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An analytical expression for the electric field in a half space conductor, due to alternating current injected at the surface, is derived. Assuming that the injected current flows in wires perpendicular to the surface of the test piece, the problem can be formulated in terms of a single, transverse magnetic, potential. Considering at first one wire, the cylindrical symmetry permits simplification of the calculation by use of the Hankel transform. The final result for a system with two current-carrying wires is obtained by superposition. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1630700]

I. INTRODUCTION

An analytical expression for the electric field in a half space conductor, due to alternating current injected at the surface, is derived. Knowledge of the electric field is useful in accurately interpreting alternating-current potential difference (ACPD) measurements. In the ACPD method, measurements are typically made using a four-point probe. Two of the four contact points pass alternating current into, and out of, the sample. The remaining two contacts form part of a high impedance circuit for measurement of potential drop. While this work is motivated by applications in ACPD, the focus of this article is the form of the electric field. In earlier work in which the ACPD method has been used for detecting and characterizing defects,[1] it has been assumed that electric current flowing in the region midway between the electrodes is approximately uniform. Theoretical interpretation of the measurements has been based on this assumption.[2] Here, a detailed description of the electric field and current in the conductor is sought.

Previous analytical work related to this subject is that by Chen *et al.*[3] in which a two-dimensional solution for current injection from shallow *p-n* junctions is derived. As in this article, cylindrical symmetry of the system is exploited and the analysis simplified by use of the Hankel transform.

II. THEORY

A. Formulation

In order to determine the electric field, $\mathbf{E}(\mathbf{r})$, in a conductive half space due to alternating current injected at the surface, the problem can be formulated as a superposition of two cylindrically symmetric systems. In one, current flows into the test piece by means of a wire contact perpendicular to the surface of the conductor, Fig. 1. In the second, the current flows out of the conductor through a similar wire.

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Analysis of the problem shown in Fig. 1 is simplified by expressing the electric field in terms of a single, transverse magnetic, potential.

Consider a time-harmonic current source varying as the real part of $\mathbf{J}(\mathbf{r})\exp(-i\omega t)$, where ω is the angular frequency of the excitation. In this case the source is essentially a wire carrying current I as shown in Fig. 1. It is assumed that the material properties are linear and that the conductor has conductivity σ_2 and scalar permeability μ_2. From Maxwell's equations, the electric field in the nonconductive region Ω_1 is a solution of

$$\nabla \times \nabla \times \mathbf{E}_1(\mathbf{r}) = i\omega\mu_0 \mathbf{J}(\mathbf{r}), \quad z < 0, \tag{1}$$

where μ_0 is the permeability of free space. The electric field in the conductive region, Ω_2, is a solution of

$$\nabla \times \nabla \times \mathbf{E}_2(\mathbf{r}) - k^2 \mathbf{E}_2(\mathbf{r}) = 0, \quad z > 0 \tag{2}$$

with $k^2 = i\omega\mu_2\sigma_2$. Equation (1) implies that $\nabla \cdot \mathbf{J} = 0$ and Eq. (2) implies that $\nabla \cdot \mathbf{E}_2 = 0$. It shall be assumed that $\nabla \cdot \mathbf{E}_1 = 0$ for $z < 0$. Then $\mathbf{E}_1$ may be written as the curl of a vector potential throughout Ω_1, including the source region.[4] The vector potential is constructed using two scalar potentials defined with respect to the direction perpendicular to the air-conductor interface[5]

$$\mathbf{E}_j(\mathbf{r}) = i\omega\mu_j \nabla \times [\hat{z}\psi_j'(\mathbf{r}) - \nabla \times \hat{z}\psi_j''(\mathbf{r})]. \tag{3}$$

In Eq. (3), the subscript j denotes either region 1 or 2, μ_j is the scalar permeability, $\hat{z}$ is a unit vector in the z direction, ψ' is a transverse electric (TE) potential and ψ'' is a transverse magnetic (TM) potential. As described in Ref. 4, uncoupled equations for the potentials may be obtained by substituting the expression for the electric field, given in Eq. (3), into Eqs. (1) and (2). Employing the transverse differential operator

$$\nabla_z \equiv \nabla - \hat{z}\frac{\partial}{\partial z},$$

the following uncoupled governing equations for the TE potential are obtained:
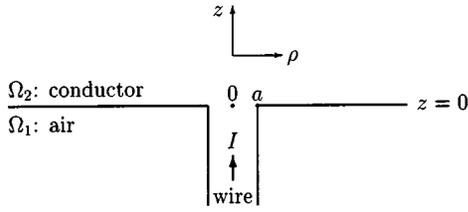
FIG. 1. Cross section of a wire, radius a, carrying current I in contact with a half space conductor. The system is cylindrically symmetric.

$$\nabla^2 \nabla_z^2 \psi_1'(\mathbf{r}) = -\nabla \cdot [\hat{z} \times \mathbf{J}(\mathbf{r})], \quad z<0, \qquad (4)$$

$$(\nabla^2 + k^2)\nabla_z^2 \psi_2'(\mathbf{r}) = 0, \quad z>0. \qquad (5)$$

For the TM potential

$$\nabla^2 \nabla_z^2 \psi_1''(\mathbf{r}) = -\hat{z} \cdot \mathbf{J}(\mathbf{r}), \quad z<0, \qquad (6)$$

$$(\nabla^2 + k^2)\nabla_z^2 \psi_2''(\mathbf{r}) = 0, \quad z>0. \qquad (7)$$

It is assumed that the potentials vanish as $|\mathbf{r}| \rightarrow \infty$.[4] For Eqs. (5) and (7) to be satisfied it is sufficient that ψ_2' and ψ_2'' satisfy the Helmholtz equation. Similarly, in source-free regions, ψ_1' and ψ_1'' satisfy the Laplace equation according to Eqs. (4) and (6).

Assume that the wires are perpendicular to the conductor surface and the radius of the wire, a, is sufficiently small that

$$\mathbf{J} = J_z \hat{z}, \quad z<0. \qquad (8)$$

This is certainly true sufficiently far from the conducting half space but it will be assumed true as $z \rightarrow 0-$. Later, the limit $a \rightarrow 0$ will be taken. Hence the assumption expressed in Eq. (8) is reasonable. Then, since $\hat{z} \times \mathbf{J} = 0$, the source term for the TE potentials in Eqs. (4) and (5) vanishes, and hence $\psi_j' = 0, j = 1,2$. Consequently, only the TM potential is required to describe the field. If the current-carrying wires are not perpendicular to the conductor surface, the TE component of $\mathbf{E}$ exists in the conductor. The cylindrical symmetry of the system is lost and, in general, $\mathbf{E}$ has three components.

A new potential is defined as follows:

$$\Psi_j = \nabla_z^2 \psi_j''. \qquad (9)$$

Then Eqs. (6) and (7) become

$$\nabla^2 \Psi_1(\mathbf{r}) = -\hat{z} \cdot \mathbf{J}(\mathbf{r}), \quad z<0, \qquad (10)$$

$$(\nabla^2 + k^2)\Psi_2(\mathbf{r}) = 0, \quad z>0. \qquad (11)$$

From Eq. (3), retaining only the TM potential, the two components of the electric field in the conductor can be expressed

$$E_z(\mathbf{r}) = i\omega\mu\Psi(\mathbf{r}), \qquad (12)$$

$$E_\rho(\mathbf{r}) = -i\omega\mu \frac{\partial^2 \psi''(\mathbf{r})}{\partial\rho\,\partial z}, \qquad (13)$$

where ρ and z are coordinates of the cylindrical system. The subscript 2 is dropped in Eqs. (12) and (13), and from here on. It is not convenient to express E_ρ in terms of Ψ. Rather, E_ρ will be obtained from Eq. (13) by means of relationship (9).

B. Scalar potential

1. Governing equation and boundary conditions

Now solve Helmholtz Eq. (11) for the scalar potential Ψ in the conductor, subject to certain boundary conditions at its surface. If Eq. (11) is written in cylindrical coordinates and Ψ is independent of azimuthal angle ϕ, then

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} + k^2 \right)\Psi(\rho,z) = 0. \qquad (14)$$

Appropriate boundary conditions can be obtained by considering the normal component of the current density at the conductor surface. At $z=0$, $J_z=0$ for $\rho>a$, outside the region of the wire contact. For $\rho \leq a$, the current density matches that in the wire

$$J_z(\rho,0) = \begin{cases} \dfrac{I}{\pi a^2}, & \rho \leq a, \\[2mm] 0, & \rho > a. \end{cases} \qquad (15)$$

In Eq. (15) it has been assumed that the current density in the wire is uniform with respect to the radial coordinate ρ. This assumption is reasonable if the radius of the wire is somewhat smaller than the electromagnetic skin depth in the wire. Later, the limit $a \rightarrow 0$ will be taken. In the limiting case it is reasonable to assume uniform current density in the wire, even for arbitrary frequency.

Now write Eq. (15) in terms of Ψ to obtain the following boundary condition:

$$\Psi(\rho,0) = C, \qquad (16)$$

where

$$C = \begin{cases} \dfrac{I}{\pi(ka)^2}, & \rho \leq a, \\[2mm] 0, & \rho > a. \end{cases} \qquad (17)$$

2. Solution

The solution of Eq. (14) subject to the boundary condition expressed in Eqs. (16) and (17) proceeds as follows. The radial variable, ρ, can be conveniently removed by application of the Hankel transform.[3] The Hankel transform of order m of a function $f(\rho)$ is given by[6,7]

$$\widetilde{f}(\kappa) = \int_0^\infty f(\rho)J_m(\kappa\rho)\rho\,d\rho. \qquad (18)$$

The Hankel transform is self-inverse, hence

$$f(\rho) = \int_0^\infty \widetilde{f}(\kappa)J_m(\kappa\rho)\kappa\,d\kappa. \qquad (19)$$

Now apply the zero-order Hankel transform to Eq. (14), making use of the following identity[6] ($f(\rho)$ is assumed to be such that the terms $\rho J_0(\kappa\rho)\partial f(\rho)/\partial\rho$ and $\rho f(\rho)\partial J_0(\kappa\rho)/\partial\rho$ vanish at both limits):

$$\int_0^\infty \left[\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} \right)f(\rho) \right]J_0(\kappa\rho)\rho\,d\rho \equiv -\kappa^2\widetilde{f}(\kappa). \qquad (20)$$

The result is a one-dimensional Helmholtz equation

$$\left(\frac{\partial^2}{\partial z^2} - \gamma^2\right)\tilde{\Psi}(\kappa,z) = 0, \quad z > 0, \tag{21}$$

wherein $\gamma^2 = \kappa^2 - k^2$. For γ the root with positive real part is taken.

The general solution of Eq. (21) is

$$\tilde{\Psi}(\kappa,z) = A(\kappa)e^{-\gamma z} + B(\kappa)e^{\gamma z} \tag{22}$$

but here $B(\kappa)$ is zero since Ψ must remain finite as $z \to \infty$. Applying the inverse transform to $\tilde{\Psi}$ then yields

$$\Psi(\rho,z) = \int_0^\infty A(\kappa)e^{-\gamma z}J_0(\kappa\rho)\kappa\,d\kappa. \tag{23}$$

$A(\kappa)$ will now be sought from the boundary condition (16). At $z = 0$,

$$C = \int_0^\infty A(\kappa)J_0(\kappa\rho)\kappa\,d\kappa. \tag{24}$$

$A(\kappa)$ is extracted from Eq. (24) by using the Fourier–Bessel integral (Ref. 8, result 6.3.62) which may be expressed as

$$f(z) = \int_0^\infty J_m(\alpha z)\alpha\,d\alpha \int_0^\infty f(\zeta)J_m(\alpha\zeta)\zeta\,d\zeta. \tag{25}$$

Multiply both sides of Eq. (24) by $\int_0^\infty J_0(\kappa'\rho)\rho\,d\rho$. Reverse the order of integration on the right-hand side and simplify, by use of Eq. (25), to give

$$A(\kappa) = C\int_0^a J_0(\kappa\rho)\rho\,d\rho. \tag{26}$$

The upper limit of integration in Eq. (26) is a since $C = 0$ for $\rho > a$. Evaluation of the integral in Eq. (26) (Ref. 9, result 9.1.30) yields the following result for $A(\kappa)$:

$$A(\kappa) = \frac{I}{\pi k^2}\frac{J_1(\kappa a)}{\kappa a}. \tag{27}$$

Now insert the above expression for $A(\kappa)$ into Eq. (23) to obtain

$$\Psi(\rho,z) = \frac{I}{\pi k^2 a}\int_0^\infty e^{-\gamma z}J_1(\kappa a)J_0(\kappa\rho)d\kappa. \tag{28}$$

The integral in Eq. (28) cannot be evaluated analytically for arbitrary z, but putting $z = 0$ it is found that (Ref. 10, result 6.512.3)

$$\Psi(\rho,0) = 0, \quad \rho > a, \tag{29}$$

in accordance with boundary condition (16).

As mentioned earlier, the limit $a \to 0$ will now be taken. The reason for this is that the radius of the contact point between the wire and the conductor surface may be assumed small in the present context, where the separation of the current-carrying wires is assumed large compared with their radius. It was necessary to include a in the calculation up to this point in order that the current density in the wire be finite.

In Ref. 9, result 9.1.7 the result

$$\lim_{z \to 0} J_\nu(z) \sim \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}$$

allows the following limit to be found:

$$\lim_{z \to 0}\frac{J_1(z)}{z} \sim \frac{1}{2}.$$

Hence,

$$\lim_{a \to 0} A(\kappa) \sim \frac{I}{2\pi k^2}. \tag{30}$$

If $A(\kappa)$, as given in Eq. (30), is now inserted into Eq. (23), the following expression for Ψ is obtained:

$$\Psi(\rho,z) = \frac{I}{2\pi k^2}\int_0^\infty e^{-\gamma z}J_0(\kappa\rho)\kappa\,d\kappa. \tag{31}$$

In expression (31) it is understood that the limit $a \to 0$ has now been taken. An analytical result for the integral in Eq. (31) is given in Ref. 7, result 8.2.23 (reproduced in the Appendix). It is found that

$$\Psi(\rho,z) = -\frac{I}{2\pi}\frac{ikz}{(ikr)^3}e^{ikr}(1 - ikr), \quad z > 0, \tag{32}$$

wherein $r^2 = \rho^2 + z^2$.

C. Electric field

Analytic forms for the two components of the electric field in the conductor, Eqs. (12) and (13), will now be obtained. It is a trivial matter to obtain E_z from result (32) since E_z and Ψ are simply related by the factor $i\omega\mu$, Eq. (12)

$$E_z(\mathbf{r}) = -\frac{i\omega\mu I}{2\pi}\frac{ikz}{(ikr)^3}e^{ikr}(1 - ikr), \quad z > 0. \tag{33}$$

From Eq. (29)

$$E_z(\rho,0) = 0, \quad \rho > 0, \tag{34}$$

as required by the boundary condition on J_z at the surface of the conductor, away from the current injection point.

To obtain E_ρ requires more work. First, apply the zero-order Hankel transform to Eq. (9) to establish

$$\tilde{\psi}''(\kappa,z) = -\frac{\tilde{\Psi}(\kappa,z)}{\kappa^2}. \tag{35}$$

Then, from Eq. (31),

$$\psi''(\rho,z) = -\frac{I}{2\pi k^2}\int_0^\infty \frac{1}{\kappa}e^{-\gamma z}J_0(\kappa\rho)d\kappa. \tag{36}$$

This integral cannot be evaluated since it diverges logarithmically. The divergent behavior is due to the fact that only one current-carrying wire is considered at this stage in the analysis. Physically, the current must flow in a closed loop. If an additional wire is considered, in which current flows out of the conductor, the integral corresponding to that in Eq. (36) is well defined. Here, the mathematical development of an expression for the electric field is simplified by considering only one wire. The process of superposition to obtain a physical result for two wires carrying opposing currents, as shown in Fig. 2, is done later.

To obtain E_ρ, it is necessary to differentiate ψ'' with respect to ρ. Applying the differential operator to Eq. (36) and reversing the order of differentiation and integration gives

$$\frac{\partial \psi''(\rho,z)}{\partial \rho} = \frac{I}{2\pi k^2} \int_0^\infty e^{-\gamma z} J_1(\kappa \rho) d\kappa. \tag{37}$$

An analytic result for this integral is available in Ref. 7, result 8.4.9, and is reproduced in the Appendix. Then

$$\frac{\partial \psi''(\rho,z)}{\partial \rho} = \frac{I}{2\pi k^2 \rho} \left(e^{ikz} - \frac{z}{r} e^{ikr} \right), \quad z \geqslant 0. \tag{38}$$

Strictly, relation (38) as obtained by use of Ref. 7, result 8.4.9 is valid only for $z > 0$. Putting $z = 0$ in Eq. (37) and invoking Ref. 10, result 6.511.1, however, shows that

$$\frac{\partial \psi''(\rho,0)}{\partial \rho} = \frac{I}{2\pi k^2 \rho}, \tag{39}$$

which may be obtained by putting $z = 0$ in Eq. (38). Hence, relation (38) holds for $z \geqslant 0$.

Referring to Eq. (13), it is necessary to take the derivative of Eq. (38) with respect to z in order to obtain E_ρ. Finally,

$$E_\rho(\mathbf{r}) = \frac{i\omega\mu I}{2\pi} \frac{1}{ik\rho} \left\{ e^{ikz} - \frac{e^{ikr}}{ikr} \left[1 + \frac{(ikz)^2}{ikr} \left(1 - \frac{1}{ikr} \right) \right] \right\}. \tag{40}$$

Considering the forms of E_z and E_ρ as given in Eqs. (33) and (40), respectively, it can be seen that the electric field exhibits correct behavior in certain simple cases. On the axis of the cylindrical system, which coincides with the axis of the current-carrying wire, $E_\rho(0,z) = 0$. E_z is symmetric with respect to ρ and both E_z and $E_\rho \to 0$ as $r \to \infty$. In the far field, the electric field is dominated by the first term in Eq. (40) and the associated current density is

$$J_\rho(\mathbf{r}) \approx \frac{k^2 I}{2\pi} \frac{e^{ikz}}{ik\rho}, \quad \text{as } r \to \infty. \tag{41}$$

If the far field current density, given in Eq. (41), is integrated over a cylindrical surface of large radius extending from $z = 0$ to ∞, the result is simply I, as it should be. Last, in the static limit of direct current, in which $k \to 0$, the current density in the conductor radiates uniformly from the point of injection

$$J_r(\mathbf{r}) = \frac{I}{2\pi r^2}. \tag{42}$$

It can be shown that the result of this article reduces to the form given in Eq. (42) by noting that $J_r = (r/\rho)J_\rho$ and letting $k \to 0$ in Eq. (40). Equivalently, Eq. (33) can be used with $J_r = (r/z)J_z$.

Consider now the system shown in Fig. 2. The electric field in the conductor can be obtained by superposition of fields separately associated with the two current-carrying wires, as determined above. In the conductor, the total electric field $\mathbf{E}^T$ is given by

$$\mathbf{E}^T(\mathbf{r}) = \mathbf{E}(\mathbf{r}_+) - \mathbf{E}(\mathbf{r}_-), \tag{43}$$
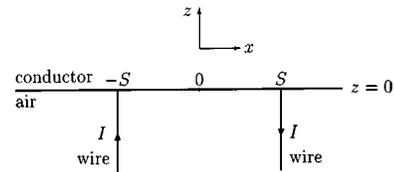

FIG. 2. Cross section of a half-space conductor and two current-carrying wires. The plane $y = 0$ is shown.

due to alternating current injected and extracted by contact wires at $x = \pm S$. In Eq. (43), $r_\pm = \sqrt{(x \pm S)^2 + y^2 + z^2}$ and the components of $\mathbf{E}$ are given in Eqs. (33) and (40).

III. CONCLUSION

An elegant analytical expression for the electric field in a half space conductor due to alternating current injected at the surface has been derived. The method of solution, in which the Hankel transform is used, suggests a solution for a layered half space, or plate, in the form of a series expansion. Explicitly, the expression for the scalar potential, Eq. (31), becomes a series of similar terms in which the exponents may include the thickness of the layer or plate. This is the subject of a future article.

ACKNOWLEDGMENT

APPENDIX

The following identities were used in evaluating the integrals which appear in Eqs. (31) and (37), respectively. From Ref. 7, result 8.2.23, for $y > 0$

$$\int_0^\infty \sqrt{x} e^{-\alpha\sqrt{x^2+\beta^2}} J_0(xy) \sqrt{xy} \, dx$$

$$= \frac{\alpha\sqrt{y}}{(y^2+\alpha^2)^{3/2}} e^{-\beta\sqrt{y^2+\alpha^2}} (1 + \beta\sqrt{y^2+\alpha^2}), \tag{A1}$$

with the restrictions Re $\alpha > 0$ and Re $\beta > 0$. Also from Ref. 7, result 8.4.9

$$\int_0^\infty \frac{1}{\sqrt{x}} e^{-\alpha\sqrt{x^2+\beta^2}} J_1(xy) \sqrt{xy} \, dx$$

$$= \frac{1}{\sqrt{y}} \left(e^{-\beta\alpha} - \frac{\alpha}{\sqrt{y^2+\alpha^2}} e^{-\beta\sqrt{y^2+\alpha^2}} \right), \tag{A2}$$

with the same restrictions on y, α and β.

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