a Cost Minimum

The existence of a unique minimum cost plays crucial roles in the economic analysis. We now look into the uniqueness of a cost minimum. The general cost-

---


10A factor of production is essential to the production of output if a positive amount of output cannot be produced without a strictly positive utilization of that input.

11The homotheticity implies that, in the two factor space, along a ray emanating from the origin the slopes of the isoquants (pertaining to a given isoquant map) are constant. For the homothetic production function, the elasticity of substitution is constant for all isoquants along a ray from the origin and is not necessarily constant along one isoquant. See Clemhout (1968, p. 91).
The minimization is:

$$\text{Minimize } C = \sum_{m=1}^{n} W_m X_m \quad (4.9)$$

subject to $f(X) \geq Q^0$ and $X \in V(Q) = \{X \mid X \in \mathbb{R}^n, (Q, -X) \in \hat{Q}\}$,

where $C$ = the total cost of production, $Q^0 \in [0, \infty)$ = the given level of output, $\hat{Q}$ = the production set, $X \in \mathbb{R}^n$ = an input vector, $W$ = a price vector and given (see the regularity conditions [H.1] and [H.2]).

4.5.0.5 Theorem 4.1: If a production function $f(X)$ is (i) continuous (by virtue of Definition 4.2); (ii) increasing by regularity condition [H.7]; (iii) strictly quasi-concave on $V(Q)$, and if $f(X)$ satisfies a regularity condition [H.9], then there exists a unique solution to the cost-minimization (4.9).

Let us prove Theorem 4.1. To do so, we need some mathematical definitions and theorems in the following order.

4.5.0.6 Definition 4.4: Let $B_r(x_0) \equiv \{x \in \mathbb{R}^n, d(x, x_0) < r\}$, where $x_0 \in \mathbb{R}^n$, $r$ is some positive real number (i.e., $r \in \mathbb{R}_{++}$), and $d(x, x_0)$ refers to the Euclidean distance between $x$ and the fixed point $x_0$, $B_r(x_0)$ is called the open ball about point $x_0$ with radius $r$. The fixed point $x_0$ is called the center of the open ball. 12

4.5.0.7 Remark 1: An open ball is always non-empty, for it contains its center.

4.5.0.8 Definition 4.5: A subset $S$ of $\mathbb{R}^n$ is said to be bounded if there exists an open ball which contains $S$.\(^{13}\)

4.5.0.9 Theorem 4.2: Every subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. (Heine-Borel theorem).\(^{14}\)

4.5.0.10 Theorem 4.3: Any closed subset of a compact set in $\mathbb{R}^n$ is compact.\(^{15}\)

4.5.0.11 Theorem 4.4: Let $f_i : T \to \mathbb{R}^n$ and $\alpha_i : T \to \mathbb{R}$ $(i = 1, 2, \ldots, M)$, where $T$ is a metric space, be continuous functions, then $f(X) \equiv \sum_{i=1}^{M} \alpha_i f_i(X)$ is also continuous in $T$.\(^{16}\)

4.5.0.12 Theorem 4.5: Let $f(X)$ be a real valued function from $S \subset \mathbb{R}^n$ to $\mathbb{R}$. If $f(X)$ is continuous on a subset $S$, and if $S$ is compact, non-empty, then $f(S)$ has a maximum and minimum in $S$ (Weierstrass theorem).\(^{17}\)

Now let us prove Theorem 4.1.

(I) First, a minimum exists:

Because a production function $f(X)$ is continuous and increasing, we can find, under the assumption [H.9], a vector $X \in V_1(Q) \ni f(X) \geq Q^0$. Thus, the feasible region is not empty. Reckon $\hat{C} = W^T \hat{X}$ at given $W$ and choose a $l > 0 \ni \hat{C} < l \cdot \min\{W_1, W_2, \ldots, W_N\}$. Define the set $L = \{X | X_m \leq l, (m = 1, 2, \ldots, N); X \in V_1(Q)\}$. Then, if $X \in L$, $W^T X > l \cdot \min\{W_1, W_2, \ldots, W_N\} > \hat{C}$. Evidently, such a

\(^{13}\)See Takayama (1988, p. 38).

\(^{14}\)See Kelly (1955, pp. 144-145) and Simmons (1963, pp. 119-120) for proof.

\(^{15}\)See Takayama (1988, p. 33).

\(^{16}\)See Takayama (1988, p. 31).

\(^{17}\)See Debreu (1959a, p. 16).
vector can not be an optimum. Consequently, the cost-minimization problem (4.9) is reduced to:

\[
\text{Minimize } C = W^T X \\
\text{subject to } f(X) \geq Q^o \text{ and } X \in L \subseteq V_1(Q).
\]

Because \( \hat{X} \in L \), the feasible region is not empty and it is closed and bounded by Definition 4.4, Remark 1, and Definition 4.5 (see Figure 4.1). Hence the feasible region is compact by Theorem 4.2 and Theorem 4.3. The elements of the vector \( X \) are continuous (see [H.2]), \( W \) is given, therefore, \( W^T X \) is a continuous function by virtue of Theorem 4.4. Now, we apply Weierstrass theorem (Theorem 4.5), which states that a continuous function defined on a compact set has a maximum and minimum, say, \( \min(C) \). Thus, there is a minimum of a cost function. The feasible region \( L = \{X|X_m \leq l, (m = 1, 2, ..., N); X \in V_1(Q)\} \cap V(Q)' = \{X|X \in V_1(Q), Q^o \in [0, \infty), f(X) \geq Q^o\} \) is shaded area shown in Figure 4.1 (in the 2-dimensional space).

(II) Secondly, in the minimum the restriction \( f(X) \geq Q^o \) is binding. In other words, \( f(X^o) = Q^o \) exists, where \( X^o \) represents the optimal demand for the factors of production. Assume that there is an optimal factors of production \( X^o \) for which the restriction \( f(X^o) \geq Q^o \) were not binding. The production function \( f(X) \) is continuous, increasing and it satisfies the ray property (we assumed these things in Section 4.4.3: [H.5], [H.7], and [H.9]). Consequently, we can find a vector \( X^{(1)} < X^o \Rightarrow f(X^{(1)}) \geq Q^o \) as shown in Figure 4.2. Because \( W > 0, \) \( [C_1 = W^T X^{(1)}] < [W^T X^o = C^o = \min(C)] \). It is a contradiction. Thus, \( f(X^o) \geq Q^o \) is binding.
(III) Third, there is a unique solution of a cost minimum under the conditions (I) and (II). Assume that there exist two optimum solutions \((X^{(1)})^\circ\) and \((X^{(2)})^\circ\). Then \(W^T(X^{(1)})^\circ = W^T(X^{(2)})^\circ = C^\circ = \min(C)\) and \(f([X^{(1)})^\circ] = f([X^{(2)})^\circ] = Q^\circ\).

Because a production function is strictly quasi-concave on \(V_1(Q)\) (see [H.4]),

\[
f(X^{(3)}) = f \left[ \theta(X^{(1)})^\circ + (1 - \theta)(X^{(2)})^\circ \right] > \min\{f([X^{(1)})^\circ], f([X^{(2)})^\circ]\} = Q^\circ, \tag{4.11}
\]

where \(\theta \in (0,1)\) and \(X^{(3)} = \theta(X^{(1)})^\circ + (1 - \theta)(X^{(2)})^\circ\). However,

\[
W^T X^{(3)} = \theta W^T(X^{(1)})^\circ + (1 - \theta)W^T(X^{(2)})^\circ = \min(C). \tag{4.12}
\]

The expressions (4.11) and (4.12) imply that the vector \(X^{(3)}\) would also be a minimal (i.e., (4.12)), but the restriction \(f(X^{(3)}) \geq Q^\circ\) would not be binding (i.e., (4.11)). This contradicts the result of (II). Thus, there is a unique cost minimum. Q.E.D.

Finally, in conclusion, we can obtain the unique solution of cost-minimization problem under the result of (I), (II), and (III). The solution can be either be an interior point, i.e., \(X^\circ \in V_1(Q)\) or a boundary point.

4.6 Cost Minimization with Equality Constraint

We have showed that the restriction \(f(X) \geq Q^\circ\) is binding at the cost minimum in the previous section. Thus, the problem of cost minimization for a competitive producer is reduced to:

\[
\text{Minimize } C = \sum_{m=1}^{n} W_m X_m \text{ subject to } f(X) = Q^\circ, \tag{4.13}
\]
Figure 4.1: 2-dimensional factor space; $R \times R$
Where \( C = \) the total cost of producing the given level of output \( Q^o \in (0, \infty) \),
\[ \mathbf{X} = (X_1, X_2, \ldots, X_n) \in \mathbb{R}^n_{++}, \]

\( \mathbf{W} = \) a vector of the strictly positive factor price: \( \mathbf{W} > 0 \).

We also showed that there exists a unique solution. The Lagrangean function for this cost-minimization is as follows:

\[
\text{Min. } \Phi(\mathbf{X}, \mu) = \mathbf{W}^T \mathbf{X} + \mu \left[ Q^o - f(\mathbf{X}) \right], \tag{4.14}
\]

where \( \mu = \) the Lagrange multiplier of the constraint.

Differentiating \( \Phi(\mathbf{X}, \mu) \) with respect to \( \mathbf{X} \) and \( \mu \) yields the equilibrium conditions (or FOCs) for the constrained cost-minimization as follows:

\[
(\text{FOC}): \text{There exists the optimal (conditional) factor demand } \mathbf{X}^o \text{ and } \mu^o \in (0, \infty) \text{ such that}
\]

\[
\mathbf{W} = \mu \mathbf{f}_x, \tag{4.15}
\]

\[
Q^o = f(\mathbf{X}), \tag{4.16}
\]

where \( \mathbf{W} = (W_1, W_2, \ldots, W_n) \) and \( \mathbf{f}_x = \left( \frac{\partial f}{\partial X_1}, \frac{\partial f}{\partial X_2}, \ldots, \frac{\partial f}{\partial X_n} \right). \)

We can write the second-order necessary condition (SONC) and the second-order sufficient condition (SOSC) as follows:

\[
(\text{SONC}) \quad \xi^T D \xi \leq 0 \quad \text{whenever } \xi^T \mathbf{f}_x = 0, \tag{4.17}
\]

\[
(\text{SOSC}) \quad \xi^T D \xi < 0 \quad \text{for all } \xi \neq 0 \quad \Rightarrow \quad \xi^T \mathbf{f}_x = 0, \tag{4.18}
\]

where \( D = \) the symmetric Hessian matrix of \( f(\mathbf{X}), \)
\[
\mathbf{f}_x^T = \left( \frac{\partial f}{\partial X_1}, \frac{\partial f}{\partial X_2}, \ldots, \frac{\partial f}{\partial X_n} \right),
\]

\( \xi \neq \mathbf{0} \in \mathbb{R}^n, \) that is, \( \xi \) is any real-valued non-null vector.

The implicit function theorem\(^\text{18}\) states that if the conditions (4.17) and (4.18)

---

are satisfied, a system of equations (4.15) and (4.16) can be solved, locally for those variables, \( X_m \) \((m = 1, 2, \ldots, n)\), being differentiated as explicit functions of the remaining variables of the system. Hence, equations (4.15) and (4.16) can be solved for \( X \) and \( \mu \) in terms of the exogenous variables \( W = (W_1, W_2, \ldots, W_n) \) and \( Q^0 \). The optimal conditional factor demand function resulting from (4.15) and (4.16) under the conditions (4.17) and (4.18) is as follows:

\[
X_m^0 = X(Q^0, W),
\]

where \( m \in [1, n] \) and \( W = (W_1, W_2, \ldots, W_n) \).

The factor demand function (4.19) implies that factor demands depend upon the level of output produced as well as on the factor prices. The factor demand function (4.19) has the following properties:

1. The factor demand function is homogeneous of degree zero in the factor prices \((W_1, W_2, \ldots, W_n)\).

2. The factor demand function has the symmetry or reciprocity conditions:

\[
\frac{\partial X_m}{\partial W_{m'}} = \frac{\partial X_{m'}}{\partial W_m}
\]

for \( m \neq m' \), through the comparative statics (see (4.41)). This means that the cross price effects must be equal.

3. The factor demand curves are downward-sloping: \( \frac{\partial X_m}{\partial W_m} < 0 \).

We can also derive the Lagrange multiplier from the system of (4.15) and (4.16) as follows:

\[
\mu^0 = \mu(Q^0, W).
\]

(4.20)
Here we can derive the minimal total cost function from substituting the optimal factor demand functions (4.19) into (4.13) as follows:

\[ C(W, Q^o) = \min \{ W^T \cdot X^o | X \in V_1(Q^o) \}, \quad (4.21) \]

where \( W=(W_1, W_2, ..., W_n) \) and \( X^T=(X_1, X_2, ..., X_n) \).

In words, the expression (4.21) implies that the minimum cost of producing a given output level \( Q^o \) is a function of the factor prices and output. Just as the production function is describing the technological possibilities of production, the cost function is describing the economic possibilities of a rational producer. Under the regularity conditions \([H.1]-[H.10]\), the minimal cost function (4.21) will satisfy the following properties:

1. (Nonnegativity): \( C^o(W, Q^o) > 0 \) for \( W > 0 \) and \( Q^o > 0 \).

2. (Monotonicity): (i) if \( W' \geq W \), then \( C^o(W', Q^o) \geq C^o(W, Q^o) \); this means that the minimal cost function \( C^o(W, Q^o) \) is nondecreasing in the factor prices \( W \). (ii) if \( Q^o' \geq Q^o \), then \( C^o(W, Q^o') \geq C^o(W, Q^o) \); the minimal cost function \( C^o(W, Q^o) \) is nondecreasing in the level of output \( Q^o \).

3. (Linear Homogeneity): \( C^o(\delta W, Q^o) = \delta C^o(W, Q^o) \) for \( \delta > 0 \); the minimum cost function \( C^o(W, Q^o) \) is homogeneous of degree one in the factor prices \( W \) for \( W > 0 \).

4. (Concavity): \( C^o\left[ (\theta W + (1 - \theta) W'), Q^o \right] > \theta C^o(W, Q^o) + (1-\theta) C^o(W', Q^o) \) for \( \theta \in (0, 1) \) and \( W \neq W' \); the minimal cost function \( C^o(W, Q^o) \) is strictly (quasi-) concave in the factor prices \( W \) for \( W > 0 \).
5. (Continuity): the minimum cost function $C^o(W, Q^o)$ is continuous as a function of the factor prices $W$ for $W > 0$.

The Lagrange multiplier (4.20) has an economic interpretation: the optimal value of the Lagrange multiplier equals the marginal cost of production, that is, \( \frac{\partial C^o(W, Q^o)}{\partial Q^o} = \mu \).

### 4.7 Comparative Statics Analysis

To analyze the effects of changes in the optimal output $Q^o$ and the factor prices $W$ on the optimal factors of production, $X^o$, we need the following comparative statics relations. First, we take the total differentials of the equilibrium conditions, i.e., FOCs. The results are as follows:

\[
\mu^{-1}dW_m = \sum_{m'}^n f_{mm'}dX_{m'} + \mu^{-1}f_m d\mu (m = 1, 2, ..., n). \tag{4.22}
\]

\[
\mu^{-1}dQ = \sum_{m=1}^n \mu^{-1}f_m dX_m, \tag{4.23}
\]

where \( f_{mm'} = \left( \frac{\partial f}{\partial X_m \partial X_{m'}} \right) \).

In matrix notation, we obtain that:

\[
\begin{bmatrix}
H & \mu^{-1}f_x \\
\mu^{-1}f_T & 0
\end{bmatrix}
\begin{bmatrix}
dX \\
d\mu
\end{bmatrix}
= \begin{bmatrix}
\mu^{-1}dW \\
\mu^{-1}dQ
\end{bmatrix}. \tag{4.24}
\]

Or,

\[
\hat{H}
\begin{bmatrix}
dX \\
d\mu
\end{bmatrix}
= \begin{bmatrix}
\mu^{-1}dW \\
\mu^{-1}dQ
\end{bmatrix}, \tag{4.25}
\]
where $H$ is the $(n \times n)$ Hessian matrix of the production function $f(X)$ and symmetric by Young's theorem, i.e., $f_{mm}^\prime = f_{m^\prime m}$. Namely,

$$H = \begin{bmatrix}
    f_{11} & f_{12} & \cdots & f_{1n} \\
    f_{21} & f_{22} & \cdots & f_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    f_{n1} & f_{n2} & \cdots & f_{nn}
\end{bmatrix}, \quad (4.26)$$

where $\tilde{H}$ is the $(n + 1) \times (n + 1)$ symmetric bordered Hessian matrix; that is,

$$\tilde{H} = \begin{bmatrix}
    H & \mu^{-1} f_x \\
    \mu^{-1} f_x^T & 0
\end{bmatrix}.$$

d$W^T = (dW_1, dW_2, \ldots, dW_n)$, $f_x^T = (\frac{\partial f}{\partial X_1}, \frac{\partial f}{\partial X_2}, \ldots, \frac{\partial f}{\partial X_n})$, and $dX^T = (dX_1, dX_2, \ldots, dX_n)$.

The matrix form (4.25) can be rewritten as follows:

$$\begin{bmatrix}
    dX \\
    d\mu
\end{bmatrix} = \tilde{H}^{-1} \begin{bmatrix}
    \mu^{-1} dW \\
    \mu^{-1} dQ
\end{bmatrix}, \quad (4.27)$$

where $\tilde{H}^{-1}$ is the inverse symmetric bordered Hessian matrix.

Now, to get the solutions, we have to evaluate the inverse of the bordered Hessian matrix in the system (4.27). From the matrix systems (4.24) and (4.25),

$$\tilde{H} = \begin{bmatrix}
    H & \mu^{-1} f_x \\
    \mu^{-1} f_x^T & 0
\end{bmatrix}. \quad (4.28)$$
Thus, the inverse matrix of (4.28) is:

\[ \tilde{H}^{-1} = \begin{bmatrix} H & \mu^{-1}f_x \\ \mu^{-1}f_x^T & 0 \end{bmatrix} \cdot (4.29) \]

\[ \tilde{H}^c = \begin{bmatrix} F & G_1 \\ G_2 & J \end{bmatrix}, \quad (4.30) \]

where \( \tilde{H}^c = \) the \([(n+1) \times (n+1)] \) symmetric cofactor matrix of the bordered Hessian matrix \( \tilde{H} \);

\( F = \) the \((n \times n) \) symmetric cofactor matrix composed of the cofactors bordered Hessian matrix \( \tilde{H} \), that is,

\[ F = \begin{bmatrix} D_{11} & D_{12} \ldots D_{1n} \\ D_{21} & D_{22} \ldots D_{2n} \\ \vdots & \vdots \\ D_{n1} & D_{n2} \ldots D_{nn} \end{bmatrix}, \quad (4.31) \]

where \( D_{mm'} = \left( -1 \right)^{m+m'} \text{det}(\tilde{H}_{m,m'}) \quad (m, m' \in [1,n]), \)

\( G_1^T = [D_{1, n+1}, D_{2, n+1}, \ldots, D_{n, n+1}], \quad G_2^T = [D_{n+1, 1}, D_{n+1, 2}, \ldots, D_{n+1, n}], \)

\( J = D_{n+1, n+1}. \)

Transposing the cofactor matrix (4.30) results in:

\[ \text{adj}(\tilde{H}) = \begin{bmatrix} F^T & G_2 \\ G_1^T & J \end{bmatrix}, \quad (4.32) \]
where \( \text{adj}(\bar{H}) = \text{adjacent} \ \bar{H} \) and symmetric.

Hence, the inverse of the bordered Hessian matrix \( \bar{H} \) can be expressed in the following way:

\[
\bar{H}^{-1} = \text{adj}(\bar{H}) \left[ \text{det}(\bar{H}) \right]^{-1}, \tag{4.33}
\]

where \( \text{det}(\bar{H}) = \text{the determinant of the bordered Hessian matrix} \ \bar{H}. \)

Substituting the expression (4.33) into the matrix form (4.27) yields

\[
\begin{bmatrix}
\frac{dX}{d\mu}
\end{bmatrix} = \text{adj}(\bar{H}) \left[ \text{det}(\bar{H}) \right]^{-1} \begin{bmatrix}
\mu^{-1}dW \\
\mu^{-1}dQ
\end{bmatrix}. \tag{4.34}
\]

By substitution of the transpose of the cofactor matrix, i.e., (4.32) into the matrix system (4.34), we obtain

\[
\begin{bmatrix}
\frac{dX}{d\mu}
\end{bmatrix} = \left[ \text{det}(\bar{H}) \right]^{-1} \begin{bmatrix}
F^T & G_2 \\
G_1^T & J
\end{bmatrix} \begin{bmatrix}
\mu^{-1}dW \\
\mu^{-1}dQ
\end{bmatrix}. \tag{4.35}
\]

Finally, we derive the fully reduced system for the solutions from the matrix system (4.35) as follows:

\[
\begin{bmatrix}
\frac{dX}{d\mu}
\end{bmatrix} = \left[ \text{det}(\bar{H}) \right]^{-1} \begin{bmatrix}
\mu^{-1}dW + G_2dQ \\
G_1^TdW + JdQ
\end{bmatrix}. \tag{4.36}
\]

where \( F^T = \text{the transpose of the} \ (n \times n) \ \text{cofactor matrix} \ (4.31). \) Therefore, the solutions for \( dX \) and \( d\mu \) can be written in determinantal notation as follows:

\[
dX_k = \mu^{-1} \left[ \text{det}(\bar{H}) \right]^{-1} \left[ \sum_{i=1}^{n} D_{ik}dW_i + D_{n+1,k}dQ \right], \tag{4.37}
\]
\[ d\mu = \mu^{-1} \left[ \det(\tilde{H}) \right]^{-1} \left[ \sum_{i=1}^{n} D_{i,n+1} dW_i + D_{n+1,n+1} dQ \right]. \] (4.38)

From the solutions (4.37), we can obtain the relationships between the optimal factors of production and the factor prices as follows:

\[ \frac{dX^{0}_{m'}}{dW_{m}} = \mu^{-1} D_{mm'} \left[ \det(\tilde{H}) \right]^{-1}, \] (4.39)

\[ \frac{dX^{0}_{m}}{dW_{m'}} = \mu^{-1} D_{m'm} \left[ \det(\tilde{H}) \right]^{-1}, \] (4.40)

where \( m, m' \in [1, n] \) and \( m \neq m' \).

We know that \( \det(H_{mm'}) = \det(H_{m'm}) \) since the cofactor matrices \( F \) and \( \tilde{H}^c \) are symmetric (See (4.31) and (4.30)). Hence,

\[ \frac{dX^{0}_{m'}}{dW_{m}} = \frac{dX^{0}_{m}}{dW_{m'}} \] for \( m \neq m' \). (4.41)

The expression (4.41) is called the reciprocity relation or the symmetry condition. This condition has an economic interpretation that the change in the \( m' \)-th factor of production, i.e., \( X^{0}_{m'} \), with respect to a change in the \( m \)-th factor price, i.e., \( W_m \), output holding constant, should be equal to the change in the \( m \)-th factor of production, \( X^{0}_{m} \), with respect to the \( m' \)-th factor price, \( W_{m'} \), output being constant.

On the other hand, as a special case of (4.41), we have the following property:

\[ \frac{dX^{0}_{m}}{dW_{m}} = \mu^{-1} D_{mm} \left[ \det(\tilde{H}) \right]^{-1} < 0. \] (4.42)

\[ [\det(\tilde{H})]^{-1} D_{mm} < 0 \] by the stability conditions (SOC). In other words, \( \det(\tilde{H}) \) and \( D_{mm} \) are of opposite sign. The same is true of \( \det(\tilde{H}) \) and \( D_{mm} \) for any \( m \).

This property (4.42) says that the utilization of a given factor of production decreases (increases) as its own factor price increases (decreases), other prices and
output holding constant. That is, the demand curve of a given factor of production is downward sloping.
5. RESPONSIVENESS OF FACTOR SUBSTITUTION

5.1 Introduction

The neo-classical theory of production and cost recognizes the possibility of substituting one factor of production for another in the production process. Many modern economists emphasize that the substitutability relations among the factors of production utilized in the production process (see Chapter 9). For our economic analysis, we need the specific quantitative measurement of the degree to which one factor can be substituted for another. Such a measure is the so-called elasticity of substitution (EOS). In other words, the elasticity of substitution expresses the degree of substitutability or complementarity among the factors of production utilized in the production of the given level of output.

Hence, we investigate that (1) the relationships among the factors of production utilized in the production process (Section 5.2); (2) (a) the concepts of elasticities: the output and the price elasticities, and (b) the relations between the elasticities and the substitutability (or complementarity) (Section 5.3); (3) (a) the notion of the EOS, (b) the relationship between the EOS and the elasticities: FPCOE and OCPE and (c) the specific derivation of the EOS (i.e., $\sigma_{mm^l}$) through the cost function (Section 5.4); (4) the relation between FPCOE and The homogeneity degree (Section 5.5).
5.2 Substitutability and Complementarity

To know the relationships among the factors of production whether they are substitutes, complements, or independents is very crucial in our study because the factors of production can be substitutes, complements, or independents in the production process and therefore they have influences on the resource allocation. Thus, we introduce those concepts.

A profit-maximizing and/or cost-minimizing producer chooses the maximum level of output with the combination of various factors of production under a particular production function. In the process of combining the factors of production, there exist three types of relationships among the productive factors: substitutability, complementarity, and independence relationships. The factors of production with substitutability, complementarity, and independence relationships are substitutes, complements, and independents, respectively. Specifically, we can define those concepts through the Slutsky-Hicks equation in the theory of production.

We can write the Slutsky-Hicks equation in the following way:

\[
\frac{\partial X^\pi_m}{\partial W^m_{m'}} = \frac{\partial X^\circ_m}{\partial W^m_{m'}} - \frac{X^\circ_{m'}}{\mu} \frac{\partial X^\pi_m}{\partial Q}.
\]

Or,

\[
K^m_{mm'} = S^m_{mm'} - H_m Q,
\]

where \( X^\pi_m \) = the profit-maximizing factor demand, \( X^\circ_m \) = the conditional (or output-constant-compensated) factor demand, \( W_{m'} \) = the price of a factor \( X_{m'} \), \( \mu \) = the marginal cost.

\( K^m_{mm'} \) (= \( \frac{\partial X^\pi_m}{\partial W^m_{m'}} \)) = the total effect term,
The notions of substitutability, complementarity, and independence are provided by the cross-substitution effect term $S_{mm'}$ in the Slutsky-Hicks equation.\(^1\)

**5.2.0.13 Definition 5.1:**

(i) If $S_{mm'} > 0 \ (m \neq m')$, then two factors $X^o_m$ and $X^o_{m'}$ are (Hicks-Allen) substitutes.

(ii) If $S_{mm'} < 0 \ (m \neq m')$, then two factors $X^o_m$ and $X^o_{m'}$ are (Hicks-Allen) complements.

(iii) If $S_{mm'} = 0 \ (m \neq m')$, then two factors $X^o_m$ and $X^o_{m'}$ are independents.\(^2\)

These notions have economic interpretations as follows: Two factors $X^o_m$ and $X^o_{m'}$ are substitutes when, as the price of a factor $X^o_{m'}$, i.e., $W_{m'}$, rises (falls), the amount of a factor $X^o_m$ utilized falls (rises), and the decrease (increase) in a factor $X^o_{m'}$ causes the amount of a factor $X^o_m$ to increase (decrease). In the opposite case, they are complements when, as $W_{m'}$ rises (falls), the amount of a factor $X^o_m$ employed falls (rises). On the other hand, if the quantity of a factor $X^o_{m'}$ utilized is unchanged even though the price of a factor $X^o_{m'}$, $W_{m'}$, varies, then two factors $X^o_m$ and $X^o_{m'}$ are independent.

\(^1\)For the concepts of substitutability, complementarity, and independence, we can use the alternative, direct definition as follows: (i) Factors $X_m$ and $X_{m'}$ are substitutes if \(\frac{\partial^2 f}{\partial X_m \partial X_{m'}} < 0\); (ii) complements if \(\frac{\partial^2 f}{\partial X_m \partial X_{m'}} > 0\); (iii) independents if \(\frac{\partial^2 f}{\partial X_m \partial X_{m'}} = 0\), where $f=f(X)$; the production function. These concepts correspond to Definition 5.1. See Silberberg (1978, p. 117).

By the symmetric condition of the cross-substitution effects, i.e., \( S_{mm'} = S_{m'm} \), both substitutability and complementarity have symmetric characteristics, respectively. In other words, if a factor \( X^o_m \) is a substitute (complement) for a factor \( X^o_{m'} \), then a factor \( X^o_{m'} \) is also a substitute (complement) for a factor \( X^o_m \).

In the modern economics literatures, many economists state that substitutability relationships among any number of the factors of production (or commodities) dominate complementarity relationships among these factors of production (or commodities). Allen (1938, p. 509) describes this point in the following way: Competition (i.e., substitutability) between factors is, on the whole, more general than complementarity. One factor, in any case, cannot be complementarity with all other factors. Hicks (1946, pp. 311-312) expounds briefly the above fact. Kamien (1964), however, proved the above statement by showing exactly the maximum number of complementarity relationships, that is, the least number of substitutability relationships in a system of \( N \) commodities, based on the properties of the complementarity and substitutability relationships which are explained in Hicks's *Value and Capital* (1946). Kamien (1964, p. 227) suggests that these results mentioned above can be applied directly to the definition of Allen's partial elasticity of substitution.

---

3Duality provides a simple proof of the symmetry of the cross-substitution effect. Consider the producer cost function \( C(W, Q) \), which is the minimal cost obtained from the cost-minimization problem. If we utilize Shephard (-Mckenzie) lemma, 
\[
\frac{\partial C}{\partial W_m} = X^o_m(W, Q) \quad \text{and} \quad \frac{\partial C}{\partial W_{m'}} = X^o_{m'}(W, Q).
\]
Thus,
\[
\frac{\partial^2 C}{\partial W_m \partial W_{m'}} = \frac{\partial X^o_{m'}}{\partial W_m} = \frac{\partial X^o_m}{\partial W_{m'}} = \frac{\partial^2 C}{\partial W_{m'} \partial W_m}.
\]
This is the symmetry condition of the cross-substitution effect: \( S_{mm'} = S_{m'm} \).

4See Allen (1959, p. 664); Bushaw and Clower (1957, p. 127); Wold (1953, pp. 104-105); Samuelson (1947, p. 184); Chambers (1988, p. 62).

5Kamien (1964, p. 227) states that at least \((N - 1)\) partial elasticities of substitu-
reasonable idea because the theory of production has strict analogies to the theory of consumer demand.

5.3 FPCOE and OCPE

The elasticities of the utilization of the factors of production have crucial significances in the theory of production and our economic analysis since the notions of substitutabilities and/or complementarities are linked with those of elasticities. In other words, we can describe the substitutability and the complementarity relationships through the measures of elasticities. Moreover, both notions represent the substitution effects among the factors of production in the production process.

The factor utilization (demand) function can be expressed by unit-free elasticity coefficients such as the output and the price elasticities. Samuelson (1947/1983, p. 125) introduces the influence of Alfred Marshall on the development of elasticity coefficient and states that mathematically an elasticity expression between two magnitudes, such as price and quantity, consists simply of the logarithm of one of those quantities differentiated with respect to the logarithm of the other. Thus, we can define the output elasticity and the price elasticities in the following manner.

First, we can express mathematically the factor price-constant output elasticity of factor utilization \( X_m^0 \) (FPCOE) as follows:

\[
N(X_m^0, Q^0) = \frac{d(lnX_m^0)}{d(lnQ^0)} = \frac{Q^0}{X_m^0} \frac{\partial X_m^0}{\partial Q^0},
\]

(5.3)

where \( X_m^0 = \) the conditional factor demand; \( Q^0 = \) the level of output.

In words, FPCOE (5.3) is the proportionate change in the utilization (demand)
for a factor $X^o_m$ in response to a proportionate change in output $Q^o$.

Secondly, the output-constant price elasticity of factor demand (OCPE) is defined as the responsiveness of a given factor demand to change in the price of a related factor, holding all other factor prices and output fixed. The OCPE is expressed mathematically in the following way:

$$
N(X^o_m, W_{m'}) = \frac{d(lnX^o_m)}{d(lnW_{m'})} = \frac{W_{m'} \partial X^o_m}{X^o_m \partial W_{m'}},
$$

(5.4)

where $m \in [1, n]$, $X^o_m = $ the conditional factor demand, $W_{m'} = $ the price of a factor $X^o_m$.

If $m \neq m'$, we call the OCPE (5.4) the output-constant cross-price elasticity of factor utilization (demand) (OCCPE), and if $m = m'$, the OCPE is called the output-constant own-price elasticity of factor utilization (OCOPE). These elasticities have economic interpretations as follows: The OCCPE represents the proportionate change in the utilization for a factor $X^o_m$ in response to a proportionate change in the price of a factor $X^o_m$, i.e., $W_{m'}$. Similarly, the OCOPE is the proportionate change in the demand for factor $X^o_m$ associated with change in its own factor price $W_m$.

Now, under Definition 5.1 and equation (5.4), we can obtain the following theorem. This theorem expresses the relation between the notions of substitutability, complementarity, and independence relationships and that of elasticity (i.e., OCCPE).

5.3.0.14 Theorem 5.1: (i) If the output-constant cross-elasticity of factor utilization is positive, that is, $N(X^o_m, W_{m'}) > 0 \ (m \neq m')$, then factors $X^o_m$ and $X^o_{m'}$ are (Allen-Hicks) substitutes, and vice versa.
(ii) If the OCCPE is negative, i.e., \( N(X_m^\circ, W_{m'}) < 0 \) (\( m \neq m' \)), then factors \( X_m^\circ \) and \( X_{m'}^\circ \) are (Allen-Hicks) complements, and *vice versa*.

(iii) If the OCCPE is equal to zero, i.e., \( N(X_m^\circ, W_{m'}) = 0 \), then factors \( X_m^\circ \) and \( X_{m'}^\circ \) are independents, and *vice versa*.

**(Proof):**

Making use of the symmetry condition of the cross-substitution effect in the Slutsky-Hicks equation and Young’s theorem, we can rewrite the expression (5.4) as:

\[
N(X_m^\circ, W_{m'}) = \left( \frac{W_{m'}}{X_m^\circ} \right) S_{mm'},
\]

where \( m, m' \in [1, n] \), \( W_{m'} > 0 \) (see \([H.1]\)), \( X_m^\circ \in (0, \infty) \).

(i) If \( S_{mm'} > 0 \), then \( N(X_m^\circ, W_{m'}) > 0 \) in equation (5.5) since \( \frac{W_{m'}}{X_m^\circ} > 0 \). Thus, since if \( S_{mm'} > 0 \), by **Definition 5.1** two factors \( X_m^\circ \) and \( X_{m'}^\circ \) are substitutes, we can say that if \( N(X_m^\circ, W_{m'}) > 0 \), factors \( X_m^\circ \) and \( X_{m'}^\circ \) are substitutes.

(ii) When \( S_{mm'} < 0 \), then \( N(X_m^\circ, W_{m'}) < 0 \) in equation (5.5) since \( \frac{W_{m'}}{X_m^\circ} > 0 \). Similarly, by virtue of **Definition 5.1**, we can say that if \( N(X_m^\circ, W_{m'}) < 0 \), factors \( X_m^\circ \) and \( X_{m'}^\circ \) are called complements.

(iii) If \( S_{mm'} = 0 \), then \( N(X_m^\circ, W_{m'}) = 0 \). We can express that factors \( X_m^\circ \) and \( X_{m'}^\circ \) are independent when \( N(X_m^\circ, W_{m'}) = 0 \). Q.E.D.

These notions of the output and the price elasticities play crucial roles in our study for analyzing the effects of price changes on the resource allocations. However, we should note the following feature of the OCCPE.

**5.3.0.15 Theorem 5.2:** \( N(X_m^\circ, W_{m'}) \neq N(X_{m'}^\circ, W_m) \) (\( m, m' \in [1, n] \)) for all \( m \neq m' \). This implies that the OCCPE is asymmetric.
5.4 Measures of Factor Substitutability

In the preceding section we mentioned that the OCCPE has a feature of asymmetry between the optimal factors of production (see Theorem 5.2). To overcome that feature, therefore, we need the elasticity of substitution (EOS) which is a more sophisticated measure of factor substitutability than that of the OCCPE. The elasticity of substitution is a quantitative measure of the degree of substitutability, complementarity, or independence relationships between the factors of production utilized in the production process, and has the symmetry property unlike the OCCPE. Thus, the elasticity of substitution can explain well, like the OCCPE, the substitution effects among the optimal factors of production utilized, but it is more convenient and stronger measure of the substitution effects between factors than that of the OCCPE in our economic analysis through a particular production function.

Specifically, the elasticity of substitution is defined as the proportionate change in the factor ratios in response to given proportionate change in the marginal rate of technical substitution (MRTS) between the optimal factors of production, holding output constant. The elasticity of substitution depends on the shape of an isoquant because the MRTS represents the curvature of an isoquant. Thus, the elasticity of substitution is a measure of the curvature of an isoquant. Mathematically, we can write the definition of the EOS as follows:

\[
\sigma_{mm'} = \frac{d\ln \left( \frac{X_{m}}{X_{m'}} \right)}{d\ln (MRTS_{mm'})} = \left[ \frac{MRTS_{mm'}}{(X_{m}/X_{m'})} \right] \left[ \frac{\partial(X_{m}/X_{m'})}{\partial(MRTS_{mm'})} \right] (m \neq m') (5.6)
\]

Since \( MRTS_{mm'} = \frac{f_{m}}{f_{m'}} = \frac{W_{m}}{W_{m'}} \), which is derived from equilibrium conditions (FOCs) of the cost-minimization problem, we can rewrite the expression (5.6) in the
The expression (5.7) is the definition of the elasticity of substitution when output and the prices of other factors of production except $X_m$ and $X_{m'}$ are held constant.

Similarly,

$$\sigma_{mm'} = \frac{d\ln\left(\frac{X_{m'}}{X_m}\right)}{d\ln\left(\frac{W_{m'}}{W_m}\right)} = \left[ \frac{\partial(X_{m'})}{\partial(W_{m'})} \right] \left[ \frac{\partial(X_m)}{\partial(W_m)} \right] \quad (m \neq m').$$

We call the expression (5.8) the direct elasticity of substitution, holding output and the quantities of all remaining factors of production constant.\(^6\) The expressions (5.7) and (5.8) imply that if we assume the cost-minimization we can define the elasticity of substitution as the ratio of the proportionate change in factor ratios to the proportionate change in relative factor prices or marginal productivities, respectively. In other words, the elasticity of substitution is a way of describing how factor choice would change in response to change in relative factor prices or marginal productivities.

For the $\nu$-degree homogeneous two-factor production function, the expression (5.8) can be rewritten as follows:

$$\sigma_{mm'} = \frac{f_{m'}f_m}{\nu Q_f + (1 - \nu)f_{m'}f_{m'}} \quad (m \neq m'),$$

where $\nu$ is the degree of homogeneity of the production function, $Q_f = f$; the given level of output or the production function.

\(^6\)The expression (5.8) is equal to the form (1) McFadden mentioned. See McFadden (1963, p. 74).
(Proof): See APPENDIX 1.

For the linear homogeneous two-factor production function, i.e., $\nu = 1$, the expression (5.9) simplifies to:

$$\sigma_{mm'} = \frac{f_m f_{m'}}{Q_0 f_{mm'}} (m \neq m'). \quad (5.10)$$

The expression (5.10) is a special case of the expression (5.9) and is the form that Hicks (1932) first introduced in his *The Theory of Wages*.

The expressions (5.9) and (5.10), however, would not hold with more than two factors of production. So, we need generalize the EOS of the two-factor production function case stated above to the multi ($n > 2$)-factor production function case. The EOS has been in several ways generalized to the case in which more than two factors of production are involved.

Uzawa (1962, pp. 291-293) has generalized the elasticity of substitution in terms of the unit cost function under the linear homogeneous multi-factor production function. Chambers (1988, pp. 35-36) has introduced the Morishima elasticity of substitution which has the asymmetry property and emphasizes the somewhat arbitrary nature of any elasticity of substitution in the multi-factor case. They are both based on Allen's (1938) generalized definition of the elasticity of substitution. Allen (1938, pp. 503-504) measures the partial elasticity of factor substitution which generalizes the elasticity of substitution of the two-factor production function as follows:

$$\sigma_{mm'} = \left( \frac{\sum_{k=1}^{n} f_k X_k}{X_m X_{m'}} \right) \left( \frac{\det(H_{mm'})}{\det(H)} \right) \text{ for } m \neq m', \quad (5.11)$$
where \( f_k = \frac{\partial f}{\partial X^0_k} \),

\[
\det(\tilde{H}) = \det \begin{bmatrix}
0 & f_1 & f_2 & \cdots & f_n \\
f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f_n & f_{n1} & f_{n2} & \cdots & f_{nn}
\end{bmatrix},
\]

(5.12)

\( \det(\tilde{H}_{mm'}) \) represents the cofactor of the element of the \( m \)-th row and the \( m' \)-th column of the bordered Hessian matrix \( \tilde{H} \).

Under the expression (5.11), we can obtain the following statement.

**5.4.0.16 Theorem 5.3:** The elasticity of factor substitution between the optimal factors \( X^0_m \) and \( X^0_{m'} \), i.e., \( \sigma_{mm'} \), is symmetric: \( \sigma_{mm'} = \sigma_{m'm} \) for \( m' \neq m \), where \( m, m' \in [1,n] \).

(Proof): See APPENDIX II.

We can also get the relation between the elasticity of substitution and the OCCPE from the definition (5.11) as follows:

**5.4.0.17 Theorem 5.4:** \( \sigma_{mm'} = \langle \gamma_{m'}^{-1} \rangle N(X^0_m, W_{m'}) \) for \( m \neq m' \). In other words, the previous formula can be rewritten as follows: \( N(X^0_m, W_{m'}) = \gamma_{m'}^{m''} \sigma_{mm'} \) for \( m \neq m' \), where \( \gamma_{m'} = [W_{m'} X^0_m] [C(W,Q)]^{-1} \); the cost share of the \( m' \)-th factor of production in the minimal total cost \( C(W,Q) \), \( \gamma_{m'} > 0 \), \( N(X^0_m, W_{m'}) = \) the OCCPE (see the expression (5.4)).

(Proof): See APPENDIX III.

The economic interpretation of Theorem 5.4 means that if the price of a factor \( X^0_{m'} \), i.e., \( W_{m'} \), increases (decreases) by one percent compared to all other factor
prices, the utilization of factor $X_m^0$ rises (falls) by $\gamma_{m'm} \sigma_{mm'}$ percent.

Theorem 5.4 leads to the relationships between factors of production $X_m^0$ and $X_{m'}^0$ in the following manner:

5.4.0.18 Theorem 5.5: (i) If the elasticity of substitution $\sigma_{mm'}$ (or $\sigma_{m'm}$) is positive, i.e., $\sigma_{mm'}$ (or $\sigma_{m'm}$) > 0 for $m \neq m'$, then factors $X_m^0$ and $X_{m'}^0$ are substitutes.

(ii) If $\sigma_{mm'}$ (or $\sigma_{m'm}$) is negative, i.e., $\sigma_{mm'}$ (or $\sigma_{m'm}$) < 0 for $m \neq m'$, then factors $X_m^0$ and $X_{m'}^0$ are called complements.

(iii) On the other hand, if $\sigma_{mm'} = 0$, then factors $X_m^0$ and $X_{m'}^0$ are independent.

(Proof): See APPENDIX IV.

The generalized definition of the elasticity of factor substitution (5.11) can be written in terms of cost function.\footnote{We call the expression (5.13) the shadow partial elasticity of substitution. See McFadden (1963), Uzawa (1962), and Mundlak (1968).}

\[
\sigma_{mm'} = \left[ CC_{mm'} \right]^{-1} \left[ C_m C_{m'} \right], \tag{5.13}
\]

where $m, m' \in [1, n], C=C(W, Q)$; the minimal total cost function,

$C_m = \frac{\partial C(W, Q)}{\partial W_m}$ and $C_{m'} = \frac{\partial C(W, Q)}{\partial W_{m'}}$,

$C_{mm'} = \frac{\partial^2 C(W, Q)}{\partial W_m \partial W_{m'}} = C_{m'm}$ \text{ (by Young's theorem)}, and $W=(W_1, W_2, ..., W_n)$.

(Proof): See APPENDIX V.

5.5 The Homogeneity Degree and FPCOE

For our economic analysis, we need to know the relationship between FPCOE and the homogeneity degree which represents the returns to scale, i.e., the scale effect.
From the expression (5.3),

\[ N(X_m^*, Q^*) = \frac{Q^*}{X_m^*} \frac{\partial X_m^*}{\partial Q^*}. \]  

(5.14)

From equilibrium conditions (FOCs) of the cost-minimization problem,

\[ f(X^*) = Q^*, \]  

(5.15)

where \( X^* = [X_1^*(W, Q^*), X_2^*(W, Q^*), \ldots, X_n^*(W, Q^*)] \).

Taking the differential of the expression (5.15) with respect to \( Q^* \), we obtain

\[ \sum_{m=1}^{n} f_m \frac{\partial X_m^*}{\partial Q^*} = 1, \]  

(5.16)

where \( f_m = \frac{\partial f}{\partial X_m^*} \), \( m \in [1, n] \). Here we assume that the production function \( f(X) \) is homogeneous of degree \( \nu \) in the factors of production. Then, by Euler’s theorem,

\[ \sum_{m=1}^{n} f_m X_m^* = \nu f(X^*) = \nu Q^*. \]  

(5.17)

From the expression (5.17), we get the following form:

\[ (\nu Q^*)^{-1} \sum_{m=1}^{n} f_m X_m^* = 1. \]  

(5.18)

Combining the expressions (5.16) and (5.18), we obtain

\[ \sum_{m=1}^{n} f_m \frac{\partial X_m^*}{\partial Q^*} = (\nu Q^*)^{-1} \sum_{m=1}^{n} f_m X_m^*. \]  

(5.19)

In matrix notation, the relation (5.19) has the following form:

\[ f_T^T \frac{\partial X^*}{\partial Q^*} = (\nu Q^*)^{-1} f_T^T X^*, \]  

(5.20)

where

\[ f_T^T = (\frac{\partial f}{\partial X_1^*}, \frac{\partial f}{\partial X_2^*}, \ldots, \frac{\partial f}{\partial X_n^*}), \]  

\( (\frac{\partial X^*}{\partial Q^*})^T = (\frac{\partial X_1^*}{\partial Q^*}, \frac{\partial X_2^*}{\partial Q^*}, \ldots, \frac{\partial X_n^*}{\partial Q^*}) \).
Thus, from the expression (5.20),

$$\frac{\partial X^o}{\partial Q^o} = (\nu Q^o)^{-1} X^o.$$  \hfill (5.21)

Or,

$$\frac{\partial X^o_m}{\partial Q^o} = (\nu Q^o)^{-1} X^o_m,$$  \hfill (5.22)

where \(m \in [1,n]\). We can derive the following relation from the expression (5.22):

$$\nu^{-1} = \frac{Q^o}{X^o_m} \frac{\partial X^o_m}{\partial Q^o},$$  \hfill (5.23)

where \(m \in [1,n]\), \(X^o_m\) = the conditional factor demand.

Substitution of the expression (5.14) into (5.23) leads to:

$$N(X^o_m, Q^o) = \nu^{-1}.$$  \hfill (5.24)

This shows that the FPCOE and the returns to scale has the reciprocal relationship.
6. THE STRUCTURE OF PRICE SYSTEM

6.1 Introduction

Adam Smith stated the role of prices as an invisible hand which coordinates the independent decisions of the rational economic units such as producers and consumers in the market mechanism. The price or market system attracts economic resources for productive objectives since the price paid to resource owners determines their income. Price plays a role in allocating economic resources among various productive objectives because economic resources will be allocated into those productive activities which offer the highest price. The resulting output of production will, in turn, be allocated among consumers by the price system since the acquisition of commodities depend upon the purchase price. The price system serves to define the incentive for people to contribute to the output of the economy.

6.2 Taxation

Taxation is one of elements exerting powerful influence on the equilibrium condition in the market economy. The imposition of taxes (or subsidies: negative taxes) on commodities and/or factors of production raises economic inefficiencies - the distortion of resource allocation - through the mechanism of the price system in the market economy. Specifically, levying taxes (or subsidies) on commodities and/or
the productive factors causes the shift-up of the supply curves of resources. In response to such a shift of the supply curve, the equilibrium conditions change in the market. As result of it, the prices of commodities rise (fall) and the quantities of commodities decrease (increase).

Like this, by the tax (or subsidy) shock, an economy is confronted with change in the market system from the initial position of equilibrium to the new position of equilibrium, *ceteris paribus*.¹ This is a negative function of taxation in the market economy. On the other hand, taxation has positive roles of reducing government budget deficits, if any, by raising government tax revenues and of redistributing incomes.² In the end, the economy including consumers and producers experience changes - losses or gains - in economic welfares resulting from the imposition of taxes (or subsidies). The economy as a whole faces unambiguous economic losses and economic inefficiencies representing the resource misallocation by the quantity changes of commodities. These economic phenomena occur since taxes (or subsidies) are, in the short run or long run, incorporated into the production costs and therefore at least partially passed on to consumers and to producers in the form of higher (lower) consumer price and lower (higher) producer price, respectively. In other words, taxes give the inflationary impact on consumers and the deflationary effect on producers.

¹See Samuelson (1947, p. 16).
²See Brashares, Speyrer, and Carlson (1988, p. 155). They say that a Federal value-added tax presents a way to reduce the budget deficit without adversely affecting savings. See De Wulf (1983, pp. 345-370). He says that the tax system is only one system that can be used to influence the income distribution. He evaluated how and to what extent taxes affect income distribution based on tax incidence analysis. He focused on the tax incidence - the incidence of personal and corporate income taxes; that of social security taxes, property taxes, domestic consumption taxes - in less industrialized countries, based on that of industrial countries.
On the other hand, subsidies - negative taxes - have just opposite impacts to the case of taxes.

Now let us examine the brief historical background of taxation and next, the impact of taxes on the economy in relation to the price change, economic efficiency, economic welfare change in detail.

6.2.1 The Historical Background of Taxation

Since the days of Adam Smith, many economists were concerned with the tax problem. Briefly reviewing, Adam Smith mentioned the full incidence of taxes on rent of a fixed immobile resource (e.g., land) and on wages in his *Wealth of Nations* (1776). David Ricardo pointed out that taxes distort the structure of prices. He also provided by intuition basic idea of the elasticity concept which determines the size of the economic tax incidence in his *Principles of Political Economy*.

John S. Mill, the author of *Principles of Political Economy* (1848), asserted that a specific or *ad valorem* tax will raise the price of good. They showed implicitly or explicitly that taxes cause the price of commodity to rise. In other words, they recognized the tax effect on the price.

Furthermore, Dupuit, who wrote *On the Measurement of the Utility of Public Works* (1844) first developed and introduced the concept of economic welfare such as the consumer surplus; Dupuit called this the relative utility in his use, the producer surplus, and the net social economic loss when he analyzed the effect of a fall in the prices of public commodities on the social benefits of publicly provided commodities (e.g., bridge). For analyzing the social benefit of public commodities, he constructed a marginal utility curve for public commodities by assuming that the government
charges the maximum price for each additional unit of public commodities, lowering
the price by small amounts as it offers additional units. Under such assumptions, he
discussed the effect of a fall in the prices of public commodities on the social benefits
as follows: The total benefits from public commodities are measured by the whole
area under the marginal utility curve. The relative utility (i.e., the consumer surplus)
equals the excess of total utility over marginal utility, multiplied by the number of
units of public commodities. Without using a supply curve, he also considered the
producers surplus from selling public commodities at a uniform price per unit. He
thought that the producers surplus equals the excess of the money received by pro-
ducers (e.g., industry) over the aggregate marginal costs. The total social benefits of
public commodities are the sum of the consumers surplus and the producers surplus. 3
He found that there was the net social loss of a rise in the prices on public goods.
That is, this amounts to the analysis of the welfare effects of the imposition of an
excise tax on consumers.

His analysis finds the problems with the measurability and the additivity of
utility. 4 He, however, contributed to providing the basic ideas for the analysis of
the effects of the tax imposition on economic welfare through the concept of the
consumers surplus and the producers surplus.

A. Marshall refined Dupuit's concept of economic welfare as the more sophisti-
cated and applied this concept to the analysis of the tax problem. His concept is the
modern apparatus popularized for analyzing the effect of the tax levy.

Furthermore, critics by Hotelling (1938) and Debreu (1954) considered the prob-

3 See Blaug (1987, pp. 319-321).
4 Even W. S. Jevons, one among the founders of marginal utility theory along with
C. Menger, L. Walras, denied later the measurability.
lem of the evaluation of economic losses due to the introduction of the tax-subsidy system. Although their approaches to such a problem differ each other, they got the same opinions that the tax-subsidy system causes economic losses in an economy. Friedman (1952), however, discussed the tax problem in terms of economic efficiency. Dixit (1970) introduced the optimal tax rate reflecting the elasticities of the demand and the supply curves of taxable commodities under the assumption that the government raises a fixed amount of tax revenue from one consumer by imposing the unit tax on $M (> 1)$ taxable commodities.

In sum, taxes raise the change in price structure, causing economic inefficiencies (i.e., the quantity change), and economic welfare losses. This means that the tax has an influence on an economy.

6.2.2 Analysis of Tax Impact

Here, we go into the specific discussion of the tax effect on the economy in detail. We use the analytical apparatus A. Marshall refined, in his standard supply-demand framework.

6.2.3 Consumers Surplus and Producers Surplus

Verbally, the consumers surplus (CS) is the difference between the value placed upon a commodity by a consumer, the potential maximum (money) value that the consumer is willing to pay for the commodity instead of going without it, the money value that the consumer actually pay for it in the market. In other words, CS is the excess of the potential maximum (money) value to be paid by the consumer over the actual (money) value paid by the consumer in the market. The producers surplus
(PS) is a measure in money value of the net gain or loss to the producer arising from a change in opportunities open to the producer. That is, PS is the excess of actual earning from a given quantity of commodity (output) over the amount that the producer would accept rather than refuse to offer his commodity.

The mathematical form supporting the verbal definition is derived as follows. Suppose we have the following demand and supply functions for a particular commodity $Q$.

$$Q^D = H(P^c), \quad H' < 0;$$  \hspace{1cm} (6.1)

$$Q^S = J(P^p), \quad J' > 0,$$  \hspace{1cm} (6.2)

where $P^c =$ the consumer price of a commodity, $P^p =$ the producer price of a commodity, $Q^D =$ the continuous demand function (or curve), $Q^S =$ the continuous supply function (or curve).

The economic equilibrium is established when supply equals demand as follows:

$$Q^D = Q^S \text{ or } H(P^c) = J(P^p).$$

Here we can obtain the equilibrium price $P^E$ and the equilibrium quantity $Q^E$ of a commodity in the market. Then, we can calculate CS and PS as follows:

$$CS = \int_0^{Q^E} P^c(Q) \, dQ - P^E Q^E,$$  \hspace{1cm} (6.3)

$$PS = P^E Q^E - \int_0^{Q^E} P^p(Q) \, dQ,$$  \hspace{1cm} (6.4)
\[ = \int_0^{Q^E} \left[ p^E - P^P(Q) \right] dQ, \]

where \( P^P(Q) \) = the inverse supply function of a commodity or, the marginal cost curve.

The graphical definitions of CS and PS are as follows: CS is the area under the demand curve minus the price-quantity rectangle. PS is the price-quantity rectangle minus the area next to the supply curve.

The concepts of CS and PS mentioned above play crucial roles in determining economic welfare losses or gains of the economic change resulting from the imposition of a tax (or subsidy). These represent the social welfare cost or economic inefficiency cost in an economy. With these concepts, we can examine the tax (or subsidy) impact on an economy.

6.2.4 Tax Impact on Prices

Let us assume the initial state of an economy as follows:

\[ Q^D = H(P^C), \quad H' < 0, \quad (6.5) \]
\[ Q^S = J(P^P), \quad J' > 0. \quad (6.6) \]

At the economic equilibrium, supply equals demand. Thus,

\[ Q^D = Q^S. \quad (6.7) \]
\[ H(P^C) = J(P^P). \quad (6.8) \]

Suppose that the unit (excise) tax \( \tau \) is levied on the producer's output (commodity) \( Q \). In other words, we now consider the impact of a tax \( \tau > 0 \) imposed on
the producer for every unit of the commodity produced. Then, by the producer’s profit-maximizing rule, the producer price $PP$; the price received by the producer, and the consumer price $PC$; the price paid by the consumer, change as follows:

$$PP = PC - \tau,$$  
(6.9)  

$$PC = PP + \tau.$$  
(6.10)

For examining the effect of the unit tax on the consumer price, we substitute (6.9) into the expression (6.8). Then we get:

$$H(PC) = J(PC - \tau).$$  
(6.11)

Taking the total differential of both sides of the expression (6.11), the result is:

$$H'dPC = J'dPC - J'dr.$$  
(6.12)

From (6.12), we can get the following result:

$$\frac{dPC}{d\tau} = \frac{J'}{J' - H'} > 0,$$  
(6.13)

where $J' - H' > 0$ since $J' > 0$ and $H' < 0$ by standard assumptions that the demand curve is downward sloping and the supply curve upward sloping. The expression (6.13) has the following economic interpretation: As the unit tax $\tau$ increases (decreases), the consumer price rises (falls). This shows the tax effect on the consumer price.

For investigating the impact of the unit tax on the producer price, we take the same method used in examining the tax effect on the consumer price. As a result of it, we obtain the following outcome:

$$\frac{dPP}{d\tau} = \frac{H'}{J' - H'} < 0,$$  
(6.14)
Figure 6.1: Tax Impact
where $J' - H' > 0$ since $J' > 0$ and $H' < 0$. This means that as the tax increases (decreases), the producer's price falls (rises).

In sum, the unit tax distorts the structures and levels of both the consumer price and the producer price (see Figure 6.1).

6.2.5 The Tax Impact on Economic Welfare

As shown in Figure 6.1, the original supply curve $S^0$ moves to the new supply curves $S^\tau$ resulting from the imposition of given unit tax $\tau$. In response to such movement of the supply curve, the economic equilibrium state changes from the initial equilibrium price-quantity combination $E : [P^E = (P^c)^E = (P^E) ; Q^E]$ to the new equilibrium price-quantity combination $E : [(P^c)^\tau > (P^c)^E ; (PP)^\tau < (PP)^E ; Q^\tau < Q^E]$, i.e., $E : [(PP)^\tau < (PP)^E = (P^c)^E < (P^c)^\tau ; Q^\tau < Q^E]$.  

6.2.5.1 (I) Consumers Surplus (CS): As mentioned above, the price-quantity combination varies with the tax imposition. Such a change affects the consumer economic welfare: CS. Specifically, consumers face the losses of economic welfare. This implies that consumers are worse off than before. We can calculate the losses of CS by the following general formula:

$$\Delta CS_\tau = \int_{(P^c)^\tau}^{P^E} Q^D(P^c) dP < 0, \quad (6.15)$$

where $P^E = (P^c)^E < (P^c)^\tau$. We should note that the negative $\Delta CS_\tau$ indicates the loss of welfare.

(Proof): Let $(P^c)^\tau = P^\tau$. From Figure 6.1,

$$\Delta CS_\tau = \int_{P^E}^{\infty} Q^D(P^c) dP - \int_{P^\tau}^{\infty} Q^D(P^c) > 0.$$
Thus, \( \Delta CS_T = \int_{P_E}^{P^\tau} Q^D(P^c) dP > 0 \). By the rule of definite integrals,

\[
\int_{P_E}^{P^\tau} Q^D(P^c) dP < 0. \quad \text{Q.E.D.}
\]

The expression (6.15) amounts to the area \( P^E EK(P^c)^\tau \) in the Figure 6.1.

Actually, the consumer welfare losses are caused by two reasons: (i) the rise in the consumer price and (ii) the decrease in quantities of output. We, therefore, can disaggregate the expression (6.15) into two parts as follows:

\[
\Delta CS_T = \left( (P^c)^\tau - P^E \right) Q^\tau + \left\{ \int_{P_E}^{(P^c)^\tau} Q^D(P^c) dP - [(P^c)^\tau - P^E]Q^\tau \right\} < 0.
\]

The first term of the right-hand side in the expression (6.16) represents the economic welfare losses of consumers caused by the rise in the consumer price. In Figure 6.1, this amounts to the area \( P^E LK(P^c)^\tau \). The second term of the right-hand side in (6.16) implies the economic welfare losses of consumers raised by the decrease in quantities supplied of output (commodity). This is the net economic loss in consumption indicating the shaded area KLE in Figure 6.1.

6.2.5.2 (II) Producers Surplus (PS): Producers like consumers also experience economic welfare losses resulting from the imposition of a unit tax. This means that producers are worse off than before. We also measure the losses of PS by the following formula:

\[
\Delta PS_T = \int_{P_E}^{(P^p)^\tau} Q^S(P^p) dP < 0,
\]

where \( P^E > (P^p)^\tau \). The expression (6.17) equals to the area \( P^E EM(P^p)^\tau \) in Figure 6.1.
(Proof): Use the same method as the proof of the expression (6.15).

The producer welfare losses are also raised by the same causes as the case of consumer welfare losses. So, the formula (6.17) is also divided into two components in the following manner:

\[ \Delta P^S_T = \left[ P^E - (PP)^T \right] Q^T \quad (6.18) \\
+ \left\{ \int_{P}^{(PP)^T} Q^S(pP) dP - [P^E - (PP)^T]Q^T \right\} < 0. \]

The first term of the right-hand side in the expression (6.18) denotes the producer economic welfare losses affected by the fall of the producer price. The second term of the right-hand side signifies the economic losses of producers influenced by the decrease in the quantities demanded of output (commodity).

6.2.5.3 (III) Government Tax Revenue (GTR):

\[ GTR_T = [(P^C)^T - (PP)^T]Q^T \quad (6.19) \]

\[ = \tau Q^T. \]

6.2.5.4 (IV) Social Welfare Loss: Using the Kaldor criterion combining the expressions (6.15), (6.17), and (6.19) together, we can obtain the net loss of the social welfare as follows:

\[ SWL_T = GTR_T + \Delta CS_T + \Delta PS_T \quad (6.20) \]

\[ = \tau Q^T + \int_{(PC)^T}^{P^E} Q^D(pC) dP + \int_{P^E}^{(PP)^T} Q^S(pP) dP < 0. \]

This expression (6.20) shows the economic welfare losses of both consumers and producers caused by decrease in the quantities of commodities which distorts economic
efficiency by causing a wrong allocation of resources in production. So, this is called the economic inefficiency cost, or dead-weight loss. This (6.20) amounts to the shaded area $KME_0$ in Figure 6.1.

### 6.2.6 Tax Impact of the Change in the Tax Rate

Let us consider the case that the unit tax rate $\tau$ increases by the increment $\Delta \tau$. $\Delta \tau$ is composed of the two parts as follows:

$$\Delta \tau = \Delta \tau^c + \Delta \tau^p,$$

(6.21)

where $\Delta \tau^c = [(P^c)^\tau + \Delta \tau - (P^c)^\tau], \Delta \tau^p = [(P^p)^\tau + \Delta \tau - (P^p)^\tau].$

$\Delta \tau^c$ and $\Delta \tau^p$ imply the increments in the tax rate $\tau$ assessed on consumers and producers, respectively. We should note that the magnitudes of $\Delta \tau^c$ and $\Delta \tau^p$ actually depend upon the elasticities of the demand for and the supply of a commodity, and that the increase in $\Delta \tau$ means the increases in both $\Delta \tau^c$ and $\Delta \tau^p$ simultaneously.

The imposition of the increased rate of a unit tax, $\tau + \Delta \tau$, forces the supply curve to shift up from $S^o$ (and $S^r$) to $S^r + \Delta \tau$. In response to the shift-up of the supply curve, the equilibrium position moves from $E^o; [P^E, Q^E]$ to $E^\tau + \Delta \tau; [P^E < (P^c)^\tau + \Delta \tau; P^E > (P^p)^\tau + \Delta \tau; Q^E > Q^\tau + \Delta \tau]$. As $\Delta \tau$ increases, $\Delta \tau$ causes the original equilibrium consumer price, $P^E = (P^c)^E$ to rise up to $(P^c)^\tau + \Delta \tau$.

$$[(P^c)^\tau + \Delta \tau] = P^E + (\tau + \Delta \tau)$$

(6.22)

$$= (P^c)^\tau + \Delta \tau^c.$$  

Also, $\Delta \tau^p$ due to the increase in $\Delta \tau$ forces the original equilibrium producer price, $P^E = (P^p)^E$, to fall down to $(P^p)^\tau + \Delta \tau$.

$$[(P^p)^\tau + \Delta \tau] = P^E - (\tau + \Delta \tau)$$

(6.23)
= (PP)^T - \Delta \tau P.

The expression (6.22) shows that as \( \Delta \tau^C \) increases, \( (P^C)^T + \Delta \tau \) increases; the more \( \Delta \tau^C \) increases, the higher the consumer price, \( (P^C)^T + \Delta \tau \), is. Similarly, the expression (6.23) demonstrates that the more \( \Delta \tau^P \) increases, the lower the producer price, \( (PP)^T + \Delta \tau \), is. This is consistent with the analysis discussed in the previous section. As results of them, the losses of CS caused by the unit tax rate \( \tau \), \( \Delta CS_T \), deteriorates more than the damages of CS forced by the unit tax rate increased, \( (\tau + \Delta \tau) \). The losses of PS also get worse with the same reasons. Formally speaking, the losses of CS changes as follows:

\[
\Delta CS_{\Delta \tau} = \int(P^c)^T_{(P^c)^T + \Delta \tau} Q^D(P^c) dP < 0. 
\] (6.24)

Thus, consumers face the following total deteriorations of CS: By combining (6.15) and (6.24),

\[
\Delta CS_{\Delta \tau + \Delta \tau} = \Delta CS_{\tau} + \Delta CS_{\Delta \tau} < 0.
\] (6.25)

The expression (6.25) indicates that consumers are much worse off by the imposition of the unit tax rate \( (\tau + \Delta \tau) > \tau \). Similarly, the PS also deteriorates due to \( \Delta \tau^P \) as follows:

\[
\Delta PS_{\Delta \tau} = \int(PP)^T_{(PP)^T + \Delta \tau} Q^S(PP) dP < 0. 
\] (6.26)

Putting (6.18) and (6.26) together, we can get the following result indicating the total losses of PS:

\[
\Delta PS_{\tau + \Delta \tau} = \Delta PS_{\tau} + \Delta PS_{\Delta \tau} < 0. 
\] (6.27)
The formulation (6.27) expresses that producers are also much worse off by the levy of the unit tax rate increased, \((\tau + \Delta \tau)\).

Next, the impact of the increased unit tax rate on the GTR has the ambiguous characteristics. In other words, we can not say exactly the variation of the GTR in relation to the change in the tax rate.\(^5\) We can examine such a fact in the following way: when the increased rate of the unit tax, \((\tau + \Delta \tau)\), is imposed on the producer commodity,

\[
GTR_{\tau + \Delta \tau} = (\Delta \tau^c + \Delta \tau^p)\frac{Q^\tau + \Delta \tau}{\tau Q^\tau + \Delta \tau}. \tag{6.28}
\]

On the other hand, the right-hand side of the expression (6.19) can be disaggregated into two terms in the following manner (see Figure 6.1):

\[
GTR_{\tau} = \tau Q^\tau + \Delta \tau + \tau (Q^\tau - Q^\tau + \Delta \tau). \tag{6.29}
\]

Subtracting (6.29) from (6.28), we get the following ambiguous outcome:

\[
GTR_{\tau + \Delta \tau} - GTR_{\tau} = \tau Q^\tau - (\tau + \Delta \tau)Q^\tau + \Delta \tau \quad \gtrless 0, \tag{6.30}
\]

where \(\tau < \tau + \Delta \tau, Q^\tau > Q^\tau + \Delta \tau\).

That is, the relation expression (6.30) can be simplified as below:

\[
GTR_{\tau + \Delta \tau} \gtrless GTR_{\tau}. \tag{6.31}
\]

The relation (6.31) supports the ambiguity of the impact of the tax rate change on the GTR. This might come from the dependence of the prices and the quantities of

\(^5\)The Laffer curve explains well this fact. The GTR increases continuously with the increase in the tax rate until the slope of the Laffer curve equals zero; at this point the tax-revenue-maximizing rate is determined. However, if the slope of the Laffer curve has a negative value, the GTR decreases even though the tax rate increases.
commodities on the elasticities of the demand for and the supply of commodities. This ambiguity of the tax impact raises the complexity of the government’s fiscal policy.

Finally, the social welfare losses are enlarged by the increase in the unit tax rate from $\tau$ to $(\tau + \Delta \tau)$. We can show mathematically such a thing as follows:

$$\text{SWL}_{\tau + \Delta \tau} = \text{SWL}_{\tau} + \Delta PS_{\Delta \tau} + \Delta CS_{\Delta \tau} - (\Delta \tau^c + \Delta \tau^p)Q^{\tau + \Delta \tau}$$

$$= \text{SWL}_{\tau} + \Delta PS_{\Delta \tau} + \Delta CS_{\tau + \Delta \tau}$$

$$- \Delta \tau Q^{\tau + \Delta \tau} < 0. \quad (6.32)$$

The expression (6.32) shows that as the unit tax rate changes to the increasing direction, the magnitude of the social economic losses gets larger.

6.2.7 The Impact of the Unit Subsidy

The imposition of the unit subsidy on producers has just opposite effects to the case of the unit tax since a subsidy is a negative tax. However, we should note that the economic inefficiency cost always occurs in both cases. In sum, the tax and subsidy have influences on an economy. Thus, the tax rate or the subsidy rate are regarded as crucial policy variables of the government.

6.2.8 Tax Policy

Taxation is one element in a complex network of institutions, practices, and relationships that characterize an economic society. Traditionally, tax economists have placed the primary emphasis among possible tax policy objectives on equity: horizontal equity and vertical equity, and efficiency. Horizontal equity requires that people in essentially similar economic circumstances should be taxed similarly, while vertical
equity objective is to control or limit inequality through suitable differentiation in the tax burdens levied on people in differing circumstances.

The objective of efficiency requires that unintended distortion through the tax system of resource allocation in the private sector should be kept to a minimum. These objectives of tax policy will, however, vary with the priority placed upon the objectives.

6.3 The Tariff System

6.3.1 Introduction

In the world of today, international trade is considered to be very important. For relatively small economies, or those with little natural resources, it is apparent that international trade may provide the only way to obtain commodities, natural resources, and other factors of production which simply cannot be produced domestically. On the other hand, for large economies, which are well endowed with natural resources and other factors of production, international trade seems to be unnecessary because large economies could, if necessary, be self-sufficient. Then, why does trade take place between countries?

Two distinct theories have been presented concerning the causes of trade between nations. The Ricardian theory of comparative advantage expounds the cause of trade as follows: Even though some country is less efficient in the production of all commodities (resources), there will be some commodity in which it has comparative factor-productivity advantage, and it will gain by specializing in and exporting that commodity in return for other commodities. Nations would trade in accord with comparative advantage. On the other hand, the Heckscher-Ohlin theory explains it
as follows: a country will have comparative advantage in the commodities which utilize the nation's abundant factor of production intensively. A country will therefore export that commodity produced by intensively using the productive factor which is relatively abundant in that country. Free trade reduces the real return to the relatively scarce factor within a country, and increases the real return to the abundant factor. Even if countries have identical technology and tastes, trade will occur between them because of differences in factor supplies. The Ricardian theory emphasizes technological differences (i.e., differences in production efficiency), whereas the Heckscher-Ohlin theory emphasizes differences in factor availability across countries. In sum, trade takes place between countries because of the gains from the consequent specialization in accord with comparative advantages.

This trade will give the gains (benefits) to countries. Such trade gains are obtained through price differences of commodities (resources) which are traded between countries. Prices of commodities (resources) traded are determined by supply and demand mechanism. Prices differ between countries when supply or demand conditions differ across countries. As a general rule, prices rise in the exporting country, as part of commodities in the exporting country leave the domestic market, and fall in the importing country, due to the increased supply of commodities on its market.

It is obvious that trade occurs because of differences in prices between countries. Also, trade tends to reduce the price differences since commodities will flow away from the market where prices are low towards the market where prices are high. The price differences between two countries will narrow but will not be eliminated because there can be trade restrictions such as tariffs which are 'artificial' impediments to trade.
6.4 The Tariff Structure

A tariff is the policy instrument utilized by governments to have an influence on the flows of commodities, services, and factors of production in the world open economy. A tariff, therefore, gives various shocks to the national and international economy.

6.4.1 The Forms of Individual Tariffs

The tariff (or customs duty) is a tax levied by a government on physical commodities as they are imported or exported. The taxation of trade is probably as old as trade itself. The Mercantilists of the eighteenth century were probably the first to make tariffs more an instrument of national control of international trade than a source of revenue. Tariffs have been used extensively ever since as a protective measure against foreign competition.

There are two basic sorts of tariffs - *ad valorem* tariff and specific tariff. The *ad valorem* tariff is expressed in terms of a percentage of the value of an imported or exported item. The specific tariff, on the other hand, is stated in terms of an amount of money per quantity of items - commodities, factors of production. A combination of the former and the latter is called a compound tariff or composite one.

6.4.2 The Functions of Tariffs

There are two functions of tariffs: the revenue function and the protection function. The former implies that tariffs play a function to collect revenue for the government. The latter means that tariffs play a function to protect the domestic producers (industries) against foreign competition. To best perform the revenue function, tariffs
are applied to commodities (resources) of wide consumption; and the rates of these tariffs are kept low enough to maximize tariff collections. The same objective may also be attained by the imposition of a uniform low rate of tariff on all commodities crossing the border either as exports, imports, or in transit. The revenue function of a tariff is a relative concept.

The protection function of the tariff depends upon a partial or complete restriction of imports. For complete protection of the domestic producers and/or consumers, a given tariff must be high enough to cover the difference in the marginal cost of production between domestic and all foreign producers, including transportation and incidental expenses of importing. On the contrary, for only partial protection the tariff must remain below this difference. When partial protection is desired, commodities will continue to be imported, but they will be imported in smaller quantities and the government will collect tariffs. The protection function, therefore, will usually afford both protection and revenue, although its objective is primarily one of protection.

The seeming incompatibility of the two functions in the same tariff does not necessarily disqualify its adoption, since most countries generally desire both protection and revenue. In the tariff schedules of nations, however, a tendency does exist to provide a certain number of generally low rates of tariffs designed essentially for revenue and other higher tariff rates for protection.

6.5 The Tariff Impact

6.5.1 Introduction

In an open economy, the impacts of a tariff depends crucially on whether the country levying it is small or large. The small country is only a world price-taker
because it cannot have significant effect on the world market - the commodity market and the factor market - as either a purchaser or a seller. On the contrary, actions of the large country can have influence on world prices of commodities and factors of production through the world market control as either a buyer or a seller possessing a major fraction of demand for and supply of commodities and/or factors of production. So, the large country has variable terms of trade - a ratio between the price of its exports and the price of its imports - and can reduce the impacts of tariff imposition. The tariff may be more advantages to a large country than to the small country. In other words, the effects of tariff levy are larger in the small country than in the large country.

This tariff impact is transmitted to the domestic production. The imposition of tariff brings on the domestic production inefficiency and also does consumption inefficiency. The tariff levy on the imported commodities utilized as factors of production makes the costs of domestic production increase through raising the price of imported factors of production. The small country under the circumstance of insufficient resources have to import resources such as natural resources and intermediate factors of production with which they produce their finished commodities. Thus, the tariff has significant impact on the economy of the small country. In the following section we examine the impact of tariff on an economy.

6.5.2 The Tariff Impact on Prices

Actually the tariff impact on prices is similar to the tax impact. The immediate effects of tariffs are those reflected in price changes and consequent adjustments in production and resource allocation. Our discussion is confined to protection tariffs:
import tariffs.

The import tariffs have influences on the market prices of the imported commodities and factors of production in different ways, depending on the condition of the market and the balance of elasticities of supply and demand at home and abroad of the imported products or imported foreign inputs. The price impact of the tariff is as follows:

(1) When the price elasticities of import supply and import demand lie between zero and infinity, then a tariff will cause the price rise by less than the tariff. This is the most common price effect of the tariffs.

(2) A tariff tends to increase the price of the imported product or the imported factor by the full amount of the tariff under conditions of perfect elasticity of import supply and less-than-perfect elasticity of import demand.

(3) A tariff may raise the price to the consumer by more than the amount of the tariff when the channel of distribution at home is lengthy and the different intermediaries add their individual profit margins, or markups, at each step of the marketing process.

6.5.3 The Impact of Tariff Changes

Kreinin (1961) examined empirically the effect of the U.S. tariff reduction program on the prices and volume of competing imports subject to tariff concessions by the GATT - General Agreement on Tariffs and Trade - negotiations, and on the welfare and the loss of employment. His analysis shows the impacts of tariff changes on the economy system. He utilized the method of a comparison of changes in the prices and the volume of imports between commodities on which tariff rates were re-
duced (the tariff-reduced commodities), and immediate substitutes which tariff rates were not subject to tariff reduction (the tariff-nonreduced products). He found the following results in his analysis:

(i) Allowing for the tariff cuts, the tariff-reduced commodities experienced more price reductions than those of the tariff-nonreduced products; (ii) the increase in the import volume of the tariff-reduced products was much larger than that of the tariff-nonreduced commodities; (iii) as a result of it, the domestic consumers benefited from the price fall and the increase in quantity (supplied) raised by the tariff reductions. Also, the foreign suppliers (exporters) benefited from the increased export prices. Such a fact explains that a part of benefit from the tariff cuts would be passed on to the domestic consumers, the remainder being obtained by foreign suppliers; (iv) the impact of the tariff reductions led to the loss of employment.

Salant (1960) and Vaccara (1960) investigated the impact of a (larger) volume of imports on domestic production and employment. They measured the order of magnitude of employment displacement. In examining the effect on domestic employment of a £1 million increase in imports in each of 72 industries, Salant concluded that the largest net decrease in employment would be 175 employees, with a median of 86. Salant's results can support the loss of employment through the increase in the volume of competing imports caused by the tariff cuts. In sum, by the causal sequences,

6 The two commodity groups were compared by choosing the tariff-nonreduced commodities in such a way as to equate it to the tariff-reduced products with respect to commodity composition and the average value of imports under comparison.

7 The results of Kreinin's analysis display that close to half of the benefit from the tariff cuts accrued to foreign exporters in the form of increased export prices.

8 The division of the total gain between the foreign producers and the domestic consumers would depend upon the relative elasticities of import demand and export supply. This effect is, in principle, similar to the effect of the tax rate.
we can boil down the impact of the tariff reductions on an economy system. In an open economy the tariff-reduction (increase) has (i) the price effect: the decrease (increase) in the domestic import price; (ii) the quantity effect: the increase (decrease) in the volume of the domestic imports; and (iii) the positive (negative) welfare effects: the increase (decrease) in the economic benefit of the domestic consumers and the foreign producers. On the contrary, the domestic producers face the negative (positive) welfare effects showing the decrease (increase) in the benefit through the price decrease (increase) and the increase (decrease) of the quantities of competing imports, respectively. Of course, the domestic production of commodities competing with imports dampens. So, the decline of the utilization of resources occurs. Here we can know that the tariff rate change induces the domestic production to dampen under the price effect of imports (including imported factors of production). Imported resources are one of crucial determinants of the production costs, especially in a small country. In other words, the tariff rates have influence on the domestic production.

As mentioned above, we can know that changes in the protection level - changes in the tariff rates - can not fail to have some impact on the economy. Consequently, we recognize that the tariff rate plays a crucial role as the government policy instrument to control and manage the economy system.

Chan (1978) showed that a tariff will have a negative overall employment effect under the assumption that the domestic commodities and the import commodities are gross substitutes. His analytical framework consists of a five-commodity-one-household open economy; an export commodity, an import commodity, money, a non-traded commodity, and labor services. The prices of import commodities and export commodities are exogenously determined in terms of the foreign currency. The export
commodities are not consumed and the import commodities are noncompetitive with
the domestic commodities. Labor services are homogeneous and freely mobile and
the only variable factor of production. The tariff revenues are redistributed to the
household. He also assumed that the foreign exchange market and the market for the
domestic commodities make clear through price adjustments, while the labor market
does not clear, but is in a state of unemployment (underemployment). In other words,
the labor market is under orthodox Keynesian unemployment.9

6.6 The System of Foreign Exchange Rate

6.6.1 Introduction

The trend of foreign exchange rate system has been changed. Historically, the
value of exchange rates has been influenced by actions of central governments. Prior
to 1914, a fixed-rate gold standard prevailed in the international currency markets
under which each country's currency was fixed in gold points. Worldwide economic
instability in the period between World War I and World War II led to the Bretton
Woods agreement10 in 1944 as an attempt to compromise between fixed and flexi­
ble exchange rates, with an emphasis on fixity. Since the post-World War II, until
1971, the major non-communist countries of the world adopted a fixed exchange rate
system. In this fixed exchange rate system (the Bretton-Woods system), most gov­
ernments committed themselves to certain policies which acted to hold constant the

9 See Malinvaud (1977).
10 The Bretton Woods agreement created the International Monetary Fund (IMF)
to set rules for maintaining the fixed exchange rates and to make loans to countries
with balance of payments problems. And the dollar was selected as the key reserve
currency in the new international monetary system.
rate of foreign exchange of their domestic currency against the U.S. dollar which is a key reserve currency in an international monetary system.

Since 1971, however, a lot of countries have chosen to allow the exchange rate to be influenced by market forces. Nevertheless, even in this flexible exchange rate system, a number of countries have frequently attempted to manipulate the exchange rate through official intervention, i.e., through Central Bank transactions in the foreign exchange market which has been called the “dirty float”. The government of Japan, for example, has tried repeatedly to hold down the dollar value of the yen in order to facilitate international marketing for Japanese sellers. Another example of a “dirty float” is the “snake” agreement of December 1971 between the governments of the European Economic Community. These countries agreed on maximum ranges of movement for the most appreciated versus the most depreciated member currency and on maximum bands within which exchange rates could change.

As the case of extremity like a fixed exchange rate system there is a freely floating exchange rate system, under which government policy actions are not designed to have an influence on the country’s exchange rate. Such a trend has an impact on international trade.

6.7 The Foreign Exchange Rate

Foreign exchange rates are of key significance in directing the flows of commodities and factors of production such as raw materials and intermediate commodities between countries.
6.7.1 The Concept of the Foreign Exchange Rate

Foreign exchange is purchased and sold in the foreign exchange market at a price that is called the rate of foreign exchange. Specifically, the foreign exchange rate is the domestic currency (money) price for a unit of a given foreign currency (money), establishing an equivalence between domestic monies and foreign monies. The daily quotations of foreign exchange are based on the price of bank (or cable) transfers, which are the quickest means of international payment.

For most currencies only a spot rate, which is the exchange rate for spot transactions of foreign exchange for immediate delivery (within one or two days) that is used in making international payments, is quoted in the foreign exchange market. For international currencies, however, the foreign exchange market also quotes forward exchange rates for forward transactions of foreign exchange that is promised for delivery at a time in the future. Actually there are several spot exchange rates for a given currency. The domestic price of bank transfers is the base rate of exchange, and all other means of international payments—bills of exchange and letters of credit—usually sell at a discount from the base rate. These discounts reflect varying delays or risks of payment compared to the bank transfer. Discounts from the base rate of exchange stem also from differences in the risk of payment. The rates of different kinds of exchange by discounts that take into account liquidity and risk factors. Although these discounts will vary with the two previous factors, the resulting pattern...
of foreign exchange rates will rise and fall with the base rate.

6.8 The Behavior of Foreign Exchange Rate

The behavior of the foreign exchange will depend upon the nature of the foreign exchange market. We now examine three types of exchange rate behavior under the assumption of the pattern of spot rates to be the base rate of exchange.

6.8.1 The Freely Flexible Exchange Rate

When the exchange rate is not stabilized or controlled by government authorities, the foreign exchange market very closely approaches the theoretical model of pure competition. The foreign exchange of any given country is one homogeneous product, except for the liquidity and risk differentials that discounts allow for. In a free, unstabilized market, the exchange rate is determined by the many individual acts of buying and selling, none of which individually is able to affect it, but all of which interact to set its level.

In a free market the exchange rate is determined by the interplay of the supply of and the demand for foreign exchanges. The foreign exchange that is demanded at any time will depend on the volume of international transactions that requires payments to foreign residents. The amount of foreign exchange in demand varies inversely with its price, *ceteris paribus*, because the exchange rate determines the domestic price of imports and thereby affects their volume and the amount of foreign exchange demanded to pay for them. A high exchange rate (depreciation in terms of foreign currency) makes imports expensive to domestic buyers because they must offer more domestic money to obtain a unit of foreign money. As a result, a high rate
of exchange reduces the volume of imports and thus lessens the amount of foreign exchange demanded by domestic residents.

The supply of foreign exchange in the foreign exchange market derives from international transactions that require money receipts from foreign residents. Unlike the amount demanded, the amount of foreign exchange supplied to the market varies directly with the exchange rate. When exchange rate is high, domestic prices appear low to foreigners since they are able to acquire a unit of domestic money. This cheapness stimulate domestic exports and thereby brings a large supply of foreign exchange into the market. Conversely, a low exchange rate restricts exports and lowers the amount of foreign exchange offered to the market.

Thus, the equilibrium exchange rate is determined when the amount of foreign exchange demanded is equal to the amount of foreign exchange supplied in the foreign exchange market. At this equilibrium exchange rate, the foreign exchange market is cleared. However, it is unlikely that this equilibrium rate of exchange will last very long, since continuing changes in demand and supply of foreign exchanges will force continuing adjustments toward new equilibrium positions. Thus, the exchange rate will fluctuate continuously.

In summary, the exchange rate in a free, unstabilized market is determined by the supply of and the demand for foreign exchange, which derive from the credit and debt items of the balance of payments. There is therefore a mutual relationship between the foreign exchange rate and the balance of payments.
6.8.2 The Stable Exchange Rate

There are strong arguments for stable exchange rates. It is forcibly argued that the exchange rate is unlike the price of an ordinary commodity and that it is illogical to view the two in the same light. When the exchange rate varies, the prices of all exports are changed for foreign buyers and, simultaneously, the prices of all imports are changed for domestic buyers. This wide spread price effects unloose a series of repercussions that extend throughout the domestic and foreign economies. This critical nature of the exchange rate rules out the unlicensed freedom of the unstabilized foreign exchange market. It is also contested that fluctuating rates invite foreign exchange speculation that may intensify balance of payments difficulties.

The stabilized market imposes no restraints on private transactions in foreign exchange; the factor of supply and demand are fully operative. Successful stabilization of the rate of foreign exchange requires that the stabilization agency be able to offset movements in market supply and demand to any desired degree. To do so, the agency must possess adequate supplies of domestic and foreign exchange. This active stabilization seeks to achieve stability in the exchange rate to correct disequilibrium in the balance of payments. This policy of “stable, yet flexible” rates attempts to secure the advantages of stability and, at the same time, to use the exchange rate as an instrument of international adjustment. In sum, exchange rates may be stabilized by compensatory purchases and sales of foreign exchange on the part of government authorities.
6.8.3 The Controlled Exchange Rates

The controlled foreign exchange market prohibits private transactions in foreign exchange not authorized by the control authority. The controlled exchange rate does not directly respond to shifts in the demand for and the supply of foreign exchanges; government rationing supersedes the allocating function of the exchange rate. When exchange controls are relaxed, the job of maintaining stable exchange rates is passed on to stabilization agencies or their counterparts. For the essential mechanism of exchange control in the foreign exchange market, we assume a completely controlled market in which the control authority is the exclusive purchaser from domestic residents and exclusive seller of foreign exchange to domestic residents. All foreign exchange must be sold to the authority at its stipulated rate, and all foreign exchange must be purchased from the authority as its stipulated. We further assume that there is only one rate of exchange.

The supply of foreign exchange is derived from the credit items of the balance of payments, and the control authorities have only a limited influence over it. The control authorities, therefore, considers the supply of foreign exchange as relatively fixed with respect to their own powers, and their main tasks are the allocations of this fixed supply among those who demand it. This is usually done by an exchange or trade - licensing system - unless a domestic resident can obtain a license, foreign exchange cannot be secured. This rationing brings about a suppressed disequilibrium between supply and demand by forcibly choking off all excess demand.

In the end, (i) the controlled exchange rate is less than the equilibrium rate of exchange; (ii) the amount of foreign exchange supplied to the market at the controlled rate is less than the amount supplied at the equilibrium rate; (iii) the market is not
truly cleared; (iv) the purchase of foreign exchange at the controlled rate is smaller than the purchase at the equilibrium rate.

In sum, controlled exchange rates result from the monopoly purchase and sale of foreign exchange by government authority. The available foreign exchange is distributed by rationing system; and, because the controlled rate is maintained below the equilibrium rate, there is an excess demand that is not satisfied. The exchange rate is determined by bureaucratic decision rather than market forces: supply and demand.

6.9 The Impact of Foreign Exchange Rate

Movements of the foreign exchange rate have influences on the domestic prices of imports and imported foreign factors of production. Specific illustration can be shown through the following formulation:

\[ W^D = eW^* , \]

where \( W^D \) = the domestic price denominated by the domestic currency, of imports, \( W^* \) = the international market price of imports, \( e \) = the rate of foreign exchange.

In the preceding section we defined the exchange rate \( e \) as the price of one country's currency measured in terms of another country's currency. In other words, the foreign exchange rate \( e \) is defined as the units of domestic currency required to purchase a unit of foreign currency. The direction of movement of the foreign exchange rate are represented by a depreciation and appreciation.\(^{12}\) The former means an increase in the number of the domestic currency needed to purchase a unit

\(^{12}\)In a stable-rate system, depreciation is commonly called devaluation because it is a government action, and appreciation is called revaluation.
of foreign currency in the currency market. It indicates a weakening of the domestic currency in terms of foreign currency. On the contrary, the latter implies a decrease in the number of the domestic currencies needed to buy a foreign currency. It indicates a strengthening of the domestic currency in terms of foreign currency.

We can see in the above expression that as the exchange rate $e$ increases, the domestic prices of imports rise, and that as the rate of foreign exchange decreases, the domestic prices of imports fall. These mean that depreciation of the domestic currency leads to an increase in the domestic price level of imports, and that appreciation leads to a decrease in the domestic price level of imports. Thus, movements of the foreign exchange rate have influences on the domestic production through cost-push.

Ishiyama (1976) analyzed the balance of payments of a small country in an open economy through four equilibrium conditions - equilibrium in the four markets: the commodity market, government bond market, currency market and commercial bank loan market - under the assumptions of constant money incomes, interest rates and prices in the foreign country. For the impact of the foreign exchange rate on the economy, he concluded that the exchange appreciation - depreciation of the domestic currency in terms of the foreign currency - raises the prices of domestic commodities and output and exchange appreciation instantly increases the domestic currency price of imports from the old rate of foreign exchange to the new rate.

6.10 The Structure of Price Changes

Taxes, tariffs (or custom duties), and the rate of foreign exchange are used as policy variables (instruments). Trade policy variables such as tariffs and the foreign exchange rate are relatively easily manipulated by government and have the
political advantage, compared to taxes, of generating real and obvious benefits for some groups. Domestic and international (or world) prices of the imported goods and the factors of production differ as a result of tariffs and taxes differences. Such taxes and tariffs lead to some major distortions in the level of production cost as we discussed in Sections 6.2.4, 6.5.2, and 6.5.3. Distortion in the production cost structure by taxes and tariffs results in inflation, i.e., a rise in the prices of outputs.

The rate of foreign exchange also has crucial influence on the domestic production cost and, in turn, the prices of outputs. Under the flexible exchange rate system, there are the sporadic disturbances in currency exchange rates. This implies variabilities or instabilities of the international currencies used as a medium of exchange and a means of payment. This instable phenomenon of the foreign exchange rate has the transmissive economic impacts on the production cost, pricing of the output, and in turn the domestic resource allocations. Specifically, the prices of (noncompetitive) factors of production imported vary with the flexibility of the foreign exchange rate, even though the international market prices of them are invariant. Such a change in the factor prices through the change in the foreign exchange rate directly affects the production cost of factor-importing domestic producers, and pricing of the output. These impacts in turn are transmitted to the other sectors associated with them, as well as consumers. The chain of such impacts coming from the change in the rate of foreign exchange has an influence on both the supply and the demand sides in a national economy as a whole. In the end, it influences and distorts product flows and the domestic resource allocations.

We briefly consider and summarize the price-structure variations caused by some taxes, tariffs, and the rate of foreign exchange.
6.10.1 Prices of the Intermediate Factors

(i) The post-tax prices are:

\[ W_g^\tau = W_g + \tau, \]  

where \( W_g \) = the original pre-tax price of an intermediate factor \( Q_{gj}, \) \( (g \in [1, G]) \),

\( W_g^\tau = \) the post-unit-tax consumer (i.e., destination-sector \( j \)) price of an intermediate factor \( Q_{gj} \).

(ii) when the \textit{ad valorem} tax \( v \) is imposed on an intermediate factor \( Q_{gj} \),

\[ W_g^v = W_g(1 + v), \]  

where \( W_g^v = \) the post-\textit{ad valorem}-tax consumer price of an intermediate factor \( Q_{gj} \).

(iii) when the composite tax \( v \) is imposed on an intermediate factor \( Q_{gj} \),

\[ W_g^c = W_g(1 + v) + \tau, \]  

where \( W_g^c = \) the post-composite-tax consumer price of an intermediate factor \( Q_{gj} \).

Similarly, we can summarize the price-structure changes for the primary factors and the imported factors as follows.

6.10.2 The Prices of the Primary Factors

For the primary factors,

(i) the unit tax:

\[ \hat{W}_m^\tau = \hat{W}_m + \hat{\tau}, \]  

(6.36)
where \( \hat{W}_{m}^\tau \) = the post-unit-tax consumer (i.e., destination-sector) price of a primary factor \( \hat{X}_{m,j} \).

(ii) the *ad valorem* tax:

\[
\hat{W}_{m}^{v} = \hat{W}_{m}(1 + \hat{\upsilon}),
\]

where \( \hat{W}_{m}^{v} \) = the post- *ad valorem*-tax consumer price of a primary factor \( \hat{X}_{m,j} \).

(iii) the composite tax:

\[
\hat{W}_{m}^{c} = \hat{W}_{m}(1 + \hat{\upsilon}) + \hat{\tau},
\]

where \( \hat{W}_{m}^{c} \) = the post-composite-tax consumer price of a primary factor \( \hat{X}_{m,j} \).

### 6.10.3 The Prices of the Imported Foreign Factors

Under the assumption that a commodity is freely tradable between two countries, the following arbitrage condition holds.

\[
W_{Z}^{D} = eW_{Z}^{*},
\]

where \( W_{Z}^{D} \) = the pre-tariff domestic price of an imported foreign factor \( X_{z,j}^{*} \), \( W_{Z}^{*} \) = the world price of an imported foreign factor \( X_{z,j}^{*} \).

e represents the foreign exchange rate measured in the domestic currency per unit foreign currency, that is, the value of the unit of the foreign currency in terms of the domestic currency or a ratio of the domestic currency to the foreign currency. (An increase in e is depreciation of the domestic currency.) Thus, when the exchange rate increases (holding \( W_{Z}^{*} \) constant), the domestic price of the imported factor, \( W_{Z}^{D} \) rises. This leads to an increase in the domestic production cost.
The above formula is known as the law of one price and shows the way converting the international (or world) market price $W^*_z$ into the domestic (or local) price $W^*_{zD}$ through the foreign exchange rate when the domestic industry utilizes the imported foreign factors such as rubber, crude oil and petroleum, uranium (ore).

(2) The post-tariff domestic prices:

(i) With the specific import tariff set at rate, $\theta^s$ the domestic tariff-inclusive price of an imported foreign factor $X^*_z$ is:

$$ (W^*_{zD})^s = eW^*_z + \theta^s, \quad (6.40) $$

where $(W^*_{zD})^s$ = the post-specific-tariff domestic price of an imported foreign factor $X^*_z$.

(ii) When the ad valorem tariff $\theta^a$ is imposed on an imported foreign factor $X^*_z$, the domestic tariff-inclusive price is:

$$ (W^*_{zD})^a = eW^*_z (1 + \theta^a), \quad (6.41) $$

where $(W^*_{zD})^a$ = the post-ad valorem-tariff domestic price of an imported foreign factor $X^*_z$.

(iii) When the composite tariff $\theta^c$ is imposed on an imported foreign factor $X^*_z$, the domestic tariff-inclusive price is:

$$ (W^*_{zD})^c = eW^*_z (1 + \theta^a) + \theta^g \quad (6.42) $$

where $(W^*_{zD})^c$ = the post-composite-tariff domestic price of an imported foreign factor $X^*_z$. 

6.11 The Impacts of Price Change

In this section we specifically explain how changes in taxes, the foreign exchange rate and tariffs exert on the (domestic) prices.

6.11.1 The Impacts of Tax Change

First, we examine the effects of taxes imposed on the intermediate factors and the primary factors on the prices of them. Taking the total differentials of (6.35) and letting \( dW_g = 0 \), we get:

\[
dW_g^C = W_g dv + d\tau. \tag{6.43}
\]

Or equivalently,

\[
dW_g^C = vW_g \tilde{v} + \tau \tilde{\tau}, \tag{6.44}
\]

where \( \tilde{\tau} = \frac{d\tau}{\tilde{v}} \); the proportional variation in the specific tax rate, \( \tilde{v} = \frac{dv}{v} \); the proportional variation in the \textit{ad valorem} tax rate \( v \). Note that \( dW_g = 0 \) since \( W_g \) is fixed.

First, suppose there is no imposition of the \textit{ad valorem} tax; \( v = 0 \) or \( dv = 0 \).

The expressions (6.43) and (6.44) then are reduced to:

\[
dW_g^T = d\tau. \tag{6.45}
\]

\[
\frac{dW_g^T}{d\tau} = 1 > 0. \tag{6.46}
\]

Or equivalently,

\[
dW_g^T = \tau \tilde{\tau}. \tag{6.47}
\]
The expressions (6.45)-(6.47) show that the magnitude of change in the price of an intermediate factor is equal to that of change in the specific tax rate $\tau$.

Second, we examine the case that no levy of the specific tax exists; $\tau = 0$ or $d\tau = 0$. Under these conditions, we obtain the followings from (6.43)-(6.44):

$$dW^g = W_g dv.$$  

(6.48)

Or equivalently,

$$dW^g = vW_g \bar{v}.$$  

(6.49)

Third, let us consider the case of the composite tax imposition on an intermediate factor; $v \neq 0$, $\tau \neq 0$ and $dv \neq 0$, $d\tau \neq 0$. Under those conditions, (6.43)-(6.44) are unchanged. In this case, let us specifically investigate the impacts of taxes on the price of an intermediate factor.

(I) Case (A): $dv > 0$ and $d\tau > 0$.

$$dW^c = W_g dv + d\tau > 0.$$  

(6.50)

(II) Case (B): $dv < 0$ and $d\tau > 0$.

$$dW^c = W_g dv + d\tau < 0,$$  

(6.51)

because $|W_g dv| > |d\tau|$, even though $|dv| > |d\tau|$.

(III) Case (C): $dv > 0$ and $d\tau < 0$.

$$dW^c = W_g dv + d\tau > 0,$$  

(6.52)

because $|W_g dv| > |d\tau|$, even though $|dv| < |d\tau|$.
(IV) Case (D): $dv < 0$ and $d\tau < 0$.

\[dW_g^C = W_g dv + d\tau < 0.\]  

(6.53)

The Cases (B)-(C) indicate that the impact of the \textit{ad valorem} tax levy on the price change is stronger than that of the specific tax imposition.

For the primary factors, we can extract the similar economic effects from utilizing the same way as the case of the intermediate factors.

6.11.2 The Impacts of the Foreign Exchange Rate and Tariffs

For convenience of overall analysis concerning the impacts of the foreign exchange rate and tariffs on the domestic price $W^D_z$, we make use of the composite tariff formula (i.e., expression (6.42)).

Totally differentiating the expression (6.42), we obtain

\[dw_P = V^F_* \cdot (1 + \epsilon^{\theta^a}) \cdot d\epsilon + \epsilon W^* \cdot d\theta^a.\]  

(6.54)

Note that we assume the international price $W^*_z$ of the foreign factor is exogenously given and unchanged (i.e., an importer is a price-taker): $dW^*_z = 0$.

First, we consider the case that there are no impositions of any tariffs on the imported foreign factor $X^*_z$, that is, $\theta^a = \theta^a = 0$ and $d\theta^a = d\theta^a = 0$. We get the following result from the expression (6.54) under $\theta^a = \theta^a = 0$ and $d\theta^a = d\theta^a = 0$.

\[dW^D_z = W^*_z \cdot d\epsilon.\]  

(6.55)

The expression (6.55) implies that as the foreign exchange rate \( \epsilon \) increases (decreases) by the absolute magnitude $d\epsilon$, the domestic price $W^D_z$ of the imported factor rises (falls) by the absolute amount $W^*_z \cdot d\epsilon$ (i.e., $d\epsilon$ multiplied by the exogenous world price
$W_{z}^{*}$ of that factor), respectively. In other words, if the rate of foreign exchange deprecies, that is, the price of foreign currency rises or the value of domestic currency decreases, then the domestic prices of the foreign factors rise. The expression (6.55) can be rewritten in another way as follows:

$$dW_{z}^{D} = \varepsilon \tilde{e} W_{z}^{*},$$

(6.56)

where $\tilde{e} = \frac{de}{e}$, the proportional change in the foreign exchange rate $e$.

In sum, the expressions (6.55) and (6.56) show the effect of the absolute or the proportional change in the foreign exchange rate on the domestic prices of the imported foreign factors under the assumptions of no tariff impositions and no changes in the world prices of the imported foreign factors. This exchange-rate-change effects would cause immediately change of the (domestic) production costs - the first effect - of producing industries utilizing the imported foreign factors and, in turn, changes of the output prices transmitted through the change of the production cost - the second effect. In other words, the exchange-rate-change effect shows the chain-of-causality effects such as the first and the second effects.

Second, the case imposing the specific tariff $\theta^{s}$ on the imported factor is examined. That is, this case corresponds to the conditions $\theta^{a} = 0$, $d\theta^{a} = 0$, and $\theta^{s} \neq 0$, $d\theta^{s} \neq 0$. The expression (6.54) then changes as follows:

$$dW_{z}^{D} = W_{z}^{*} de + d\theta^{s}.$$  

(6.57)

Or equivalently,

$$dW_{z}^{D} = \tilde{e} W_{z}^{*} + \theta^{s} \tilde{\theta}^{s},$$

(6.58)

where $\tilde{\theta}^{s} = \frac{d\theta^{s}}{\theta^{s}}$; the proportional change in the specific tariff $\theta^{s}$. 


From the expression (6.57), we can see five impacts of the foreign exchange rate $e$ and/or the specific tariff $\theta^s$ on the domestic price $W^D_z$.

(i) Case I: $de = 0$.

$$dW^D_z = d\theta^s.$$ (6.59)

Or,

$$\frac{dW^D_z}{d\theta^s} = 1 > 0.$$ (6.60)

The expressions (6.59) and (6.60) show that the domestic price $W^D_z$ of the imported factor $X^{*j}$ rises (falls) as the specific tariff $\theta^s$ increases (decreases) when the foreign exchange rate $e$ is unchanged.

(ii) Case II: $e > 0$ and $d\theta^s > 0$.

$$dW^D_z = W^*_z de + d\theta^s > 0.$$ (6.61)

This implies that as both the foreign exchange rate and the specific tariff increase, there is a rise in the domestic price $W^D_z$ of the imported factor.

(iii) Case III: $de > 0$ and $d\theta^s < 0$.

$$dW^D_z = W^*_z de + d\theta^s > 0.$$ (6.62)

The expression (6.62) represents that even though the rate of foreign exchange increases and, on the other hand, the specific tariff decreases, the domestic price $W^D_z$ would still rise.

(iv) Case IV: $de < 0$ and $d\theta^s > 0$.

$$dW^D_z = W^*_z de + d\theta^s < 0.$$ (6.63)
This case shows that if the foreign exchange rate decreases and the specific tariff increases, the domestic price $W^D_z$ would fall.

Both Case III and Case IV indicate that when the world price $W^*_z$ of the imported factor is exogenously given, the exchange-rate-change effect on $W^D_z$ is stronger than the specific-tariff-change effect on it, regardless of whether $de > d\theta^s$ or $de = d\theta^s$ or $de < d\theta^s$.

(v) Case V: $de < 0$ and $d\theta^s < 0$.

$$dW^D_z = W^*_z de + d\theta^s < 0.$$  \hfill (6.64)

The expression (6.64) obviously expounds the fact that when both the foreign exchange rate and the specific tariff decrease simultaneously, the domestic price $W^D_z$ would fall.

Under the assumption that the world prices of the imported foreign factors are exogenously given and unchanged, we can get the following outcomes from five Cases I-V analyzed above. First, when both the foreign exchange rate $e$ and the specific tariff $\theta^s$ shift to the same direction, the impact direction of the domestic price $W^D_z$ of the imported factor $X^*_z$ follows the same direction as that of both $e$ and $\theta^s$ (see Case II and Case V).

Second, when both the foreign exchange rate $e$ and the specific tariff $\theta^s$ move to the different direction, the impact direction of the domestic price $W^D_z$ of the imported factor $X^*_z$ follows the direction of change in the foreign exchange rate $e$, since the degree or the magnitude of the exchange-rate-change in the domestic price $W^D_z$ is greater than that of the specific-tariff-change effect (see Case III and Case IV).
Finally, let us investigate the case imposing the *ad valorem* tariff $\theta^a$. This case can be explained under the conditions $\theta^b = 0$, $d\theta^b = 0$, and $\theta^a \neq 0$, $d\theta^a \neq 0$. Making use of these conditions, we can derive the following expression (6.65) from the expression (6.54).

\[ dW^D_z = W^*_z (1 + \theta^a) de + eW^*_z d\theta^a. \]  

(6.65)

Or equivalently,

\[ dW^D_z = W^*_z (1 + \theta^a) \hat{e} + eW^*_z (\theta^a \hat{\theta}^a). \]  

(6.66)

where $\hat{e} = \frac{de}{e}$; the proportional change in the foreign exchange rate $e$.

$\hat{\theta}^a = \frac{d\theta^a}{\theta^a}$; the proportional change in the *ad valorem* tariff $\theta^a$.

For analyzing the impact of the *ad valorem* tariff $\theta^a$, we can use the same way as that adopted in the case of the specific tariff. The results derived from (6.65) are as follows:

(i) Case [1]: $de = 0$.

\[ dW^D_z = eW^*_z d\theta^a. \]  

(6.67)

Or,

\[ \frac{dW^D_z}{d\theta^a} = eW^*_z > 0. \]  

(6.68)

(ii) Case [2]: $de > 0$ and $d\theta^a > 0$.

\[ dW^D_z = W^*_z (1 + \theta^a) de + eW^*_z d\theta^a > 0. \]  

(6.69)

(iii) Case [3]: $de > 0$ and $d\theta^a < 0$.

\[ dW^D_z = W^*_z \left[ (1 + \theta^a) de + ed\theta^a \right] < 0, \]  

(6.70)
where \((1 + \theta^a) \ll e\) (in a (small) country with the weaker-power currency).

(iv) Case \([4]\): \(de < 0\) and \(d\theta^a > 0\).

\[
dW_z^D = W_z^* \left[ (1 + \theta^a)de + ed\theta^a \right] > 0
\]  

(v) Case \([5]\): \(de < 0\) and \(d\theta^a < 0\).

\[
dW_z^D = W_z^*(1 + \theta^a)de + eW_z^* d\theta^a < 0.
\]

Cases \([1]-[2]\) and \([5]\) correspond to Cases I-II and V. Thus, the economic impacts (but not the magnitudes of the impacts) of the specific tariff \(\theta^s\) and the \textit{ad valorem} tariff \(\theta^a\) on the domestic price \(W_z^D\) of the imported factor are same. However, Cases \([3]\) and \([4]\) show, on the contrary to Cases III-IV, that the exchange-rate-change effect on \(W_z^D\) is \textit{weaker} than the \textit{ad valorem}-tariff-change effect on it, when the world market price \(W_z^*\) of the imported factor is exogenously given.
7. THE SEPARABLE TWO-STAGE LEVEL CES-TYPE PRODUCTION FUNCTION

7.1 Aggregation Problem

7.1.1 Aggregation and Consistency

In a production process, the production function has to include a large number of factors of production showing various types of the productive factors. Aggregation is a process whereby a part of the information available for the solution of a problem is sacrificed for the purpose of making the problem more easily manageable. Aggregation takes the form of replacing a set of numbers (for example, quantities of the productive factors) by a single number or a smaller set of numbers or aggregates (e.g., quantity indices). And aggregation will be said to be consistent when the use of information more detailed than that contained in the aggregates would make no difference to the results of the analysis of the problem at hand. In the theory of production it is commonly assumed that firms or industries purchase that collection of factors of production - intermediate inputs, primary factors, and natural resources - which maximizes output subject to a budget constraint or minimizes costs subject to a given output. The number of different factors of production is generally so large that considerable interest attaches to the conditions in which it is possible to reduce
the number of variables by grouping together some of the productive factors, representing the quantities of the members of each group by a quantity index. Group quantity indices can be defined in such a way that output may be written as a function of such indices. It, therefore, is possible to argue that no two factor units can be alike in all respects. We shall assume that there is a degree of disaggregation at which it is legitimate to assume the perfect substitutability of elements treated as units of a given factor of production. We call those elements units of elementary factors. Elementary inputs can be aggregated as follows: let us suppose that the following procedure for maximizing output or for minimizing cost is available. The elementary variables are grouped, and for each group a price-index is defined as a function of the factor prices of members of the group. First, the optimal distribution of a given total cost among the groups is determined by reference to the factor price-indices alone. The cost thus allocated to each group is then distributed among the members of the group on the basis of their individual prices. Moreover, the quantity of each elementary input determined by this two-stage procedure is identical with the amount which would have been purchased if output, regarded as a function of all the elementary variables, had been maximized with reference to all the individual factor prices, without any grouping. The aggregation of the elementary variables into groups is then consistent, according to the definition of consistency mentioned above.

7.2 Functional Separability

7.2.1 Weak Separability and Strong Separability

The aggregation problem mentioned in the previous section is associated with the separability of variables. In the production process, the production function must
include many factors representing various types of factors. However, a certain degree of aggregation is essential in making such a production function operationally manageable. A number of factors of production must be aggregated into a single index. Hence we aggregate different types of the productive factors into a composite factor. The condition that must be satisfied for aggregation is the functional separability.

Leontief (1947) discussed the concepts of functional separability through the set theory and general function. On the basis of the Leontief's concepts of functional separability, many modern economists such as Strotz (1957, 1959), Gorman (1959), Houthakker (1960), Pearce (1961), and Green (1964) analyzed the structure of consumers preference fields in demand analysis. Goldman and Uzawa (1964) first stated necessary and sufficient conditions for an aggregation (grouping) of commodities to be separable in terms of utility functions. We can apply their analyses of the concepts of functional separability to the production functions.

We assume the $N \ (N \in (2, \infty))$-factor homothetic production function $f(X)$ characterized by the regularity conditions [H.2]-[H.10]. The production function $f(X)$ has a finite number $N$ of the factors of production $X$. The set of all $N$ productive factors is represented by $R$:

$$R = \{1, 2, \ldots, N\}.$$ 

Any aggregation of the $N$ productive factors is expressed by a partition of the set $R$ into $D$ mutually disjoint and exhaustive subsets, $\{R_1, R_2, \ldots, R_D\}$:

$$R = \{1, 2, \ldots, N\} = \bigcup_{d=1}^{D} R_d, \quad R_d \cap R_{d'} = \emptyset \text{ for } d \neq d', \quad (7.1)$$

where $R_d$ is of size $N_d$ and $\sum_{d=1}^{D} N_d = N$.

The vector of productive factors, $X = (X_1, X_2, \ldots, X_N)$, is correspondingly par-
tioned into a set of subvectors \( X = \{X_1, X_2, ..., X_N\} = \left[ X_{(1)}, X_{(2)}, ..., X_{(D)} \right] \):

\[
X = (X_1, X_2, ..., X_N) = \left[ X_{(1)}, X_{(2)}, ..., X_{(D)} \right],
\]

(7.2)

where the subvector \( X_{(d)} \) is composed of \( X_k, k \in R_d \).

### 7.2.1.1 (I) Weak Functional Separability:

Let \( \{R_1, R_2, ..., R_D\} \) be a partition of the finite set \( R \) and \( f(X) \) be a twice-continuously differentiable, strictly quasi-concave production function with a finite number of the factors of production, each having a strictly positive marginal product. The production function \( f(X) \) has the characteristic of weak functional separability with respect to the partition \( \{R_1, R_2, ..., R_D\} \) if

\[
\frac{\partial}{\partial X_k} \left[ \frac{\partial f(X)}{\partial X_m} \cdot \frac{\partial f(X)}{\partial X_{m'}} \right] = 0, \quad \text{for all } m, m' \in R_d \text{ and } k \notin R_d,
\]

(7.3)

where \( m \neq m' \) and \( d \in [1, D] \).

Or alternatively,

\[
f_{mk} f_{m'} - f_m f_{m'k} = 0,
\]

(7.4)

where \( f_i = \frac{\partial f}{\partial X_i}, f_{ik} = \frac{\partial f_i}{\partial X_k} \), and \( i = m, m' \).

This means that the expression (7.3) or (7.4) is a necessary and sufficient condition for the grouping of production factors that the marginal rate of technical substitution between any two factors, \( X_m \) and \( X_{m'} \), in the same subset (group) \( R_d \) is independent of the quantities of productive factors \( X_k \) in the different subset (group) outside of \( R_d \).

From the viewpoint of the above mentioned, we can state the fundamental result on weak functional separability.
7.2.1.2 Theorem 7.1: A production function \( f(X) \) is weakly separable with respect to a partition \( \{R_1, R_2, \ldots, R_D\} \) if and only if the form of the production function \( f(X) \) is:

\[
f(X) = G \left[ f_1(X(1)), f_2(X(2)), \ldots, f_D(X(D)) \right],
\]

(7.5)

where \( G(f_1, f_2, \ldots, f_D) \) is a function of \( D \) variables, and for each \( d \), \( f_d(X(d)) \) is a function of subvector \( X(d) \) alone and \( X(d) \) is a function of the elements of \( R_d \).

7.2.1.3 (II) Strong Functional Separability: Let \( \{R_1, R_2, \ldots, R_D\} \) be a partition of the set \( \mathbb{R} \) and \( f(X) \) be a twice-continuously differentiable, strictly quasi-concave production function. The production function \( f(X) \) has strong functional separability with respect of the partition \( \{R_1, R_2, \ldots, R_D\} \) if:

\[
\frac{\partial}{\partial X_k} \left[ \frac{\partial f(X)}{\partial X_m} \frac{\partial f(X)}{\partial X_{m'}} \right] = 0 \quad \text{for all} \quad m \in R_d, \ m' \in R_{d'},
\]

\[
\text{and} \quad k \notin [R_d \cup R_{d'}], \quad \text{(7.6)}
\]

where \( d, d' \in [1, D] \) are integers.

Or alternatively,

\[
f_{mk}f_{m'k} - f_{m}f_{m'k} = 0, \quad \text{(7.7)}
\]

where \( f_i = \frac{\partial f}{\partial X_i}, \ f_{ik} = \frac{\partial f_i}{\partial X_k}, \) and \( i = m, m' \).

The expression (7.6) or (7.7) says that the marginal rate of technical substitution between any two factors, \( X_m \) and \( X_{m'} \), from subsets (factor groups) \( R_d \) and \( R_{d'} \), respectively, does not depend upon the quantities of the factors of production, \( X_k \), outside of subsets \( R_d \) and \( R_{d'} \).
7.2.1.4 Theorem 7.2: Let \( \{ R_1, R_2, \ldots, R_D \} \) be a partition of \( \mathbb{R} = \{1, 2, \ldots, N\} \) with \( D > 2 \). A production function \( f(X) \) is strongly separable with respect to the partition \( \{ R_1, R_2, \ldots, R_D \} \) if and only if the production function \( f(X) = [X_1, X_2, \ldots, X_D] \) has the following additive form:

\[
f(X) = H \left[ \sum_{d=1}^{D} f_d(X_{(d)}) \right],
\]

where \( H(z) \) is a monotonic transformation function of one variable \( z \), and for each \( d \), \( f_d(X_{(d)}) \) is a function of subvector \( X_{(d)} \), and \( X_{(d)} \) is a function of the elements of \( R_d \) only. Note that strong functional separability implies weak functional separability, but weak separability implies strong separability only when the partition \( \mathbb{R} = \{ R_1, R_2, \ldots, R_D \} \) is limited to two subsets, i.e., \( \mathbb{R} = \{ R_i, R_j \} \) \((i, j \in [1, D]) \).\(^1\)

---

\(^1\)See Green (1964) for further discussions and proof of this.
8. THE INTERNAL STRUCTURE OF PRODUCTION

8.1 Introduction

As we have seen in the conventional input-output system, the system analyzes the national economy through the most rigid type of production function - this is usually called the Marx-Leontief production function - with fixed technological (input-output) coefficients. This rigid-type production function just has the invariant linear relation between each factor of production utilized and output produced. It therefore does not describe the substitutabilities among the factors of production in the production process.

Leontief (1941/1951, pp. 38-39) recognized this fact in his utilization of the rigid type of production function as follows: "It [The choice of particular (rigid) type of production function] means no less than a formal rejection of the marginal productivity theory; ...all factors appear to be strictly complementary or limitational. ...our production functions exclude technical substitutability of factors within the framework of any given production process." He also expounded the weak-point of the rigid-type production function, in his *Wages, Profit, and Prices* (1946b), as follows: "The assumption of fixed technical coefficients can be questioned from the point of view of general theory of production. Insofar as the proportion in which the separate factors can be combined within the same production function are variable, these
proportions will most probably vary with every change in their relative prices.” This indicates that there can be the substitutabilities between the factors of production in the production process. Leontief, in spite of it, emphasizes only the complementarity between the factors of production in the production process, excluding the substitutability between them.

However, all factors of production utilized in the production process cannot be complementary. In the modern economics literatures, many economists emphasize the substitutabilities among the factors of production. They expound the substitutability through the relations among capital, labor, and natural resources (products). They state that capital (equipment), labor, and natural resources are significantly substitutable as the factors of production. For instances, Barnett and Morse (1963) describe several processes whereby capital and labor may be substituted for natural resources within a given state of technical knowledge (information). In the technologically progressive sectors, Hagen (1953), Barnett and Morse (1963), Adler (1963) state that the substitution between capital and natural resources often accompanies improvements in technical knowledge and, indeed, that innovation is often directed toward increasing such substitutability.

Meier (1961) emphasizes the substitution possibilities that the substitutability of one resource for another is augmented by expansion of the stock of technical knowledge. His viewpoint implies that increases in the stock of information in society should facilitate the substitutability of both capital and labor for natural resources. Humphrey and Moroney (1975) reaches the general conclusion in their Substitution

1Natural resource products are themselves the first-stage results in the physical transformation of pure or primitive natural resources. Actually, some of them play a role of intermediate inputs.
among Capital, Labor, and Natural Resource Products in American Manufacturing

that natural resource products are not strictly complementary in production with either capital or labor, regardless of the underlying causes of factor substitution (that is, whether it is technology induced or price induced). In other words, they imply that labor and capital are substitutable for natural resource products in the process of production. In addition, they also state that capital and labor are substitutes. On the other hand, Kamien (1964) stated and proved that the substitutability relationships among any number of the factors of production (or commodities) dominate the complementarity relationships among the factors of production by showing exactly the maximum number of complementarity relationships, or conversely showing the least number of the substitutability relationships in the system of \( N \) factors of production (or commodities), based on the properties of the complementarity and the substitutability relationships. Kamien (1964, p. 226) stated that there must be a minimum of \((N - 1)\) substitutes in a system of \( N \) factors of production (or commodities). Economists such as Hudson and Jorgenson (1974) and Berndt and Wood (1975) showed in their econometric studies that the elasticities of substitution between the primary factor and the intermediate factor are significantly different from zero.

Finally, in the light of the above mentioned, we can conclude that all factors cannot be complementary in the production process, and rather, the substitutability dominates the complementarity. Thus, for our economic analysis associated with the resource allocation, we need the production function which reflects the interactions - the substitutabilities and/or the complementarities - among the factors of production.
8.2 The Production Function Framework

As mentioned in the preceding section, the interactions among the factors of production in the production process are very crucial. We therefore need the appropriate, specific production function reflecting the interactions among the productive factors. Such production functions are the CES-type functional forms. We therefore utilize the CES-type production function for our economic analysis about the resource allocation in the national economy. The another reasons why we utilize the CES-type production function in our study related to the input-output relation are as follows: First, the parameters of the CES-type production function are relatively easy to calculate. Second, the CES-type functional form imposes relatively few a priori restrictions on them. Third, it is possible for us to utilize the self-dual cost function of the CES-type production function so as to calculate the parameters of production.\(^2\) This is possible under our regularity conditions. For the choice of the appropriate, specific CES-type production function for our study, we examine the properties of some types of specific production function related to the CES-type production functions.

The first development of a specific production function - the Cobb-Douglas function - had been taken by C. W. Cobb and Paul Douglas (1928). The Cobb-Douglas function has a unitary elasticity of substitution between factors: labor and capital. Arrow, Chenery, Minhas, and Solow (1961) also developed a specific production function with the property of the constant elasticity of substitution in response...
to an empirical test they carried out to see if the factor rewards were indeed constant as implied by the Cobb-Douglas function. Their production function is called the constant-elasticity-of-substitution (CES) or S.M.A.C. production function. The S.M.A.C. (CES) production function in two factors of production introduced by them has the following mathematical form:

\[ V = \gamma \left[ \delta X_1^{-\rho} + (1 - \delta)X_2^{-\rho} \right]^{-\frac{1}{\rho}}, \quad (8.1) \]

where \( V = \) the value-added, \( X_i = \) the quantity of factor of production \( i \) (e.g., capital and labor), \( \gamma = \) the efficiency parameter which serves as an indicator of the state of technology, \( \delta = \) the distribution parameter which has to do with the relative factor shares in the output; \( \delta \in (0,1) \), \( \rho = \) the substitution parameter which determines the value of the elasticity of substitution.

This S.M.A.C. (CES) production function has the properties of (i) the linear homogeneity; (ii) the constant elasticity of substitution, i.e., \( \sigma = \frac{1}{1+\rho} \), regardless of the factors of production and (iii) the possibility of different elasticities for different industries.

The two-factor S.M.A.C. (CES) production function can be generalized to the \( N \quad (N \in (2, \infty)) \)-factor CES production function. Uzawa (1962) introduced the generalized CES production function of the the two-factor CES production function as

\[ \text{See Arrow, Chenery, Minhas, and Solow (1961, p. 230). Solow (1956b) first introduced the production function of the form (8.1) as an example to illustrate his model of economic growth. The production function of the form (8.1) is known as a mean of order } \rho \text{ in the mathematics literature. See Hardy, Littlewood, and Pólya (1934).} \]
below:

\[
f(X) = \left[ \sum_{i=1}^{N} \alpha_i X_i^{-\beta} \right]^{-\frac{1}{\beta}}, \tag{8.2}
\]

where \( X=(X_1, X_2, ..., X_N) \), \( \alpha_i > 0 \) (\( i = 1, 2, ..., N \)), \( \beta > -1 \), and \( \sigma = \frac{1}{1+\beta} \).

This is obviously a straightforward generalization of the two-factor S.M.A.C. (CES) production function to the multi-factor CES functional form, provided that the factors of production function can be partitioned into separate subsets (groups).

The CES production function (8.2) has the elasticities of factor substitution: \( \sigma_{ij} = \sigma \), for all \( i \neq j \). This implies that if the production function is of the form (8.2), then the partial elasticity of substitution \( \sigma_{ij} \) are independent of factor prices and are identical for all pairs of two factors of production, and vice versa.\(^4\) Uzawa's generalized CES production function is homogeneous of degree one in the factors of production. This functional form would not describe the economic effects of change in returns to scale and technological change.

Mukerji (1963) also generalized the two-factor CES production function to the multi-factor functional form which keeps the ratios of Allen partial elasticities of substitution constant, in the following way:

\[
y = \gamma \left[ \sum_{r=1}^{m} \delta_r X_r^{-\rho r} \right]^{-\frac{1}{\rho}}. \tag{8.3}
\]

Mukerji's generalized CES production function is in general non-homogeneous function and has the characteristic of linear homogeneity only when \( \rho_r \)'s are all equal to \( \rho \). That is, we can see that when the expression (8.3) has the conditions that all \( \rho \)'s

\(^4\)Uzawa (1962, p. 294) proved that if the elasticities of factor substitution \( \sigma_{ij} \) are all constant and identical for different pairs of the factors of production, then the production function is of the form (8.2).
are equal and $\gamma = 1$, Mukerji's CES production function coincides with Uzawa's CES production function. Mukerji's nonhomogeneous production function also would not state the change in returns to scale except when $\rho_r = \rho$.

Mukerji also introduced the following form of the generalized CES production function showing the linear homogeneity:

$$f(X) = \gamma \left[ \sum_{n=1}^{N} \beta_n X_n^{-\rho} \right]^{-\frac{1}{\rho}},$$

where $X = (X_1, X_2, \ldots, X_N)$; a vector of factors of production,

$\gamma = $ the efficiency parameter; $\gamma > 0$, $\beta = $ the distribution parameter, $\rho = $ the substitution parameter.

The functional form (8.4) is closely related to the forms (8.2) and (8.3).

The CES-type production function includes several other well-known production functions such as the Cobb-Douglas function, the Marx-Leontief production function, and the perfect-substitution linear production function as special cases. We examine this things more in detail through the CES-type functional form (8.4).

(1) The Cobb-Douglas production function: [case: $(\rho \to 0 \Leftrightarrow \sigma \to 1)$].

8.2.0.5 Theorem 8.1: L'Hôpital's rule

Let both $H(X)$ and $G(X)$ tend to have a limit of zero as $X \to 0$. Then, if the ratio $\frac{H'(X)}{G'(X)}$ exists, $\lim_{x \to 0} \left[ \frac{H(X)}{G(X)} \right] = \lim_{x \to 0} \left[ \frac{H'(X)}{G'(X)} \right]$.

This rule implies that the limit of the ratio of the two functions, if it exists, equals the ratio of the derivatives of $H(X)$ and $G(X)$, respectively.

The case $[\rho \to 0 \Leftrightarrow \sigma \to 1]$ implies that $\rho = 0$ yields a unitary elasticity of substitution; $\sigma = 1$. However, as $\rho \to 0$, the right-hand side of the CES production
function (8.4) becomes an indeterminate form. We therefore make use of L'Hôpital's rule.

Taking the natural logarithm of the production function (8.4), we get

\[
\ln f(X) - \ln \gamma = \frac{\ln[\sum_{n=1}^{N} \beta_n X_n^{-\rho}]}{-\rho}.
\]  

(8.5)

Let

\[
\frac{\ln[\sum_{n=1}^{N} \beta_n X_n^{-\rho}]}{-\rho} = \frac{H(\rho)}{G(\rho)}.
\]  

(8.6)

Since when \( y = a^{-x}, y' = -a^{-x} \ln a, (a > 0) \), we can derive the following results from taking the derivatives of the numerator and denominator of the expression (8.6) with respect to \( \rho \):

\[
H'(\rho) = -\frac{\sum_{n=1}^{N} \beta_n (\ln X_n) X_n^{-\rho}}{\sum_{n=1}^{N} \beta_n X_n^{-\rho}}.
\]  

(8.7)

\[
G'(\rho) = -1.
\]  

(8.8)

Taking the limit of the expressions (8.7) and (8.8), we get

\[
\lim_{\rho \to 0} H'(\rho) = -\frac{\sum_{n=1}^{N} \beta_n \ln X_n}{\sum_{n=1}^{N} \beta_n} \quad \text{since} \quad \sum_{n=1}^{N} \beta_n = 1.
\]  

(8.9)

and

\[
\lim_{\rho \to 0} G'(\rho) = -1.
\]  

(8.10)

Thus, as \( \rho \to 0 \), we can obtain the below outcome by virtue of (8.5), (8.9), and (8.10),

\[
\ln f(X) - \ln \gamma = \ln \left[ \prod_{n=1}^{N} X_n^{\beta_n} \right].
\]  

(8.11)
From (8.11), we can yield the final result

\[ f(X) = \gamma \left[ \prod_{n=1}^{N} X_n^{\beta_n} \right]. \tag{8.12} \]

Thus, the limiting functional form of the generalized CES production function (8.4) at \( \rho = 0 \) is indeed the Cobb-Douglas production function form. In sum, the EOS for the Cobb-Douglas production function is constant along the whole range of any isoquant and equal to 1, i.e., \( \sigma = 1 \). Thus, the Cobb-Douglas production function (8.12) is a special case of production functions which exhibit the constant elasticity of substitution along any isoquant.

(2) The Marx-Leontief Production Function: [case: \( \rho \to \infty \iff \sigma \to 0 \)]

Let us measure units so that \( \beta_1 = \beta_2 = \ldots = \beta_N \). Then, the CES production function (8.4) has the form:

\[ f(X) = \gamma \left[ \sum_{n=1}^{N} X_n^{-\rho} \right]^{-\frac{1}{\rho}}. \tag{8.13} \]

Let us suppose that \( X_k = \min \left[ X_1, X_2, \ldots, X_N \right] \) \( (k=1, 2, \ldots, N) \).

We want to show that

\[ X_k = \lim_{\rho \to \infty} \left[ \left( \sum_{n=1}^{N} X_n^{-\rho} \right) \right]^{-\frac{1}{\rho}}. \tag{8.14} \]

Note that

\[ X_k^{-\rho} \leq \left( \sum_{n=1}^{N} X_n^{-\rho} \right). \tag{8.15} \]

So,

\[ X_k \geq \left( \sum_{n=1}^{N} X_n^{-\rho} \right)^{-\frac{1}{\rho}}. \tag{8.16} \]
Since $\rho > 0$ and $X_k = \min \left[ X_1, X_2, \ldots, X_N \right]$, we have

$$\frac{\sum_{n=1}^{N} X_n^{-\rho}}{N} \leq X_k^{-\rho}.$$

(8.17)

So,

$$N^{\frac{1}{\rho}} \left[ \frac{\sum_{n=1}^{N} X_n^{-\rho}}{\rho} \right]^{-1} \geq X_k.$$

(8.18)

Approaching $\rho$ to $\infty$ yields

$$X_k \leq \lim_{\rho \to \infty} \left[ \frac{\sum_{n=1}^{N} X_n^{-\rho}}{\rho} \right]^{-1},$$

(8.19)

which, combined with (8.16), yields

$$X_k = \lim_{\rho \to \infty} \left[ \frac{\sum_{n=1}^{N} X_n^{-\rho}}{\rho} \right]^{-1}.$$

(8.20)

Finally,

$$f(X) = \gamma \min \left[ X_1, X_2, \ldots, X_N \right]$$

$$= \min \left[ \frac{X_1}{\gamma^{-1}}, \frac{X_2}{\gamma^{-1}}, \ldots, \frac{X_N}{\gamma^{-1}} \right],$$

(8.21)

where $\gamma^{-1}$ is constant.

This Marx-Leontief production function, for example, represents a system of right-angled isoquants (i.e., L-shaped isoquants) in the $R^2$ factor space. This means that there is no substitution between the factors of production in the production process. In other words, the marginal product of each factor utilized is 0 in the Marx-Leontief production function unless it is combined a fixed proportion with the other factor. In sum, the Marx-Leontief production function does not describe the substitutability except the complementarity.
(3) The perfect-substitution linear production function: [case: \(\rho \to -1, \sigma \to \infty\)].

By Substitution \(\rho = -1\) into the expression (8.4), we get the following linear function:

\[
f(X) = \gamma \left[ \sum_{n=1}^{N} \beta_n X_n \right]. \tag{8.22}
\]

This perfect-substitution production function has the configurations of the straight-line isoquants showing an infinite elasticity of factor substitution. The factors of production are perfect substitutes.

In the light of the facts mentioned above, the CES-type production function includes the characteristics of the Cobb-Douglas functional form, the Marx-Leontief type, and the perfect-substitution production function. In other words, we can expound those functions through the CES-type functional form. We therefore exclude the special cases mentioned above: (1), (2), and (3).

### 8.3 Two-Stage Level CES-Type Production Function

In this section we discuss the production function - the two-stage level CES-type functional form - we will utilize in our study, on the basis of functional separability. First, we consider the relationship between functional separability and the elasticity of substitution. Second, from the results of such relation we derive the two-stage level CES-type production function.
8.3.1 Functional Separability and Elasticity of Substitution (EOS)

We discussed the elasticities of substitution and functional separability as different or separate ways of characterizing a the functional relationship among economic variables. In Chapter 5 we discussed the EOS related to comparative statics analysis of the cost-minimizing factor demand and changes in factor shares. On the other hand, we examined in the Section 7.2 functional separatibilities which explain that a function of many economic variables (arguments) could be separated into subfunctions or subgroups. In this section we discuss the relationship between functional separability and the partial EOS.

If the functional form has strong separability, we can get the following fundamental statement.

8.3.1.1 Theorem 8.2: Let \( f(X) \) be homothetic and let \( (R_1, \ldots, R_D) \) be a partition of the set \( R \) (see (7.1)); The two partial elasticities of substitution, \( \sigma_{mk} \) and \( \sigma_{m'k} \), are equal each other, i.e., \( \sigma_{mk} = \sigma_{m'k} = \sigma \) (constant) \( (m \in R_d, m' \in R_{d'}, \text{ and } k \in R_{d''} \notin [R_d \cup R_{d''}]) \) if and only if the production function \( f(X) \) has strong functional separability with respect to the partition \( \{R_1, R_2, \ldots, R_D\} \) at any point in factor space.

This implies the equality and constancy of all Allen partial elasticities of substitution among different subsets (groups) of the partition \( R = \{R_1, R_2, \ldots, R_D\} \) in factor space. In other words, this means that if the production function is strongly separable with respect to the partition \( \{R_1, R_2, \ldots, R_D\} \), the partial EOS between a factor \( X_m \) within a subset (group) \( R_d \) and a factor \( X_k \) within a subset (group) \( R_{d''} \) is equal to the partial EOS between a factor \( X_{m'} \) within a subset (group) \( R_{d'} \).
and a factor $X_\kappa$ within a subset (group) $R_\mu$ and they are constant.

### 8.3.1.2 Proof:

As in the case of examining functional separability in Section 7.2 in Chapter 7, we assume the homothetic production function $f(X)$ characterized by the regularity conditions [H.2]-[H.10].

From (5.11), Allen partial EOS between factors $X_m$ and $X_k$ is:

$$
\sigma_{mk} = \left[ \sum_{i=1}^{N} f_i \frac{X_i^0}{X_m^0 X_k^0} \right] \left[ \frac{\text{det}(\bar{H}_{mk})}{\text{det}(\bar{H})} \right] \text{ for } m \neq k,
$$

(8.23)

where $f_i = \frac{\partial f}{\partial X_i^0} (i \in [1, n])$.

Similarly, Allen partial EOS between factors $X_{m'}$ and $X_k$ is:

$$
\sigma_{m'k} = \left[ \sum_{i=1}^{N} f_i \frac{X_i^0}{X_{m'}^0 X_k^0} \right] \left[ \frac{\text{det}(\bar{H}_{m'k})}{\text{det}(\bar{H})} \right] \text{ for } m' \neq k.
$$

(8.24)

Making use of (4.38) or (4.39), we can obtain the following forms:

$$
\text{det}(\bar{H}_{mk}) \left[ \text{det}(\bar{H}) \right]^{-1} = \mu \left( \frac{dX_m^0}{dW_k} \right)
$$

(8.25)

and

$$
\text{det}(\bar{H}_{m'k}) \left[ \text{det}(\bar{H}) \right]^{-1} = \mu \left( \frac{dX_{m'}^0}{dW_k} \right),
$$

(8.26)

where $m \neq k$ and $m' \neq k$.

Putting the expression (8.25) into (8.23) yields

$$
\sigma_{mk} = \left( \sum_{i=1}^{N} (\mu f_i) \frac{X_i^0}{X_m^0 X_k^0} \right) \left( \frac{dX_m^0}{dW_k} \right) \text{ for } m \neq k,
$$

(8.27)

where $f_i = \frac{\partial f}{\partial X_i^0} (i \in [1, N])$. 
Similarly, Allen partial EOS between factors $X_{m'}$ and $X_k$ is derived from making
use of (8.24) and (8.26) as follows:

$$
\sigma_{m'k} = \left( \sum_{i=1}^{N} \left( \frac{\mu f_i}{X_{m'} X_k} \right) \right) \left( \frac{dX^\circ_{m'}}{dW_k} \right) \text{ for } m' \neq k. \quad (8.28)
$$

From the equilibrium conditions (4.14) derived from the cost-minimization rule,

$$
W_i = \mu f_i, \quad (8.29)
$$

where $i \in [1, N]$ and $f_i = \frac{\partial f}{\partial X_i}$.

Putting the expression (8.29) into (8.27) leads to:

$$
\sigma_{mk} = \left( \sum_{i=1}^{N} \frac{W_i X_i^\circ}{X_m X_k} \right) \left( \frac{dX^\circ_m}{dW_k} \right) \text{ for } m \neq k, \quad (8.30)
$$

where $f_i = \frac{\partial f}{\partial X_i^\circ} (i \in [1, N])$.

Similarly, Allen partial EOS between factors $X_{m'}$ and $X_k$ is derived from plugging (8.29) into (8.28) as below:

$$
\sigma_{m'k} = \left( \sum_{i=1}^{N} \frac{W_i X_i^\circ}{X_{m'} X_k} \right) \left( \frac{dX^\circ_{m'}}{dW_k} \right) \text{ for } m' \neq k. \quad (8.31)
$$

We know that if the production function $f(X)$ has the homotheticity, the corresponding cost function\(^5\) has the form of

$$
C(W, Q^\circ) = G(Q^\circ)J(W). \quad (8.32)
$$

Applying Shephard's lemma to the homothetic cost function, and then differentiating the results with respect to $W_k$, respectively, we yield:

$$
X_m^\circ = \frac{\partial C(W, Q^\circ)}{\partial W_m} = G(Q^\circ)J_m. \quad (8.33)
$$

\[
X_{m'}^0 = \frac{\partial C(W, Q^0)}{\partial W_{m'}} = G(Q^0) J_{m'}, \tag{8.34}
\]
\[
X_k^0 = \frac{\partial C(W, Q^0)}{\partial W_k} = G(Q^0) J_k. \tag{8.35}
\]
And
\[
\frac{\partial X_{m'}^0}{\partial W_k} = G(Q^0) J_{m'k}, \tag{8.36}
\]
\[
\frac{\partial X_k^0}{\partial W_k} = G(Q^0) J_{m'k}. \tag{8.37}
\]

where \( J_{i,k} = \frac{\partial J(W)}{\partial W_i \partial W_k} \) \( (i = m, m') \).

Substituting the expressions (8.32)-(8.33) and (8.35)-(8.36) into (8.30) leads to:
\[
\sigma_{mk} = \frac{JJ_{mk}}{J_{m'k}}, \ m \neq k. \tag{8.38}
\]

Similarly, putting the expressions (8.32), (8.34)-(8.35), and (8.37) into (8.31), we get:
\[
\sigma_{m'k} = \frac{JJ_{m'k}}{J_{m'k}}, \ m' \neq k. \tag{8.39}
\]

Combining (8.38) with (8.39), we get:
\[
\sigma_{mk} J_{m'k} - \sigma_{m'k} J_{m'k} J_{mk} = 0. \tag{8.40}
\]

From (8.40), we can know that \( \sigma_{mk} = \sigma_{m'k} \) if and only if
\[
J_{m} J_{m'k} = J_{m'k} J_{mk}. \tag{8.41}
\]

Following Berndt and Christensen (1973, pp. 404-405), the expression (8.41) is the condition for strong separability of \( J(W) \) in the homothetic cost function.\(^6\)

\(^6\)L. J. Lau (1969) has shown that \( J(W) \) is strongly (weakly) separable with respect to the partition \( R \) in the factor prices \( W \) if and only if \( f(X) \) is strongly (weakly) separable with respect to the same partition \( R \) in the quantities of the factors of production.
Q.E.D.

8.3.2 Two-Stage Level CES-Type Functional Form

Following Berndt and Christensen (1973, p. 408), the homothetic production function $f(X)$ characterized by (i) the strong functional separability (i.e., additive form) and/or (ii) the equality and constancy of some (but not necessarily all) partial elasticities of substitution, that is, $\sigma_{mm'} = \sigma (m \neq m')$ for $m \in R_d$ and $m' \in R_{d'}$, can be written in the form:

$$f(X) = K \left[ \sum_{d=1}^{D} a_d(X(d))^s \right], \quad (8.42)$$

where $f(X)$ is any twice-differentiable monotone increasing function, $s$ and $a_d$ are constants, $X(d)$ is a consistent aggregate quantum index of the elementary variables in the subset $R_d$. In other words, all $X(d)$ are consistent aggregate indexes of the input subsets.

If the aggregate quantum index $X(d)$ represents complete strong separability at every point in factor space, then the production function $f(X)$ can be the two-stage level CES-type functional form as follows. We assume that each aggregate quantum index of $\{X(d)\}$ is a CES function with the linear homogeneity in all arguments in subvector $\{X(d)\}$. We then get:

$$X(d) = \left[ \sum_{d=1}^{D} b_{\ell(d)} X_{\ell(d)}^{-\rho_d} \right]^{-\frac{1}{\rho_d}}, \quad (8.43)$$

where $d \in [1, D]$ and $\rho_d = \frac{1-\sigma_d}{\sigma_d}$. We call the expression (8.43) the first-stage level CES-type function.
Then, we can think of the CES-type function composing of all aggregate quantum indexes in the following way:

$$Q^o = f(X) = \psi \left( X_{(d)} \right) = \left[ \sum_{d=1}^{D} h_d X_{(d)}^{-\rho} \right]^{-\frac{1}{\rho}},$$  

(8.44)

where $h_d > 0$ and $\rho = \frac{1-\sigma}{\sigma}$. By substituting the expression (8.43) into (8.44), we can obtain the specific second-stage level CES-type production function as follows:

$$Q^o = f(X) = \left[ \sum_{d=1}^{D} h_d \left( \sum_{\ell=1}^{L_d} b_{\ell (d)} X_{\ell (d)}^{-\rho_d} \right)^{\frac{\rho}{\rho_d}} \right]^{-\frac{1}{\rho}},$$  

(8.45)

where $X$ = a vector of the productive factors, $h_d$, $b_{\ell (d)}$ = the distribution parameters, and $\rho_d$, $\rho$ = the substitution parameters, $L_d$ = the number of elementary inputs in factor group $d$.

Both the first-stage level and the second-stage level CES-type functions\(^7\) show the linear homogeneity in the factors of production. This functional form is not able to describe the change in the technical change. We therefore modify the functional form (8.45) to the functional form to be able to reflect the technical change effects, without loss of generality in the following way:

$$Q^o = f(X) = \gamma \left[ \sum_{d=1}^{D} h_d \left( \sum_{\ell=1}^{L_d} b_{\ell (d)} X_{\ell (d)}^{-\rho_d} \right)^{\frac{\rho}{\rho_d}} \right]^{-\frac{1}{\rho}},$$  

(8.46)

Where $\gamma$ = the efficiency parameter.

\(^7\)The reason why we call $X_{(d)}$ the first-stage level function and $f(X)$ the second (or two)-stage level function is as follows. $X_{(d)}$ is a function in elementary arguments $\{X\}$ and in turn $f(X)$ is a function in the aggregate quantum indexes $\{X_{(d)}\}$ which are functions in elementary variables $\{X\}$. Hence, we call $X_{(d)}$ the first-stage level function and $f(X)$ the two-stage level function.
Finally, the functional form (8.46) also can be rewritten without loss of generality as follows:

\[ Q^0 = f(X) = \gamma \left[ \sum_{d=1}^{N} \left( \sum_{\ell=1}^{L_d} \beta_{\ell(d)} X_{\ell(d)}^{-\rho_d} \right)^{\frac{\rho}{\rho_d}} \right]^{-\frac{1}{\rho}}, \quad (8.47) \]

where \( \beta_{\ell(d)} = \left[h_{\ell(d)}\right]^{\rho} b_{\ell(d)}. \)
9. THE PRODUCTION OF EACH INDUSTRY: THE NEO-CLASSICAL THEORY OF PRODUCTION AND COST

9.1 Two-Stage Level CES-Type Production Function and Input-Output Framework

In the preceding chapter 8, we have discussed the general concept of the two-stage level CES-type production function (see (8.47)). We will utilize that type of production function in our study. The production function (8.47) shows that the first-stage level CES-type function is homogeneous of degree one in the factors of production, and that the second-stage level CES-type function also has the linear homogeneity in the factors of production.

Here we will specifically connect such a concept of the production function (8.47) with the system of input-output relation. For this, we assume that (i) in a national economy, there are \( G \) industries which each of them plays roles both as a producing and selling industry (seller) \( g \) of its outputs (products) and as a purchasing industry (a buyer; a price-taker) \( j \) of the productive factors\(^1\); (ii) there are \((G + M + Z)\) production factors: that is, \( G \) intermediate factors (or outputs), \( M \) primary factors, and \( Z \) imported foreign factors; (iii) each of the producing industries yields its single

\(^1\)In relation to the framework of input-output relation, for convenience of economic analysis, we can classify the productive factors into three categories: the intermediate factors, the primary factors, and the imported foreign factors.
output through the two-stage CES-type production function (8.47).

We then can express the form of the production function (8.47) in terms of the intermediate factors, the primary factors, and the imported foreign factors in the following way. We assume that each producing industry \( j \) utilizes \( G \) intermediate factors \( Q_{gj} \), \( M \) primary factors \( \dot{X}_{m_j} \), and \( Z \) imported foreign factors \( X^*_{zj} \) in its production process. We then can in general write the internal structure of production utilized by each producing sector \( j \) so as to yield the output \( Q^o_j \) as follows:

\[
Q^o_j = F \left[ (Q_{1j}, \ldots, Q_{Gj})(\dot{X}_{1j}, \ldots, \dot{X}_{Mj})(X^*_{1j}, \ldots, X^*_{Zj}) \right],
\]

where \( Q_{gj} \) = the amount of an intermediate factor \( Q_g \) utilized by each producing industry (destination-industry) \( j \). This represents an elementary factor within the intermediate factor group.

\( \dot{X}_{m_j} \) = the amount of a primary factor \( \dot{X}_m \) utilized by each producing industry (destination-industry) \( j \). This expresses an elementary factor within the primary factor group.

\( X^*_{zj} \) = the amount of an imported foreign factor \( X^*_z \) utilized by each producing industry \( j \). This denotes an elementary factor within the imported foreign factor group.

Actually, in the real world, the internal structure of production includes many economic factors in the production process. Moreover, such factors of production contribute to the output through the interactions representing the substitutability, the complementarity, or the independence relationships among them. For example, in the production process, an elementary production factor within a certain factor group interacts another elementary production factors within the same factor group, on the one hand, and other production factors within the different groups, on the
The economic effects of such interactions among the productive factors utilized have to be reflected in the output produced. It however seems to be complicated and inconvenient to individually reflect such interaction effects. We thus consider, for convenience, a way through a consistent (sub-) aggregate index of the factors of production suggested by Leontief (1946, 1947) and Solow (1956a) and many others in relation to the functional separability.

We then can rewrite the internal production structure of each producing industry (9.1) as below:

\begin{equation}
Q_j^0 = \left[ q_{1j}, q_{2j}, q_{3j} \right],
\end{equation}

where $q_{1j} = (Q_{1j}, Q_{2j}, ..., Q_{Gj}, ..., Q_{G_j})$; this represents the consistent (sub-) aggregate quantum index of the intermediate factor from all $G$ intermediate factors, or the first subfunction (group) in the function $F(X)$.

$q_{2j} = (\tilde{X}_{1j}, \tilde{X}_{2j}, ..., \tilde{X}_{Mj}, ..., \tilde{X}_{M_j})$; this denotes the consistent (sub-) aggregate quantum index of the primary factor from all $M$ primary factors $\tilde{X}_{m_j}$, or the second subfunction (group) in the function $F(X)$.

$q_{3j} = (X_{1j}^*, X_{2j}^*, ..., X_{zj}^*, ..., X_{Z_j}^*)$; this expresses the consistent (sub-) aggregate quantum index of the imported foreign factor from all $Z$ imported foreign factors $X_{zj}^*$, or the third subfunction (group) in the function $F(X)$.

In order to reflect the economic impacts of the interactions among the elementary production factors within each group (subset) to the corresponding (sub-) aggregates to which the elementary production factors belong, and for the purpose of specific calculation we assume that each subfunction has complete strong separability, i.e., the additive functional form and is the CES-type function which has the linear homogeneity in the elementary production factors. Making use of the form (8.43), the
subfunctions can be written in the forms:

\[ q_{1j} = \left[ \sum_{g=1}^{G} b_{gj} Q_{gj}^{-\rho_{1j}} \right]^{-\frac{1}{\rho_{1j}}}, \quad (9.3) \]

\[ q_{2j} = \left[ \sum_{m=1}^{M} b_{mj} X_{mj}^{-\rho_{2j}} \right]^{-\frac{1}{\rho_{2j}}}, \quad (9.4) \]

\[ q_{3j} = \left[ \sum_{z=1}^{Z} b_{zj} X_{zj}^{-\rho_{3j}} \right]^{-\frac{1}{\rho_{3j}}}. \quad (9.5) \]

These functions are the first-stage level CES-type functions. We need the more specific production function to be able to reflect the interactions among the productive factors in our study for analyzing the specific resource allocations. We therefore assume that the main production function representing the internal structure of production of each producing industry \( j \) has strong functional separability with respect to the subfunctions, that is, the expressions (9.3)-(9.5). Then, by making use of (8.44), we can write the production function as follows:

\[ Q_j^o = \left[ \sum_{i=1}^{3} h_i q_{ij}^{-\rho_j} \right]^{-\frac{1}{\rho_j}}. \quad (9.6) \]

Following (8.47), the production function (9.6) can be rewritten without loss of generality in the following functional form2:

\[ Q_j^o = \gamma_j \left[ \sum_{i=1}^{3} Y_{ij}^{-\rho_j} \right]^{-\frac{1}{\rho_j}}, \quad (9.7) \]

where

\[ Y_{1j} = \left[ \sum_{g=1}^{G} b_{gj} Q_{gj}^{-\rho_{1j}} \right]^{-\frac{1}{\rho_{1j}}}, \]

\[ Y_{2j} = \left[ \sum_{m=1}^{M} b_{mj} X_{mj}^{-\rho_{2j}} \right]^{-\frac{1}{\rho_{2j}}}, \]

\[ Y_{3j} = \left[ \sum_{z=1}^{Z} b_{zj} X_{zj}^{-\rho_{3j}} \right]^{-\frac{1}{\rho_{3j}}}. \]

Note that \( \beta_{gj} = h_1 b_{gj}, \beta_{mj} = h_2 b_{mj}, \beta_{zj} = h_3 b_{zj}, \) and \( \beta_{ij} = h_i b_{ij}. \)
\[ Y_{3j} = \left( \sum_{z=1}^{Z} \beta_{zj}^* X_{zj} - \rho_{3j} \right)^{-\frac{1}{\rho_{3j}}}, \]

\[ \gamma_j = \text{the efficiency parameter which serves as an indicator of the state of technology}, \]

\[ \beta's = \text{the distribution parameters which has to do with the relative factor shares in the output}, \]

\[ \rho_j, \rho_{ij} = \text{the substitution parameters which determines the value of the elasticity of substitution}; \rho \in (0, \infty), \]

\[ Q_{gj} = \text{the amount of an intermediate factor } Q_g \text{ utilized (purchased) by each producing industry (destination-industry) } j \text{ from an origin-industry } g, \]

\[ \hat{X}_{mj} = \text{the amount of a primary factor } \hat{X}_m \text{ utilized by each producing industry (destination-industry) } j, \]

\[ X_{zj}^* = \text{the amount of an imported foreign factor } X_z^* \text{ utilized by each producing industry } j. \]

\[ Y_{1j} = \text{the contribution of the aggregate of the intermediate factors to the output } Q_{jt}, \text{ or the aggregate quantum index of the intermediate factor from all } G \text{ intermediate factors } Q_{gj}, \]

\[ Y_{2j} = \text{the contribution of the aggregate of the primary factors to the output } Q_j, \text{ or the aggregate quantum index of the primary factor from all } M \text{ primary factors } \hat{X}_{mj}, \]

\[ Y_{3j} = \text{the contribution of the aggregate of the imported foreign factors to the output } Q_j^0, \text{ or the aggregate quantum index of the imported foreign factor from all } Z \text{ imported foreign factors } X_{zj}^*. \]

Thus, we have \( d = 3 \) in relation to the functional form (8.47). The production
function (9.7) is the specific two-stage level CES-type production function we will utilize in our study. We assume Hicks-neutrality\(^3\) in the technical change since the neutralities of technological progress both in the Harrodian and the Solowan senses are often violated in the CES function.\(^4\)

The production function (9.7) is closely associated with multi-factor CES production functions introduced by Uzawa (1962), Mukerji (1963), and Sato (1967). As mentioned in the preview section, Theil and Tilanus (1964) utilized the Uzawa-type or modified Mukerji-type CES production function in their price effect study (see the preview section for the limitations they have). Sato (1967) discussed the substitutability between capital (i.e., physical capital: equipment and structure) and labor by utilizing that functional form.

### 9.2 Methodology: Two-Stage Optimization

We have discussed the two-stage level CES-type production function with strong functional separability, considering the input-output relation. For our analyses of the economic impacts of taxes, imported tariffs, and the foreign exchange rate on the domestic resource allocations, we need the optimal factor demand functions and the minimal total cost function of each of the domestic producing industries. Such functions play crucial roles in calculating the production parameters (i.e., EOS) of the two-stage level CES-type production function characterized by strong functional separability. So, in this section, we are concerned with the optimization procedures of

\(^3\)Technical change is completely Hicks-neutral for all factors simultaneously if and only if all \(\gamma_j\) change at the same rate.

\(^4\)For more details, see Heathfield and Wibe (1987, pp. 121-125).
each producing industry \( j \) based on the neo-classical theory of production and cost.

We will calculate the optimal factor demand functions and the minimal total cost function through the two-stage cost minimization procedure in Section 9.2.1. We then will show how the production parameters (i.e., EOS) can be calculated by making use of the optimal factor demand functions and the minimal total cost function of each producing industry in Section 9.3.

### 9.2.1 Two-Stage Optimal Functions

We now examine the derivation of the optimal factor demand functions and the minimal total cost function through the two-stage cost-minimization process because we assume that each producing industry \( j \) utilizes the two-stage level CES-type production function. When the prices of all factors of production and output are given (see [H.1]), the sector \( j \) (price-taker) facing perfect competition in the factor and the commodity markets\(^5\) can determine its optimum combination of the productive factors to yield the given level of output \( Q_j^0 \) by the cost-minimization (or the profit-maximization) rule based on the basic theorem of the neo-classical theory of production and cost.

The production function we utilize is the two-stage level CES-type functional form characterized by strong functional separability. Since the production function has the characteristic of strong functional separability, efficiency in production can be realized by sequential optimization. For example, in production decisions, relative factor ratios among the production factors can be optimized within each separable

\(^5\)This implies that there is no monopsony in the factor market and no monopoly in the commodity market.
factor subset (group), and then optimal factor ratios can be attained by holding the intra-subset (group) factor ratios fixed and optimizing the inter-subset (group) factor ratios. Thus, the choice of the cost-minimizing factor combination can be effectively separated into two stages reflecting aggregation (or grouping) consistency. In other words, production decisions of a producing sector \( j \) are broken up into two stages. The primary stage is the determination of the total production cost in the production process. The secondary stage is the allocation (or distribution) of the total cost among the various types of the productive factors.

Since we assume that each producing industry \( j \) \((j \in [1, G])\) utilizes \( G \) intermediate factors, \( M \) primary factors, and \( Z \) imported foreign factors, the total cost \( C_j \) of each producing industry (firms) \( j \) is composed of (i) the intermediate cost \( c_{1j} \); (ii) the primary factor cost \( c_{2j} \); (iii) the imported foreign factor cost \( c_{3j} \).

Formally, the total cost function of each producing industry \( j \) is as follows:

\[
C_j = \sum_{i=1}^{3} c_{ij}.
\]

Equivalently,

\[
C_j = \sum_{g=1}^{G} W_{gj} Q_{gj} + \sum_{m=1}^{M} \hat{W}_{mj} \hat{X}_{mj} + \sum_{z=1}^{Z} W_{zj}^{D} X_{zj}^*, \tag{9.8}
\]

where \( j \in [1, G] \) are integers, \( W_{gj} \) is the price of an intermediate factor \( Q_{gj} \) utilized by each producing industry \( j \), \( \hat{W}_{mj} \) is the price of a primary factor \( \hat{X}_{mj} \) utilized by each producing industry \( j \), \( W_{zj}^{D} \) is the domestic price of an imported foreign factor \( X_{zj}^* \) utilized by each producing industry \( j \).

From (9.7), the separable two-stage level CES-type production exhibiting the
linear homogeneity (constant returns to scale) is:

\[ Q^o_j = \gamma_j \left[ \sum_{i=1}^{3} Y_{ij}^{-\rho_j} \right]^{-\frac{1}{\rho_j}}, \quad (9.9) \]

where

\[ Y_{1j} = \left[ \sum_{g=1}^{G} \beta_{gj} Q_{gj}^{-\rho_{1j}} \right]^{-\frac{1}{\rho_{1j}}}, \quad (9.10) \]
\[ Y_{2j} = \left[ \sum_{m=1}^{M} \hat{\beta}_{mj} X_{mj}^{-\rho_{2j}} \right]^{-\frac{1}{\rho_{2j}}}, \quad (9.11) \]
\[ Y_{3j} = \left[ \sum_{z=1}^{Z} \beta_{zj} X_{zj}^{-\rho_{3j}} \right]^{-\frac{1}{\rho_{3j}}}, \quad (9.12) \]

where \( j \in [1, G] \) are integers.

Under our regularity conditions \([H.1]-[H.10]\), we assume that:

1. \( \gamma_j > 0 \),
2. \( \rho_j, \rho_{ij} > 0 \), specifically, \( 0 < \sigma_j < \sigma_{ij} \) \((i \in [1, 3])\),
3. \( \beta_{gj}, \hat{\beta}_{mj}, \beta_{zj}^* > 0 \), for \( g \in [1, G], m \in [1, M], z \in [1, Z] \); \((g, m, z)\) are integers.

Each producing industry \( j \) then can minimize the total cost \( C_j = \sum_{i=1}^{3} c_{ij} \) subject to its two-stage level CES-type production function \( Q^o_j \). The Lagrangean function by the fundamental theorem of the neo-classical cost-minimization theory is:

\[ \text{Min. } \Phi = \sum_{i=1}^{3} c_{ij} + \lambda_j \left[ Q^o_j - \gamma_j \left( \sum_{i=1}^{3} Y_{ij}^{-\rho_j} \right)^{-\frac{1}{\rho_j}} \right], \quad (9.13) \]

where \( Q^o_j \) is the given level of output produced by each producing industry \( j \).

\[ c_{1j} = \sum_{g=1}^{G} W_{gj} Q_{gj}, \quad (9.14) \]
Each producing industry $j$ can utilize the two-stage cost-minimization because the production function $Q_j^0$ has the separability characteristic. The sequential processes of the cost-minimization have the following procedures:

[I] In the first stage, each producing industry $j$ perform the cost-minimizations as below:

\[
\text{Min. } \Phi_1 = c_{1j} + \mu_{1j} \left[ Y_{1j} - \left( \sum_{g=1}^{G} \beta_{gj} Q_{gj}^{-\rho_{1j}} \right)^{-\frac{1}{\rho_{1j}}} \right].
\]

(9.17)

\[
\text{Min. } \Phi_2 = c_{2j} + \mu_{2j} \left[ Y_{2j} - \left( \sum_{m=1}^{M} \beta_{mj} \hat{X}_{mj}^{-\rho_{2j}} \right)^{-\frac{1}{\rho_{2j}}} \right].
\]

(9.18)

\[
\text{Min. } \Phi_3 = c_{3j} + \mu_{3j} \left[ Y_{3j} - \left( \sum_{z=1}^{Z} \beta_{zj} X_{zj}^{-\rho_{3j}} \right)^{-\frac{1}{\rho_{3j}}} \right],
\]

(9.19)

where $\mu_{ij}$ ($i \in [1,3]$) and $j \in [1,G]$ are the Lagrange multipliers in the first stage.

From (9.17) through (9.19), each producing industry $j$ can derive its first-stage minimal cost functions $c_{1j}^0(W,Y_{1j})$, $c_{2j}^0(W,Y_{2j})$, $c_{3j}^0(W^D,Y_{3j})$, where $c_{1j}^0$, $c_{2j}^0$, $c_{3j}^0$, are the first-stage minimum cost functions for the contributions of the aggregate quantum indexes, $Y_{1j}$, $Y_{2j}$, and $Y_{3j}$ of the intermediate factors, the primary factors, and the imported foreign factors, respectively, to the output $Q_j^0$.

[II] In the second stage, each producing industry $j$ can obtain its optimal cost-minimizing factor demand functions, $Q_{gj}^0$, $\hat{X}_{mj}^0$, and $X_{zj}^0$, and its minimum total
cost function \( C^o_j \), to produce its output \( Q^o_j \) by the following cost-minimization rule:

\[
\text{Min. } \Phi = C^o_j + \lambda_j \left[ Q^o_j - \gamma_j \left( \sum_{i=1}^{3} Y_{ij} - \rho_j \right) \right],
\]

(9.20)

where \( \lambda_j \) is the Lagrange multiplier in the second stage.

Now we will specifically derive the optimal factor demand functions and the minimal total cost function of each producing industry \( j \) from the first-stage and the second-stage cost-minimization processes.

9.2.1.1 (1) The First-Stage Optimization Procedure First, each producing industry \( j \) can derive its first-stage cost-minimizing factor demand functions and its minimal total cost function from the Lagrangean function (9.17). Taking the partials of the Lagrangean function (9.17) with respect to \( Q_{gj} \) \( g \in [1, G] \) and a Lagrange multiplier \( \mu_{1j} \) and setting them equal to zero yields the following first-order conditions for constrained minimization:

\[
W_{gj} = \mu_{1j} \left[ \beta_{gj} Q_{gj}^{-(1+\rho_{1j})} \right] \left[ \sum_{g=1}^{G} \beta_{gj} Q_{gj}^{-\rho_{1j}} \right]^{-(1+\rho_{1j})} \rho_{1j},
\]

(9.21)

where \( g, j \in [1, G] \) are integers.

\[
Y_{1j} = \left[ \sum_{g=1}^{G} \beta_{gj} Q_{gj}^{-\rho_{1j}} \right]^{\frac{-1}{\rho_{1j}}},
\]

(9.22)

From the elasticity of substitution \( \sigma_j = \frac{1}{1+\rho_j} \), we can obtain:

\[
1 + \rho_{1j} = \frac{1}{\sigma_{1j}},
\]

(9.23)

\[
\rho_{1j} = \frac{1 - \sigma_{1j}}{\sigma_{1j}}.
\]

(9.24)
Substituting the expressions (9.23)-(9.24) into the expressions (9.21) and (9.22), respectively, leads to the following results:

\[ W_{gj} = \mu_{1j} \left[ \beta_{gj} Q_{gj} \right] \left[ \sum_{g=1}^{G} \beta_{gj} Q_{gj} \right]^{1-\sigma_{1j}} \frac{1}{\sigma_{1j}-1}, \tag{9.25} \]

\[ Y_{1j} = \left[ \sum_{g=1}^{G} \beta_{gj} Q_{gj} \right]^{1-\sigma_{1j}} \frac{\sigma_{1j}}{\sigma_{1j}}, \tag{9.26} \]

where \( g, j \in [1, G] \) are integers.

From the expression (9.26), we yield:

\[ \left[ \sum_{g=1}^{G} \beta_{gj} Q_{gj} \right]^{1-\sigma_{1j}} = \left( Y_{1j} \right)^{\sigma_{1j}}. \tag{9.27} \]

Substitution of (9.27) into (9.25) leads to:

\[ Q_{gj}^{\circ}(W, Y_{1j}) = Y_{1j} \left[ \frac{\sigma_{1j}}{\sigma_{1j}} \right]^{1-\sigma_{1j}} \mu_{1j}^{1-\sigma_{1j}} \tag{9.28} \]

where \( g, j \in [1, G] \) are integers, \( W=(W_{1j}, W_{2j}, ..., W_{Gj}) \), and

\[ \mu_{1j} = \left[ \sum_{g=1}^{G} \beta_{gj} Q_{gj} \right]^{1-\sigma_{1j}} \frac{1}{1-\sigma_{1j}}. \tag{9.29} \]

Note that \( \mu_{1j} \) is derived by putting (9.28) into (9.25) or (9.26).

The expression (9.28) shows the optimal cost-minimizing factor demand functions of the intermediate inputs \( Q_{gj}^{\circ} \), optimized in the first stage. Each producing industry \( j \) can also derive its first-stage minimal cost function, \( c_{1j}^{\circ} \), from substituting
the first-stage optimal factor demand functions (9.28) of the intermediate inputs \( Q_{gj}^o \) into the total cost function (10.14):

\[
c_{ij}^o(W, Y_{1j}) = \mu_{1j} Y_{1j},
\]

where \( W=(W_{1j}, W_{2j}, ..., W_{Gj}) \) and \( j \in [1, G] \) are integers.

Second, the first-stage optimum factor demand and minimal total cost functions of the primary factors \( X_{m,j}^c \) are derived from the Lagrangean function (9.18).

Utilizing the same method as that of the intermediate factor case, each producing industry can obtain the following outcomes:

\[
X_{m,j}^c(\hat{W}, Y_{2j}) = Y_{2j} \left[ \hat{\beta}_{m,j} \frac{\sigma_{2j} W_{m,j}^{-\sigma_{2j}}}{\mu_{2j}} \right]^{\sigma_{2j}},
\]

\[
c_{ij}^o(\hat{W}, Y_{2j}) = \mu_{2j} Y_{2j},
\]

where \( m \in [1, M], j \in [1, G]; m, j \) are integers, \( \hat{W}=(\hat{W}_{1j}, \hat{W}_{2j}, ..., \hat{W}_{Mj}) \),

\[
\sigma_{2j} = \frac{1}{1+\rho_{2j}}, \text{ and}
\]

\[
\mu_{2j} = \left[ \sum_{m=1}^{M} \hat{\beta}_{m,j} \frac{1-\sigma_{2j}}{\sigma_{2j} W_{m,j}^{1-\sigma_{2j}}} \right]^{\frac{1}{1-\sigma_{2j}}},
\]

The expression (9.31) represents the first-stage optimum demand functions of the primary factors \( \hat{X}_{m,j} \). The expression (9.32) means the first-stage minimum cost function in terms of the primary factors.

Through the same method as that used in the cases of both the intermediate factor and the primary factor, each producing industry yields its first-stage optimal
factor demand functions of the imported foreign factors and its first-stage minimum cost function in terms of the imported foreign factors as below:

\[ X_{zj}^*(W^D, Y_{3j}) = Y_{3j} \left[ \beta_{zj}^{*\sigma_{3j}}(W_{zj}^D)^{-\sigma_{3j}} \right]^{\sigma_{3j}} \mu_{3j} \]

\[ = Y_{3j} \left[ \frac{\mu_{3j}\beta_{zj}^{*\sigma_{3j}}}{W_{zj}^D} \right]^{\sigma_{3j}}. \]  \hspace{1cm} (9.34)

\[ c_{3j}^2(W^D, Y_{3j}) = \mu_{3j} Y_{3j}, \]  \hspace{1cm} (9.35)

where \( z \in [1, Z], j \in [1, G]; z, j \) are integers, \( W^D = (W_{1j}^D, W_{2j}^D, ..., W_{Zj}^D) \),

\[ \sigma_{3j} = \frac{1}{1 + \rho_{3j}}, \] and

\[ \mu_{3j} = \left[ \sum_{z=1}^{Z} \beta_{zj}^{*\sigma_{3j}}(W_{zj}^D)^{1-\sigma_{3j}} \right]^{\frac{1}{1-\sigma_{3j}}}. \]  \hspace{1cm} (9.36)

9.2.1.2 (2) The Second-Stage Optimization Procedure

Now, we start on the second-stage performances of the cost-minimization with the results of the first-stage: Minimize the total cost \( C_j = \sum_{i=1}^{3} c_{ij}^2 \) subject to the separable two-stage level CES-type production function \( Q_j^0 \).

Substituting (9.30), (9.32), and (9.35) into (9.13) or (9.20), the Lagrangean function is reduced to:

\[ \text{Min. } \Phi = \left[ \sum_{i=1}^{3} \mu_{ij} Y_{ij} \right] + \lambda_j \left[ Q_j^0 - \gamma_j \left( \sum_{i=1}^{3} Y_{ij}^{-\rho_j} \right) \right], \]  \hspace{1cm} (9.37)

where

\[ \mu_{1j} = \left[ \sum_{g=1}^{G} \beta_{gj}^{*\sigma_{1j}} W_{gj}^{1-\sigma_{1j}} \right]^{\frac{1}{1-\sigma_{1j}}}, \]

\[ \mu_{2j} = \left[ \sum_{m=1}^{M} \beta_{mj}^{*\sigma_{2j}} W_{mj}^{1-\sigma_{2j}} \right]^{\frac{1}{1-\sigma_{2j}}}, \]
\[ \mu_{3j} = \left[ \sum_{z=1}^{Z} \beta_{zj}^\ast \sigma_{3j} (W_j)^{1-\sigma_{3j}} \right]^{1-\sigma_{3j}}, \text{ and } j \in [1, G] \text{ are integers.} \]

Taking the differentials of (9.37) with respect to \( Y_{1j}, Y_{2j}, Y_{3j}, \) and \( \lambda_j, \) we can set up the equilibrium conditions (FOCs) as follows:

[FOC]:

\[
\mu_{\ell j} = \lambda_j \left[ \gamma_j Y_{\ell j}^{-(1+\rho_j)} \left( \sum_{i=1}^{3} Y_{ij}^{-\rho_j} \right)^{-\lambda_j} \right],
\]

(9.38)

where \( \ell \in [1, 3] \) are integers and \( \lambda_j \) is the Lagrange multiplier.

\[
Q_j^0 = \gamma_j \left[ \sum_{i=1}^{3} Y_{ij}^{-\rho_j} \right]^{-\frac{1}{\rho_j}}.
\]

(9.39)

From the production function (9.39), we can obtain:

\[
\left( \sum_{i=1}^{3} Y_{ij}^{-\rho_j} \right) = \gamma_j^j (Q_j^0)^{-\rho_j} = \left( \frac{Q_j^0}{\gamma_j^j} \right)^{-\rho_j}.
\]

(9.40)

Putting (9.40) into (9.38), we can get:

\[
\mu_{\ell j} = \lambda_j H_j Y_{\ell j}^{-(1+\rho_j)},
\]

(9.41)

where \( H_j = \gamma_j^{-\rho_j} (Q_j^0)^{1+\rho_j} \) and \( \ell \in [1, 3] \) are integers.

From the expression (9.41), we get:

\[
Y_{\ell j} = \left( \lambda_j H_j \right)^{\frac{1}{1+\rho_j}} \left( \mu_{\ell j} \right)^{-\frac{1}{1+\rho_j}}, \quad \ell \in [1, 3].
\]

(9.42)

Substituting (9.42) into (9.39) or (9.40), we obtain:

\[
\lambda_j = \left( \frac{Q_j^0}{\gamma_j^j} \right)^{1+\rho_j} H_j^{-1} \left[ \sum_{i=1}^{3} \frac{\rho_j}{\mu_{ij}^{1+\rho_j}} \right]^{\frac{1+\rho_j}{\rho_j}}.
\]
where $H_j = \gamma_j \rho_j (Q_j^o)^{1+\rho_j}$.

Equivalently,

$$\lambda_j = \gamma_j^{-1} \left[ \sum_{i=1}^{3} \mu_{ij} \right]^{1-\sigma_j}, \quad (9.43)$$

where $\sigma_j = \frac{1}{1+\rho_j}$, i.e., $\rho_j = \frac{1-\sigma_j}{\sigma_j}$. This $\lambda_j$ is the Lagrange multiplier and has an economic interpretation implying that it is equal to the average cost and the marginal cost (in the long-run).

Plugging (9.43) into (9.42), we get:

$$Y_{\ell j} = \left( \frac{Q_j^o}{\gamma_j^2} \right)^{-\sigma_j} \left[ \sum_{i=1}^{3} \mu_{ij} \right]^{1-\sigma_j}, \quad (9.44)$$

where $\sigma_j = \frac{1}{1+\rho_j}$, i.e., $\rho_j = \frac{1-\sigma_j}{\sigma_j}$ and $\ell \in [1, 3], j \in [1, G]$; $\ell, j$ are integers.

More specifically speaking, the expression (9.44) can be rewritten as:

$$Y_{1j} = \left( \frac{Q_j^o}{\gamma_j^2} \right)^{-\sigma_j} \left[ \sum_{i=1}^{3} \mu_{ij} \right]^{1-\sigma_j}, \quad (9.45)$$

$$Y_{2j} = \left( \frac{Q_j^o}{\gamma_j^2} \right)^{-\sigma_j} \left[ \sum_{i=1}^{3} \mu_{ij} \right]^{1-\sigma_j}, \quad (9.46)$$

$$Y_{3j} = \left( \frac{Q_j^o}{\gamma_j^2} \right)^{-\sigma_j} \left[ \sum_{i=1}^{3} \mu_{ij} \right]^{1-\sigma_j}, \quad (9.47)$$

Substitution of (9.45) into (9.28) leads to the optimal cost-minimizing factor demand functions of the intermediate factors $Q_{gij}$ utilized by each producing industry (firms) $j$. Similarly, putting (9.46) into (9.31), and (9.47) into (9.34) lead to the optimal cost-minimizing demand functions of the primary factors $X_{mij}$ and the imported foreign factors $X_{zij}^*$ utilized by each producing industry (firms) $j$, respectively. The results are as follows:
Let
\[ \Lambda_j = \left[ \sum_{i=1}^{3} \mu_{ij} \right]^{1-\sigma_j} \left( \frac{1}{1-\sigma_j} \right) (j \in [1, G]). \] (9.48)

Then,
\[ Q_{g,j}(\gamma_j, W, Q_j^0) = \left( \frac{Q_j^0}{\gamma_j^0} \right) \left( \frac{\beta_{g,j}}{W_{g,j}} \right)^{\sigma_{1,j}} \mu_{1,j}^{\sigma_{1,j} - \sigma_j} \Lambda_j, \] (9.49)

where \( W=(W, \hat{W}, W^D), \mu_{1,j} = \left[ \sum_{g=1}^{G} \beta_{g,j}^{\sigma_{1,j}} W_{g,j}^{1-\sigma_{1,j}} \right]^{1-\sigma_{1,j}}, \) and \( g, j \in [1, G] \) are integers.

The expression (9.49) represents the optimum demand functions of the intermediate factors \( Q_{g,j} \) utilized by each producing industry \( j \), which is derived from the two-stage cost-minimization rule. The respective optimal factor demand functions (9.49) are homogeneous of degree zero in the prices of all factors of production \( W=(W, \hat{W}, W^D) \).

\[ \hat{X}_{m,j}(\gamma_j, W, Q_j^0) = \left( \frac{Q_j^0}{\gamma_j^0} \right) \left( \frac{\beta_{m,j}}{W_{m,j}} \right)^{\sigma_{2,j}} \mu_{2,j}^{\sigma_{2,j} - \sigma_j} \Lambda_j, \] (9.50)

where \( W=(W, \hat{W}, W^D), \mu_{2,j} = \left[ \sum_{m=1}^{M} \beta_{m,j}^{\sigma_{2,j}} W_{m,j}^{1-\sigma_{2,j}} \right]^{1-\sigma_{2,j}}, m \in [1, M] \) and \( j \in [1, G]; m, j \) are integers.

The expression (9.50) represents the optimum demand functions of the primary factors \( \hat{X}_{m,j} \) utilized by each producing industry \( j \), which is derived from the two-stage cost-minimization rule. The respective optimal factor demand functions (9.50) are homogeneous of degree zero in the prices of all factors of production \( W=(W, \hat{W}, W^D) \).

\[ X_{z,j}(\gamma_j, W, Q_j^0) = \left( \frac{Q_j^0}{\gamma_j^0} \right) \left( \frac{\beta_{z,j}}{W_{z,j}} \right)^{\sigma_{3,j}} \mu_{3,j}^{\sigma_{3,j} - \sigma_j} \Lambda_j, \] (9.51)
where \( W = (W, \dot{W}, W^D) \), 
\[
\mu_{3j} = \left[ \sum_{z=1}^{Z} \beta_{zj}^{\sigma_{3j}} (W^D_{zj})^{1-\sigma_{3j}} \right]^{1-\sigma_{3j}}, \quad z \in [1, Z],
\]
and \( j \in [1, G] \); \( z, j \) are integers.

The expression (9.51) denotes the optimum demand functions of the imported foreign factors \( X^*_{zj} \), utilized by each producing industry \( j \), which is derived from the two-stage cost-minimization rule. The respective optimal factor demand functions (9.51) are also homogeneous of degree zero in the prices of all factors of production \( W = (W, \dot{W}, W^D) \).

These optimal cost-minimizing factor-resource demand functions (9.49)-(9.51) also show that the symmetry condition representing that the cross price effects are equal and the own price effects are negative.

Next, we can also calculate the minimum total cost function of each producing industry \( j \) from substitution of (9.49)-(9.51) into (9.8) as follows:
\[
C^*_{ij}(\gamma_j, W, Q^*_j) = \left( \frac{Q^*_j}{\gamma_j} \right) \Lambda_j.
\]
(9.52)

where \( W = (W, \dot{W}, W^D) \) and \( j \in [1, G] \) are integers.

This minimal total cost function (9.52) preserves the linear homogeneity in the factor prices \( (W, \dot{W}, W^D) \). We can also see that the minimum total cost decreases when there are technical improvements. The minimal total cost function (9.52) and the two-stage CES-type production function (9.9) has the self-dual relationship each other. By this duality, therefore, we can use the minimal total cost function in calculating the production parameters. In the next section we will discuss and examine the derivation of the values of the production parameters, i.e., the elasticities of factor substitution.
9.3 The Production Parameters (EOS)

9.3.1 Introduction

In the preceding section we have derived the optimal factor demand functions and the minimal total cost function from the two-stage level CES-type production function through the two stage cost-minimization processes. As mentioned in the previous section, the production function and the cost function have the dual relationship. We therefore can calculate the production parameters by making use of the minimal cost function.

Specifically, we can derive the partial elasticity of substitution between two factors by the formula (5.13) associated with the cost function. The formula (5.13) is associated with cost functions. The elasticity of factor substitution is derived from distributing the total cost through two stages: First, the total cost $C_j^\circ$ is distributed to proportions of the cost of the aggregate quantum indexes of the intermediate factors, $C^i$, the primary factors, $C^p$, and the imported foreign factors, $(C^*)^f$, respectively. Second, these three aggregated costs are also allocated to each elementary factor of production, i.e., $Q^o_{gj}$, $X^o_{mj}$, and $X^o_{zj}$, in which they include, respectively.

Formally, we can express the two stages as follows:

$$S^\ell_{C_j} = S^\ell C^\ell C^h [ S^h C_j ] .$$

Equivalently,

$$S_{\ell j} = S_{\ell h} [ S_{hj} ] ,$$

(9.53)

where $(C^g, C^m, C^*^z) \in C^\ell$ and $(g, m, z) \in \ell$,

$(C^i, C^p, C^f) \in C^h$ and $(i, p, f) \in h,$
\[ S_{tj} = S^{C_{j}}_{C_{j}} = \text{the cost (value) share of each elementary factor } \ell, C_{j} \text{, in the total cost } C_{j}^{o} \text{ of each producing industry,} \]

\[ S_{th} = S^{C_{h}}_{C_{h}} = \text{the cost share of each elementary factor } \ell, C_{h} \text{ in each of the factor groups (the aggregated quantum indexes of the intermediate factors, the primary factors, and the imported foreign factors) in the total cost } C_{j}^{o} \text{ of each producing industry,} \]

\[ S_{hj} = S^{C_{h}}_{C_{j}} = \text{the cost share of each of the factor groups (the aggregated factors } h) C_{h} \text{ in the total cost } C_{j}^{o} \text{ of each producing industry.} \]

We now examine specifically how the elasticities of factor substitution can be calculated.

[I] The cost share of each of the factor groups (the aggregated factors) in the total cost } C_{j}^{o} \text{ of each producing industry can be calculated in the following way.

(1) The cost share form of the aggregate quantum index of the intermediate factors in the total cost is as below:

\[ S_{ij} = \frac{\sum_{g=1}^{G} W_{gj} Q_{gj}^{o}}{C_{j}^{o}}. \quad (9.54) \]

Substitution of (9.49) and (9.52) into (9.54) leads to

\[ S_{ij} = \Lambda_{j}^{1-\sigma_{j}} - \mu_{1j}. \quad (9.55) \]

(2) The cost share form of the aggregate quantum index of the primary factors in the total cost is derived from the same way as the case of (1):

\[ S_{pj} = \frac{\sum_{m=1}^{M} W_{mj} X_{mj}^{o}}{C_{j}^{o}}. \quad (9.56) \]
We can yield the following result by putting (9.50) and (9.52) into (9.56):

\[ \hat{S}_{pj} = \Lambda_j^{-1} \sigma_j^{1-\sigma_j} \mu_2^{j-1}. \]  

(9.57)

Similarly, the cost share form of the aggregate quantum index of the imported foreign factors is expressed in the following manner:

\[ S^*_f = C_j^{-1} \left[ \sum_{z=1}^{Z} W_{zj}^D X_{zj}^* \right]. \]  

(9.58)

Plugging (9.51)-(9.52) into (9.58),

\[ S^*_f = \Lambda_j^{-1} \sigma_j^{1-\sigma_j} \mu_3^j. \]  

(9.59)

[II] The cost share of each elementary factor \( i \) in the aggregated cost to which an elementary factor \( \ell \) belongs:

1. The cost share form of each elementary intermediate factor \( g \) in the aggregated cost of all \( G \) intermediate factors to which an elementary factor \( g \) is included can be obtained by making use of (9.49):

\[ S_{gi} = \left( W_{gj} Q_{gj}^o \right) \left[ \sum_{g'=1}^{G} W_{g'j} Q_{g'j}^o \right]^{-1}, \]  

(9.60)

\[ S_{gi} = \Omega_{gj} \sigma_{1j}^{1-\sigma_{1j}}, \]  

(9.61)

where

\[ \Omega_{gj} = \left[ \beta_{gj}^{1j} W_{gj}^{-\sigma_{1j}} \right]. \]  

(9.62)

2. The cost share form of an elementary primary factor \( m \) in the aggregated cost of all \( M \) primary factors in which an individual factor \( m \) is involved is:

\[ \hat{S}_{mp} = \left( \hat{W}_{mj} \hat{X}_{mj}^o \right) \left[ \sum_{m'=1}^{M} \hat{W}_{m'j} \hat{X}_{m'j}^o \right]^{-1}, \]  

(9.63)
\[
\hat{s}_{mp} = \hat{\Omega}_{mj} \mu_{2j}^{-1}
\]  
(9.64)

(by virtue of (9.50)), where

\[
\hat{\Omega}_{mj} = \left[ \beta_{mj}^{\sigma_{2j}} \hat{w}_{mj}^{1-\sigma_{2j}} \right].
\]  
(9.65)

(3) Analogously, the cost share form of an elementary imported foreign factor \( z \) in the aggregated cost of all \( Z \) imported foreign factors is derived from making use of (9.51):

\[
\hat{s}_{zf} = \left( \frac{W_{zj}^D X_{zj}^*}{\sum_{z'=1}^{Z} W_{zj}^D X_{zj'}^*} \right)^{-1}
\]  
(9.66)

\[
\hat{s}_{zf} = \Omega_{zj}^* \mu_{3j}^{-1},
\]  
(9.67)

where

\[
\Omega_{zj}^* = \left[ \beta_{zj}^{*\sigma_{3j}} (W_{zj}^D)^{1-\sigma_{3j}} \right].
\]  
(9.68)

[III] We can derive the cost share of each elementary factor in the total cost from the formulation (9.53).

(1) The cost share form of each intermediate factor \( g \) in the total cost \( C_j^O \) is:

\[
\hat{s}_{gj} = \left[ W_{gj}^O \Omega_{gj}^O \right] \left[ C_j^O \right]^{-1}.
\]

Substitution of the expressions (9.55) and (9.61) into (9.53) leads to

\[
\hat{s}_{gj} = \Omega_{gj} \mu_{1j}^{-\sigma_{1j}} \Lambda_{1j} \mu_{j}^{-1}.
\]  
(9.69)

(2) The cost share form of each primary factor \( m \) in the total cost is:

\[
\hat{s}_{mj} = \left[ \hat{w}_{mj}^O \hat{x}_{mj}^O \right] \left[ C_j^O \right]^{-1}.
\]
Making use of the expressions (9.57) and (9.64), we obtain

\[
\hat{S}_{mj} = \hat{\Omega}_{mj} \mu_{2j}^{\sigma j - \sigma j} \Lambda_j^{\sigma j - 1}.
\]  
(9.70)

(3) The cost share form of each imported foreign factor \( z \) in the total cost is:

\[
S_{zj}^* = \left[ W_{zj} D X_{zj} \right] \left[ C_j^\circ \right]^{-1}.
\]

By putting the expression (9.59) and (9.67) into (9.53), we get

\[
S_{zj}^* = \Omega_{zj}^* \mu_{3j}^{\sigma j - \sigma j} \Lambda_j^{\sigma j - 1}.
\]  
(9.71)

### 9.3.2 The Specific Elasticities of Substitution

Now, we can formulate the partial elasticities of factor substitution under the discussions of the preceding Section 9.3.

(I) The elasticities of factor substitution between different intermediate inputs, \( Q_{gj}^\circ \) and \( Q_{g'j}^\circ \) (\( g \neq g' \)), within the intermediate-factor category can be derived as follows.

Shephard (-Mckenzie) lemma holding for the minimum total cost function (9.52) leads to:

\[
C_{g}^\circ = Q_{gj}^\circ,
\]  
(9.72)

where \( C_{g}^\circ = \frac{\partial C_{gj}^\circ}{\partial W_{gj}} \).

By virtue of (9.49), the expression (9.72) can be written as:

\[
C_{g}^\circ = \left( \frac{Q_{gj}^\circ}{\gamma_j} \right) \left( \frac{\beta_{gj}}{W_{gj}} \right)^{\sigma j - \sigma j} \mu_{1j}^{\sigma j - \sigma j} \Lambda_j^{\sigma j},
\]  
(9.73)

where \( C_{g}^\circ = \frac{\partial C_{gj}^\circ}{\partial W_{gj}} \).
Also, by Shephard (McKenzie) lemma holding for (9.52), we can obtain:

\[
C_{g'}^o = Q_{g'}^o.
\]  

(9.74)

Or,

\[
C_{g'}^o = \left( \frac{Q_{j}^o}{\gamma_j} \right) \left( \frac{\beta_{g'j}}{W_{g'j}} \right)^{\sigma_{1}j} \mu_{1j}^{\sigma_{1j} - \sigma_j} \Lambda_j^{\sigma_j},
\]  

(9.75)

where \( C_{g'}^o = \frac{\partial C_{g'}^o}{\partial W_{g'j}} \).

From the expression (9.73), we get:

\[
C_{gg'}^o = \left( \frac{Q_{j}^o}{\gamma_j} \right) \Gamma_1 \left( \mu_{1j}^{2\sigma_{1j} - \sigma_j - 1} \right) \Lambda_j^{2\sigma_j - 1} \Gamma_2,
\]  

(9.76)

where \( C_{gg'}^o = \frac{\partial C_{gg'}^o}{\partial W_{gj} \partial W_{g'j}} \), \( \Gamma_1 = \left( \frac{\beta_{gj}}{W_{gj}} \right)^{\sigma_{1j}} \left( \frac{\beta_{g'j}}{W_{g'j}} \right)^{\sigma_{1j}} \), and

\[
\Gamma_2 = \left[ (\sigma_{1j} - \sigma_j) \Lambda_j + \sigma_j \mu_{1j}^{1 - \sigma_j} \right].
\]

Making use of (9.52) and (9.76), we can yield:

\[
C_{gg'}^o = \left( \frac{Q_{j}^o}{\gamma_j} \right)^2 \Gamma_1 \left( \mu_{1j}^{2\sigma_{1j} - \sigma_j - 1} \right) \Lambda_j^{2\sigma_j} \Gamma_2.
\]  

(9.77)

From (9.73) and (9.75), we can derive:

\[
C_{gg'}^o = \left( \frac{Q_{j}^o}{\gamma_j} \right)^2 \Gamma_1 \left( \mu_{1j}^{2(\sigma_{1j} - \sigma_j)} \right) \Lambda_j^{2\sigma_j}.
\]  

(9.78)

Dividing (9.77) by (9.78) leads to:

\[
\sigma_{g'j}^j = \frac{\sigma_j^{-1}}{\mu_{1j}^{\sigma_{1j} - \sigma_j}} \left[ (\sigma_{1j} - \sigma_j) \Lambda_j + \sigma_j \mu_{1j}^{1 - \sigma_j} \right].
\]  

(9.79)

We can derive the following expression (9.80) from (9.55):

\[
\mu_{1j}^{1 - \sigma_j} = \Lambda_j^{1 - \sigma_j} S_{ij}.
\]  

(9.80)
Finally, we can obtain the elasticity of factor substitution between two different intermediate factors by putting (9.80) into (9.79) as below:

\[ \sigma_{gg'}^j = \left[ \sigma_j + (\sigma_{1j} - \sigma_j)(S_{ij})^{-1} \right], \tag{9.81} \]

where \( g, g' \in [1, G] \) and \( g \neq g' \); \( g \) and \( g' \) are integers.

\( \sigma_j \) denotes the elasticities of substitution among the different categories (groups) of the factors of production utilized by each producing industry \( j \),

\( \sigma_{1j} \) stands for the elasticities of factor substitution within the intermediate-factor category (group) which is composed of the elementary intermediate factors utilized by each producing industry \( j \),

The expression (9.81) implies the elasticity of factor substitution between different intermediate inputs, \( Q_{gj}^0 \) and \( Q_{g'j}^0 \), within the category of the intermediate factors of production.

Similarly, the elasticity of factor substitution between different primary factors, \( \hat{X}_{m,j}^0 \) and \( \hat{X}_{m',j}^0 \), within the group of the primary factors of production is:

\[ \sigma_{mm'}^j = \left[ \sigma_j + (\sigma_{2j} - \sigma_j)(\hat{S}_{pj})^{-1} \right], \tag{9.82} \]

where \( m, m' \in [1, M] \) and \( m \neq m' \); \( m \) and \( m' \) are integers.

\( \sigma_{2j} \) denotes the elasticities of factor substitution within the primary-factor category (group) which is composed of the elementary primary factors utilized by each producing industry \( j \),

The elasticity of factor substitution between different foreign factors imported, \( X_{zj}^* \) and \( X_{z'j}^* \), can be derived from the following formula:

\[ \sigma_{zz'}^j = \left[ \sigma_j + (\sigma_{3j} - \sigma_j)(S_{fj}^*)^{-1} \right], \tag{9.83} \]
where $z, z' \in [1, Z]$ and $z \neq z'$; $z$ and $z'$ are integers.

$\sigma_{3j} = \text{the elasticities of factor substitution within the imported foreign factor category (group) which is composed of the elementary imported factors utilized by each producing industry } j$.

In sum, substitution between two factors in the same category depends upon the substitution effect within the category and with other categories (see (9.81)-(9.83)).

(II) The elasticity of factor substitution between different inputs in different categories, i.e., an intermediate factor and a primary factor, $Q_{gj}^o$ and $\hat{X}_{mj}^o$ can be derived as follows.

By Shephard (-Mckenzie) lemma holding for the minimum cost function (9.52), we get:

$$C_m^o = \hat{X}_{m,j}^o,$$  \hspace{1cm} (9.84)

$$C_m^o = \Gamma_3 Q_{gj}^o \left( \frac{\hat{\beta}_{mj}}{W_{mj}} \right)^{\sigma_{2j}} \mu_j^{2\sigma_j - \sigma_j} \Lambda_j^\sigma_j,$$ \hspace{1cm} (9.85)

where $C_m^o = \frac{\partial C_j^o}{\partial W_{mj}}$, $m \in [1, M]$; $m$ is an integer, $\Gamma_3 = \left( \frac{Q_{gj}^o}{\gamma_j} \right)$.

From the expression (9.73), we can derive the following outcome (under the same method utilized in (I)).

$$C_{gm}^o = \Gamma_4 \mu_j^{2\sigma_j - \sigma_j} \sigma_{1j}^{\sigma_j - 1},$$ \hspace{1cm} (9.86)

where $C_{gm}^o = \frac{\partial C_j^o}{\partial W_{mj} \partial W_{gj}}$, $g \in [1, G]$ and $m \in [1, M]$; $g$ and $m$ are integers.

$$\Gamma_4 = \sigma_{1j} \left( \frac{Q_{gj}^o}{\gamma_j} \right) \left( \frac{\hat{\beta}_{gj}}{W_{gj}} \right)^{\sigma_{1j}} \left( \frac{\hat{\beta}_{mj}}{W_{mj}} \right)^{\sigma_{2j}}.$$
Utilizing (9.52) and (9.86), we can obtain:

\[ C_{gm}^o = \sigma_j \Gamma_5 \mu_{1j}^{\sigma_j - \sigma_j} \mu_{2j}^{\sigma_j - 2\sigma_j} \Lambda_j^{2\sigma_j}. \]  

(9.87)

where \( \Gamma_5 = \left( \frac{Q_j^o}{V_j} \right)^2 \left( \frac{\beta_{gj}}{W_{gj}} \right)^{\sigma_{1j}} \left( \frac{\beta_{mj}}{W_{mj}} \right)^{\sigma_{2j}}. \)

From (9.73) and (9.85), we yield

\[ C_{gm}^o = \Gamma_5 \mu_{1j}^{\sigma_j} \mu_{2j}^{\sigma_j} \Lambda_j^{2\sigma_j}. \]  

(9.88)

Dividing (9.87) by (9.88), we obtain

\[ \sigma_{gm}^j = \sigma_j = \sigma_{mg}^j \]  

(9.89)

by the symmetry condition, where \( g \in [1, G] \) and \( m \in [1, M] \); \( g, m \) are integers.

This expression (9.89) means that the elasticity of factor substitution between an intermediate input \( Q_{gj}^o \) and a primary factor \( X_{mj}^o \) is constant.

Similarly, the elasticities of factor substitution between a primary input \( X_{mj}^o \) and an imported foreign factor \( X_{zj}^{*o} \), and between an intermediate factor \( Q_{gj}^o \) and a foreign factor imported \( X_{zj}^{*o} \) are as follows:

\[ \sigma_{mj}^j = \sigma_j = \sigma_{zm}^j, \]  

(9.90)

\[ \sigma_{gj}^j = \sigma_j = \sigma_{zg}^j. \]  

(9.91)

In general,

\[ \sigma_{vv'}^j = \sigma_j \text{ if } v \in i, v' \in p; \ v \neq v'. \]  

(9.92)

This formulation implies that the elasticity of factor substitution between two different inputs, each factor belonging to the different categories, are constant and equal to the elasticity of substitution between the different factor categories (groups).
(III) The elasticity of factor substitution of an intermediate factor \( g \) in response to its own price can be derived from the formulation (5.13) in the following way.

By making use of (9.48), (9.62), and (9.73), we get:

\[
C_{gg}^\circ = \left( \frac{Q_j^c}{y_j} \right)^2 \left( \frac{\beta_{gj}}{W_{gj}} \right)^2 \frac{2\sigma_{1j} - \sigma_j - 1}{\mu_{1j}^\circ} 2\sigma_j - 1 \Lambda_j \Theta_2, \tag{9.93}
\]

where

\[
\Theta_2 = -\sigma_{1j} \left( \Omega_{gj} \right)^{-1} \mu_{1j}^\circ \Lambda_j + (\sigma_{1j} - \sigma_j) \Lambda_j + \sigma_j \mu_{1j}^\circ.
\]

Making use of (9.52) and (9.93), we get:

\[
C_{gg}^\circ C_{gg}^\circ = \left( \frac{Q_j^c}{y_j} \right)^2 \left( \frac{\beta_{gj}}{W_{gj}} \right)^2 \frac{2\sigma_{1j} - \sigma_j - 1}{\mu_{1j}^\circ} 2\sigma_j \Theta_2. \tag{9.94}
\]

Utilizing (9.73), we obtain:

\[
C_{gg}^\circ C_{gg}^\circ = \left( \frac{Q_j^c}{y_j} \right)^2 \Lambda_j^\circ \left( \frac{\beta_{gj}}{W_{gj}} \right)^2 \mu_{1j}^\circ 2(\sigma_{1j} - \sigma_j). \tag{9.95}
\]

Dividing (9.94) by (9.95), we get:

\[
\sigma_{gg}^j = \sigma_j + (\sigma_{1j} - \sigma_j) \left[ \frac{\sigma_{1j} - 1}{\mu_{1j}^\circ} \Lambda_j \right] - \sigma_{1j} \left[ (\Omega_{gj})^{-1} \mu_{1j}^\circ (\sigma_{1j} - \sigma_j) \Lambda_j \right]. \tag{9.96}
\]

From (9.55),

\[
S_{ij}^{-1} = \mu_{1j}^\circ \Lambda_j^\circ 1 - \sigma_j. \tag{9.97}
\]

From (9.69),

\[
S_{jg}^{-1} = \Lambda_j (\Omega_{gj})^{-1} \mu_{1j}^\circ (\sigma_{1j} - \sigma_j). \tag{9.98}
\]
By substituting (9.97) and (9.98) into (9.96), we obtain the below result:

$$\sigma_{gg} = \sigma_j + (\sigma_{1j} - \sigma_j)(S_{ij})^{-1} - \sigma_{1j}(S_{gj})^{-1}. \quad (9.99)$$

Equivalently,

$$\sigma_{gg}' = \sigma_{gg}' - \sigma_{1j}(S_{gj})^{-1}. \quad (9.100)$$

Similarly, for the elasticity of substitution of a primary factor, we can yield:

$$\sigma_{mm} = \sigma_j + (\sigma_{2j} - \sigma_j)(S_{pj})^{-1} - \sigma_{2j}(S_{mj})^{-1}. \quad (9.101)$$

Equivalently,

$$\sigma_{mm}' = \sigma_{mm}' - \sigma_{2j}(S_{mj})^{-1}. \quad (9.102)$$

For the elasticity of substitution of an imported foreign factor, we can also obtain:

$$\sigma_{zz} = \sigma_j + (\sigma_{3j} - \sigma_j)(S_{fj})^{-1} - (\sigma_{3j})(S_{zj})^{-1}. \quad (9.103)$$

Equivalently,

$$\sigma_{zz}' = \sigma_{zz}' - (\sigma_{3j})(S_{zj})^{-1}. \quad (9.104)$$
10. THE GENERAL SYSTEM IN THE INPUT-OUTPUT THEORY

10.1 The General Case

In relation to the input-output theory, the separable two-stage level CES-type production function exhibiting the linear homogeneity (constant returns to scale) can generally be rewritten as follows:

\[ Q_j^o = \gamma_j \left[ \sum_{i=0}^{N_j} \left( \sum_{k=1}^{K_{ij}} \beta_{ikj} X_{ikj}^{-\rho_j} \right) \frac{\rho_j}{\rho_{ij}} \right]^{-\frac{1}{\rho_j}}, \quad (10.1) \]

where \( X_{ikj} \) is the quantity of the production factor \( k \) belonging to block (factor group) \( i \), which is utilized in producing a commodity \( Q_j^o; j=1,2,...,G, i=0,1,...,N_j, \) and \( k=1,2,...,K_{ij}. \) \( Q_j^o \) is the gross output of a commodity \( j. \) We assume that all intermediate-good factors show up in just one block, i.e., block 0; that is, \( X_{0kj} = Q_k^o, \) which represents flow of an intermediate factor from an origin industry \( k \) to a producing industry \( j; k,j = 1,2,...,G. \)

We also need the following constraint associated with the framework of the input-output theory.

\[ F_k = Q_k^o - \sum_{j=1}^{G} Q_{kj}^o, \quad (10.2) \]

where \( k = 1,2,...,G. \)
Equivalently,

\[ Q_k^o = \sum_{j=1}^{G} Q_{kj}^o + F_k, \]  

(10.3)

where \( Q_k^o \) is the gross equilibrium output of each producing industry \( k \). \( Q_{kj}^o \) is the optimal quantity of the intermediate factor (output) \( k \) utilized in producing the \( j \)-th industry's output \( Q_j^o \). So, the absolute value of the second term of the right-hand side of the expression (10.2) (or the first term of the right-hand side of the expression (10.3)) represents the total intermediate demand in \( G \) producing industries for the \( k \)-th origin-industry's output (commodity) which is utilized in the production of other commodities. \( F_k \) is the final demand for a commodity \( k; k = 1, 2, \ldots, G \). This final demand represents the amount of the output of the \( k \)-th industry which is not consumed in the process of production itself.

The constraint (10.2) or (10.3) states that in equilibrium, the output (supply) of each commodity is equal to the demand for that commodity. This is the so-called material balance equation.

For deriving the optimal factor demand functions and the prices of the intermediate factors, we minimize the total cost of the non-intermediate factors subject to the final demand constraint (10.2) or (10.3) as follows: Let

\[ C = \sum_{j=1}^{G} C_j, \]  

(10.4)

where \( C_j = \sum_{i=1}^{N_j} \sum_{k=1}^{K_{ij}} W_{ikj} X_{ikj} \). \( C_j \) is the total cost of the non-intermediate factors utilized in producing a commodity \( Q_j^o \) and \( W_{ikj} \) is the price of each of the production factors (such as the primary factors and the imported foreign factors).
Note that the total cost $C$ does not include the intermediate factor cost, i.e., $i \neq 0$. In effect, this says that the final demand (bill) of commodities $(F_1, F_2, ..., F_G)$ is produced at the minimal cost.

The Lagrangean function then is as follows:

$$\text{Min. } \Phi = C + \sum_{j=1}^{G} \left\{ P_j \left[ F_j + \sum_{k=1}^{G} Q_{jk}^{\circ} - \gamma_j \left( \sum_{i=0}^{N_j} V_{ij} - \rho_j \right) \right] \right\},$$

where $V_{ij} = \left( \sum_{k=1}^{K_{ij}} \beta_{ikj} X_{ikj}^{-\rho_{ij}} \right)^{-\frac{1}{\rho_{ij}}}$. $P_j$ $(j = 1, 2, ..., G)$ are the Lagrange multipliers of the final demand constraints and are the shadow-prices of the final demands $F_j$ $(j = 1, 2, ..., G)$. Taking the partials of the Lagrangean function (10.5) with respect to $X_{ikj}$ and a Lagrange multiplier $P_j$ and setting them equal to zero yields the following first-order conditions for constrained minimization:

$$\text{[FOC]:} \quad W_{ikj} = P_j(\gamma_j - \rho_j) \beta_{ikj} \left[ \frac{X_{ikj}}{V_{ij}} \right]^{-(1+\rho_{ij})} \left[ \frac{Q_{ij}^{\circ}}{V_{ij}} \right]^{1+\rho_j},$$

where $j = 1, 2, ..., G$ and

$$V_{ij} = \left[ \sum_{k=1}^{K_{ij}} \beta_{ikj} X_{ikj}^{-\rho_{ij}} \right]^{-\frac{1}{\rho_{ij}}}.$$  \hspace{1cm} (10.7)

$$F_j + \sum_{k=1}^{G} Q_{jk}^{\circ} = \gamma_j \left[ \sum_{i=0}^{N_j} \left( \sum_{k=1}^{K_{ij}} \beta_{ikj} X_{ikj}^{-\rho_{ij}} \right) \frac{\rho_j}{\rho_{ij}} \right]^{-\frac{1}{\rho_j}}.$$  \hspace{1cm} (10.8)

Note that these equilibrium conditions hold for all $(i, k, j)$ with $i > 0$, and that they also hold for $i = 0$ if we define $W_{0kj} = P_k$ ($P_j$ when $k = j$).
Equivalently, since $\sigma_j = \frac{1}{1 + \rho_j}$, i.e., $\rho_{ij} = \frac{1 - \sigma_{ij}}{\sigma_{ij}}$ and $\rho_j = \frac{1 - \sigma_j}{\sigma_j}$ from the context of the elasticity of substitution, the expressions (10.6)-(10.7) can be rewritten respectively as follows:

$$W_{ikj} = P_j \left( \gamma_j \right)^{\frac{\sigma_{ij} - 1}{\sigma_j}} \beta_{ikj} \left[ \frac{X_{ikj}}{V_{ij}} \right]^{\frac{1}{\sigma_{ij}}} \left[ \frac{Q^i_{ij}}{V_{ij}} \right]^{\frac{1}{\sigma_j}}, \tag{10.9}$$

where $j = 1, 2, \ldots, G$ and

$$V_{ij} = \left[ \sum_{k=1}^{K_{ij}} \beta_{ikj} X_{ikj} \right]^{\frac{\sigma_{ij}}{\sigma_{ij} - 1}}. \tag{10.10}$$

10.1.1 The Derivation of Intermediate Factor Prices

We here will derive the simultaneous equation system for getting the prices of the intermediate factors which are the endogenous variables, from utilizing the equilibrium conditions (FOCs) (10.8) and (10.9).

From the equilibrium condition (10.9), we get:

$$\left( \frac{X_{ikj}}{V_{ij}} \right)^{\sigma_{ij}} = P_j \left( \gamma_j \right)^{\frac{\sigma_{ij}}{\sigma_j}} \beta_{ikj} \sigma_{ij} \left( \frac{Q^i_{ij}}{V_{ij}} \right)^{\frac{\sigma_{ij}}{\sigma_j}}. \tag{10.11}$$

We, on the other hand, derive the expression (10.12) from dividing both sides of the expression (10.10) by $V_{ij}$ as follows:

$$1 = \sum_{k=1}^{K_{ij}} \beta_{ikj} \left( \frac{X_{ikj}}{V_{ij}} \right)^{\frac{\sigma_{ij} - 1}{\sigma_{ij}}}. \tag{10.12}$$

Substituting the expression (10.11) into the expression (10.12) and arranging it lead to the following result:

$$V_{ij} = Q^i_{ij} \gamma_j^{\sigma_j} \left( \frac{P_j}{\mu_{ij}} \right)^{\sigma_j}, \tag{10.13}$$
where \( i = 0, 1, \ldots, N_j \); \( j = 1, 2, \ldots, G \) and
\[
\mu_{ij} = \left[ \sum_{k=1}^{K_{ij}} (\beta_{ikj})^\sigma_{ij} w_{ikj} \right]^{1-\sigma_{ij}}.
\] (10.14)

Substituting the expressions (10.3) and (10.7) into (10.8) and arranging it, we get:
\[
\left( \frac{\gamma_j}{Q_j^o} \right)^{\rho_j} = \left[ \sum_{i=0}^{N_j} V_{ij}^{-\rho_j} \right].
\] (10.15)

Multiplying both sides of (10.15) by \( (Q_j^o)^{\rho_j} \), we obtain:
\[
\left( \frac{\gamma_j}{Q_j^o} \right)^{1-\sigma_j} = \left[ \sum_{i=0}^{N_j} \left( V_{ij} \right)^{\sigma_j-1} \right],
\] (10.16)
where \( \sigma_j = \frac{1}{1+\rho_j} \) or \( \rho_j = \frac{1-\sigma_j}{\sigma_j} \).

By substituting (10.13) into (10.16), we can finally derive the simultaneous equation system for the prices of the intermediate factors as follows:
\[
\left( \gamma_j P_j \right)^{1-\sigma_j} = \left[ \sum_{i=0}^{N_j} \mu_{ij}^{1-\sigma_j} \right],
\] (10.17)
where \( j = 1, 2, \ldots, G \).

Equivalently,
\[
\left( \gamma_j P_j \right)^{1-\sigma_j} = \left[ \sum_{k=1}^{G} (\beta_{0kj})^{\sigma_0j} p_k^{1-\sigma_0j} \right]^{1-\sigma_0j} + \sum_{i=1}^{N_j} \mu_{ij}^{1-\sigma_j},
\] (10.18)
since \( \mu_{0j} = \left[ \sum_{k=1}^{G} (\beta_{0kj})^{\sigma_0j} p_k^{1-\sigma_0j} \right]^{1-\sigma_0j} \) from (10.14), where \( j = 1, 2, \ldots, G \).

From the simultaneous equation system (10.17) or (10.18), we can get the prices of the intermediate factors which are the endogenous variables and depend on the
technology indicator and the prices of the non-intermediate production factors (such as the primary factors and the imported foreign factors). Here we should note the following: First, the prices $P_j$ of the intermediate factors do not depend on the final demands $E_k$ $(k = 1, 2, \ldots, G)$. This is a form of non-substitution theorem. Second, if $\sigma_0j = \sigma_j = \sigma$ for all $j$ $(j = 1, 2, \ldots, G)$, each of the price equations in the simultaneous equation system is linear in $p^{1-\sigma}$.

10.1.2 The Optimal Factor Demand Functions

Now, we can derive the optimal demand functions of the production factors from the first-order conditions of the Lagrangean function (10.5) as follows:

$$X^*_i k j = Q^*_i \sigma^*_i j^{-1} P_j \sigma^*_j \left( \frac{\beta_{i k j}}{W_{i k j}} \right)^{\sigma^*_i j - \sigma^*_j} \mu_{i j},$$

(10.19)

where $\mu_{i j} = \left[ \sum_{k=1}^{K_{i j}} \sigma^*_{i j} W_{i k j}^{1-\sigma^*_{i j}} \right]^{1-\sigma^*_{i j}}$, and $j = 1, 2, \ldots, G; i = 0, 1, \ldots, N_j$; and $k = 1, 2, \ldots, K_{i j}$. Specifically, substituting (10.13)-(10.14) into (10.9) and arranging it lead to (10.19). The expression (10.19) states the optimal demand function of each of the production factors utilized by each producing industry $j$ in an economy.

10.1.3 The flexible Technological Coefficients

In preceding sections, we have discussed the derivation of the optimal factor demand functions and of the prices of the intermediate factors which are endogenous and depend on the exogenous economic variables such as the technology indicator and the prices of the non-intermediate factors. Now, since the input-output coefficients are very crucial elements within the context of input-output analysis, we here define the technological input-output ratios for the production factors utilized by each of
the producing industries as:

\[ \phi_{ikj} = \frac{X_{ikj}^0}{Q_j^0}, \]  

(10.20)

where \( \phi_{ikj} \geq 0; i = 0, 1, ..., N_j; j = 1, 2, ..., G; k = 1, 2, ..., K_{ij} \). We can specifically calculate the values of the technological input-output coefficients for the production factors from (10.19) (considering (10.20)) under the assumptions that the prices of the intermediate factors and the output level are given.

The expression (10.20) represents the ratio of the physical quantity of each of the production factors to the physical quantity of output in the production process of each producing industry \( j \). In particular, for the intermediate factors (i.e., \( i = 0 \)) we can obtain:

\[ \phi_{kj} = \frac{X_{0kj}^0}{Q_j^0} = \frac{Q_{kj}^0}{Q_j^0} 
\]

\[ = \left( \frac{P_j^\sigma_j}{P_k^\sigma_j} \right) \gamma_j^{-1} (\beta_{0kj})^{\sigma_0j} (\mu_{0j})^{\sigma_0j-\sigma_j} \]  

(10.21)

so that

\[ Q_k^0 = \sum_{j=1}^{G} \phi_{kj}Q_j^0 + F_k, \]  

(10.22)

where \( k = 1, 2, ..., G \).

The expression (10.21) shows the following: First, all the technological input-output coefficients depend on the exogenous economic variables such as the technology indicator and the prices (relative prices) of the non-intermediate factors (we know that the prices \( P_j \) of the intermediate factors (or outputs) derived from (10.18) depend on the exogenous economic variables such as the technological indicator and
the prices of the non-intermediate factors) and thus are flexible according to the changes in the technological indicator and/or the prices of the non-intermediate factors. Second, the technological coefficients do not depend on the final demands $F_k$.

## 10.2 Analysis of a Special Case

In preceding Section 10.1 we have discussed the general price system, the optimal factor demand functions, and the flexible technological coefficients on the basis of the general two-stage level CES-type production function. Here we derive the system for the intermediate factors (commodities) and the general open input-output system under the assumption that $\sigma_0j = \sigma_j = \sigma$ for all $j$. If we assume that $\sigma_0j = \sigma_j = \sigma$ for all $j$ ($j = 1, 2, ..., G$) (here $\sigma_{ij}$ may have any values for $i > 0$), then the simultaneous equation system (10.18) is transformed into:

$$
\left( \gamma_j P_j \right)^{1-\sigma} = \left[ \sum_{k=1}^{G} (\beta_0k_j)^{\sigma} P_k^{1-\sigma} \right] + \sum_{i=1}^{N_j} \mu_i^{1-\sigma},
$$

(10.23)

where $j = 1, 2, ..., G$.

Equivalently,

$$
P_j^{1-\sigma} = \gamma_j^{\sigma-1} \left[ \sum_{k=1}^{G} (\beta_0k_j)^{\sigma} P_k^{1-\sigma} \right] + \gamma_j^{\sigma-1} \left[ \sum_{i=1}^{N_j} \mu_i^{1-\sigma} \right],
$$

(10.24)

where $j = 1, 2, ..., G$.

In matrix notation, the expression (10.24) can be written as follows:

$$
P^{1-\sigma} = \left[ I - \Psi \right]^{-1} R,
$$

(10.25)

where $P^{1-\sigma}$ is a $(G \times 1)$ column vector; $\left[ P_1^{1-\sigma}, P_2^{1-\sigma},..., P_G^{1-\sigma} \right]^T$. 

I is a \((G \times G)\) identity matrix.

\[ \Psi = \begin{bmatrix} \alpha_{kj} \end{bmatrix} \] is a \((G \times G)\) square matrix, where \(\alpha_{kj} = (\beta_{0kj})^{\sigma - 1} (0 < \alpha_{kj} < 1)\) \((k, j = 1, 2, ..., G)\).

\[ R = \begin{bmatrix} R_1, R_2, ..., R_G \end{bmatrix}^T, \] where \(R_j = \sum_{i=1}^{N_j} \mu_{ij}^{1-\sigma} \gamma_j^{\sigma - 1} (j = 1, 2, ..., G)\).

The prices \(P_j\) derived from (10.25) depend on the exogenous economic variables such as the technological indicator \(\gamma_j\) and the prices of the non-intermediate factors, i.e., \(W_{ikj} (i > 0)\).

Under the assumption that \(\sigma_0 = \sigma_j = \sigma\) for all \(j (j = 1, 2, ..., G)\), the technological coefficients (10.21) are changed as below:

\[
\phi_{kj} = \frac{X_{0kj}}{Q_j^0} = \frac{Q_{kj}^0}{Q_j^0} = \left( \frac{P_j}{P_k} \right)^{\sigma \gamma_j^{\sigma - 1} (\beta_{0kj})^{\sigma}} = \left( \frac{P_j}{P_k} \right)^{\sigma} \alpha_{kj}, \quad (10.26)
\]

where \(k, j = 1, 2, ..., G\).

The expression (10.26) shows that the technological coefficients are flexible in accordance with the changes in the technological indicator and the relative prices. However, the technological input-output coefficients (10.26) are completely determined by the intermediate-factor prices \(P_j\) or \(P_k\) \((k, j = 1, 2, ..., G)\) derived from (10.25).

Under the results mentioned above, we can set the system for our economic analysis in the following way. We know that the fundamental open input-output system (10.22) is:

\[ Q^0 = \left[ I - \Psi \right]^{-1} F, \quad (10.27) \]
where $Q^o$ is a $(G \times 1)$ column vector; $Q^o = \left[ Q^o_1, Q^o_2, \ldots, Q^o_G \right]^T$.

$F$ is a $(G \times 1)$ column vector; $F = \left[ F_1, F_2, \ldots, F_G \right]^T$.

$\Psi$ represents $(G \times G)$ square technological matrix.

$I$ is a $(G \times G)$ identity matrix.

Here, let $\Psi = \left[ \phi_{kj} \right]$ : a $(G \times G)$ square matrix and $A = \left[ \alpha_{kj} \right]$ : a $(G \times G)$ square matrix; that is,

$$
\Psi = \begin{bmatrix}
\phi_{11} & \phi_{12} & \cdots & \phi_{1G} \\
\phi_{21} & \phi_{22} & \cdots & \phi_{2G} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{G1} & \phi_{G2} & \cdots & \phi_{GG}
\end{bmatrix}
$$

and

$$
A = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1G} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2G} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{G1} & \alpha_{G2} & \cdots & \alpha_{GG}
\end{bmatrix}
$$

Then, since $\phi_{kj} = \left( \frac{P_j}{P_k} \right)^{\sigma} \alpha_{kj}$ by (10.26), we can get:

$$
\Psi = P^{-\sigma}AP^\sigma,
$$

where $P^{-\sigma}$ is a $(G \times G)$ diagonal matrix with $P^{-\sigma}_j$ ($j = 1, 2, \ldots, G$) on its main diagonal and $P^\sigma$ is a $(G \times G)$ diagonal matrix with $P^\sigma_j$ ($j = 1, 2, \ldots, G$) on its main diagonal. Hence, the Leontief inverse becomes:

$$
\left[ I - \Psi \right]^{-1} = P^{-\sigma} \left( I - A \right)^{-1} P^\sigma.
$$

As a result, the fundamental open input-output system (10.27) can be rewritten as:

$$
Q^o = \left[ P^{-\sigma} \left( I - A \right)^{-1} P^\sigma \right] F.
$$
The simultaneous equation system (10.30) can also be written specifically as follows:

\[
\begin{bmatrix}
P_1^\sigma Q_1^o \\
P_2^\sigma Q_2^o \\
\vdots \\
P_G^\sigma Q_G^o
\end{bmatrix} = \begin{bmatrix}
I - A
\end{bmatrix}^{-1}
\begin{bmatrix}
P_1^\sigma F_1 \\
P_2^\sigma F_2 \\
\vdots \\
P_G^\sigma F_G
\end{bmatrix}.
\]

(10.31)

The system (10.30) or (10.31) is the general input-output system reflecting the technological improvements, the factor substitutabilities, and the relative prices to the demands for the intermediate factors (or outputs).

From the general open input-output system (10.30) or (10.31), we can get the optimal demand for each of the outputs. As a result, we can derive the total optimal demand for each of the intermediate factors \(Q_k^o\) or \(Q_j^o\), which depend on the exogenous economic variables such as the technological indicators and the prices of the non-intermediate factors such as the primary factors and the imported foreign factors. Using the equilibrium outputs derived from (10.30) or (10.31), the optimal demand functions of the non-intermediate factors such as the primary factors and the imported foreign factors can be obtained from (10.19). Such optimal demand functions show the allocation of domestic resources in an economy.

10.3 The Special Cases

10.3.1 The Cobb-Douglas Case

In Section 10.1, we have discussed the general case through the two-stage level CES-type production function (10.1). And, in Section 10.2, under the assumption that \(\sigma_0 j = \sigma_j = \sigma\) for all \(j\), we have derived (i) the price system for the intermediate
factors (outputs), (ii) the optimal factor demand functions, (iii) flexible input-output coefficients and (iv) finally developed the general open input-output system.

Here we more specifically examine the special case, i.e., the Cobb-Douglas case where $\rho_0j=\rho_j=\rho=0$ (or $\sigma_0j=\sigma_j=\sigma=1$). As shown in Chapter 8, the original form of the two-stage level CES-type production function (10.1) (see (8.47)) actually is of the type:

$$Q_j^o = \gamma_j \left[ \sum_{i=0}^{N_j} h_{ij} V_{ij}^{-\rho_j} \right]^{-\frac{1}{\rho_j}}, \quad (10.32)$$

where

$$V_{ij} = \left[ \sum_{k=1}^{K_{ij}} b_{ikj} X_{ikj}^{\beta_{ikj}} \right]^{-\frac{1}{\rho_{ij}}}.$$

$b_{ikj}$ represents the distribution (contribution) parameters of the elementary intermediate factors $k$ in the factor group $i$ utilized in producing output $Q_j^o$.

$h_{ij}$ denotes the distribution (contribution) parameter of each of the quantum indexes for the production factors used in yielding output $Q_j^o$.

Utilizing L'Hôpital theorem (see Theorem 8.1) as done in Section 8.2 in Chapter 8 under the assumption (condition) that $\rho_j=\rho (= 0)$, we can finally derive the Cobb-Douglas production function from (10.32) as follows:

$$Q_j^o = \gamma_j \left[ \left( \prod_{k=1}^{G} X_{0kj}^{\beta_{0kj}} \right) \left( F_j(X_{ikj}) \right)^{\theta_j} \right]. \quad (10.33)$$

Or equivalently,

$$Q_j^o = \gamma_j \left[ \left( \prod_{k=1}^{G} Q_{kj}^{o} \beta_{0kj} \right) \left( F_j(X_{ikj}) \right)^{\theta_j} \right], \quad (10.34)$$

where we write $X_{0kj}=Q_{kj}^{o}$ (see Section 10.1) and $F_j(X_{ikj})$ is any arbitrary two-stage CES-type functional form with constant returns to scale, not involving intermediate goods ($i \geq 0$); in other words, $F_j(X_{ikj})$ arises from any two-stage CES production.
function of the linear homogeneity in the primary and the imported foreign factors. Note that we get the following restriction from the assumption of constant returns to scale:

\[ \sum_{k=1}^{G} \beta_{0kj} + \theta_j = 1. \] (10.35)

The expression (10.33) or (10.34) represents the (two-stage) Cobb-Douglas function derived from the general two-stage level CES-type production function (8.47) or (10.1) under the assumption that \( \rho_{0j} = \rho_j = \rho = 0 \).

Now, let \( Z_j = F_j(X_{i,kj}) \) in (10.33) or (10.34) and \( W_j^\gamma \) is the unit cost of producing \( Z_j \) as determined by the analysis (i.e., two-stage optimization) of Chapter 9. Then, we minimize the total cost \( \left( \sum_{j=1}^{G} W_j^\gamma Z_j \right) \) of the non-intermediate factors subject to the two-stage Cobb-Douglas function (10.33) or (10.34) and the material balance relation (equation) (10.3). The Lagrangean function for the cost-minimization is as follows:

\[
\text{Min. } \Phi = \sum_{j=1}^{G} W_j^\gamma Z_j + \sum_{j=1}^{G} \left[ P_j \left( F_j + \sum_{k=1}^{G} Q_{jk} - Q_{j}^o \right) \right],
\] (10.36)

where \( Q_{j}^o \) is (10.33) or (10.34). \( P_j \) \((j = 1, 2, ..., G)\) are the Lagrange multipliers of the final demand constraints and are the shadow-prices of the final demands \( F_j \) \((j = 1, 2, ..., G)\). Taking the partials of the Lagrangean function (10.36) with respect to \( Z_j \) and \( Q_{jk}^o \) \((= X_{0,kj})\) and a Lagrange multiplier \( P_j \) and setting them equal to zero yields the following first-order conditions for constrained minimization:

[FOC]:

\[
W_j^\gamma = \left[ \frac{\theta_j P_j Q_{jk}^o}{Z_j} \right],
\] (10.37)
where \( j = 1, 2, \ldots, G \).

\[
P_k = \left( \frac{\beta_{0k_j}}{Q^o_{kj}} \right) \left( P_j Q^o_j \right).
\]

\[
Q^o_j = \gamma_j \left[ \left( \prod_{k=1}^{G} Q^o_{kj}^{\beta_{0k_j}} \right) Z_j^{\theta_j} \right].
\]

### 10.3.2 The Price System for the Intermediate Factor

We here will derive the simultaneous equation system for getting the endogenous prices of the intermediate factors from utilizing the equilibrium conditions (FOCs) (10.37)-(10.39).

From the expressions (10.37) and (10.38), we can obtain:

\[
\frac{\partial}{\partial P_j} \left( P_j Q^o_j \right) = \frac{\beta_{0k_j}}{P_k} \left( P_j Q^o_j \right).
\]

Substituting (10.40) and (10.41) into (10.39) and arranging it, we can get:

\[
Q_j^o = \gamma_j (P_j Q^o_j)^{\delta} \left[ \prod_{k=1}^{G} \left( \frac{\beta_{0k_j}}{P_k} \right)^{\beta_{0k_j}} \right] \left( \frac{\theta_j}{W_j^e} \right),
\]

where \( \delta = \sum_{k=1}^{G} \beta_{0k_j} + \theta_j \).

Since \( \delta = 1 \) by the restriction (10.35), the expression (10.42) becomes:

\[
1 = \gamma_j P_j \left[ \prod_{k=1}^{G} \left( \frac{\beta_{0k_j}}{P_k} \right)^{\beta_{0k_j}} \right] \left( \frac{\theta_j}{W_j^e} \right).
\]
Taking the natural logarithm of (10.43) and re-arranging, we can obtain:

$$\ln P_j - \sum_{k=1}^{G} \beta_{0kj} (\ln P_k) = \hat{R}_j,$$

(10.44)

where \( \hat{R}_j = \theta_j \ln (\frac{W_j^2}{\frac{\partial^2}{\partial j^2}}) - \ln \gamma_j - \sum_{k=1}^{G} \beta_{0kj} (\ln \beta_{0kj}) \) (\( j = 1, 2, ..., G \)). Note that \( \ln \beta_{0kj} < 0 \) because \( 0 < \beta_{0kj} < 1 \).

In matrix notation, the expression (10.44) can be written as follows:

$$\hat{P} \left[ I - B \right] = \hat{R},$$

(10.45)

where \( \hat{P} = \left[ \ln P_1, \ln P_2, ..., \ln P_G \right] \). \( I \) is a \((G \times G)\) identity (unit) matrix.

\( B = \left[ \beta_{0kj} \right]^T \) is a \((G \times G)\) square matrix, where \( 0 < \beta_{0kj} < 1 \) (\( k, j = 1, 2, ..., G \)).

\( \hat{R} \) is a \((G \times 1)\) row vector; \( \left[ \hat{R}_1, \hat{R}_2, ..., \hat{R}_G \right] \).

In the expression (10.45), all column sums of a square matrix \( B \) are less than one since \( \sum_{k=1}^{G} \beta_{0kj} = 1 - \theta_j < 1 \) (see (10.35)). As a result, Hawkins-Simon condition is always satisfied; \([ I - B ]^{-1} \geq 0 \). Finally, we can obtain the endogenous prices \( P_j \) of the intermediate factors (outputs) by solving the simultaneous equation system (10.45). The expression (10.45) can be rewritten as:

$$\hat{P} = \hat{R} \left[ I - B \right]^{-1}.$$

(10.46)

Note that \( P_j \) is derived from the anti-log of \( \ln P_j \) which is a solution of (10.46).

### 10.3.3 The Optimal Factor Demand Functions

In preceding section we have derived the price system for the intermediate factors (commodities) (see (10.46)). Given the endogenous prices \( P_j \) derived from (10.46)
and the output level (we shall discuss later), we can derive the optimal demand functions of the production factors from (10.40)-(10.41) as follows:

\[ Z_j = \left( \frac{\theta_j}{W_j} \right) \left( P_j Q_j^\circ \right). \tag{10.47} \]

\[ Q_{kj}^\circ = X_{0kj} = \left( \frac{\beta_{0kj}}{P_k} \right) \left( P_j Q_j^\circ \right), \tag{10.48} \]

where \( k, j = 1, 2, \ldots, G \). The expressions (10.47)-(10.48) state the optimal demand function of each of the production factors utilized by each producing industry \( j \) in an economy under the two-stage Cobb-Douglas technology.

10.3.4 The Flexible Technological Coefficients

In preceding sections, we have examined the derivation of the optimal factor demand functions and of the endogenous prices of the intermediate factors on the basis of the two-stage Cobb-Douglas technology. Following the definition of the input-output coefficient (10.20), we can get the technological input-output coefficients for the intermediate production factors from (10.48) under the assumptions that the prices of the intermediate factors and the output level are given as below:

\[ \phi_{kj} = \left( \frac{P_j}{P_k} \right) \beta_{0kj}, \tag{10.49} \]

where \( k, j = 1, 2, \ldots, G \).

10.3.5 The Open Input-Output System of the Cobb-Douglas Case

Given the endogenous prices (see (10.58)) of the intermediate factors and the flexible technological coefficients (see (10.49)), we can derive the open input-output
system of the Cobb-Douglas case. First, in the basic open input-output system (10.22) or (10.27), we let \( \Psi = \left[ \begin{array}{c} \phi_{kj} \end{array} \right] \): a \((G \times G)\) square matrix and \( \mathbf{B} = \left[ \begin{array}{c} \beta_{0kj} \end{array} \right]^T \): a \((G \times G)\) square matrix; that is,

\[
\Psi = \begin{bmatrix}
\phi_{11} & \phi_{12} & \cdots & \phi_{1G} \\
\phi_{21} & \phi_{22} & \cdots & \phi_{2G} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{G1} & \phi_{G2} & \cdots & \phi_{GG}
\end{bmatrix}
\quad \text{and} \quad
\mathbf{B} = \begin{bmatrix}
\beta_{011} & \beta_{012} & \cdots & \beta_{01G} \\
\beta_{021} & \beta_{022} & \cdots & \beta_{02G} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{0G1} & \beta_{0G2} & \cdots & \beta_{0GG}
\end{bmatrix}.
\]

Then, since \( \phi_{kj} = \left( \frac{P_j}{P_k} \right) \beta_{0kj} \) (see (10.49)), we can obtain:

\[
\Psi = \mathbf{P}^{-1} \mathbf{BP},
\]

where \( \mathbf{P}^{-1} \) is a \((G \times G)\) diagonal matrix with \( P_j^{-1} \) \((j = 1, 2, \ldots, G)\) on its main diagonal and \( \mathbf{P} \) is a \((G \times G)\) diagonal matrix with \( P_j \) \((j = 1, 2, \ldots, G)\) on its main diagonal. Hence, the Leontief inverse becomes:

\[
\left[ \mathbf{I} - \Psi \right]^{-1} = \mathbf{P}^{-1} \left( \mathbf{I} - \mathbf{B} \right)^{-1} \mathbf{P}.
\]

Finally, the open input-output system (10.52) can be derived from substituting (10.51) into (10.22) or (10.27) as follows:

\[
\mathbf{Q}^0 = \left[ \mathbf{P}^{-1} \left( \mathbf{I} - \mathbf{B} \right)^{-1} \mathbf{P} \right] \mathbf{F},
\]

where \( \mathbf{P}^{-1} \left( \mathbf{I} - \mathbf{B} \right)^{-1} \mathbf{P} \geq 0 \) because \( \mathbf{P} \) is positive diagonal and \( \left( \mathbf{I} - \mathbf{B} \right)^{-1} \geq 0 \).

The expression (10.52) represents the open input-output system derived on the basis of the two-stage level Cobb-Douglas technology. We can get the equilibrium output \( \mathbf{Q}^0 \geq 0 \) from the system (10.52). Using the solutions of (10.52), we can get
the specific optimal demand functions of the production factors from (10.47)-(10.48). Such optimal demand functions also states the allocations of the domestic resources in an economy.

10.3.6 The Marx-Leontief Case

In Section 10.3.1 we have examined the special case, i.e., the Cobb-Douglas case where $\rho_{0j}=\rho_{j}=\rho=0$ (or $\sigma_{0j}=\sigma_{j}=\sigma=1$). Similarly, we can also discuss the Marx-Leontief case, where $\rho_{0j}=\rho_{j}=\rho=\infty$ (or $\sigma_{0j}=\sigma_{j}=\sigma=0$). Using the same method as done in Section 8.2 in Chapter 8, under the condition $\rho_{0j}=\rho_{j}=\rho=\infty$ (or $\sigma_{0j}=\sigma_{j}=\sigma=0$), we can obtain the two-stage Marx-Leontief-type production function from (10.1) or (10.32) as shown below:

$$Q_j^0 = \gamma_j \min_{k=1,\ldots,G} \left( \frac{Q_{kj}^0}{\beta_{0kj}} , F_j(X_{ikj}) \right),$$

(10.53)

where $i > 0$ and $j = 1, 2, \ldots, G$. $F_j(X_{ijk})$ is any arbitrary two-stage CES-type functional form with constant returns to scale. This Marx-Leontief production function can be rewritten as:

$$Q_{kj}^0 = X_{0kj} = \dot{\phi}_{kj} Q_j^0,$$

(10.54)

where $\dot{\phi}_{kj} = \frac{\beta_{0kj}}{\gamma_j}$; this is the flexible technological input-output coefficients. The functional form (10.53) describes the technical setup of each industry by a series of as many homogeneous linear equations as there are separate cost factors involved. The form (10.53) therefore is the most rigid type of production function than other production functions such as the CES-type production function and the Cobb-Douglas production function.
From (10.54), we can get the input-output system which is very similar to the traditional input-output system as follows:

\[ Q_k^o = \sum_{k=1}^{G} \phi_{kj} Q_j^o + F_k, \]  

(10.55)

where \( F_k \) is the final demand for the output \( Q_k^o; k = 1, 2, \ldots, G \).

The simultaneous equation system (10.55) can be written in matrix notation as shown below:

\[
\begin{bmatrix}
I - \hat{\Psi}
\end{bmatrix} Q^o = F,
\]  

(10.56)

where \( Q^o = \begin{bmatrix} Q_1^o, Q_2^o, \ldots, Q_G^o \end{bmatrix}^T \). \( I \) is a \((G \times G)\) identity (unit) matrix.

\( \hat{\Psi} = \begin{bmatrix} \beta_{0k_j} \\ \gamma_{ij} \end{bmatrix}^T \) is a \((G \times G)\) square matrix; \((k, j = 1, 2, \ldots, G)\).

\( F \) is a \((G \times 1)\) column vector; \( \begin{bmatrix} F_1, F_2, \ldots, F_G \end{bmatrix}^T \).

Solving (10.57) for \( F \) leads to:

\[
Q^o = \left[ I - \hat{\Psi} \right]^{-1} F.
\]  

(10.57)

From (10.57), we can produce the equilibrium outputs \( Q_k^o \) (or \( Q_j^o \) when \( k = j \)) which are nonnegative. Once the equilibrium outputs \( Q_j^o \) are determined, it is clearly optimal to take \( F_j(X_i, k_j) = Q_j^o \) from (10.53).

10.4 Remarks

In this chapter, on the bases of the neo-classical production theory and the traditional (Leontief) input-output theory, we have discussed the following.
First, we derived the general simultaneous equation system (10.18) (and the restricted one (10.25) under the assumption that $\sigma_{ij}=\sigma_j=\sigma$) for obtaining the endogenous prices of the intermediate factors (commodities). The endogenous prices yielded from such simultaneous equation system are functions of the technology (efficiency) indicator and the exogenous, non-intermediate factor prices. They of course are affected by the factor substitutabilities. On the other hand, they do not depend on the final demands - this is the non-substitution theorem.

Second, we showed the derivation of the optimal factor demand functions through the minimization of the total cost of the non-intermediate factors in an economy subject to the material balance relation (equation) associated with the input-output analysis. The optimal factor demand functions depend on the technology indicator, the output level, and on the non-intermediate factor prices (and the prices of the intermediate factors (outputs)). Furthermore, they exhibit the zero-degree homogeneity in the exogenous prices of the non-intermediate factors.

Third, we stated flexible technological input-output coefficients which differ from fixed (or constant) ones having been used in the traditional Leontief input-output theory. They vary with the changes in the technological efficiency, the prices of the non-intermediate factors, (and in turn the prices of the intermediate factors (outputs)) (see (10.21)). They therefore have significant influences on the equilibrium output level and the factor allocations because, unlike fixed input-output coefficients, the flexible ones reflect the technological improvements, the factor substitutabilities, and the relative prices in the equilibrium output level and then in the optimal factor demand (i.e., the allocations of the domestic resources).

Fourth, we developed the general open input-output system (10.30) (though it
is restricted one) which can reflect the technology effect and the price effect. Fifth, we examined the open input-output systems of the special cases such as the Cobb-Douglas case and the Marx-Leontief case which have the restrictions $\rho_0 j = \rho_j = \rho = 0$ (or $\sigma_0 j = \sigma_j = \sigma = 1$) and $\rho_0 j = \rho_j = \rho = \infty$ (or $\sigma_0 j = \sigma_j = \sigma = 0$), respectively.
11. THE ANALYTICAL SMALL MODEL

11.1 Introduction

On the basis of the results discussed in the preceding chapters, we here analyze the economic impacts of the variations in the exogenous economic variables such as taxes, tariffs, and the foreign exchange rate which cause the changes in the prices of the production factors, on the domestic resource allocations with a (3 x 3) small model associated with the input-output relation. These results can be extended to the general case.

For these economic analyses, it is first assumed that there are three industries, which each industry produces only one homogeneous commodity in the national economy. In other words, three different commodities are produced and existed in the national economic world. Specifically, we assume that there are two different commodities, $Q_1^O$ and $Q_2^O$, which play roles both as the intermediate-good factors utilized in the production of the outputs of producing industries and as the final commodities. We will call these commodities dual commodities (as mentioned in footnote 3 in Chapter 3). The remaining commodity $Q_3^O$ is assumed to be treated as a pure intermediate commodity which is utilized only by producing industries. In other words, a pure intermediate resource is totally used up in producing final products. The final demand for a pure intermediate commodity, therefore, does not exist in the
framework of the input-output relations.

Second, we assume that there are three factor groups (blocks), i.e., \( i = 0, 1, 2 \); the intermediate-factor group (\( i = 0 \)), the primary-factor group (\( i = 1 \)), and the imported-foreign-factor group (\( i = 2 \)). Specifically speaking, the intermediate-factor group is composed of three intermediate-good factors, \( Q^0_{k,j} (= X_{0k,j}) (k, j = 1, 2, 3) \); \( K_{0j} = 3 \) in (11.1). The primary-factor group comprises three primary factors (such as labor, (physical) capital, and natural resources); \( K_{1j} = 3 \) in (11.1). The imported-foreign-factor group consists of two imported foreign factors: \( X_{21j} \) and \( X_{22j} \); \( K_{2j} = 2 \) in (11.1). Third, there are perfect competitions in the demand sides of both the commodity and the factor markets. Fourth, we assume the free mobility of the factors of production among industries. Fifth, supply and demand are equal in equilibrium. Sixth, we assume that all producing industries are in small country which has no power to change the world market prices of the imported materials. So, all producing industries importing the foreign factors are just price-takers in the world factor market.

11.2 The Model Components

11.2.1 The Internal Structure of Production

As discussed in Chapter 9 (see the expression (9.1) in Chapter 9), we utilize the separable two-stage level CES-type production function with the linear homogeneity and with the technological indicator \( \gamma_j \).

\[
Q^2_{j} = \gamma_j \left[ \sum_{i=0}^{2} \left( \sum_{k=1}^{K_{ij}} \beta_{ikj} X_{ikj} - \rho_{ij} \right) \frac{\rho_{ij}}{\beta_{ij}} \right]^{\frac{1}{\rho_{ij}}},
\]  

(11.1)
11.2.2 The Total Cost Function

The total cost $C_j$ in each producing industry (firms) $j$ is composed of the non-intermediate factor costs (social costs) such as the primary factor cost and the imported foreign factor cost. Formally, the total cost function of each producing industry $j$ is as follows:

$$C_j = \sum_{i=1}^{2} \sum_{k=1}^{K_{ij}} W_{ikj} X_{ikj},$$  \hspace{1cm} (11.2)

where $K_{1j} = 3$ and $K_{2j} = 2$.

The total cost $C$ comprising the non-intermediate factors of a national economy then becomes:

$$C = \sum_{j=1}^{3} C_j.$$ \hspace{1cm} (11.3)

This total cost means social costs which come from the use of the primary factors (including the imported foreign factors).

11.2.3 The Final Demand Constraint

Under the assumptions that $Q^o_k$ ($k = 1, 2, 3$) are scarce commodities and fully employed, we can use the following final demand constraint. This constraint comes from the material balance equation.

$$F_k = Q^o_k - \sum_{j=1}^{3} Q^o_{k_j},$$ \hspace{1cm} (11.4)
where \( k = 1, 2, 3 \). Note that a pure intermediate factor (commodity) \( Q_3^0 \) has no final demand, i.e., \( F_3 = 0 \).

This final demand constraint implies social benefits which come from final consumption (demand).

11.2.4 The Intermediate Factor Prices

Following the cost-minimization rule based on the fundamental theorem of the neo-classical theory of production shown in Chapter 9, we can get the simultaneous equation system for the intermediate-factor prices as follows:

\[
\left( \gamma_j P_j \right)^{1-\sigma_j} = \left[ \sum_{k=1}^{3} (\beta_{0kj})^2 \sigma_j 0_j P_k^{1-\sigma_0j} \right]^{1-\sigma_j} + \sum_{i=1}^{2} \mu_{ij}^{1-\sigma_j}, \quad (11.5)
\]

where \( \mu_{ij} = \left[ \sum_{k=1}^{3} (\beta_{ikj}) \sigma_{ij} W_{ikj}^{1-\sigma_{ij}} \right]^{1-\sigma_{ij}} \) (\( K_{1j} = 4 \) and \( K_{2j} = 2 \)) and

\[
\sigma_j = \frac{1}{1+\rho_j}, \quad \sigma_{ij} = \frac{1}{1+\rho_{ij}}, \quad j = 1, 2, 3.
\]

From this price equation system, we can yield the prices \( P_k (k = 1, 2, 3) \) of the intermediate factors (outputs), which are endogenous and functions of the technological indicator \( \gamma_j \) and the prices \( W_{ikj} (i > 0) \) of the non-intermediate factors.

11.2.5 The Optimal Factor Demand functions

Utilizing the cost-minimization rule based on the Lagrange function (10.5) discussed in Chapter 10, we can get the cost-minimizing demand functions of the production factors which are utilized in producing a commodity \( Q_j^0 \) as follows.

(I) The optimal demand functions of the dual commodities utilized by each
producing industry $j$ are:

$$Q_{kj}^j = X_{0kj} = Q_j^j(\gamma_j)^{\sigma_j-1} P_j^{\sigma_j} \left( \frac{\beta_{0kj}}{P_k} \right)^{\sigma_{0j}} (\mu_{0j})^{\sigma_{0j}-\sigma_j}, \quad (11.6)$$

where $\mu_{0j} = \left[ \sum_{k=1}^{3} (\beta_{0kj})^{\sigma_{0j}} (P_k)^{1-\sigma_{0j}} \right]^{1-\sigma_{0j}}$ and $j = 1, 2; k = 1, 2, 3$.

(II) The optimal demand functions of the pure intermediate commodity utilized by each producing industry $j$ are:

$$Q_{k3}^j = X_{0k3} = Q_3^j(\gamma_3)^{\sigma_3-1} P_3^{\sigma_3} \left( \frac{\beta_{0k3}}{P_k} \right)^{\sigma_{03}} (\mu_{03})^{\sigma_{03}-\sigma_3}, \quad (11.7)$$

where $\mu_{03} = \left[ \sum_{k=1}^{3} (\beta_{0k3})^{\sigma_{03}} (W_{0k3})^{1-\sigma_{03}} \right]^{1-\sigma_{03}}$ and $k = 1, 2, 3$.

Note that if we write that $W_{0kj} = P_k (P_j$ when $k = j)$, we can derive the optimal demand functions (11.6)-(11.7) of the intermediate factors utilized by producing industries in an economy from the Lagrange function.

(III) The optimal demand functions of the primary factors utilized by each producing industry $j$ are:

$$X_{1kj} = Q_j^j(\gamma_j)^{\sigma_j-1} P_j^{\sigma_j} \left( \frac{\beta_{1kj}}{W_{1kj}} \right)^{\sigma_{1j}} (\mu_{1j})^{\sigma_{1j}-\sigma_j}, \quad (11.8)$$

where $\mu_{1j} = \left[ \sum_{k=1}^{3} (\beta_{1kj})^{\sigma_{1j}} (W_{1kj})^{1-\sigma_{1j}} \right]^{1-\sigma_{1j}}$ and $j = 1, 2, 3; k = 1, 2, 3$.

(IV) The optimal demand functions of the imported foreign factors utilized by each producing industry $j$ are:

$$X_{2kj} = Q_j^j(\gamma_j)^{\sigma_j-1} P_j^{\sigma_j} \left( \frac{\beta_{2kj}}{W_{2kj}} \right)^{\sigma_{2j}} (\mu_{2j})^{\sigma_{2j}-\sigma_j}, \quad (11.9)$$

where $\mu_{2j} = \left[ \sum_{k=1}^{2} (\beta_{2kj})^{\sigma_{2j}} (W_{2kj})^{1-\sigma_{2j}} \right]^{1-\sigma_{2j}}$ and $j = 1, 2, 3; k = 1, 2$. 
Each of the optimal factor demand functions shown in the expressions (11.6) through (11.9) amounts to each entry (cell) located in intersecting the corresponding rows and columns (except final demand block) in the input-output table (see Table 3.1). Each of the entries of the non-GNP (intermediate demand) part and the factor-cost part in the input-output table shows the allocations of the production factors to the producing industries (destination-industries).

11.2.6 The Flexible Input-Output Coefficients

The technological input-output coefficients for the intermediate factors can be obtained from (11.6)-(11.7) by using the results derived in (11.5) as follows:

\[ \phi_{kj} = \left( \frac{\sigma_{kj}}{\sigma_{0j}} \right) \gamma_{j}^{-1} (\beta_{0kj}) (\mu_{0j})^{\sigma_{0j} - \sigma_{j}}, \]  

(11.10)

where \( k, j = 1, 2, 3 \).

This expression (11.10) shows that the technological coefficients \( \phi_{kj} \) \( k, j = 1, 2, 3 \) are flexible because, if the prices of the production factors (as well as the technological indicator) change, they vary with such changes. And, the technological coefficients do not depend on the final demands \( F_k \); this is a form of the non-substitution theorem (see Dorfman, Arrow, and Samuelson (1958)).

11.3 The Economic Impact Analysis in the Input-Output Framework

11.3.1 Introduction

In the preceding sections, we have reviewed the model components on the basis of the \( 3 \times 3 \) small model. For our purpose of economic analysis, we assume more details
as follows. First, all domestic producing industries utilize all of three intermediate factors in their production processes. The producing industries 1-3 employ all of primary factors in producing their outputs (or products). Third, we assume about the imported foreign factors as below: (i) the producing industry 1 utilizes two of them, (ii) the producing industry 2 does not utilize the imported foreign factors at all, (iii) the industry 3 producing a pure intermediate good utilizes $X_{213} > 0$, but not $X_{223} = 0$.

11.3.2 The Derivation of the Endogenous Prices

Sato (1967) says that the isoquant surface is convex to the origin as long as the elasticities of substitution are positive, i.e., $\sigma_{ij}, \sigma_j > 0$. Such statement support many economists such as Allen (1938), Hicks (1946), and Kamien (1964) who say that the substitutability relations among the factors of production dominate the complementarity relationships among them.

We here first derive the prices of the intermediate factors (outputs) specifically under the assumption that $\sigma_{0j} = \sigma_j = \sigma$ for all $j (j = 1,2,3)$. This assumption makes the simultaneous equation system for the intermediate factor prices, i.e., (11.5) simplified and easy to solve it. Following the assumptions mentioned above, the simultaneous equation system (11.5) can be written in matrix notation as below:

$$P^{1-\sigma} = \left[ I - A \right]^{-1} R,$$

(11.11)

As mentioned in footnote 5 in Section 3.2 of Chapter 3, we assume that the technological coefficients $\phi_{ij}$ are not necessary zeros since there are the possibilities that some (but not all) destination-industries utilize parts of their outputs as necessary factors in their production processes.
where \( \mathbf{P}^{1-\sigma} \) is a \((3 \times 1)\) column vector; 
\[
\mathbf{P} = \left[ \begin{array}{c} P_1^{1-\sigma}, P_2^{1-\sigma}, P_3^{1-\sigma} \end{array} \right]^T.
\]

\( \mathbf{I} \) is a \((3 \times 3)\) identity matrix.

\( \mathbf{A} = \left[ \begin{array}{ccc} \alpha_{kj} \end{array} \right] \) is a \((3 \times 3)\) square matrix, where 
\[
\alpha_{kj} = (\beta_{0kj})^{\sigma} \gamma_j^{\sigma-1} (k, j = 1, 2, 3).
\]

\( \mathbf{R} \) is a \((3 \times 1)\) column vector; 
\[
\mathbf{R} = \left[ \begin{array}{c} R_1, R_2, R_3 \end{array} \right]^T, \quad \text{where} \quad R_j = \sum_{i=1}^{N_j} \mu_{ij}^{1-\sigma} \gamma_j^{\sigma-1}.
\]

Solving the price system (11.11) for \( \mathbf{P} \), we obtain each of the endogenous prices of the intermediate factors (commodities) as follows:

\[
P_k^{1-\sigma} = \sum_{j=1}^{3} H_{kj} R_j \quad (k = 1, 2, 3), \tag{11.12}
\]

where 
\[
H_{kj} = \frac{D_{kj}}{\Delta} \quad (\Delta = |\mathbf{I} - \mathbf{A}|). \quad D_{kj} \text{ is an element of the transposed cofactor matrix of } [\mathbf{I} - \mathbf{A}].
\]

Or equivalently,

\[
P_k = \left[ \sum_{j=1}^{3} H_{kj} R_j \right]^{1-\sigma} \quad (k = 1, 2, 3), \tag{11.13}
\]

The expression (11.12) or (11.13) represents the endogenous prices \((P_1, P_2, P_3)\) of the intermediate factors (commodities) \((Q_1^o, Q_2^o, Q_3^o)\), respectively, in the framework of the input-output analysis. Such prices have been influenced on the technology indicator, the factor substitutabilities, and the prices of the non-intermediate factors (because \(H^t\)s depend on \(\alpha^t\)s and \(R^t\)s are functions of the prices of the non-intermediate factors, i.e., \(W_{ikj} (i > 0)\)). Note that, in terms of the prices of the non-intermediate factors, that is, \(W_{ikj} (i > 0)\), the prices of the primary factors, i.e., \(W_{1kj}\), include the taxes (specific and/or ad valorem) and the domestic prices of the imported foreign factors, i.e., \(W_{2kj}\), are composed of the tariffs (specific and/or ad
valorem) and the foreign exchange rate which are assumed to be exogenous. We can also know that the intermediate-factor prices do not depend on the final demands $F_k$.

### 11.3.3 The Equilibrium Outputs

In the preceding subsection 11.3.2, we have derived the endogenous prices of the intermediate factors (commodities) in the framework of the input-output theory. Given those results, we can yield the gross equilibrium outputs of all three producing industries in the following way. First, since we assume that $\sigma_0_j = \sigma_j = \sigma$ for all $j$ ($j = 1, 2, 3$), the technological input-output coefficients (see (11.10)) are simplified like (10.26) shown below:

$$\phi_{kj} = \frac{X_{0kj}}{Q_{j}^0} = \frac{Q_{k}^0}{Q_{j}^0} = \left( \frac{P_j}{P_k} \right)^\sigma \alpha_{kj},$$

where $\alpha_{kj} = (\beta_{0kj})^\sigma \gamma_{j}^{\sigma - 1}$ and $k, j = 1, 2, 3$.

Second, given the prices of the intermediate factors and the technological coefficients, we can have the general open input-output system (following (10.27)-(10.29)) as below:

$$Q^0 = \left[ P^{-\sigma} \left( I - A \right)^{-1} P^\sigma \right] F,$$

where $Q^0$ is a $(3 \times 1)$ column vector; $Q^0 = \left[ Q_1^0, Q_2^0, Q_3^0 \right]^T$.

$F$ is a $(3 \times 1)$ column vector; $F = \left[ F_1, F_2, 0 \right]^T$. $I$ is a $(3 \times 3)$ identity matrix.

$A = \left[ \alpha_{kj} \right]$ is a $(3 \times 3)$ square matrix; $k, j = 1, 2, 3$.
\( P^{-\sigma} \) is a \((3 \times 3)\) diagonal matrix with \( P_j^{-\sigma} \) \((j = 1, 2, 3)\) on its main diagonal and \( P^\sigma \) is a \((3 \times 3)\) diagonal matrix with \( P_j^\sigma \) \((j = 1, 2, 3)\) on its main diagonal.

Finally, we can derive the total optimal supply of (or demand for) the respective outputs yielded by three producing industries from solving the system (11.15) for \( Q^0 \) as shown below:

\[
Q_k^0 = \sum_{j=1}^{2} S_{kj} F_j \quad (k = 1, 2, 3),
\]

(11.16)

where \( S_{kj} = H_{kj} \left( \frac{P_j}{P_k} \right)^{\sigma} \).

In sum, we can see, from the equilibrium solutions (11.16), the facts that: First, each \( Q_k^0 \) of the gross equilibrium outputs yielded is linear in terms of the given final demands \( (F_1, F_2) \) for the dual commodities (since the final demand for the pure intermediate factor does not exist, i.e., \( F_3 = 0 \)). This implies that the total of any equilibrium output needed to yield a given (or preassigned) target of the final demands can be built up by adding the separate outputs needed to yield each of the given final demands. Second, each of all the multipliers (i.e., \( S'\)'s) in the right-hand sides of the optimal solutions (11.16) states the gross output \( Q_k^0 \) (or \( Q_j^0 \)) needed to yield 1 unit of \( F_j \) (or \( F_k \)) alone. Third, the multipliers \( S'\)'s depend on \( \alpha'\)'s and the relative prices, that is, \( \left( \frac{P_j}{P_k} \right) \). Specifically speaking, each of the equilibrium outputs, \( Q_k^0 \) is affected by the technological indicator, the factor substitutabilities, and the non-intermediate factor prices.

11.3.4 The Optimal Allocation of Production Resources

Now, we here will examine the allocation, in an optimal situation, of each of the production factors utilized in the production processes of all producing industries.
From the optimal factor demand functions (11.6)-(11.9), we can see that each factor demand function is governed by the endogenous prices (which actually depend on the non-intermediate factor prices and the optimal level of the output produced by it). Thus, each factor demand function can be obtained through the endogenous prices and the optimal level of its corresponding output, which come from (11.12) (or (11.13)) and (11.16), respectively.

In particular, the optimal demand functions of the intermediate factors (commodities) can be derived simply from (11.14), given the prices of the intermediate factors and the output level, as follows:

\[ Q_{kj}^2 = \left( X_{0kj} \right) = Q_j^2 \left( \frac{P_j}{P_k} \right)^\sigma \alpha_{kj}, \quad (11.17) \]

where \( k, j = 1, 2, 3 \) and \( Q_j^2 = \sum_{k=1}^{2} S_{kj} F_k \left( S_{kj} = H_{kj} \left( \frac{P_j}{P_k} \right)^\sigma \right). \)

We can also obtain the optimal demand functions of the imported foreign factors as below:

\[ X_{1kj} = Q_j^2 \left( \gamma_j \right)^{\sigma-1} P_j \left( \frac{\beta_{1kj}}{W_{1kj}} \right)^{\sigma 1j} \left( \mu_{1j} \right)^{\sigma 1j-\sigma}, \quad (11.18) \]

where \( k = 1, 2, 3; j = 1, 2, 3; \mu_{1j} = \left[ \sum_{k=1}^{3} \left( \beta_{1kj} \right)^{\sigma 1j} (W_{1kj})^{1-\sigma 1j} \right]^{1-\sigma 1j}. \)

We can also obtain the optimal demand functions of the non-intermediate factors as below:

\[ X_{2kj} = Q_j^2 \left( \gamma_j \right)^{\sigma-1} P_j \left( \frac{\beta_{2kj}}{W_{2kj}} \right)^{\sigma 2j} \left( \mu_{2j} \right)^{\sigma 2j-\sigma}, \quad (11.19) \]
where \( k = 1, 2; j = 1, 2, 3; \mu_{2j} = \left[ \sum_{k=1}^{2} (\beta_{2kj}) \sigma_{2j} (W_{2kj})^{1-\sigma_{2j}} \right]^{1-\sigma_{2j}} \)

\[ Q_{j}^o = \sum_{k=1}^{2} S_{kj} F_{k} (S_{kj} = H_{kj} \left( \frac{P_{i}}{P_{k}} \right)^{\sigma}) P_{k} = \left[ \sum_{j=1}^{3} H_{kj} R_{j} \right]^{1/1-\sigma}. \]

The expressions (11.17)-(11.19) show that all those optimal factor demand functions actually depend on the prices of the non-intermediate factors, the technological indicator, the factor substitutabilities, and the (given) final demands. Such optimal factor demand functions have the linear homogeneity in the prices, \( W_{ikj} (i > 0) \), of the non-intermediate factors.

### 11.4 Hypothetical and Numerical Analysis

In the preceding section we developed the theoretical input-output system with flexible technological coefficients. In this section we numerically analyze how the optimal domestic resource allocations in a small importing country changes with the variations in the world market price of the imported material (for example, oil). Table 11.1 through Table 11.5 include the parameter values that are utilized throughout our numerical calculations and constitute our ‘base’ case.

#### 11.4.1 The Prices of the Intermediate Factors

We calculate the prices of the intermediate factors (shadow-prices of final demands) by using the values of the flexible input-output parameters (see Table 11.5) as follows:

\[
A = \begin{bmatrix}
0.214 & 0.311 & 0.297 \\
0.327 & 0.270 & 0.262 \\
0.391 & 0.270 & 0.198
\end{bmatrix}.
\]
\[
\begin{bmatrix}
I - A
\end{bmatrix} =
\begin{bmatrix}
0.786 & -0.311 & -0.297 \\
-0.327 & 0.730 & -0.262 \\
-0.391 & -0.270 & 0.802
\end{bmatrix}.
\tag{11.21}
\]

\[
\Delta = |I - A| = 0.18.
\tag{11.22}
\]

The cofactor matrix of (11.21) becomes:
\[
D =
\begin{bmatrix}
0.518 & 0.365 & 0.374 \\
0.330 & 0.514 & 0.334 \\
0.298 & 0.303 & 0.472
\end{bmatrix}.
\tag{11.23}
\]

Transpose of \(D\) in (11.23) can be written as:
\[
D^T =
\begin{bmatrix}
0.518 & 0.330 & 0.298 \\
0.365 & 0.514 & 0.303 \\
0.374 & 0.334 & 0.472
\end{bmatrix}.
\tag{11.24}
\]

Then, we can get the following from the price equation system (11.12) or (11.13):
\[
\begin{bmatrix}
p_1^{0.5} \\
p_2^{0.5} \\
p_3^{0.5}
\end{bmatrix} = \frac{1}{0.18}
\begin{bmatrix}
0.518 & 0.330 & 0.298 \\
0.365 & 0.514 & 0.303 \\
0.374 & 0.334 & 0.472
\end{bmatrix}
\begin{bmatrix}
R_1 \\
R_2 \\
R_3
\end{bmatrix}.
\tag{11.25}
\]

Equivalently,
\[
\begin{bmatrix}
p_1^{0.5} \\
p_2^{0.5} \\
p_3^{0.5}
\end{bmatrix} =
\begin{bmatrix}
2.878 & 1.833 & 1.656 \\
2.028 & 2.856 & 1.683 \\
2.078 & 1.856 & 2.622
\end{bmatrix}
\begin{bmatrix}
R_1 \\
R_2 \\
R_3
\end{bmatrix}.
\tag{11.26}
\]
Hence, we get the following from (11.26):

\[ P_i^{0.5} = (2.878)^1 + (1.833)^2 + (1.656)^3. \]  

\[ (11.27) \]

\[ P_2^{0.5} = (2.028)^1 + (2.856)^2 + (1.683)^3. \]  

\[ (11.28) \]

\[ P_3^{0.5} = (2.078)^1 + (1.856)^2 + (2.622)^3. \]  

\[ (11.29) \]

On the other hand, we can calculate \( R_j = (= 7j - 1, 2, 3) \) by using the values in Table 11.4 and the prices\(^2\) of the non-intermediate factors in Table 11.3 as follows:

\[ R_1 = \gamma_j^{\sigma - 1} \sum_{i=1}^{2} \mu_{ij}^{1 - \sigma}. \]  

\[ (11.30) \]

\[ \mu_{11} = \Gamma_{1}^* = (10.258)^{2.857} = 773.76. \]  

\[ (11.31) \]

where

\[ \Gamma_{1}^* = [(0.105)^{0.65}(200)^{0.35} + (0.262)^{0.65}(1500)^{0.35} + (0.157)^{0.65}(1000)^{0.35}]^{0.35}. \]

Note that \( \mu_{ij} = \frac{K_{ij}}{\sum_{k=1}^{K} \rho_{ikj} W_{ikj}} \) and \( K_{1j} = 3, K_{2j} = 2. \)

Using (11.31), we obtain:

\[ \mu_{11}^{1 - 0.5} = (773.76)^{0.5} = 27.817. \]  

\[ (11.32) \]

Similarly, we can calculate \( \mu_{21} \) as below:

\[ \mu_{21} = \left[ (0.369)^{0.55}(300)^{0.45} + (0.104)^{0.55}(400)^{0.45} \right]^{0.45} \]

\[ = (11.795)^{2.222} \]

\[ = 240.61 \]  

\[ (11.33) \]  

\(^2\)Each value in Table 11.3 stands for the hypothetical price of each of the non-intermediate factors. Actually, we don’t know such prices in the traditional input-output table because it is recorded in monetary terms. However, we assume that we can know such prices.
Utilizing (11.33), we get:

\[ \mu_{21}^{1-0.5} = (240.61)^{0.5} = 15.512. \]  

(11.34)

Thus, we can get \( R_1 \) as shown below:

\[ R_1 = (0.98)^{-0.5}[27.817 + 15.512] = 43.762. \]  

(11.35)

Analogously, we calculate \( R_2 \) in the following way.

\[ R_2 = \gamma_2^{\sigma-1} \mu_{12}^{1-\sigma}. \]  

(11.36)

\[ \mu_{12} = \left[ (0.398)^{0.7}(300)^{0.3} + (0.309)^{0.7}(1250)^{0.3} + (0.177)^{0.7}(1000)^{0.3} \right]^{0.3} \]

\[ = (9.001)^{3.33} \]

\[ = 1505.873. \]  

(11.37)

So, we get:

\[ \mu_{12}^{1-0.5} = (1505.873)^{0.5} = 38.806. \]  

(11.38)

We then can obtain \( R_2 \) as below:

\[ R_2 = (1.03)^{-0.5}(38.806) = 38.224. \]  

(11.39)

By using the same procedure, \( R_3 \) can be calculated as follows:

\[ R_3 = \gamma_3^{\sigma-1} \sum_{i=1}^{2} \mu_{13}^{1-\sigma}. \]  

(11.40)

\[ \mu_{13} = \Gamma^* = (17.496)^{2.5} = 1280.404. \]  

(11.41)
where
\[ \Gamma_2^* = [(0.197)^{0.6}(250)^{0.4} + (0.512)^{0.6}(1000)^{0.4} + (0.079)^{0.6}(1000)^{0.4}]^{0.4}. \]

Using (11.41), we obtain:
\[ \mu_{13}^{1-0.5} = (1280.404)^{0.5} = 35.783. \] (11.42)

Similarly,
\[ \mu_{23} = \left[(0.109)^{0.51}(250)^{0.49}\right]^{0.49} = (4.831)^{0.49} = 24.856. \] (11.43)

Using (11.43), we get:
\[ \mu_{23}^{1-0.5} = (24.856)^{0.5} = 4.986. \] (11.44)

Thus, we can obtain \( R_3 \) as shown below:
\[ R_3 = (1.02)^{-0.5}[35.783 + 4.986] = 40.361. \] (11.45)

We then can derive the prices of the intermediate factors (commodities) from (11.26) by using (11.35), (11.39), and (11.45) as follows:
\[ P_1^{0.5} = (2.878)(43.762) + (1.833)(38.224) + (1.656)(40.361) \]
\[ = 262.849. \] (11.46)

\[ P_2^{0.5} = (2.028)(43.762) + (2.856)(38.224) + (1.883)(40.361) \]
\[ = 265.845. \] (11.47)
\[ p^0.5 = (2.078)(43.762) + (1.856)(38.224) + (2.622)(40.361) \]
\[ = 267.708. \]  

Finally, we can get the prices as follows:
\[ P_1 \approx 69090, \ P_2 \approx 70674, \text{ and } P_3 \approx 71668. \]  

These results show and prove rationality of the price equation system (11.12) or (11.13) because the nonnegative (shadow) prices of the intermediate factors can be derived from (11.12) or (11.13).

### 11.4.2 Equilibrium Domestic Output

We have developed the general open input-output system (10.30) in Chapter 10. Utilizing \((P_1, P_2, P_3)\) in (11.49) and (11.22), (11.24), we can obtain the equilibrium domestic output from (10.30) as follows:

\[
Q^0 = \left[ P^{-0.5} \left( I - A \right)^{-1} P^{0.5} \right] F, \]  

where \( Q^0 = \begin{bmatrix} Q_1^0, Q_2^0, Q_3^0 \end{bmatrix}^T \) and \( F = \begin{bmatrix} F_1, F_2, 0 \end{bmatrix}^T \).

\[
P^{-0.5} = \begin{bmatrix}
0.0038 & 0 & 0 \\
0 & 0.0038 & 0 \\
0 & 0 & 0.0037
\end{bmatrix},
\]

\[
P^{0.5} = \begin{bmatrix}
262.849 & 0 & 0 \\
0 & 265.845 & 0 \\
0 & 0 & 267.708
\end{bmatrix}.
Rearranging (11.50) leads to:

\[
(I - A)^{-1} = \begin{bmatrix}
2.878 & 1.833 & 1.656 \\
2.028 & 2.856 & 1.683 \\
2.078 & 1.856 & 2.622 \\
\end{bmatrix}
\]

From (11.51), we get:

\[
Q_1^o = (2.891)F_1 + (1.861)F_2. \\
Q_2^o = (2.103)F_1 + (2.924)F_2. \\
Q_3^o = (2.103)F_1 + (1.861)F_2. \\
\]

Assuming that the preassigned final demands are \( F_1 = 700 \) and \( F_2 = 900 \), we can predict the equilibrium domestic outputs \( Q_1^o, Q_2^o, \) and \( Q_3^o \) to be produced as follows:

\[
Q_1^o = (2.891)(700) + (1.861)(900) = 3,699. \\
Q_2^o = (2.103)(700) + (2.924)(900) = 4,104. \\
Q_3^o = (2.103)(700) + (1.861)(900) = 3,147. \\
\]

11.4.3 The Optimal Factor Demands

In the preceding sections we calculated the prices of the intermediate factors (commodities) (see the expression (11.49)) and the equilibrium domestic outputs (see the expressions (11.55)-(11.57)) to be produced under the assumption that the
preassigned final demands are $F_1 = 700$ and $F_2 = 900$. Utilizing such results and the values in Table 11.1 through Table 11.5, we can numerically calculate the optimal factor demands from (11.17)-(11.19). Table 11.6 shows the final outcomes calculated.

11.4.4 The Economic Effects of the World Market Price

11.4.4.1 Rise in the World Market Price Here we specifically examine the economic effects of the changes in the world market prices of the imported materials (e.g., oil) on the domestic resource allocations in a small importing country. In particular, we investigate how the optimal demand for the imported material (e.g., oil) is affected by its own-price change.

For those purposes, we assume as follows: (1) Only the world market prices $W_{211}^*$ and $W_{213}^*$ of the imported materials (e.g., oil) $X_{211}$ and $X_{213}$ increase by 20 percent simultaneously and other factor prices are held constant. (2) All parameters in Table 11.1 through Table 11.5 except for $W_{211}^*$ and $W_{213}^*$ in Table 11.3 are constant. (3) The preassigned final demands are unchanged; $F_1 = 700$ and $F_2 = 900$. (4) The foreign exchange rate $e$ is fixed at $e = 10$.

Under the assumptions mentioned above and the expression (6.39) in Section 6.10.3 in Chapter 6, the domestic prices of the imported materials $X_{211}$ and $X_{213}$ become:

$$W_{211} = eW_{211}^* = (10)(36) = 360.$$  \hspace{1cm} (11.58)

$$W_{213} = eW_{213}^* = (10)(30) = 300.$$  \hspace{1cm} (11.59)

The expressions (11.58) and (11.59) state that the domestic prices $W_{211}$ and $W_{213}$
of the imported materials \( X_{211} \) and \( X_{213} \) in a small importing country also increase by 20 percent as the world market prices \( W_{211}^* \) and \( W_{213}^* \) of such imported materials rise by 20 percent when the foreign exchange rate \( e \) is unchanged. So the production costs (i.e., the prices of the outputs produced) will be higher as a result of the rises in the world market prices.

Therefore, we here calculate the prices of outputs (intermediate factors) through the method used in Section 11.4.1. However, by assumption (1), we can calculate them more conveniently as follows:

\[
\mu_{21} = \left[ (0.369)^{0.55}(360)^{0.45} + (0.104)^{0.55}(400)^{0.45} \right]^{0.45} \\
= (12.438)^{2.222} \\
= 270.73. \tag{11.60}
\]

Utilizing (11.60), we get:

\[
\mu_{21}^{1-0.5} = (270.73)^{0.5} = 16.454. \tag{11.61}
\]

Using (11.60) and (11.61), we can obtain \( R_1 \) as shown below:

\[
R_1 = (0.98)^{-0.5}[27.817 + 16.454] = 44.714. \tag{11.62}
\]

Similarly,

\[
\mu_{23} = \left[ (0.109)^{0.51}(300)^{0.49} \right]^{0.49} \\
= (5.283)^{2.04} \\
= 29.831. \tag{11.63}
\]

Using (11.63), we get:

\[
\mu_{23}^{1-0.5} = (29.831)^{0.5} = 5.462. \tag{11.64}
\]
Thus, we can calculate $R_3$ by using (11.42) and (11.64) as shown below:

$$R_3 = (1.02)^{-0.5}[35.783 + 5.462] = 40.833. \quad (11.65)$$

On the other hand, $R_2$ of (11.39) is not changed because $R_2$ doesn't depend on the prices of the imported materials. In the end, the prices of the intermediate factors (commodities) are derived from substituting (11.39), (11.62), and (11.65) into (11.27)-(11.29) as follows:

$$P_1^{0.5} = (2.878)(44.714) + (1.833)(38.224) + (1.656)(40.833)$$
$$= 266.371. \quad (11.66)$$

$$P_2^{0.5} = (2.028)(44.714) + (2.856)(38.224) + (1.683)(40.833)$$
$$= 268.570. \quad (11.67)$$

$$P_3^{0.5} = (2.078)(44.714) + (1.856)(38.224) + (2.622)(40.833)$$
$$= 270.924. \quad (11.68)$$

Finally, we get the prices as follows:

$$P_1 \approx 70954, \ P_2 \approx 72130, \text{ and } P_3 \approx 73400. \quad (11.69)$$

The values in (11.69) show that the prices of each of the intermediate factors (commodities) are higher than ones in (11.49) when the world market prices of imported materials (e.g., oil) increase.

Next, we calculate the equilibrium domestic outputs responding to the rises in the world market prices of the imported materials. As we did in Section 11.4.2, we
can get the equilibrium domestic outputs from (10.30) as below:

\[ Q^o = \left[ P^{-0.5} \left( I - A \right)^{-1} P^{0.5} \right] F, \]  

(11.70)

where \( Q^o = \left[ Q_1^o, Q_2^o, Q_3^o \right]^T \) and \( F = \left[ F_1, F_2, 0 \right]^T \).

\[ P^{-0.5} = \begin{bmatrix}
0.0038 & 0 & 0 \\
0 & 0.0037 & 0 \\
0 & 0 & 0.0037
\end{bmatrix}. \]

\[ P^{0.5} = \begin{bmatrix}
266.371 & 0 & 0 \\
0 & 268.570 & 0 \\
0 & 0 & 270.924
\end{bmatrix}. \]

\( (I - A)^{-1} = \begin{bmatrix}
2.878 & 1.833 & 1.656 \\
2.028 & 2.856 & 1.683 \\
2.078 & 1.856 & 2.622
\end{bmatrix}. \)

Rearranging (11.70) leads to:

\[ \begin{bmatrix}
Q_1^o \\
Q_2^o \\
Q_3^o
\end{bmatrix} = \begin{bmatrix}
2.930 & 1.880 & 1.626 \\
2.134 & 2.954 & 1.626 \\
2.134 & 1.880 & 2.709
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2 \\
0
\end{bmatrix}. \]

(11.71)

From (11.71), we get:

\[ Q_1^o = (2.930)F_1 + (1.880)F_2. \]

(11.72)

\[ Q_2^o = (2.134)F_1 + (2.924)F_2. \]

(11.73)

\[ Q_3^o = (2.134)F_1 + (1.880)F_2. \]

(11.74)
Assuming that the preassigned final demands are $F_1 = 700$ and $F_2 = 900$, we can predict the equilibrium domestic outputs $Q_1^0$, $Q_2^0$, and $Q_3^0$ to be produced as follows:

$$Q_1^0 = (2.930)(700) + (1.880)(900) = 3,743.$$  \hfill (11.75)  

$$Q_2^0 = (2.134)(700) + (2.954)(900) = 4,152.$$  \hfill (11.76)  

$$Q_3^0 = (2.134)(700) + (1.880)(900) = 3,186.$$  \hfill (11.77)  

The above results state that if the final demands are assumed to be unchanged, the equilibrium domestic outputs in a small importing country increase even though the world market prices of the imported materials (e.g., oil) rise.

Finally, we can numerically calculate the optimal factor demands responding to the rises in the world market prices of the imported materials by plugging the outcomes derived above into (11.17)-(11.19). Table 11.7 demonstrates the final results calculated.

**11.4.4.2 Fall in the World Market Price**  In Section 11.4.4.1, we numerically showed (i) the prices of the intermediate factors (outputs), (ii) the equilibrium domestic outputs, and (iii) the optimal demand for each of the production factors responding to the rises in the world market prices of the imported materials (e.g., oil). In this section, we will show the opposite case; that is, the case of the falls in the world market prices of the imported foreign factors. The assumptions (2) through (4) used in Section 11.4.4.1 still hold in this case. However, we change the assumption (1) used in Section 11.4.4.1 into that (1)' only the world market prices $W^*_{211}$ and $W^*_{213}$ of the imported materials (e.g., oil) $X_{211}$ and $X_{213}$ fall by 20 percent simultaneously and other factor prices are held constant.
Under such assumptions and (6.39) in Section 6.10.3 in Chapter 6, the domestic prices of the imported materials (e.g., oil) $X_{211}$ and $X_{213}$ become:

\[
W_{211} = eW_{211}^* = (10)(24) = 240. 
\]

\[
W_{213} = eW_{213}^* = (10)(20) = 200. 
\]

The expressions (11.78) and (11.79) state that the domestic prices $W_{211}$ and $W_{213}$ of the imported materials $X_{211}$ and $X_{213}$ in a small importing country also decrease by 20 percent as the world market prices $W_{211}^*$ and $W_{213}^*$ of such imported materials fall by 20 percent when the foreign exchange rate $e$ is unchanged.

Using (11.78)-(11.79) along with the same method utilized in Section 11.4.4.1 under the assumptions mentioned above, we can get the following outcomes.

1. Finally, we can get the prices of the intermediate factors as follows:

\[
\mu_{21} = \left[ (0.369)^{0.55}(240)^{0.45} + (0.104)^{0.55}(400)^{0.45} \right]^{0.45} 
= (11.076)^{2.222} 
= 209.228. 
\]

\[
\mu_{21}^{1-0.5} = (209.228)^{0.5} = 14.465. 
\]

Using (11.32) and (11.81), we can obtain $R_1$ as shown below:

\[
R_1 = (0.98)^{-0.5}[27.817 + 14.465] = 42.705. 
\]

Similarly,

\[
\mu_{23} = \left[ (0.109)^{0.51}(200)^{0.49} \right]^{0.49} 
= (4.331)^{2.04} 
= 19.890. 
\]
Thus, we can calculate $R_3$ by using (11.42) and (11.84) as shown below:

$$R_3 = (1.02)^{-0.5}[35.783 + 4.460] = 39.841.$$  \hfill (11.85)

On the other hand, $R_2$ of (11.39) is not changed because $R_2$ doesn’t depend on the prices of the imported materials. Hence, the prices of the intermediate factors (commodities) are derived from substituting (11.39), (11.82), and (11.85) into (11.27)-(11.29) as follows:

$$P_1^{0.5} = (2.878)(42.705) + (1.833)(38.224) + (1.656)(39.841)$$
$$= 258.946.$$  \hfill (11.86)

$$P_2^{0.5} = (2.028)(42.705) + (2.856)(38.224) + (1.683)(39.841)$$
$$= 262.826.$$  \hfill (11.87)

$$P_3^{0.5} = (2.078)(42.705) + (1.856)(38.224) + (2.622)(39.841)$$
$$= 264.148.$$  \hfill (11.88)

Finally, we get the prices as follows:

$$P_1 \cong 67053,\ P_2 \cong 69078,\ \text{and}\ P_3 \cong 69774.$$  \hfill (11.89)

The values in (11.89) show that the prices of the intermediate factors (commodities) are lower as a result of the falls in $W_{211}^*$ and $W_{213}^*$ than the prices in (11.49).

(2) The equilibrium domestic outputs to be produced are calculated as follows:

$$Q^0 = \left[ P^{-0.5} \left( I - A \right)^{-1} P^{0.5} \right] F,$$  \hfill (11.90)
where \( Q^o = \begin{bmatrix} Q_1^o, Q_2^o, Q_3^o \end{bmatrix}^T \).

\[
P^{-0.5} = \begin{bmatrix}
0.00386 & 0 & 0 \\
0 & 0.00380 & 0 \\
0 & 0 & 0.00379
\end{bmatrix}.
\]

\[
P^{0.5} = \begin{bmatrix}
258.946 & 0 & 0 \\
0 & 262.826 & 0 \\
0 & 0 & 264.148
\end{bmatrix}.
\]

\[
(I - A)^{-1} = \begin{bmatrix}
2.878 & 1.833 & 1.656 \\
2.028 & 2.856 & 1.683 \\
2.078 & 1.856 & 2.622
\end{bmatrix}.
\]

Rearranging (11.90) leads to:

\[
\begin{bmatrix}
Q_1^o \\
Q_2^o \\
Q_3^o
\end{bmatrix} = \begin{bmatrix}
2.848 & 1.840 & 1.585 \\
2.072 & 2.891 & 1.585 \\
2.072 & 1.840 & 2.641
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2 \\
0
\end{bmatrix}. \quad (11.91)
\]

From (11.91), we get:

\[
Q_1^o = (2.848)F_1 + (1.840)F_2. \quad (11.92)
\]

\[
Q_2^o = (2.072)F_1 + (2.891)F_2. \quad (11.93)
\]

\[
Q_3^o = (2.072)F_1 + (1.840)F_2. \quad (11.94)
\]

Assuming that the preassigned final demands are \(F_1 = 700\) and \(F_2 = 900\), we can predict the equilibrium domestic outputs \(Q_1^o, Q_2^o,\) and \(Q_3^o\) to be produced as
follows:

\[
Q_1 = (2.848)(700) + (1.840)(900) = 3,650. \tag{11.95}
\]

\[
Q_2 = (2.072)(700) + (2.891)(900) = 4,052. \tag{11.96}
\]

\[
Q_3 = (2.072)(700) + (1.840)(900) = 3,106. \tag{11.97}
\]

The results (11.95)-(11.97) state that if the final demands are assumed to be unchanged, the equilibrium domestic outputs in a small importing country decrease even though the world market prices of the imported materials (e.g., oil) fall.

(3) The optimal factor demands are given in Table 11.8.

11.4.5 Derivation of Graph for the Own-Price Effect

From Table 11.6 through Table 11.8, we can derive figures which show the downward-sloping demand curves. Figure 11.1 and Figure 11.2 demonstrate the own-price effects of the imported materials.

11.4.6 Remarks

In Section 11.4.4, we calculated (i) the prices of the intermediate factors (commodities), (ii) the equilibrium domestic outputs to be produced to meet the presigned final demands, (iii) the optimal factor demands for (ii) under the assumptions that: (1) Only the world market prices \( W_{211}^* \) and \( W_{213}^* \) of the imported materials (e.g., oil) \( X_{211} \) and \( X_{213} \) increase by 20 percent simultaneously and other factor prices are held constant. Or (1)' Only the world market prices \( W_{211}^* \) and \( W_{213}^* \) of the imported material (e.g., oil) \( X_{211} \) and \( X_{213} \) decrease by 20 percent simultaneously and other factor prices are held constant. (2) All parameters in Table 11.1
through Table 11.5 except for $W_{211}$ and $W_{213}$ in Table 11.3 are constant. (3) The preassigned final demands are unchanged; $F_1 = 700$ and $F_2 = 900$. (4) The foreign exchange rate $e$ is fixed at $e = 10$.

The results calculated in Section 11.4.4 show the following. First, as the world market prices of the imported materials (e.g., oil) rise (fall), the domestic prices of such imported production factors also increase (decrease) in the same proportion to the rises (falls) in the world market prices of the imported materials (e.g., oil) (see (11.55)-(11.59) and (11.75)-(11.79)). As a result, the prices (production costs) of the intermediate factors (commodities) increase (decrease) in accordance with the rises (falls) in the world market prices of the imported materials (e.g., oil) (compare (11.49) with (11.69) and (11.90), respectively).

Second, in particular, we can see that the multipliers (coefficients) of the right-hand sides of the equilibrium output equations (see (11.52)-(11.54), (11.72)-(11.74), and (11.92)-(11.94)) vary in response to the changes in the prices of the intermediate factors which actually depend on the variations in the world market prices of the imported materials (e.g., oil) (in this numerical case). These result from the flexible technological coefficients come from (10.26).

Third, the equilibrium domestic outputs to be produced to meet the preassigned final demands in a small importing country (would) increase (decrease) even though the world market prices of the imported materials (e.g., oil) rise (fall). (Compare (11.55)-(11.57) with (11.75)-(11.77) and (11.95)-(11.97), respectively).

Fourth, we can also see that the optimal demand for each of the imported materials (e.g., oil) decreases (increases) when the world market price of the corresponding imported production factor rises (falls) (see boldfaced values in Table 11.6 through
Table 11.8 and Figure 11.1 and Figure 11.2). On the other hand, the hypothetical numerical analysis of Section 11.4.4 shows that other production factors increase (decrease) as the world market prices $W_{211}^*$ and $W_{213}^*$ of the imported production factors (e.g., oil) $X_{211}^*$ and $X_{213}^*$ rise (fall) (see Table 11.6 through Table 11.8).

So far, under the restricted assumptions (used in Section 11.4.4.1 and Section 11.4.4.2), we showed the numerical analysis based on the hypothetical parameter values (as an example) because the estimation of the parameter values by the econometric method is beyond the scope of this study. However, the economic effects discussed in this section on the basis of the hypothetical parameter values would not lose the generality except for the magnitudes (sizes) of the specific values calculated. In particular, the own-price effects on the corresponding imported production factors mentioned above are always the obvious facts in an economy.

Finally, we should note that the preassigned final demands are crucial roles in our theoretical input-output system with flexible technological coefficients (as well as the traditional input-output system with fixed input-output coefficients). Thus, we need the final demands which are more accurately preassigned (or forecasted). Then, our theoretical input-output system with flexible technological coefficients plays a significant role in predicting the equilibrium (optimal) domestic outputs and in turn the optimal factor demands.
### Table 11.1: Elasticity of substitution

<table>
<thead>
<tr>
<th>$\sigma_{ij}$</th>
<th>Producing Industries j</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>EOS</td>
<td></td>
</tr>
<tr>
<td>$\sigma_{0j}$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\sigma_{1j}$</td>
<td>0.65</td>
</tr>
<tr>
<td>$\sigma_{2j}$</td>
<td>0.55</td>
</tr>
<tr>
<td>$\sigma_{j}$</td>
<td>0.5</td>
</tr>
</tbody>
</table>

### Table 11.2: Technology Parameter

<table>
<thead>
<tr>
<th>Technology</th>
<th>Producing Industries j</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$\gamma_{ij}$</td>
<td>0.98</td>
</tr>
</tbody>
</table>

### Table 11.3: Non-intermediate Factor Prices

<table>
<thead>
<tr>
<th>$W_{ikj}$</th>
<th>Producing Industries j</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<tr>
<td>Primary Factors</td>
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</tr>
<tr>
<td>$W_{11j}$</td>
<td>200</td>
</tr>
<tr>
<td>$W_{12j}$</td>
<td>1,500</td>
</tr>
<tr>
<td>$W_{13j}$</td>
<td>1,000</td>
</tr>
<tr>
<td>Imported Factors</td>
<td></td>
</tr>
<tr>
<td>$W_{21j}$</td>
<td>300</td>
</tr>
<tr>
<td>$W_{22j}$</td>
<td>400</td>
</tr>
</tbody>
</table>
### Table 11.4: Distribution Parameters

<table>
<thead>
<tr>
<th></th>
<th>$\beta_{ikj}$ Producing Industries j</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Intermediate Factors</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{0kj}$</td>
<td>0.045</td>
<td>0.1</td>
<td>0.09</td>
<td></td>
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<tr>
<td></td>
<td>0.105</td>
<td>0.075</td>
<td>0.07</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.15</td>
<td>0.075</td>
<td>0.04</td>
<td></td>
</tr>
<tr>
<td><strong>Primary Factors</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{1kj}$</td>
<td>0.105</td>
<td>0.398</td>
<td>0.197</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.262</td>
<td>0.309</td>
<td>0.512</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.157</td>
<td>0.177</td>
<td>0.079</td>
<td></td>
</tr>
<tr>
<td><strong>Imported Factors</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{2kj}$</td>
<td>0.369</td>
<td>0</td>
<td>0.109</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.104</td>
<td>0</td>
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</tr>
</tbody>
</table>

### Table 11.5: Input-Output Parameter

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_{kj}$ Producing Industries j</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Parameter</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_{1j}$</td>
<td>0.214</td>
<td>0.311</td>
<td>0.297</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{2j}$</td>
<td>0.327</td>
<td>0.270</td>
<td>0.262</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{3j}$</td>
<td>0.391</td>
<td>0.270</td>
<td>0.198</td>
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</table>
Table 11.6: Original Optimal Factor Demands

<table>
<thead>
<tr>
<th></th>
<th>Destination sectors</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Intermediate Factors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q_{1j}^i$</td>
<td>792</td>
<td>1,290</td>
</tr>
<tr>
<td>$Q_{2j}^i$</td>
<td>1,196</td>
<td>1,108</td>
</tr>
<tr>
<td>$Q_{3j}^i$</td>
<td>1,420</td>
<td>1,100</td>
</tr>
<tr>
<td>Primary Factors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{11j}^i$</td>
<td>18,473</td>
<td>46,433</td>
</tr>
<tr>
<td>$X_{12j}^i$</td>
<td>10,695</td>
<td>14,183</td>
</tr>
<tr>
<td>$X_{13j}^i$</td>
<td>7,778</td>
<td>9,819</td>
</tr>
<tr>
<td>Imported Factors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{21j}^i$</td>
<td>32,085</td>
<td>0</td>
</tr>
<tr>
<td>$X_{22j}^i$</td>
<td>14,205</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 11.7: Factor Demands After World Prices Rise

<table>
<thead>
<tr>
<th></th>
<th>Destination sectors</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Intermediate Factors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q_{1j}^i$</td>
<td>801</td>
<td>1,302</td>
</tr>
<tr>
<td>$Q_{2j}^i$</td>
<td>1,214</td>
<td>1,291</td>
</tr>
<tr>
<td>$Q_{3j}^i$</td>
<td>1,439</td>
<td>1,111</td>
</tr>
<tr>
<td>Primary Factors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{11j}^i$</td>
<td>18,944</td>
<td>47,949</td>
</tr>
<tr>
<td>$X_{12j}^i$</td>
<td>10,967</td>
<td>14,496</td>
</tr>
<tr>
<td>$X_{13j}^i$</td>
<td>7,976</td>
<td>10,036</td>
</tr>
<tr>
<td>Imported Factors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{21j}^i$</td>
<td>30,988</td>
<td>0</td>
</tr>
<tr>
<td>$X_{22j}^i$</td>
<td>14,655</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 11.8: Factor Demands After World Prices Fall

<table>
<thead>
<tr>
<th></th>
<th>$X_{ikj}$</th>
<th>Destination sectors j</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Intermediate Factors</td>
<td>$Q_{ij}^1$</td>
<td>781</td>
<td>1,279</td>
</tr>
<tr>
<td></td>
<td>$Q_{ij}^2$</td>
<td>1,176</td>
<td>1,094</td>
</tr>
<tr>
<td></td>
<td>$Q_{ij}^3$</td>
<td>1,399</td>
<td>1,089</td>
</tr>
<tr>
<td>Primary Factors</td>
<td>$X_{ij}^0$</td>
<td>17,958</td>
<td>45,794</td>
</tr>
<tr>
<td></td>
<td>$X_{ij}^{12}$</td>
<td>10,397</td>
<td>13,845</td>
</tr>
<tr>
<td></td>
<td>$X_{ij}^{13}$</td>
<td>7,561</td>
<td>9,585</td>
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<tr>
<td>Imported Factors</td>
<td>$X_{ij}^{21}$</td>
<td>34,908</td>
<td>0</td>
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<tr>
<td></td>
<td>$X_{ij}^{22}$</td>
<td>13,340</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 11.1: Demand Curve of $X_{211}$ (Oil)

Figure 11.2: Demand Curve of $X_{213}$ (Oil)
12. SUMMARY AND CONCLUSIONS

Input-output analysis is a study of the mutual interdependence of the different sectors of the economy and one of the analytical tools often utilized to analyze an economy in a simplified general equilibrium framework. This input-output analysis makes it possible to systematically calculate and evaluate both direct and indirect economic effects of a change in the economic behavior on an economy.

The idea of input-output analysis is dated back to Quesnay's *Tableau Économique* which illustrates the phenomenon of mutual interdependence among industries by a simple zigzag diagram. W. W. Leontief (1936) contributed to the development and the systematic formulation of the input-output system on the bases of the simplified Walrasian general equilibrium system as well as Quesnay's idea under the assumptions of fixed technological coefficients and constant returns to scale, as reviewed in Chapter 2.

There are two major types of the Leontief input-output system (see Sections 3.2.1-3.2.2 in Chapter 3). One is the open system and another is the closed system. The main difference between two systems is that the former includes the autonomous sector (i.e., exogenous final demand) which is unexplained within the system, but the latter does not contain the autonomous sector. In the closed system, all sectors are endogenous. Hence, the optimal solutions derived from the Leontief open system are
functions of exogenous final demands. Specifically, those reduced solutions depend on fixed technological coefficients and exogenous final demands.

In relation to the internal production structure, the distinct characteristic of the fundamental Leontief input-output system is that the so called Marx-Leontief rigid-type production function with fixed technological coefficients is utilized. That rigid-type production function shows no substitutabilities among the factors of production utilized in the production process. However, we adopted and utilized a specific production function reflecting the factor substitutabilities among the production factors used, which satisfies with the assumptions or the regularity conditions mentioned in Section 4.4.3 in Chapter 4.

With relevance to productions and the resource allocations within the internal production structures of the domestic industries, the factor substitution relations are very crucial because this relation would affect and distort the structures of them. So the factor substitutabilities should be specifically reflected in an economy as a specific quantitative measurement of the substitution degree. The typical method for this purpose is performed through the elasticity of substitution (EOS) which is a quantitative measure of the degree of substitutability (and/or complementarity) among the factors of production utilized in the production process for producing the given level of output. Although the FPCOE and the OCPE also can expound the factor substitution, the utilization of EOS is more convenient and stronger measure because the EOS includes the concepts of FPCOE and OCPE and it has the symmetry property unlike the OCCPE (see Theorem 5.2-5.3).

For our economic analyses, we need the EOS related to the multi-factor production function because the internal production function structure concerning the
input-output system includes many productive factors. In that sense, we utilized the EOS (see the expression (5.13)) derived from Allen's partial EOS (see the expression (5.11)) by making use of the minimal cost function. On the other hand, we get the relationship between the FPCOE and the homogeneity degree of the production function from the general production function and the results of comparative static analysis discussed in Section 4.7 in Chapter 4. They have the reciprocal relationship each other.

In the market mechanism, the price system plays a crucial role in allocating economic resources to the various productive objectives because the allocations of economic resources first correspond to the variations of the prices of economic resources. That is, the variation of the price system causes the distortion of the resource allocation in an economy. As mentioned in Chapter 6, such a variation of the price system comes from the changes of major economic elements such as the world market prices of the imported materials (e.g., oil), taxes, tariffs, and the rate of foreign exchange. These economic element changes have influences on the resource allocation in an economy through the price system. In the end, the distortion of economic resource allocation caused by the change of the price system exerts on economic welfares in an economy because the resource allocation is associated with the distribution of income.

As is stated in Chapter 9, the internal production structure of each of producing industries employs many factors such as the intermediate factors, the primary factors, and the imported foreign factors in the production process. Under taking consideration of it, we utilized the two-stage level CES-type production function (see the expression (9.2)). The two-stage level CES-type function we used has the following
characteristics.

First, it has strong functional separability (see Section 7.2 in Chapter 7). Second, it can include the characteristics of the Cobb-Douglas production function, the Marx-Leontief rigid-type production function, and the perfect-substitution production function which are the specific production functions as shown in Section 8.2 in Chapter 8. Third, it can show the interactions among the productive factors through measuring the EOS by making use of the self-dual cost function of the corresponding production function, i.e., the two-stage level CES-type function (see Section 9.3 in Chapter 9).

So as to analyze the economic impacts of economic element in an optimal situation, we first derived the optimal factor-resource demand functions, the minimal cost function of each producing industry through the two-stage optimization methodology in Chapter 9 or the minimization of the total cost (social cost) of the non-intermediate factors utilized by all domestic producing industries (in Chapter 11). The minimal cost function (see (9.52) in Chapter 9) and the factor-resource demand functions (see Chapter 9 and Chapter 10) derived from the traditional cost-minimization are functions of all factor prices and the level of output and show the homogeneity of degree one and zero, respectively, in the exogenous prices of the non-intermediate factors. By utilizing such functions, we derived the elasticities which are nonnegative. The nonnegativity condition is very crucial in our economic analyses because it means that the production factors utilized in yielding outputs are substitutes.

In Chapter 10, on the basis of outcomes we discussed in Chapter 7 through Chapter 9, we first derived the general price system for the intermediate factors in the framework of input-output analysis. The endogenous prices obtained from such
general system depend on the technology indicator and the exogenous prices of the non-intermediate factors. Such exogenous prices can be changed by the variations in the world market prices of the imported materials (e.g., oil) (taxes (specific and/or *ad valorem*), import tariffs, and the foreign exchange rate). We next developed the general open input-output system with *flexible* technological coefficients which differs from the traditional Leontief input-output system with *fixed* (or *constant*) input-output coefficients. The general open input-output system we developed has a notable characteristic that it can expound the economic impacts of the variations in technological improvements, factor substitutabilities, and relative prices (which are caused by the changes in the world market prices of the imported materials (e.g., oil) and in the rate of taxes and/or import tariffs and in the foreign exchange rate) on the domestic resource allocations.

In Chapter 11, we analyzed the economic effects of the changes in the world market prices of the imported materials (e.g., oil) on the output prices, the equilibrium domestic outputs, and the domestic resource allocations through the ($3 \times 3$) small model based on the general input-output system with *flexible* input-output coefficients. In sum, as results of our economic analyses, we came to the following conclusions. In Section 11.4.4, we calculated (i) the prices of the intermediate factors (commodities), (ii) the equilibrium domestic outputs to be produced to meet the preassigned final demands, (iii) the optimal factor demands for (ii) under the assumptions that: (1) Only the world market prices $W_{211}^*$ and $W_{213}^*$ of the imported materials (e.g., oil) $X_{211}$ and $X_{213}$ increase by 20 percent simultaneously and other factor prices are held constant. Or (1)' Only the world market prices $W_{211}^*$ and $W_{213}^*$ of the imported material (e.g., oil) $X_{211}$ and $X_{213}$ decrease by 20 percent
simultaneously and other factor prices are held constant. (2) All parameters in Table 11.1 through Table 11.5 except for $W_{211}^*$ and $W_{213}^*$ in Table 11.3 are constant. (3) The preassigned final demands are unchanged; $F_1 = 700$ and $F_2 = 900$. (4) The foreign exchange rate $e$ is fixed at $e = 10$.

The results calculated in Section 11.4.4 showed the following. First, as the world market prices of the imported materials (e.g., oil) rise (fall), the domestic prices of such imported production factors also increase (decrease) in the same proportion to the rises (falls) in the world market prices of the imported materials (e.g., oil) (see (11.58)-(11.59) and (11.78)-(11.79)). As a result, the prices (production costs) of the intermediate factors (commodities) increase (decrease) in accordance with the rises (falls) in the world market prices of the imported materials (e.g., oil) (compare (11.49) with (11.69) and (11.90), respectively).

Second, in particular, we can see that the multipliers (coefficients) of the right-hand sides of the equilibrium output equations (see (11.52)-(11.54), (11.72)-(11.74), and (11.92)-(11.94)) vary in response to the changes in the prices of the intermediate factors which actually depend on the variations in the world market prices of the imported materials (e.g., oil) (in this numerical case). These result from the flexible technological coefficients come from (10.26).

Third, the equilibrium domestic outputs to be produced to meet the preassigned final demands in a small importing country (would) increase (decrease) even though the world market prices of the imported materials (e.g., oil) rise (fall). (Compare (11.55)-(11.57) with (11.75)-(11.77) and (11.95)-(11.97), respectively).

Fourth, we can also see that the optimal demand for each of the imported materials (e.g., oil) decreases (increases) when the world market price of the corresponding
imported production factor rises (falls) (see boldfaced values in Table 11.6 through Table 11.8 and Figure 11.1 and Figure 11.2). On the other hand, the hypothetical numerical analysis of Section 11.4.4 shows that other production factors increase (decrease) as the world market prices $W_{211}^*$ and $W_{213}^*$ of the imported production factors (e.g., oil) $X_{211}^*$ and $X_{213}^*$ rise (fall) (see Table 11.6 through Table 11.8).

Under the restricted assumptions (used in Section 11.4.4.1 and Section 11.4.4.2), we showed the numerical analysis based on the hypothetical parameter values (as an example) because the estimation of the parameter values by the econometric method is beyond the scope of this study. However, the economic effects discussed in Section 11.4.4 on the basis of the hypothetical parameter values would not lose the generality except for the magnitudes (sizes) of the specific values calculated. In particular, the own-price effects on the corresponding imported production factors mentioned above are always the obvious facts in an economy.

Finally, we should note that the preassigned final demands are crucial roles in our theoretical input-output system with flexible technological coefficients (as well as the traditional input-output system with fixed input-output coefficients). Thus, we need the final demands which are more accurately preassigned (or forecasted). Then, our theoretical input-output system with flexible technological coefficients plays a significant role in predicting the equilibrium (optimal) domestic outputs and in turn the optimal factor demands.

In addition, in the future, the economic effects of the economic variables such as taxes, tariffs, and the foreign exchange rate on an economy will be analyzed on the basis of the practical parameter values under our theoretical input-output system with flexible technological coefficients.
13. APPENDIX

13.1 APPENDIX I: [Proof of (5.9)]

Consider the two-factor production function \( Q^o = f(X_m, X_{m'}) \), where \( Q^o \) is the given level of output. We assume that the production function is strictly quasi-concave and homogeneous of degree \( \nu \). This means that the isoquant \( Q^o = f(X_m, X_{m'}) \) is convex to the origin.

Taking the total differential of the production function \( Q^o = f(X_m, X_{m'}) \),

\[
dQ^o = f_m dX_m + f_{m'} dX_{m'},
\]

(13.1)

where \( f_m = \frac{\partial f}{\partial X_m} \) and \( f_{m'} = \frac{\partial f}{\partial X_{m'}} \).

Since \( dQ^o = 0 \), we can derive the expression (13.2) from the expression (13.1) as follows:

\[
- \left( \frac{dX_{m'}}{dX_m} \right) = \frac{f_m}{f_{m'}} \quad (= \text{MRTS}_{mm'}).
\]

(13.2)

From the form (5.8),

\[
\sigma_{mm'} = \frac{d \ln \left( \frac{X_m}{X_{m'}} \right)}{d \ln \left( \frac{f_m}{f_{m'}} \right)} = \frac{\frac{f_m}{f_{m'}} \cdot \frac{\partial \left( \frac{X_m}{X_{m'}} \right)}{\partial \left( \frac{f_m}{f_{m'}} \right)}}{\left( \frac{X_m}{X_{m'}} \right) \cdot \frac{\partial \left( \frac{f_m}{f_{m'}} \right)}{\partial \left( \frac{f_m}{f_{m'}} \right)}} \quad (m \neq m').
\]

(13.3)

\[
d \left( \frac{f_m}{f_{m'}} \right) = \left( \frac{f_{mm'} f_{m'm'} - f_{m} f_{m'm'}}{f_{m'}^2} \right) dX_m + \left( \frac{f_{m'm} f_{mm'} - f_{m} f_{m'm'}}{f_{m'}^2} \right) dX_{m'}.
\]
Equivalently, 
\[ d\left( \frac{f_m}{f_{m'}} \right) = dX_m \left[ \left( \frac{f_{m'm'} - f_m f_{m'm}}{f_{m'}^2} \right) + \left( \frac{f_{m'm'} f_m - f_m f_{m'm'} f_{m'}}{f_{m'}^2} \right) \left( \frac{dX_m}{dX_{m'}} \right) \right] \] (13.4)

Substituting the expression (13.2) into the expression (13.4) yields
\[ d\left( \frac{f_m}{f_{m'}} \right) = dX_m \left[ \left( \frac{f_{m'm'} - f_m f_{m'm}}{f_{m'}^2} \right) + \left( \frac{f_{m'm'} f_m - f_m f_{m'm'} f_{m'}}{f_{m'}^2} \right) \left( \frac{f_m}{f_{m'}} \right) \right] \] (13.5)

By making use of Young's theorem, i.e., \( f_{m'm'} = f_m f_{m'} \) and arranging, we can obtain:
\[ d\left( \frac{f_m}{f_{m'}} \right) = \Delta \left( \frac{dX_m}{f_{m'}^3} \right) \] (13.6)

where \( \Delta = f_{m'm'} f_{m'}^2 - 2 f_m f_{m'm'} f_{m'} + f_{m'm'} f_{m'}^2 \).

\[ d\left( \frac{X_m}{X_{m'}} \right) = \frac{X_{m'} dX_m - X_m dX_{m'}}{X_{m'}^2} \]
\[ = \frac{dX_{m'}}{X_{m'}} \left( \frac{dX_m}{X_m} - \frac{X_m}{X_{m'}} \right) \] (13.7)

From the expression (13.2),
\[ \frac{dX_m}{dX_{m'}} = \left( -\frac{f_{m'}}{f_m} \right)^{-1} \]
\[ = \left( -\frac{f_m}{f_{m'}} \right) \] (13.8)

Putting (13.8) into (13.7) leads to:
\[ d\left( \frac{X_m}{X_{m'}} \right) = \frac{dX_{m'}}{X_{m'}} \left( -\frac{f_m}{f_{m'}} - \frac{X_m}{X_{m'}} \right) \]
\[ = -\frac{dX_{m'}}{f_m X_{m'}^2} \left( f_m X_m + f_{m'} X_{m'} \right) \] (13.9)
Substituting (13.5) and (13.9) into (13.3), we get:

\[ \sigma_{mm'} = (-\frac{dX_m'}{dX_m}) \left[ \frac{f_m^2(f_mX_m + f_m'X_m')}{X_mX_m'\Delta} \right]. \tag{13.10} \]

(13.2) \rightarrow (13.10):

\[ \sigma_{mm'} = -\frac{f_mf_m'(f_mX_m + f_m'X_m')}{X_mX_m'\Delta}, \tag{13.11} \]

where \( \Delta < 0 \) by the assumption that the production function \( f(X) \) is strictly quasi-concave (see [H.4]).

Now we assume that the production function \( Q^\circ = f(X) \) has the \( \nu \)-degree homogeneity. Then, by Euler's theorem, we get:

\[ f_mX_m + f_m'X_m' = \nu f(X_m, X_m') = \nu Q^\circ. \tag{13.12} \]

Since the production function \( f \) is homogeneous of degree \( \nu \), the marginal productivities of two factors, \( f_m \) and \( f_m' \), are homogeneous of degree \( (\nu - 1) \). Hence, application of Euler's theorem to \( f_m \) leads to:

\[ f_{mm}X_m + f_{mm'}X_m' = (\nu - 1)f_m. \tag{13.13} \]

Or,

\[ f_{mm} = \frac{(\nu - 1)f_m}{X_m} - f_{mm'} \left( \frac{X_m'}{X_m} \right). \tag{13.14} \]

Similarly, we can get the following result by applying Euler's theorem to \( f_m' \).

\[ f_{m'm}X_m + f_{m'm'}X_m' = (\nu - 1)f_m'. \tag{13.15} \]

Or,

\[ f_{m'm} = \frac{(\nu - 1)f_m'}{X_m'} - f_{m'm'} \left( \frac{X_m}{X_m'} \right). \tag{13.16} \]
Let the denominator of (13.11) be

\[ J = X_mX_{m'}\Delta. \quad (13.17) \]

Factoring out \((X_mX_{m'})\) and \(f_{mm'}\) after substituting the expressions (13.14) and (13.16) into (13.17) leads to:

\[ J = f_{mm'}\left\{ \left( (\nu - 1)\frac{f_{mm'}}{f_{mm'}}(f_mX_m + f_{m'}X_{m'}) \right) - (f_mX_m + f_{m'}X_{m'})^2 \right\} \quad (13.18) \]

By substituting the expression (13.12) into (13.18), we yield:

\[ J = \nu Q^o [(\nu - 1)f_mf_{m'} - \nu Q^o f_{mm'}]. \quad (13.19) \]

Substitution of the expressions (13.12) and (13.19) into (13.11) leads to:

\[ \sigma_{mm'} = \frac{f_mf_{m'}}{\nu Q^o f_{mm'} + (\nu - 1)f_mf_{m'}}. \quad \text{Q.E.D.} \quad (13.20) \]

13.2 APPENDIX II: [Proof of Theorem 5.2]

Consider the expression (5.11):

\[ \sigma_{mm'} = \frac{\sum_{k=1}^{n} f_kX_k}{X_mX_{m'} \det(\hat{H})} \quad \text{for} \quad m \neq m'. \quad (13.21) \]

Let

\[ \alpha = \left( \sum_{k=1}^{n} f_kX_k \right)(X_mX_{m'})^{-1}(\det(\hat{H}))^{-1} \quad \text{for} \quad m \neq m'. \quad (13.22) \]

(13.22) \rightarrow (13.21):

\[ \sigma_{mm'} = \alpha \det(\hat{H}_{mm'}). \quad (13.23) \]

Or,

\[ \sigma_{mm'} = \alpha \det(\hat{H}_{m'm}). \quad (13.24) \]
Since the cofactor matrix $F$ is symmetric (see (4.30)), $\det(\tilde{H}_{mm'}) = \det(\tilde{H}_{m'm})$.

Thus, combining (13.23) and (13.24) leads to:

$$\sigma_{mm'} = \sigma_{m'm} \ (m \neq m'). \ \text{Q.E.D.}$$

### 13.3 APPENDIX III: [Proof of Theorem 5.3]

From the expression (4.11):

$$\sigma_{mm'} = \frac{\sum_{k=1}^{n} f_k X_k^o \det(\tilde{H}_{mm'})}{X_m^o X_{m'}^o \det(\tilde{H})} \text{ for } m \neq m'. \ \ (13.25)$$

From equilibrium conditions (FOCs) of the cost-minimization problem,

$$f_k = \mu^{-1} W_k, \ \ (13.26)$$

where $k \in [1, n]$.

(13.26) $\rightarrow$ (13.25):

$$\sigma_{mm'} = \frac{\mu^{-1} \sum_{k=1}^{n} W_k X_k^o \det(\tilde{H}_{mm'})}{X_m^o X_{m'}^o \det(\tilde{H})} \text{ for } m \neq m'. \ \ (13.27)$$

From the minimal total cost function,

$$\sum_{k=1}^{n} W_k X_k^o = C(W, Q). \ \ (13.28)$$

From the expression (4.40),

$$\mu \frac{d X_m^o}{d W_{m'}} = \left[ \det(\tilde{H}) \right]^{-1} \det(\tilde{H}_{m'm}), \ \ (13.29)$$

where $m, m' \in [1, n]$.

(13.28) and (13.29) $\rightarrow$ (13.27):

$$\sigma_{mm'} = \frac{C(W, Q) \ d X_m}{X_m^o X_{m'}^o \ d W_{m'}}. \ \ (13.30)$$
Multiplying the left hand side of the expression (13.30) by \( \frac{W_m'}{W_m'} \),

\[ \sigma_{mm'} = \left( \frac{C(W, Q)}{X_m'} \right) \left( \frac{W_m'}{X_m'} \right) \frac{dX_m}{dW_m'}. \]  (13.31)

From the expression (5.4),

\[ N(X_m^\circ, W_m') = \frac{W_m'}{X_m^\circ} \frac{\partial X_m^\circ}{\partial W_m'}. \]  (13.32)

where \( m \in [1, n] \),

\[(13.32) \rightarrow (13.31):\]

\[ \sigma_{mm'} = \left( \frac{C(W, Q)}{X_m'} \right) N(X_m^\circ, W_m'), (m \neq m'). \]  (13.33)

Let

\[ \Upsilon_{m'} = \frac{(W_m'X_m^\circ)}{C(W, Q)}. \]  (13.34)

\[(13.34) \rightarrow (13.33):\]

\[ \sigma_{mm'} = \Upsilon_{m'}^{-1} N(X_m^\circ, W_m') \text{ for } m \neq m', \]

where \( \Upsilon_{m'} = (W_m'X_m^\circ) \). Q.E.D.

13.4 APPENDIX IV: [Proof of Theorem 5.4]

\[ \sigma_{mm'} = \Upsilon_{m'}^{-1} N(X_m^\circ, W_m'), (m, m' \in [1, n], \text{ for } m \neq m', \]  (13.35)

where \( \Upsilon_{m'} > 0 \).
(i) \( N(X_m^o, W_{m'}) > 0 \Rightarrow \sigma_{mm'} > 0 \). According to Theorem 5.1, if \( \sigma_{mm'} > 0 \), then factors \( X_m \) and \( X_{m'} \) are substitutes.

(ii) \( N(X_m^o, W_{m'}) < 0 \Rightarrow \sigma_{mm'} < 0 \). According to Theorem 5.1, if \( \sigma_{mm'} < 0 \), then factors \( X_m \) and \( X_{m'} \) are complements.

(iii) \( N(X_m^o, W_{m'}) = 0 \Rightarrow \sigma_{mm'} = 0 \). According to Theorem 5.1, if \( \sigma_{mm'} = 0 \), then factors \( X_m \) and \( X_{m'} \) are independents. Q.E.D.

13.5 APPENDIX V: [Proof of the Expression (5.13)]

From the expression (13.30) of APPENDIX III,

\[
\sigma_{mm'} = \frac{C(W, Q)}{X_m^o X_{m'}^o} \frac{dX_m}{dW_{m'}}. \quad (13.36)
\]

By Shephard’s lemma or the envelope theorem,

\[
\frac{\partial C(W, Q)}{\partial W_m} = X_m^o, \quad (13.37)
\]

and

\[
\frac{\partial C(W, Q)}{\partial W_{m'}} = X_{m'}^o, \quad (13.38)
\]

and

\[
\frac{dX_m}{dW_{m'}} = \frac{\partial^2 C(W, Q)}{\partial W_m \partial W_{m'}} = C_m m' m (by \ Young's \ \text{theorem}). \quad (13.39)
\]

(13.37)-(13.39) \( \rightarrow \) (13.36):

\[
\sigma_{mm'} = \frac{C(W, Q)(\frac{\partial^2 C}{\partial W_m \partial W_{m'}})}{(\frac{\partial C}{\partial W_m})(\frac{\partial C}{\partial W_{m'}})}. \quad (13.40)
\]

Let \( C_m = \frac{\partial C}{\partial W_m}, C_m' = \frac{\partial C}{\partial W_m'}, \) and \( C_{mm'} = C_m m' m = \frac{\partial^2 C}{\partial W_m \partial W_{m'}}. \)
Then,

$$\sigma_{mm'} = \left[ CC_{mm'} \right] \left[ C_m C_{m'} \right]^{-1}, \quad (13.41)$$

where \( m, m' \in [1, n] \) and \( m \neq m' \). Q.E.D.


Company.


Koopmans, Tjalling C. (1951). “Analysis of Production as an Efficient Combination


