Information order in monotone decision problems under uncertainty*

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Abstract

This paper examines the robustness of Lehmann's ranking of experiments (Lehmann, 1988) for decisionmakers who are uncertainty-averse à la Cerreia-Vioglio et al. (2011). We show that, assuming commitment, for all uncertainty-averse indices satisfying some mild assumptions, Lehmann's informativeness ranking is equivalent to the induced uncertainty-averse value ranking of experiments for all agents with single-crossing vNM utility indices (Theorem 1). Moreover, Lehmann ranking can also be detected by varying the uncertainty-averse indices for a fixed finite collection of vNM utility indices (Theorem 2). Our findings suggest that Lehmann's ranking can be a useful enrichment of Blackwell's ranking for monotone decision problems even if ambiguity is present. We apply our results to social aggregation of information preferences and investment decision problems.

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1 Introduction

Blackwell's (1951a) theorem says Experiment I is more (Blackwell-)informative than Experiment II if and only if Experiment I is more valuable than Experiment II *for all* expected-utility (EU) maximizers.

A common critique levelled on Blackwell's informativeness ranking is that the order is very coarse.[1] This is exemplified by how it often fails to compare experiments with intuitively ranked information content. For instance, consider two experiments $H = \omega + \epsilon_H$ and $F = \omega + \epsilon_F$, where $\epsilon_H \sim Uniform[-a, a]$ and $\epsilon_F \sim Uniform[-b, b]$ are independent from ω with $0 < a < b$. It is natural to think a signal from experiment H is less noisy than that from F, yet these two experiments cannot be Blackwell-ranked unless b/a is a positive integer.[2] As a result, Blackwell's ranking can be too restrictive for application.

In many economic questions related to information, the decision problem is often not arbitrary but has a certain structure. A fruitful approach is to focus on the class of *monotone decision problems* proposed by Karlin and Rubin (1956), which imposes certain payoff complementarity between action and state. Thus, the optimal decision is monotone in some state relevant variable (Athey, 2002), which is then applied to economic problems such as information acquisitions in auctions and production decisions. This motivates the need for a less demanding order of information characterized by the monotone decision class.

Lehmann (1988) proposes an information order that is equivalent to a higher ranked experiment to be more valuable for all monotone decision problems, which is a natural enrichment of Blackwell's order. Some follow-up papers apply Lehmann's order to economic problems and consider more primitive assumptions on the payoff functions leading to the monotone decision problems. For instance, Athey and Levin (2018) consider supermodular preferences. Quah and Strulovici (2009) examine an even larger family of interval dominance order (IDO) preferences, which includes all the single-crossing preferences.[3]

Recent experimental findings have indicated the prevalence of non-EU preferences in decisionmaking under uncertainty.[4] These findings motivate a re-examination of

[1] This is noted by Blackwell himself in Blackwell and Girshick (1954). See also Lehmann (1988).

[2] See Example 3.

[3] Lehmann order has been applied elsewhere in economic theory. Persico (2000) applies the notion to information acquisition in first and second price auctions. Bergemann and Välimäki (2002) apply to mechanism design. Jewitt (2007) applies to agency problems. Athey and Levin (2018) apply to production with cost uncertainty. Bond (2016) applies the order to study information content of actions or prices in financial market.

[4] See references in Li and Zhou (2016).

the appropriateness of existing information order for non-EU decisionmaker (DM)s, and particularly, uncertainty-averse DMs. For the case of Blackwell's ranking, Çelen (2012) and Li and Zhou (2016) consider the robustness of Blackwell's equivalence theorem for maxmin EU and uncertainty-averse agents, respectively. While Quah and Strulovici (2009) show that Experiment I is more Lehmann informative than Experiment II implies the former is more valuable than the latter for any maxmin EU DM with IDO utility indices, it is largely unknown whether the reverse statement—can Lehamnn order be equivalently characterized by the value ranking within the IDO family— holds for the maxmin EU family or other well-known ambiguity-averse families. The main goal of this paper is to provide a complete answer to this question in the most general case of *uncertainty-averse* DMs.

Throughout the paper, we assume the DM can commit to any signal-contingent strategy and the experiments have monotone likelihood ratio (MLR)-ranked distributions. An immediate observation (Corollary 1), is that a more Lehmann informative experiment is more valuable for all uncertainty-averse agents with vNM utility indices observing IDO.[5] Consequently, any ambiguity-averse DM à la Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) will not choose a less Lehmann informative experiment over a more Lehmann informative one.

In Section 4, we provide two different characterizations of the reverse direction, by exploring variation in either preferences (vNM indices) or beliefs (uncertainty-averse indices). Theorem 1 says that, fixing an uncertainty-averse index G satisfying mild technical assumptions, the Lehmann ranking of any pair of experiments can be distinguished by their value ranking for all single-crossing vNM utility indices. Theorem 2 says that, fixing a finite collection of indices $\{u^1, \ldots, u^I\}$ that satisfy a criterion called *diversity*, the Lehmann ranking can also be detected by value ranking for at least one vNM index in the collection as one varies the uncertainty-averse indices. To link these two Theorems, we show that changing beliefs has the same effect as state-wise reweighting the vNM indices. Consequently, a one-to-one mapping between the vNM index chosen in Theorem 1 and the prior belief chosen in Theorem 2 can be explicitly described for the binary state case (Section 4.1).

Furthermore, in Section 5.1, we apply Theorem 2 to the social ranking of experiments, by reinterpreting a diverse collection of vNM indices $\{u^1, \ldots, u^I\}$ as a society of individuals whose ambiguous beliefs are unspecified. Then Lehmann ranking corresponds to the social ranking of experiments under unanimity aggregation rule in a diverse society. To satisfy diversity, the number of individuals in a society is at least the number of states minus one.

Our main results confirm the robustness of Lehmann's order to ambiguity-averse pref-

[5]This follows directly from Blackwell (1951b) and (Quah and Strulovici, 2009). See Section 3.

erences. The single crossing condition rules out some vNM utility indices, nevertheless, this family is still broad enough for many economic applications. The ambiguity-averse preferences we consider in this paper are a very general class in the literature, which also suggests wide applicability. Since any of the well-known ambiguity-averse preference families[6] leads to the same information rank—the original Lehmann order that was derived in the EU case—the validity of our characterization theorem for Lehmann's order reveals that single-crossing utility rather than the expected utility assumption is the key determinant of the ranking over information structures.

Also, we make a technical contribution in showing that Lehmann's informativeness order can be characterized by comparing the value for all single crossing utility indices, i.e., the "only if" direction of the equivalence theorem. The usual proof of such equivalence theorem, for the case of Blackwell's order, relies crucially on the separating hyperplane theorem to find a suitable vNM index u (Blackwell, 1951a; de Oliveira, 2018; Lehrer et al., 2013). However, the same technique does not apply in our setting, as one cannot guarantee the identified u is single crossing. Instead, for any prior with full support, in Proposition 1, we explicitly construct a single crossing vNM index u, which is pinned down by a suitable transformation of the prior and a few appropriately chosen parameters. The proof technique may be applicable to settings with similar restrictions on the vNM indices.

We provide a formal discussion on the conditions needed for an information order to be prior-free. As most orders on information structures can be derived from comparing their values for certain classes of vNM utility indices, for a Bayesian DM, transforming the utility index state-wise has the same effect as re-weighting the prior belief. In light of this observation, we formally prove that the corresponding information order associated with a class of utility indices must be prior-free as long as the class is closed under state-wise weighting and the prior considered is of full support (Proposition 4). This result helps reconcile why classic Lehmann and Blackwell orders are prior-free; while some other orders[7] are prior-dependent.

The rest of the paper is organized as follows. Section 2 introduces the set-up and relevant concepts. Section 3 and 4 present two characterization Theorems, followed by two economic applications in Section 5. Section 6 discusses the key assumption, special cases of ambiguity preferences, related literature, and relaxing the commitment assumption. Proofs not provided in the main text are relegated to the appendix.

[6]That is, the maxmin EU (Gilboa and Schmeidler, 1989), the variational preferences (Maccheroni, Marinacci, and Rustichini, 2006), the multiplier preferences (Hansen and Sargent, 2001), the confidence preferences (Chateauneuf and Faro, 2009), the smooth preferences (Klibanoff, Marinacci, and Mukerji, 2005), and the second-order EU (Grant, Polak, and Strzalecki, 2009).

[7]Such as the MIO-ND order studied by Athey and Levin (2018).

2 Framework

We first introduce the notation. Let $\Omega \subseteq \mathbb{R}$ be a finite state space and $\pi \in \Delta\Omega$ denote a generic prior belief. An *experiment* or *information structure* generates a signal that is correlated with the states according to some probability. Experiments are denoted by tuples (S, H) and (S, F), where S is the set of signals and assumed to be some compact interval in $\mathbb{R}$ and $H(\cdot|\omega)$, $F(\cdot|\omega) : S \mapsto [0, 1]$ are the cumulative distribution functions (CDFs) of signals conditional on each state. Assume all $H(\cdot|\omega)$ and $F(\cdot|\omega)$ have strictly positive densities $h(\cdot|\omega)$ and $f(\cdot|\omega)$ on S.

Let $A \subseteq \mathbb{R}$ be a compact action space. The DM has payoff function/vNM utility index $u : A \times \Omega \mapsto \mathbb{R}$ that depends on the action taken and the state of nature. For all $a_1, a_2 \in A$, $[a_1, a_2]$ is *an interval of* A if for all element $a \in A$ such that $a_1 < a < a_2$, $a \in [a_1, a_2]$. Say a vNM utility index $u(\cdot, \cdot)$ satisfies the *interval dominance order (IDO)* if for all $a_2 > a_1$ and $w_2 > w_1$, if $u(a_2, \omega_1) \geq u(a, \omega_1)$ for all $a \in [a_1, a_2]$, then $u(a_2, \omega_1) \geq (>) u(a_1, \omega_1)$ implies $u(a_2, \omega_2) \geq (>) u(a_1, \omega_2)$. In this paper, we consider monotone decision problems by focusing on the vNM utility indices that belong to the IDO family (Quah and Strulovici, 2009). A special case of the IDO preference family is the *single crossing* preference family, where for all $a_2 > a_1$ and $\omega_2 > \omega_1$, $u(a_2, \omega_1) - u(a_1, \omega_1) \geq (>) 0$ implies $u(a_2, \omega_2) - u(a_1, \omega_2) \geq (>) 0$.

Given an experiment (S, H), the DM can choose any measurable signal-contingent mixed strategy $\sigma : S \mapsto \Delta A$.[8] We use σ_s to denote the mixed strategy in ΔA if s occurs. Pure strategies, mappings from $S \mapsto A$, are denoted by ϕ, ψ. We are ultimately interested in the set of *monotone pure strategies*, denoted Φ, in which $\phi(\cdot) \in \Phi$ is an increasing function on S.

For an expected utility (EU) maximizing DM with prior π, his EU from committing to a strategy σ under information structure H is

$$U^{EU}(H, \pi, \sigma, u) = \int_\Omega \int_S \int_A u(a, \omega) d\sigma_s(a) dH(s|\omega) d\pi(\omega) \tag{1}$$

The value of information structure H is

$$V^{EU}(H, \pi, u) = \max_{\sigma \in (\Delta A)^S} U^{EU}(H, \pi, \sigma, u).$$

Uncertainty-averse Preferences (UAP)

[8] Let $\mathcal{S}$ be the Boral σ-algebra and μ be the Lebesgue measure of S. The mixed strategy space can be treated as a bounded subset of $\mathcal{L}^\infty(S, \mathcal{S}, \mu, \mathbb{R}^{|A|})$. The dual of $\mathcal{L}^1(S, \mathcal{S}, \mu, \mathbb{R}^{|A|})$ is exactly $\mathcal{L}^\infty(S, \mathcal{S}, \mu, \mathbb{R}^{|A|})$. Endowed with the weak-* topology on $\mathcal{L}^\infty(S, \mathcal{S}, \mu, \mathbb{R}^{|A|})$, the set of mixed strategy is thus compact (recall that A is finite, and S is a compact subset of $\mathbb{R}$.).

We consider DM with uncertainty-averse preferences à la Cerreia-Vioglio et al. (2011), whose ex-ante utility if he takes strategy σ is[9]

$$U^{UA}(H, G, \sigma, u) = \min_{\pi \in \Delta\Omega} G\left(U^{EU}(H, \pi, \sigma, u), \pi\right), \tag{2}$$

where $U^{EU}(H, \pi, \sigma, u)$ is given by (1), function $G : \mathcal{X} \times \Delta(\Omega) \mapsto (-\infty, +\infty]$ is the index of uncertainty aversion, and $\mathcal{X} \subseteq \mathbb{R}$ is the range of u, which is assumed to be unbounded either above or below.

Following Cerreia-Vioglio et al. (2011), G satisfies the following properties: (i) G is quasi-convex and lower semi-continuous; (ii) $G(\cdot, \pi)$ is increasing for all $\pi \in \Delta\Omega$; (iii) $\inf_{\pi \in \Delta(\Omega)} G(x, \pi) = x$ for all $x \in \mathcal{X}$. The uncertainty-averse preferences are monotone and convex. They nest most well-known ambiguity-averse preferences as special cases.[10] Let $\mathcal{G}$ be the family of all uncertainty-averse indices. For all $\pi \in \Delta\Omega$, let $domG(\cdot, \pi) = \{x \in \mathcal{X} : G(x, \pi) < +\infty\}$. We further assume $domG(\cdot, \pi)$ is either empty or $\mathcal{X}$ for all π.[11] Let $M := \{\pi \in \Delta\Omega : domG(\cdot, \pi) = \mathcal{X}\}$, note that M is closed and convex and is interpreted as the set of plausible priors held by the DM.[12]

For a UAP DM with vNM index u and uncertainty-averse index G, maximizing over all mixed strategies, the *ex ante* value of the experiment (S, H) is

$$V^{UA}(H, G, u) = \max_{\sigma \in (\Delta A)^S} \min_{\pi \in \Delta\Omega} G(U^{EU}(H, \pi, \sigma, u), \pi).$$

In the expression above, we assume the DM can commit to any signal-contingent strategy, to abstract away from concerns about dynamic inconsistency due to ambiguity aversion.[13]

Lehmann's order

Fix an information structure $\{H(\cdot|\omega)\}_{\omega \in \Omega}$. Let $\{F(\cdot|\omega)\}_{\omega \in \Omega}$ be another information structure.

Assume the likelihood distributions $\{H(\cdot|\omega)\}_{\omega \in \Omega}$ are *monotone likelihood ratio (MLR)-ranked*, that is, for any fixed $\omega_2 \geq \omega_1$, $\frac{h(s|\omega_2)}{h(s|\omega_1)}$ is an increasing function of s.

Definition 1 (Lehmann's order)**.** Let function $T : S \times \Omega \mapsto S$ such that $H(T(s, \omega)|\omega) =$

[9]For expositional simplicity, we assume the minimizing probability exists; otherwise, we could just replace min by inf and the same argument follows.

[10]See Section 6.3 for the six familiar ambiguity-averse families that are special cases of UAP.

[11]The assumption is very mild and verified in the six special cases we discuss in Section 6.3.

[12]Closeness follows from lower-semi continuity of G and convexity follows from quasi-convexity of G. See Section 6.1 for detailed interpretation of M.

[13]See Li and Zhou (2016) for a discussion on this matter.

$F(s|\omega)$.[14] We say H is *more Lehmann-informative (accurate) than* F, denoted $H \succsim_L F$, if T is an increasing function of ω, for each fixed s.

Figure 1 illustrates the CDFs of two experiments that are Lehmann ranked.

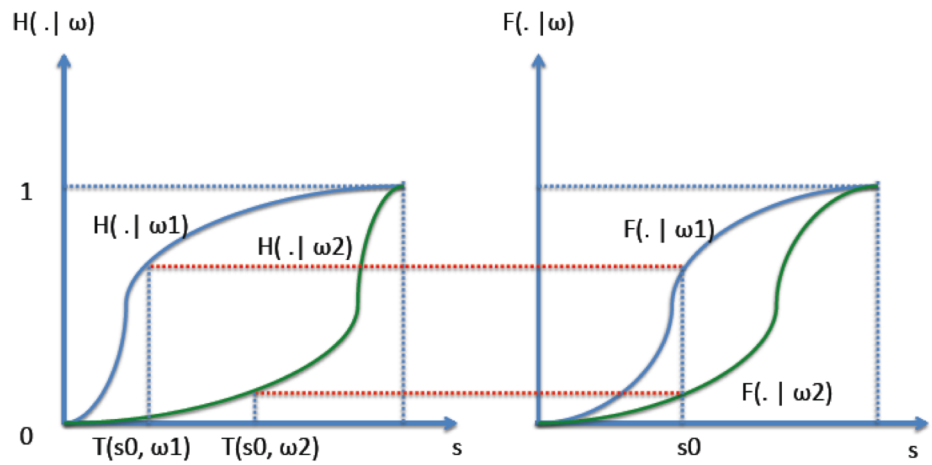

Figure 1: The CDF comparison of two experiments, where H is more Lehmann-informative than F. For fixed s_0, the function $T(s_0, \omega)$, given by $H^{-1}(F(s_0|\omega))$, is increasing in ω.

Below are some examples illustrating Lehmann orders.

Example 1 (Location experiment)**.** Suppose $H(s|\omega) = H_0(s-\omega), F(s|\omega) = F_0(s-\omega)$ for two CDFs H_0 and F_0. Assume both distributions have log-concave densities, then the monotone likelihood ratio property (MLRP) is satisfied. H is more Lehmann-informative than F if and only if [15]

$$\frac{H_0^{-1}[F_0(s')] - H_0^{-1}[F_0(s)]}{s' - s} \leq 1, \quad \text{for all } s < s'.$$

Example 2 (Gaussian experiment)**.** Suppose the signal is generated by adding normal noise to the state: for each θ, the distribution of signal $S = \omega + \rho\epsilon$, where ϵ is the standard normal distribution $\mathcal{N}(0,1)$, and parameter $\rho > 0$. Consider two signals H,

[14] The distribution functions, $\{H(\cdot|\omega)\}_\Omega$ and $\{F(\cdot|\omega)\}_\Omega$, are assumed to have strictly positive densities on S, therefore they are strictly increasing in s. Hence, T is always well-defined and unique.

[15] See Lehmann (1988) Theorem 5.2.

F with parameters ρ_h, ρ_f respectively. Then

$$H(s|\omega) = \bar{\Phi}\left(\frac{s-\omega}{\rho_h}\right), F(s|\omega) = \bar{\Phi}\left(\frac{s-\omega}{\rho_f}\right),$$

where $\bar{\Phi}$ is the cumulative distribution function of $\mathcal{N}(0,1)$. Then the corresponding T-function associated with the two signals is $T(s,\omega) = (1-\frac{\rho_h}{\rho_f})\omega + \frac{\rho_h}{\rho_f}s$. Therefore, H is more Lehmann-informative than F if and only if $\rho_h < \rho_f$.[16]

Example 3. Suppose $H(\cdot|\omega) \sim U[\omega - \frac{1}{2}, \omega + \frac{1}{2}]$, and $F(\cdot|\omega) \sim U[\omega - \frac{\tau}{2}, \omega + \frac{\tau}{2}]$, where $U[a,b]$ denote the uniform distribution on the interval $[a,b]$. Then the corresponding T-function associated with the two signals is $T(s,\omega) = \tau s + (1-\tau)\omega$, and, hence, H is more Lehmann-informative than F for any $\tau \in (0,1)$. However, H is more Blackwell-informative than G if and only if $1/\tau = k$ for some positive integer k.[17]

3 More Lehmann-informative implies more valuable

In this section, we show the "if" direction of Lehmann's equivalence for UAP DMs, that is, a more Lehmann-informative information structure is more valuable for all UAP DMs with IDO preferences, as long as the information structures are MLR-ranked. The statement follows from two complete class theorems (Blackwell, 1951b; Quah and Strulovici, 2009) and a property about Lehmann-ranked information structures shown by Quah and Strulovici (2009).

For any information structure H and strategy σ, the induced state-wise expected utility vector is

$$\mathbf{u}_{H,\sigma} := \int_S \int_A u(a,\omega) d\sigma_s(a) dH(s|\omega) \in \mathbb{R}^{|\Omega|}.$$

A family of strategies $\Psi \subseteq \Delta A^S$ is an *essentially complete class* if for all strategy $\sigma \in \Delta A^S$, there exists a strategy ψ from the family Ψ such that the expected utility vector induced by ψ dominates that of σ state by state, i.e., $\mathbf{u}_{H,\psi}(\omega) \geq \mathbf{u}_{H,\phi}(\omega), \forall \omega \in \Omega$.

By Blackwell (1951b), when there are finitely many distributions over S, where each distribution is atomless, then the set of pure strategies forms an essentially complete class. In our case, each likelihood distribution in the finite set $\{H(\cdot|\omega)\}_{\omega\in\Omega}$ has strictly positive density on S and hence is atomless. Furthermore, we have the following complete class theorem from Quah and Strulovici (2009).

[16] In this example, H is more Blackwell-informative than F if and only if $\rho_h < \rho_f$.

[17] See Lehmann (1988) Theorem 3.1.

Theorem (Quah and Strulovici, 2009, Theorem 4)**.** *Suppose $\{u(\cdot,\omega)\}_{\omega\in\Omega}$ is an IDO family and $\{H(\cdot|\omega)\}_{\omega\in\Omega}$ is MLR-ordered. Then the set of monotone pure strategies Φ forms an essentially complete class (among all the pure strategies).*

Hence, it is without loss to focus on the monotone pure strategies Φ. Moreover, the next result from Quah and Strulovici (2009) says the following: for any monotone pure strategy under the less Lehmann-informative experiment, there exists another monotone pure strategy under the more Lehmann-informative experiment, such that the utility vector induced by the latter state-wise dominates that induced by the former.

Proposition (Quah and Strulovici, 2009, Proposition 9)**.** *Suppose $\{u(\cdot,\omega)\}_{\omega\in\Omega}$ is an IDO family and H is more Lehmann-informative than F. Then for any monotone pure strategy $\psi : S \mapsto A$ under F, there is a monotone pure strategy $\phi : S \mapsto A$ under H such that, for all $\omega \in \Omega$,*

$$\mathbf{u}_{H,\phi}(\omega) \geq \mathbf{u}_{F,\psi}(\omega).$$

With these results, we have the "if" direction for the UAP case.

Corollary 1. *Suppose the information structures H and F are MLR-ranked, and H is more Lehmann-informative than F. Then for all uncertainty averse index G, and for all vNM utility index u obeying the interval dominance order,*

$$V^{UA}(H,G,u) \geq V^{UA}(F,G,u). \tag{3}$$

Proof. [18] Fix H, G and u. By Blackwell (1951b), Quah and Strulovici's (2009) Theorem 4 and Proposition 9, pick any (mixed) strategy σ under F, there is a monotone pure strategy ψ under F, and a monotone pure strategy ϕ under H such that

$$\mathbf{u}_{H,\phi}(\omega) \geq \mathbf{u}_{F,\psi}(\omega) \geq \mathbf{u}_{F,\sigma}(\omega), \quad \forall \omega \in \Omega.$$

Since $U^{EU}(H,\pi,\phi,u) = \int_\Omega \mathbf{u}_{H,\phi} d\pi$ and $U^{EU}(F,\pi,\sigma,u) = \int_\Omega \mathbf{u}_{F,\sigma} d\pi$, we have

$$U^{EU}(H,\pi,\phi,u) \quad \geq \quad U^{EU}(F,\pi,\sigma,u), \forall \pi.$$

Since $G(\cdot,\pi)$ is increasing, the above statement implies

$$G(U^{EU}(H,\pi,\phi,u),\pi) \quad \geq \quad G(U^{EU}(F,\pi,\sigma,u),\pi), \forall \pi,$$

which implies

$$\min_{\pi\in\Delta\Omega} G(U^{EU}(H,\pi,\phi,u),\pi) \quad \geq \quad \min_{\pi\in\Delta\Omega} G(U^{EU}(F,\pi,\sigma,u),\pi).$$

[18] We thank a referee who suggested this simpler proof.

Since σ is chosen arbitrarily, we have

$$\max_{\sigma' \in (\Delta A)^S} \min_{\pi \in \Delta\Omega} G(U^{EU}(H, \pi, \sigma', u), \pi) \geq \max_{\sigma \in (\Delta A)^S} \min_{\pi \in \Delta\Omega} G(U^{EU}(F, \pi, \sigma, u), \pi).$$

□

Remark 1. Clearly, the same result in Corollary 1 applies to the family of vNM utility indecies satisfying the single-crossing property, which is a special case of IDO.

Remark 2. An uncertainty-averse DM can often strictly benefit from mixing between two equally attractive pure strategies, since the UAP utility functional is quasi-concave (in the state-wise utility vector). Nevertheless, since in our setting signal distributions are atomless, the set of state-wise utilities induced by all the pure strategies is convex, thus (ex-post) mixing between any two equally attractive pure strategies does not further benefit the DM.

4 More valuable implies more Lehmann-informative

In this section, we prove the "only if" direction of Lehmann's equivalence theorem for the UAP case, that is, Lehmann's informativeness order can be equivalently defined by comparing the induced values of experiments for UAP DMs with single-crossing utility indices.

We first impose some technical assumptions on G. For any $\delta > 0$, let $\Delta^\delta := \{\pi \in \Delta\Omega : \pi(\omega) \geq \delta, \forall \omega\}$.

Assumption 1. $\{G(\cdot, \pi)\}_{\pi \in M}$ is uniformly equi-continuous on $\mathcal{X}$.[19]

Assumption 2. (i) There exists some $\delta > 0$ such that $M \subseteq \Delta^\delta$. (ii) For all $\pi \in M$, $G(\cdot, \pi)$ is strictly increasing. (iii) For all $x \in \mathcal{X}$, $G(x, \cdot)$ is continuous on M.

The main restriction of the two assumptions is $M \subseteq \Delta^\delta$, which requires the set of priors considered plausible by the DM must assign at least $\delta > 0$ weight to every state. See Section 6.1 for a discussion on how to interpret this requirement.

The next theorem says for any UAP DM with index G satisfying the two technical assumptions, the induced value of an information structure characterized by the family of single-crossing utility indices implies Lehmann's order.

[19]See Cerreia-Vioglio et al. (2011) Theorem 7.

Theorem 1. *Fix some uncertainty averse index G that satisfies Assumption 1 and 2. For any information structures F and H that are MLR-ranked, if H is not Lehmann more informative than F, then there exists a single crossing u such that*

$$V^{UA}(F, G, u) > V^{UA}(H, G, u).$$

A sketch of the proof of Theorem 1, in several steps, is as follows.

Suppose H is not more Lehmann-informative than F.

1. In Proposition 1 in Section 4.2, for any prior π with full support, we can explicitly construct a single-crossing index u (that depends on the prior π) satisfying

$$V^{EU}(H, \pi, u) < V^{EU}(F, \pi, u).$$

2. The value of experiment F for a UAP DM is

$$V^{UA}(F, G, u) = \max_{\sigma \in (\Delta A)^S} \min_{\pi \in \Delta\Omega} G(U^{EU}(F, \pi, \sigma, u), \pi). \tag{4}$$

   For each fixed u, after showing that Minimax Theorem holds for the problem in (4), the maxmin value in (4) equals the minmax value, and thus can be obtainable at some saddle point (σ^*, π^*). Moreover, the set of saddle points, although depending on u, is always convex and exchangeable.

3. We show there exists a particular pair π^*, u^* such that, (i) for such π^*, u^* satisfies the requirement in the first step, and (ii) for such u^*, π^* is a part of some saddle point (σ^*, π^*) in the second step. The existence of such pair is proved using Kakutani's fixed point theorem (see Lemma 3).

4. Since $G(\cdot, \pi^*)$ is strictly increasing, the strict inequality found in the EU case at π^* at u^* can be passed on to the UAP case to get $V^{UA}(H, G, u^*) < V^{UA}(F, G, u^*)$.

Theorem 1 characterizes Lehmann order for all pairs of experiments, by fixing a uncertainty-averse index and varying all single-crossing vNM indices. Alternatively, one can characterize the same information order by fixing a vNM index and varying all uncertainty-averse indices. Theorem 2 below takes the other approach.

Definition 2. Suppose $A = \{a_1, a_2\}$. A finite collection of single-crossing utility indices $\{u^1, \ldots, u^I\}$ satisfies *diversity*, if, for any pair of states $\omega < \omega'$, there exists some $i \in \{1, 2, \cdots, I\}$ such that

$$u^i(a_1, \omega) > u^i(a_2, \omega), \quad u^i(a_2, \omega') > u^i(a_1, \omega').$$

Intuitively, diversity is violated if there exists a pair of states, $\omega' < \omega''$ such that for every i, either $u^i(a_1, \omega) > u^i(a_2, \omega)$ for both $\omega = \omega'$ and ω'', or $u^i(a_1, \omega) < u^i(a_2, \omega)$ for both $\omega = \omega'$ and ω''.

Theorem 2. *Fix some finite collection of single crossing utility indices $\{u^1, \ldots, u^I\}$ that satisfies* diversity. *For any information structures F and H that are MLR-ranked, if H is not more Lehmann informative than F, then there exists some $G \in \mathcal{G}$ and u^i from $\{u^1, \ldots, u^I\}$ such that*

$$V^{UA}(F, G, u^i) > V^{UA}(H, G, u^i). \tag{5}$$

Proof. Suppose H is not Lehmann more informative than F, then by Lemma 1 (in Section 4.2), there exists $\bar{x}, \bar{y} \in S$ and states $\omega < \omega'$ such that

$$H(\bar{x}|\omega) \leq F(\bar{y}|\omega), \quad H(\bar{x}|\omega') > F(\bar{y}|\omega').$$

Suppose $A = \{a_1, a_2\}$. Then by diversity of $\{u^1, \cdots, u^I\}$, there exists some element u^i such that

$$u^i(a_1, \omega) > u^i(a_2, \omega), \quad u^i(a_2, \omega') > u^i(a_1, \omega').$$

Fix this u^i. Consider all the uncertainty-averse indices G that belong to EU family with priors supported on $\{\omega, \omega'\}$. Then Theorem 2 becomes a case with EU preferences, two-states, and two actions that are non-ordered with respect to u, which is proved in Proposition 11 of Quah and Strulovici (2009). □

Remark 3. Theorem 2 remains valid, if we focus on G that belongs to the EU family, i.e.,

$$G^{\pi^0}(x, \pi) = \begin{cases} x, & \text{if } \pi = \pi^0; \\ +\infty, & \text{otherwise.} \end{cases}$$

for all $\pi^0 \in \Delta(\Omega)$. Since the inequality in (5) is strict, by continuity, we can further require π^0 to be of full support.

Remark 4. For the collection of single crossing utility indices $\{u^1, \ldots, u^I\}$ to satisfy diversity, I is at least $(|\Omega| - 1)$. For the binary state case, one such u suffices.

Remark 5. In the EU case with binary states, our Theorem 2 becomes Quah and Strulovici's (2009) Proposition 11. That is, for any *fixed* vNM utility index u, whenever H is not Lehmann more informative than F, one can construct some prior π^u at which the value of experiment F is strictly higher than that of H. In our Theorem 1 and Proposition 1, we consider any *fixed* prior π with full support and *construct* a specific form of vNM utility index u^π for which the value of experiment F is strictly higher than that of F when H is not Lehmann more informative than F.

4.1 Binary state case

In this subsection we illustrate the connection between our Theorem 1 and Theorem 2 (and thus Quah and Strulovici's (2009) Proposition 11) using the EU case with binary states.

Consider the EU case with binary states $\{\omega_1, \omega_2\}$, and binary action $A = \{a_1, a_2\}$. Let $\mathcal{U}$ be the class of vNM utility indices satisfying the following

$$\mathcal{U} = \{u | u(a_1, \omega_1) > u(a_2, \omega_1), \;\; u(a_2, \omega_2) > u(a_1, \omega_2)\}.$$

To compare different experiments, without any loss of generality, we could restrict the class on $\mathcal{U}$.[20] For any $u \in \mathcal{U}$, define the ratio

$$\chi(u) = \left(\frac{u(a_1, \omega_1) - u(a_2, \omega_1)}{u(a_2, \omega_2) - u(a_1, \omega_2)} \right) \in (0, \infty)$$

For this u, the prior belief on Ω to make the DM indifferent between a_1 and a_2 is given by $\left(\frac{1}{1+\chi(u)}, \frac{\chi(u)}{1+\chi(u)} \right)$.

Given π and u, we define $(D^\pi u)(a, \omega) = u(a, \omega)\pi(\omega)$ as the prior-weighted utility. Clearly, for any π with full support, $u \in \mathcal{U}$ if and only if $D^\pi u \in \mathcal{U}$, moreover,

$$\chi(D^\pi u) = \chi(u) \frac{\pi(\omega_1)}{\pi(\omega_2)}. \tag{6}$$

We have two simple observations:[21]

(O_1) Take $u, \tilde{u} \in \mathcal{U}$ with $\chi(u) = \chi(\tilde{u})$, then for two experiments H^1 and H^2,

$$V^{EU}(H^1, \pi, u) \geq V^{EU}(H^2, \pi, u) \;\text{ if and only if }\; V^{EU}(H^1, \pi, \tilde{u}) \geq V^{EU}(H^2, \pi, \tilde{u}).$$

(O_2) Suppose $D^\pi u = D^{\pi'} u'$, then for any experiment H,

$$V^{EU}(H, \pi, u) = V^{EU}(H, \pi', u'). \tag{7}$$

[20] With binary actions and binary states, for any u, either a_1 weakly dominates a_2 for any ω, or a_2 weakly dominates a_1 for any ω, or u belongs to $\mathcal{U}$, or belongs to $\mathcal{U}$ after relabeling actions. For the former two cases, such u is useless to check informativeness of experiments. Note that any $u \in \mathcal{U}$ is single crossing.

[21] See Appendix Section A.1 for the proofs. While O_1 requires binary state assumption, O_2 holds for any finite state space.

Observation 1 states that the threshold $\chi(u)$ is a sufficient statistic for comparing a pair of experiments. Observation 2 states that, for the EU case, the prior-weighted utility is a sufficient statistic for the value of any experiment.

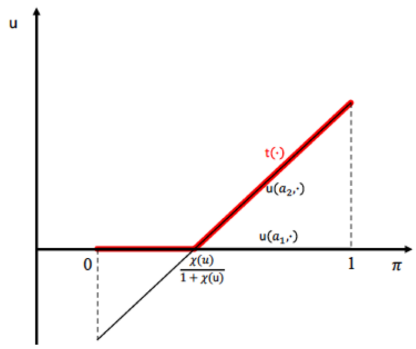

Figure 2: $\chi(u)$

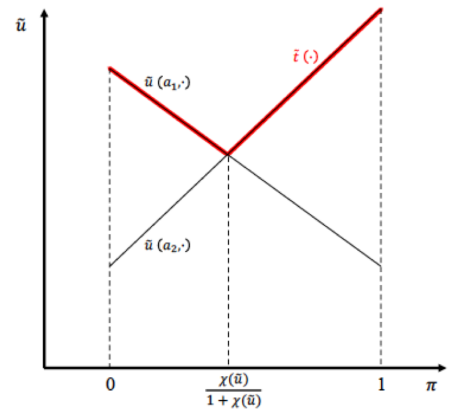

Figure 3: $\chi(\tilde{u})$

The intuition for observation 1 can be illustrated by Figures 2 and 3. For fixed u, the ranking of E^1 and E^2 is given by the expected values of a test function $t(\cdot)$, which is the upper envelop of the linear expected utility functions of the posterior π under a_1 and a_2. Figure 2 is a basic test function $t(\cdot)$ under our utility construction (by Equation (8) below). Figure 3 is an arbitrary test function $\tilde{t}(\cdot)$ under a generic $\tilde{u}$ such that a_1 and a_2 are non-ordered. Suppose the two test functions have the exact same cutoff so $\frac{\chi(u)}{1+\chi(u)} = \frac{\chi(\tilde{u})}{1+\chi(\tilde{u})}$. Then, graphically, $\tilde{t}(\cdot) = l(\cdot) + \beta t(\cdot)$ for some linear function $l(\cdot)$ and some scalar $\beta > 0$. Since for a fixed prior the ranking of two experiments is invariant to such transformations, the two test functions will rank experiments the same way as long as their cutoffs are the same.

These two observations are useful for connection our Theorem 1 and Theorem 2 (or Quah and Strulovici's (2009) Proposition 11).

In the proof of Theorem 1, we fix any prior π^1 and construct the following u^1 with

$$u^1(a_1, \omega_1) = u^1(a_1, \omega_2) = 0, \quad u^1(a_2, \omega_1) = -k, u^1(a_2, \omega_2) = 1 - k. \tag{8}$$

For any $k \in (0, 1)$, such $u^1 \in \mathcal{U}$. Moreover, $\chi(D^{\pi^1} u^1) = \frac{\pi^1(\omega_1)}{\pi^1(\omega_2)} \frac{k}{(1-k)}$, which can obtain any positive number as k varies in $(0, 1)$.

In our Theorem 2 (or Quah and Strulovici's (2009), Proposition 11), for a fixed u^2 in $\mathcal{U}$, we vary the prior distribution π^2. By Equation (6), $\chi(D^{\pi^2} u^2)$ can obtain any positive number as the prior π^2 varies.

For fixed u^2 and π^1, the equation

$$\chi(D^{\pi^1}u^1) = \chi(D^{\pi^2}u^2) \tag{9}$$

defines a one-to-one mapping between π^2 and k (in our construction of u^1). This equation (9) connects our Theorem 1 and Theorem 2 (Quah and Strulovici's (2009) Proposition 11).

Formally, let π^0 be the uniform prior, $\hat{u}^1$ be $D^{\pi^0}\hat{u}^1 = D^{\pi^1}u^1$, and $\hat{u}^2$ be $D^{\pi^0}\hat{u}^2 = D^{\pi^2}u^2$. By Equation (9), $\chi(D^{\pi^0}\hat{u}^1) = \chi(D^{\pi^0}\hat{u}^2)$. Then observation 1 implies

$$V^{EU}(H, \pi^0, \hat{u}^1) < V^{EU}(F, \pi^0, \hat{u}^1) \text{ if and only if } V^{EU}(H, \pi^0, \hat{u}^2) < V^{EU}(F, \pi^0, \hat{u}^2).$$

Together with observation 2, we have

$$V^{EU}(H, \pi^1, u^1) < V^{EU}(F, \pi^1, u^1) \text{ if and only if } V^{EU}(H, \pi^2, u^2) < V^{EU}(F, \pi^2, u^2).$$

4.2 Proof of Theorem 1

The formal result of the first step is summarized in the following Proposition.

Proposition 1. *Fix an arbitrary prior π of full support. For any information structures H and F that are MLR-ranked, if H is not Lehmann more informative than F, then we can find a vNM index u such that (1) u is single crossing; (2) u is uniformly bounded by 1; and (3) $V^{EU}(H, \pi, u) < V^{EU}(F, \pi, u)$.*

The proof this proposition is in Appendix A.3.

The following lemma provides an equivalent statement of Lehmann's order.

Lemma 1. *$H \succsim_L F$ if and only if for all $x, y \in S$, $F(x|\omega) - H(y|\omega)$ is single crossing in ω.*

The proof of this Lemma is in Appendix A.2.

Suppose H is not Lehmann more informative than F, by Lemma 1, there exists $\bar{x}, \bar{y} \in S$ and state $\omega_0 < \omega_1$ such that [22]

$$H(\bar{x}|\omega_0) \leq F(\bar{y}|\omega_0), \quad H(\bar{x}|\omega_1) > F(\bar{y}|\omega_1). \tag{10}$$

[22] Another case of violation of Lehmann order is $H(\bar{x}|\omega_0) < F(\bar{y}|\omega_0)$, $H(\bar{x}|\omega_1) \geq F(\bar{y}|\omega_1)$. The proof is similar, hence omitted.

Below we describe how the vNM index u in Proposition 1 is constructed.

Let $A = \{0, 1\}$. We will construct some u that satisfies the single-crossing property as follows. Pick any $\hat{\omega} \in (\omega_0, \omega_1)$. Define a step function

$$\psi(\omega) = \begin{cases} 0 & \text{if } \omega \leq \hat{\omega} \\ 1 & \text{otherwise} \end{cases}.$$

Clearly ψ is monotone, hence single crossing in ω. Define

$$g(\omega) = \begin{cases} 1 & \text{if } \omega \in \{\omega_0, \omega_1\}, \\ \epsilon & \text{if } \omega \notin \{\omega_0, \omega_1\}. \end{cases} \tag{11}$$

for some small $\epsilon \in (0, 1)$. Obviously g is positive for any $\epsilon > 0$, and it is uniformly bounded by 1. Now we define a utility function on $A \times \Omega$ as

$$u_\epsilon^\pi(a, \omega) = a(\psi(\omega) - k_\epsilon^\pi)g(\omega), \tag{12}$$

where u_ϵ^π is single crossing and k_ϵ^π is chosen such that $\mathbb{E}[(\psi(\omega) - k_\epsilon^\pi)g(\omega))|X = \bar{x}] = 0$, with expectation taken with respect to H and prior π. From the construction of k, the optimal decision rule under H is to choose $a = 1$ if and only if $x \geq \bar{x}$. Clearly, k_ϵ^π lies in $[0, 1]$.

After some calculations, we show that

$$V^{EU}(F, u_\epsilon^\pi, \pi) - V^{EU}(H, u_\epsilon^\pi, \pi) \geq -2\epsilon + (1 - k_\epsilon^\pi)\pi(\omega_1) \underbrace{[H(\bar{x}|\omega_1) - F(\bar{y}|\omega_1)]}_{>0 \text{ by } (10)} \tag{13}$$

As $\epsilon \to 0$, the parameter k_ϵ^π converges to some number in $(0, 1)$, and hence the right-hand-side term is strictly positive for small enough ϵ. This proves Proposition 1.

We could further strengthen Proposition 1 by choosing a small $\epsilon_1 > 0$ such that the right-hand-side of (13) is still strictly uniformly positive for all $\pi \in \Delta^\delta$ for some fixed $\delta > 0$. We denote $u^\pi := u_{\epsilon_1/2}^\pi$ henceforth.

Now we move on to the second step, where we consider the UAP case:

$$V^{UA}(F, G, u) = \max_{\sigma \in (\Delta A)^S} \min_{\pi \in \Delta\Omega} G(U^{EU}(F, \pi, \sigma, u), \pi). \tag{4}$$

For a fixed utility u^π, the next lemma verifies that we can turn our maxmin problem into a minmax problem. This lemma utilizes the convexity of UAP, with which we can apply Sion's (1958) minimax theorem.

Lemma 2. *For all uncertainty-averse index G satisfying Assumption 1, fix any u^π constructed in Proposition 1, we have*

$$\max_{\sigma\in(\Delta A)^S}\min_{\tilde{\pi}\in\Delta\Omega} G(U^{EU}(F,\tilde{\pi},\sigma,u^\pi),\tilde{\pi}) = \min_{\tilde{\pi}\in\Delta\Omega}\max_{\sigma\in(\Delta A)^S} G(U^{EU}(F,\tilde{\pi},\sigma,u^\pi),\tilde{\pi}). \tag{14}$$

For fixed π and u^π, denote by $S(\pi)$ the set of saddle points of (14). Clearly it is nonempty for all $\pi \in \Delta^\delta$ following Lemma 2. By Lemma 5 in Appendix A.6, $S(\pi)$ has a product structure. Thus we can project the set of saddle points $S(\pi)$ to $\Delta\Omega$. By Assumption 2, the projected area belongs to $M \subseteq \Delta^\delta$. Hence, we can define a correspondence $\Sigma : \Delta^\delta \rightrightarrows \Delta^\delta$ where

$$\Sigma(\pi) := \{\tilde{\pi}^* \in \Delta^\delta | \exists \sigma^* \in (\Delta A)^S \text{ such that } (\tilde{\pi}^*, \sigma^*) \in S(\pi)\}. \tag{15}$$

Lemma 5 implies that $S(\pi)$ is convex for all $\pi \in \Delta^\delta$. Since $\Sigma(\pi)$ is the projection of $S(\pi)$ to Δ^δ, it is nonempty and convex.[23] The continuity of $G(U^{EU}(F,\tilde{\pi},\sigma,u^\pi),\tilde{\pi})$ in π implies Σ is upper hemi-continuous.[24] By Kakutani's fixed point theorem (see Appendix A.7), we have the following lemma.

Lemma 3. *Suppose Assumptions 1 and 2 hold. Then correspondence Σ has a fixed point in Δ^δ.*

Now, we present the proof of Theorem 1.

Proof of Theorem 1. Let $\pi^* \in \Delta^\delta$ be a fixed point of the correspondence Σ. Then it could be used simultaneously to construct u^{π^*} and be part of a saddle point for the maxmin problem (14) using $u^\pi = u^{\pi^*}$. Let σ^* be the mixed strategy component of this saddle point. By definition, σ^* maximizes utility at π^*. Then,

$$\begin{aligned}
V^{UA}(F,G,u^{\pi^*}) &= G(U^{EU}(F,\pi^*,\sigma^*,u^{\pi^*}),\pi^*) && (16)\\
&= G(V^{EU}(F,\pi^*,u^{\pi^*}),\pi^*) && (17)\\
&> G(V^{EU}(H,\pi^*,u^{\pi^*}),\pi^*) && (18)\\
&= \max_{\sigma\in(\Delta A)^S} G(U^{EU}(H,\pi^*,\sigma,u^{\pi^*}),\pi^*) && (19)\\
&\geq \min_{\pi}\max_{\sigma\in(\Delta A)^S} G(U^{EU}(H,\pi,\sigma,u^{\pi^*}),\pi) && (20)\\
&= V^{UA}(H,G,u^{\pi^*}). && (21)
\end{aligned}$$

[23]The full statement of Lemma 5 shows additional properties, such as a product structure, of the set $S(\pi)$, which could be of independent interest. Hence, it is relegated to Appendix A.6.

[24]Recall that from the construction of u^π in (12), the value k_ϵ^π depends continuously on both π and ϵ. As discussed above, one can pick some $\epsilon(\delta) = \epsilon_1/2(\delta)$ that works uniformly for all $\pi \in \Delta^\delta$. Put together, we have that u^π and hence $G(U^{EU}(F,\tilde{\pi},\sigma,u^\pi),\tilde{\pi})$ are continuous in π for all $\pi \in \Delta^\delta$.

Here (16) uses the fact that $\pi^* \in \Sigma(\pi^*)$ and (σ^*, π^*) is a saddle point for the maxmin problem with F, u^{π^*}; (17) and (19) follow from $G(\cdot, \pi^*)$ is (strictly) increasing; (18) follows from that $G(\cdot, \pi^*)$ is (strictly) increasing, as well as the "only if" direction of Lehmann's equivalence theorem in the EU case (Lemma 4 in appendix A.4); and (21) uses the minimax equation from Lemma 2. □

Remark 6. Lehman's original proof showed that, for a fixed u, if H is not more Lehmann-informative than F, then there exists some prior π (supported at the two points ω_0 and ω_1 found at condition (10)) such that the expected value of F is higher than H. Here we want to prove that for any fixed prior with full support, there exists some u where the expected value of F is strictly higher than H. To bridge the difference, we multiply the payoff function by a $g : \Omega \mapsto (0, 1]$ function (11) that shifts most of the weight to ω_0, ω_1. We can do so because the single-crossing property of the original payoff function is preserved after being multiplied by a strictly positive function g. The parameter $\epsilon > 0$ can be arbitrarily small but not zero, as otherwise u is no longer single crossing.[25]

Remark 7. A challenge in proving Theorem 1 is that the minimizing prior π^* identified in (4) depends on the vNM index u, yet in step one the construction of u (rendering a strictly higher value for F) also depends on the initial prior π. A coincidence of $\pi^* = \pi$ is required here. So if Σ is a correspondence mapping from an initial prior π to the index u constructed in the first step and further from u to the set of minimizing priors from the second step, then π^* should be a fixed point of Σ. The existence of such a fixed point is obtained by applying Kakutani's fixed point theorem.

5 Applications

5.1 Social ranking of experiments

A society consists of I individuals. Each i has a single-crossing vNM index u^i. We consider a social planner's aggregation problem, where she knows the individuals' risk preferences but does not know their uncertainty-averse indices (in particular, their prior beliefs). The social risk preference profile, $(u^1, \ldots, u^I)$, is called *diverse*, if the collection $\{u^1, \ldots, u^I\}$ satisfies *diversity*.

[25]Formally, pick any positive function g, i.e., $g(\omega) > 0$ for any ω. Suppose $u(a, \omega)$ is single crossing, then $u(a, \omega)g(\omega)$ is also single crossing. However, if $u(a, \omega)$ is supermodular, $u(a, \omega)g(\omega)$ may not be supermodular. The proof is straightforward, hence omitted.

Suppose information structures F and H are MLR-ranked. Say individual i prefers H to F, $H \succsim_i F$, if

$$V^{UA}(H, G, u^i) \geq V^{UA}(F, G, u^i), \quad \forall G \in \mathcal{G}.$$

Say the society *unanimously* prefers H to F, if $H \succsim_i F$ for all $i \in \{1, 2, \cdots, I\}$.

Corollary 2. *Suppose the social risk preference profile is diverse. Pick two arbitrary information structures F and H that are MLR-ranked, H is Lehmann more informative than F if and only if the society unanimously prefers H to F.*

Proof. The statement follows directly from Theorem 2. □

As an immediate implication of Corollary 2, two societies with different compositions of risk preferences can have the same social ranking of experiments, as long as the risk preference profiles in these societies both satisfy diversity.

5.2 Investment decisions

Next we will apply our result to an investment problem studied by Bond (2016). [26]

An uninformed (individual) investor is making an investment decision, where $\omega \in \Omega \subseteq \mathbb{R}$ is a payoff relevant state such that the expected return of investment is increasing in ω. The state space Ω is assumed to be compact. This uninformed investor cannot directly observe ω; he infers ω from some economic outcome a that is, in this case, the investment decision made by an informed (institutional) investor who observes the state. The decision a depends on the state ω and some unobserved random variable, $t \sim U[0, 1]$, that is irrelevant to the uninformed investor's decision. After observing a, the uninformed investor makes an investment decision $b \in B \subseteq \mathbb{R}$. His payoff from decision b in state ω is $u^U(b, \omega)$.

The informed investor directly observes ω and chooses an investment level a. She is the beneficiary of a government bailout policy—with probability $1 - \psi \in [0, 1]$, the government bails out this investor if her investment project fails. Let $(1 + r)$ be the gross return of the investment when it succeeds, and 0 is the gross return of the investment when it fails. The probability of success is

$$q(\omega, \psi) = \omega + (1 - \omega)(1 - \psi),$$

[26] We examine the first model in Bond (2016) to illustrate how our result is applicable. The same analysis can be applied to the other two models studied there.

where $\omega \in [0,1]$ is the actual success rate of the investment without bailout, and conditional on failure there is a $(1-\psi)$ probability of bailout. Assume $\omega r \geq 1$ for all ω. The informed investor's vNM index is $u^I(c;t)$ with monetary outcome c and some parameter t. Her problem is

$$\max_a q(\omega,\psi)u^I(ar;t) + (1-q(\omega,\psi))u^I(-a;t),$$

Corollary 3. *Suppose the informed investor is an EU maximizer and the uninformed investor is uncertainty averse with index G. Assume that the uninformed investor's vNM index $u^U(b,\omega)$ satisfies (i) u^U is continuous in b and satisfies the single-crossing property in (b,ω); (ii) For any $b'' > b'$, $u^U(b'',\omega) - u^U(b',\omega)$ is quasi-concave in ω. The informed investor's vNM index $u^I(\cdot,t)$ is strictly increasing, strictly concave, and satisfies $\frac{\partial}{\partial t}\left(\frac{u^I_{cc}}{u^I_c}\right) < 0$. Then a higher bailout probability makes the uncertainty-averse uninformed investor worse off.*

Proof. By Proposition 4 in Bond (2016), a higher probability of bailout reduces the Lehmann-informativeness of the informed investor's action a. Then the result follows from our Corollary 1. □

6 Discussion

6.1 Assumption 2

Recall that M is a closed and convex subset of $\Delta\Omega$ that includes all the priors in the domain of G. In Section 4, we have assumed $M \subseteq \Delta^\delta$ for the "only if" direction of Lehmann's equivalence theorem.

According to Cerreia-Vioglio et al. (2011), M is interpreted as the set of priors the DM considers relevant. The set is identified behaviorally as follows: take the initial preference relation $\succsim$ over all Anscombe-Aumann acts, which satisfies all the axioms of UAP, and let the partial order $\succsim^*$ be the subset of $\succsim$ on which the independence axiom holds. Then $f \succsim^* g$ is interpreted as f is unambiguously preferred to g. This order $\succsim^*$ admits a Bewley representation, i.e., there is some vNM index u (unique up to a positive affine transformation) and a unique closed and convex set of priors $C \subseteq \Delta\Omega$ such that

$$f \succsim^* g \Leftrightarrow \int_\Omega u(f)d\pi \geq \int_\Omega u(g)d\pi \quad \forall \pi \in C$$

for all acts f, g. Hence C is naturally interpreted as the revealed ambiguous priors by an UAP agent. By Cerreia-Vioglio et al. (2011), $C = cl(dom_\Delta(G)) = M$.

Assumption 2(i) requires that every relevant prior must assign at least a probability δ to each state, where $\delta > 0$ is a uniform lower bound. This assumption is stronger than requiring all relevant priors to have full support, since it rules out the case that M is the set of all priors with full support. Nevertheless, since the number of states is finite and δ can be arbitrarily small, we still consider it as a relatively mild assumption.

6.2 Alternative proofs with weaker assumptions

Regrettably, our Theorem 1 does not apply to multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), an important sub-family of UAP:

$$V^{MP}(H, \pi_0, \theta, u) = \max_{\sigma \in (\Delta A)^S} \min_{\pi \in \Delta\Omega} U^{EU}(H, \pi, \sigma, u) + \theta R(\pi || \pi_0),$$

where $R(\pi||\pi_0)$ is the relative entropy of π with respect to $\pi_0 \in \Delta^\delta$ and $\theta \in (0, +\infty]$. This is because the proof relies on the assumption that $M \subseteq \Delta^\delta$for some $\delta > 0$, yet for the multiplier preferences $M = \Delta$. Hence a weaker assumption can be desirable for certain subfamilies of UAP.[27]

In this section, we provide alternative proofs for the "only if" direction for smooth preferences and SOEU cases, which relax Assumption 2 and use a different technique.

Smooth preferences

The proof uses an idea similar to the proof in the paper by Li and Zhou (2016) for the Blackwell ranking case. The main technique is to scale down uniformly the vNM utility index, and, provided that the scale parameter is sufficiently small, any local change in state-dependent utility vector would lead to a first order change in risk premium but only a second order change in ambiguity premium.

For the smooth preferences (Klibanoff et al., 2005), the value takes the following form:

$$V^S(H, \mu, \phi, u) = \max_{\sigma \in (\Delta A)^S} \phi^{-1}\left(\int_{\Delta\Omega} \phi(U^{EU}(H, \pi, \sigma, u)) d\mu(\pi)\right),$$

where $\mu \in \Delta(\Delta\Omega)$ and $\phi : \mathbb{R} \mapsto \mathbb{R}$ is a continuous, strictly increasing and concave function.

[27] Assumption 1 always holds for the six special families considered.

Proposition 2. *For smooth ambiguity case, assume that $\pi^0 := \int_{\Delta(\Omega)} \pi d\mu(\pi)$ is of full support, ϕ is strictly concave, then the implication of Theorem 1 holds: i.e., there exists a single crossing utility u such that the value under H is strictly smaller than that under F:*

$$V^S(H, \mu, \phi, u) < V^S(F, \mu, \phi, u).$$

The proof is given in Appendix A.8.

SOEU

Recall that for second order expected utility (Grant et al., 2009),

$$U^{SO}(H, \pi^0, \phi, u) = \max_{\sigma \in (\Delta A)^S} \phi^{-1}\left(\int_\Omega \int_S \phi\left(\int_A u(a, \omega) d\sigma_s(a)\right) dH(s|\omega) d\pi^0(\omega)\right),$$

where $\phi : \mathbb{R} \mapsto \mathbb{R}$ is some continuous, concave, and strictly increasing function. When $\phi(z) = z$, we are back to the EU case.

Proposition 3. *For SOEU case, assume that the prior π^0 is of full support, ϕ is strictly concave, then the implication of Theorem 1 holds: i.e., there exists a single crossing utility u such that the value under H is strictly smaller than that under F:*

$$V^{SO}(H, \pi^0, \phi, u) < V^{SO}(F, \pi^0, \phi, u).$$

The proof is quite similar to that of Proposition 2. We scale down the utility index appropriately to make sure that the result for the EU case carries over to the SOEU case. When u is sufficiently small, the state-dependent utility vector can only vary within in a small neighborhood around 0, which corresponds to the case of local ambiguity neutrality. See Appendix A.9 for the proof.

Recall that for multiplier preferences,

$$V^{MP}(H, \pi^0, \theta, u) = \max_{\sigma \in (\Delta A)^S} \min_{\pi \in \Delta\Omega} U^{EU}(H, \pi, \sigma, u) + \theta R(\pi || \pi^0),$$

where $R(\pi||\pi^0)$ is the relative entropy of π with respect to $\pi^0 \in \Delta^\delta$ and $\theta \in (0, +\infty]$.

Note that MP is a special case of SOEU with

$$\phi_\theta(x) = \begin{cases} -\exp\left(-\frac{x}{\theta}\right) & \text{for } 0 < \theta < \infty, \\ x & \text{for } \theta = \infty. \end{cases}$$

the multiplier preferences to have full support.

The next corollary is a direct consequence of Proposition 3.

Corollary 4 (Multiplier preferences case)**.** *For multiplier preferences, assume that the reference prior π^0 is of full support, then the implication of Theorem 1 holds: i.e., there exists a single crossing utility u such that the value under H is strictly smaller than that under F:*

$$V^{MP}(H, \pi^0, \theta, u) < V^{MP}(F, \pi^0, \theta, u). \tag{22}$$

6.3 Six special cases

Uncertainty-averse preferences nest the following six well-known ambiguity averse preferences as special cases (Cerreia-Vioglio et al., 2011). Previously, we have discussed three special families: smooth preferences, SOEU preferences, and multiplier preferences. The remaining three special cases are:

1. Maxmin EU (Gilboa and Schmeidler, 1989), where for some convex and closed set $C \subseteq \Delta\Omega$, $V^M(H, C, u) = \max_{\sigma \in (\Delta A)^S} \min_{\pi \in C} U^{EU}(H, \pi, \sigma, u)$.

2. Variational preferences (Maccheroni, Marinacci, and Rustichini, 2006), where for some convex, lower-semi continuous, and grounded cost function $c : \Delta(\Omega) \mapsto [0, \infty]$, $V^V(H, c, u) = \max_{\sigma \in (\Delta A)^S} \min_{\pi \in \Delta\Omega} U^{EU}(H, \pi, \sigma, u) + c(\pi)$.

3. Confidence preferences (Chateauneuf and Faro, 2009). If $range(\mathbf{u}) = \mathbb{R}_+^{|\Omega| \times |A|}$, then $V^C(H, \phi, \alpha, u) = \max_{\sigma \in (\Delta A)^S} \min_{\{\pi : \phi(\pi) \geq \alpha\}} \frac{1}{\phi(\pi)} U^{EU}(H, \pi, \sigma, u)$ for some confidence level $\alpha \in (0, 1)$ and some quasi-concave and upper semi-continuous confidence function $\phi : \Delta(\Omega) \mapsto [0, 1]$.

Corollary 5. *Lehmann's equivalence results hold for the following preference families, with Assumption 2 taking specified form:*

1. *Maxmin EU (MEU), with $C \subseteq \Delta^\delta$;*

2. *Variational preferences (VP), with $dom(c) \subseteq \Delta^\delta$ and $c(\cdot)$ being continuous on $dom(c)$;*

3. *Multiplier preferences (MP), with $\pi^0 \in \Delta^\delta$;*

4. *Confidence preferences (CP), with $\{\pi : \phi(\pi) \geq \alpha\} \subseteq \Delta^\delta$ and ϕ being continuous on $\{\pi : \phi(\pi) \geq \alpha\}$;*

5. *Smooth preferences (SP), with $\pi^0 := \int_{\Delta(\Omega)} \pi d\mu(\pi) \in \Delta^\delta$ and ϕ being strictly concave;*

6. *Second-order expected utility (SOEU), with $\pi^0 \in \Delta^\delta$ and ϕ being strictly concave.*

Proof. The "if" direction follows directly from Corollary 1. For the "only if" direction, note that Assumption 1 is clearly satisfied in all six special cases. The cases of SP, SOEU, and MP follow from Propositions 2, 3, and Corollary 4. The MEU, VP, and CP cases follows from Theorem 1 and a straightforward adaption of Assumption 2.[28] □

6.4 Relation to the literature

The Blackwell's informativeness order can be characterized by comparing the value of information of two experiments by an expected utility maximizing DM, for all vNM utility indices. This dual characterization motivates several studies to identity finer ranking of Blackwell by either restricting the set of utility functions, and/or the set of experiments. For example, among pairs of MLR-ranked experiments, Lehmann (1988) considers utility indices that are quasi-concave in action with optimal actions ("peaks") that are increasing in the state;[29] Athey and Levin (2018) study supermodular vNM indices; and Quah and Strulovici (2009) explore the most general family of interval dominance order vNM indices (IDO-family).

Another direction is to re-evaluate the comparative value of experiment by assuming that the DM is not expected utility maximizer, but exhibits ambiguity aversion. For example, assuming commitment and allowing for all vNM indices, Çelen (2012) shows the same ranking holds for maxmin EU, and Li and Zhou (2016) further generalize it to all uncertainty-averse preferences, thus showing the robustness of Blackwell's informativeness ranking.[30]

This paper lies in the intersection of the two directions. To derive the richer information ranking, we restrict to the family of single-crossing vNM utility indices; and to examine robustness of Lehmann's ranking, we generalize the ex-ante value of information to almost all uncertainty-averse preferences. Again, our results confirm that

[28] More precisely, the corresponding conditions for Assumption 2 are listed as follows:

- For Maxmin EU, $\mathcal{X} = \mathbb{R}$ and $G(x, \pi) = \begin{cases} x, & \text{if } \pi \in C; \\ +\infty, & \text{otherwise.} \end{cases}$ and $C \subseteq \Delta^\delta$.
- For VP, $\mathcal{X} = \mathbb{R}$ and $G(x, \pi) = x + c(\pi)$; we assume $dom(c) \subseteq \Delta^\delta$ and c is continuous on $dom(c)$.
- For CP, $\mathcal{X} = \mathbb{R}_+$ and $G(x, \pi) = \begin{cases} \frac{x}{\phi(\pi)}, & \text{if } \phi(\pi) \geq \alpha; \\ +\infty, & \text{otherwise.} \end{cases}$; we assume $\{\pi : \phi(\pi) \geq \alpha\} \subseteq \Delta^\delta$ and ϕ is continuous on $\{\pi : \phi(\pi) \geq \alpha\}$.

[29] Also see the monotone decision class studied by Karlin and Rubin (1956).

[30] See also Gensbittel et al. (2015) for a discussion on the ambiguous information case.

the characterization of Lehmann's ranking depends crucially on the single-crossing (or similar) property of risk preferences but is robust to varying assumptions regarding uncertainty preferences.

It is worth noting that the "only if" direction may fail when preferences are *not* convex. For example, for a decision maker who perceives extreme ambiguity about ω and views every prior in $\Delta\Omega$ as possible, and moreover she is optimistic and evaluates a strategy by the best-case scenario. Then any two experiments have the same value to her for any utility function. Therefore the convexity of UAP is necessary.

6.5 Information orders and prior belief

In this subsection, we present a simple observation to explain why Blackwell's order and Lehmann's order are prior-free. Within the EU framework, the value of a decision problem depends on the vNM utility index u and the experiment available. The comparison of experiments can be reflected by comparing the values on certain subsets of vNM utility indices.

Fixing a prior π, for given set of vNM utility indices $\mathbb{U}^*$, we can define a partial order $\succeq_{\mathbb{U}^*}$ on experiments such that:

$$H \succeq_{\mathbb{U}^*} F \iff \max_{\sigma} U^{EU}(H, \pi, \sigma, u) \geq \max_{\sigma} U^{EU}(F, \pi, \sigma, u), \text{ for all u in } \mathbb{U}^*. \tag{23}$$

For any u, for any positive function $\bar{g} : \Omega \to \mathbf{R}_+ = (0, \infty)$, define a new vNM utility index $u^{\bar{g}}$ such that $u^{\bar{g}}(a, \omega) := u(a, \omega)\bar{g}(\omega)$ for any a, ω.

Definition 3. A set of utility indices $\mathbb{U}^*$ are called *closed under state-wise weighting*, if $u \in \mathbb{U}^*$ implies $u^{\bar{g}} \in \mathbb{U}^*$ for any positive function $\bar{g}$.

Proposition 4. *Assume $\mathbb{U}^*$ is closed under state-wise weighting, then $\succeq_{\mathbb{U}^*}$ does not depend on the choice of π as long as π is of full support.*

Suppose we take $\mathbb{U}^* = \mathbb{U}^{all}$ as the set of all vNM utility indices, which is clearly closed under state-wise weighting. The corresponding rank is exactly Blackwell's informativeness order. On the other hand, suppose we take $\mathbb{U}^* = \mathbb{U}^{sc}$ as the set of all single-crossing vNM utility indices, it is easy to check that the single-crossing utility family is also closed under state-wise weighting (see footnote 25). The corresponding ranking is exactly Lehmann's order. By Proposition 4, both partial orders must be prior free. Similarly, if $\mathbb{U}^* = \mathbb{U}^{ido}$, the set of all IDO vNM utility indices (Quah and Strulovici, 2009), it is also closed under state-wise weighting. By Proposition 4, the

corresponding rank is also prior free. In fact, both $\mathbb{U}^{ido}$ and $\mathbb{U}^{sc}$ induce the same partial order on experiments, i.e., the Lehmann's order.

However, if the information partial rank $\succeq_{\mathbb{U}^*}$ defined in (23) is prior dependent, then the set of utility indices $\mathbb{U}^*$ can not be closed under state-wise weighting. For example, Athey and Levin (2018) use $\mathbb{U}^* = \mathbb{U}^{spm}$, the set of all supermodular vNM utility indices to define an order on experiments called **MIO-ND**.[31] Since supermodularity is not preserved under state-wise weighting, **MIO-ND** is indeed prior dependent. Therefore applying this order **MIO-ND** requires knowledge of the prior belief.

Also, it is necessary that π is of full support, otherwise $\bar{g}$ is either not well-defined, or not positive valued.

Remark 8. Consider $\mathbb{U}^* = \{u^0\} \cup \mathbb{U}^{sc}$, where u^0 is an (arbitrary) element in $\mathbb{U}^{ido} \backslash \mathbb{U}^{sc}$. By construction, this $\mathbb{U}^*$ is clearly not closed under state-wise weighting. Since both $\mathbb{U}^{sc}$ and $\mathbb{U}^{ido}$ induce Lehmann's order, the induced ranking $\succeq_{\mathbb{U}^*}$ is also Lehmann's order, hence prior-free. As shown by this example, the reverse implication of Proposition 4 does not always hold.

6.6 Dynamically consistent updating rule

So far we have focused on the ex-ante value of information for a DM who can commit. In this subsection we show that our main results also apply to an UAP agent who only makes decisions ex-post but follows some dynamically consistent updating rule (Hanany and Klibanoff, 2009).

Fix an experiment (S, H). A prior $\pi \in \Delta(\Omega)$ induces a joint density $p(s, \omega) = h(s|\omega)\pi(\omega)$ in the product space $S \times \Omega$, i.e., $p \in \Delta(S \times \Omega)$. Let $\pi_{\cdot s} \in \Delta\Omega$ denote its Bayesian posterior conditional on event $\{s\} \times \Omega$. Denote by $\Delta_H(S \times \Omega)$ the set of joint probabilities induced by H and all priors π.

Let $\succsim$ be the ex-ante UA preference we considered, which has UAP representation $(\tilde{G}, u)$ with $\tilde{G}$ naturally extended from G to domain $\mathcal{X} \times \Delta(S \times \Omega)$.[32] Let σ^* be an ex-ante optimal strategy from the set $(\Delta A)^S$. A conditional event is $\{s\} \times \Omega$.

[31] MIO-ND stands for "Monotone Information Order for payoff functions with NonDecreasing incremental returns", see Athey and Levin (2018).

[32] More precisely, $\tilde{G} : \mathcal{X} \times \Delta(S \times \Omega) \mapsto (-\infty, +\infty]$ is obtained from G through the following:

$$\tilde{G}(x, p) = \begin{cases} G(x, \pi), & \text{if there exists } \pi \text{ such that } p(s, \omega) = h(s|\omega)\pi(\omega), \forall s, \omega; \\ +\infty, & \text{otherwise.} \end{cases}$$

See Li and Zhou (2016) footnote 8.

Let $\{\succsim_{s,\sigma^*}\}_{s\in S}$ denote the conditional preferences system with UAP representation $\{(G_{s,\sigma^*}, u)\}_{s\in S}$ derived from some updating rule.

The DM follows some *dynamically consistent* updating rule if for any ex-ante optimal σ^*, conditionally $\sigma^* \succsim_{s,\sigma^*} \sigma$ for all σ such that $\sigma = \sigma^*$ on $S\backslash\{s\}$.

Following Hanany and Klibanoff (2009), we consider the set of probabilities supporting the conditional optimality of σ^* in $(\Delta A)^S$,i.e., [33]

$$Q^{s,\sigma^*} = \left\{ p \in \Delta_H(S\times\Omega) | \int_{S\times\Omega}\int_A u d\sigma^*_{s'} dp \geq \int_{S\times\Omega}\int_A u d\sigma_{s'} dp, \quad \forall \sigma = \sigma^* \text{ on } S\backslash\{s\} \right\}.$$

Let $Q^{s,\sigma^*}_s \subseteq \Delta\Omega$ be the set of Bayesian posteriors of Q^{s,σ^*} conditional on event $\{s\}\times\Omega$.

Clearly, if the DM is dynamically consistent then our main results extend to the case without commitment.

Corollary 6. *For a given optimal strategy σ^* under H, the DM follows some dynamically consistent updating rule if for all signal $s \in S$,*

$$\arg\min_{\pi_s\in\Delta\Omega} G_{s,\sigma^*}\left(\int_\Omega\int_A u d\sigma^*_s d\pi_s, \pi_s\right) \cap Q^{s,\sigma^*}_s \neq \emptyset. \tag{24}$$

For any pair of experiments H and F that are MLR-ranked, suppose (24) holds for the optimal strategies under both H and F. Then the results in Corollary 1 and Theorem 1 extend to the case of no commitment.

Proof. The first part follows from Proposition 4.1 in Hanany and Klibanoff (2009). The rest follows from the definition of dynamic consistency. □

7 Conclusion

This paper considers the ex-ante value of information for DMs who are uncertainty averse but has the ability to commit. We find that under weak technical assumptions, classic characterization of Lehmann's order is robust to consideration of uncertainty-averse agents. This suggests broader applicability of Lehmann's order for economic problems with payoff complementarity.

[33] Since we assume H has strictly positive density, $p(\{s\}\times\Omega) > 0$.

Appendix

A Proofs

A.1 Proof of the two observations in Section 4.1

Proof. (1) Let $P^i, i = 1, 2$, be the distribution of the posterior beliefs given experiment H^i and the prior π. By Bayes' rule, P^1 and P^2 have the same mean that equals the prior, i.e.,

$$\int x dP^1(x) = \int x dP^2(x).$$

For a given u, let $t(\cdot)$ the highest expected utility for a given posterior belief $x \in [0, 1]$, i.e.,

$$t(x) = \max_{a \in A}((1 - x)u(a, \omega_1) + xu(a, \omega_2)).$$

The value of experiment H for such u is

$$V^{EU}(H^i, \pi, u) = \int t(x) dP^i(x)$$

Since there are only two actions, $t(x)$ is a piece-wise linear, convex function of x with a unique kink at $\frac{\chi(u)}{1+\chi(u)}$, so

$$t(x) = l(x) + \beta \max\left(0, x - \frac{\chi(u)}{1 + \chi(u)}\right),$$

where $l(x)$ is linear in x, and β is a positive scalar (See Figure 2 for an example of $t(\cdot)$).

As a consequence,

$$V^{EU}(H^1, \pi, u) - V^{EU}(H^2, \pi, u) = \beta \int \max\left(0, x - \frac{\chi(u)}{1 + \chi(u)}\right) d(P^1(x) - P^2(x)).$$

Similarly,

$$V^{EU}(H^1, \pi, \tilde{u}) - V^{EU}(H^2, \pi, \tilde{u}) = \tilde{\beta} \int \max\left(0, x - \frac{\chi(\tilde{u})}{1 + \chi(\tilde{u})}\right) d(P^1(x) - P^2(x)).$$

Recall that $\beta, \tilde{\beta}$ are both positive. So when $\chi(u) = \chi(\tilde{u})$,

$$V^{EU}(H^1, \pi, u) \geq V^{EU}(H^2, \pi, u) \text{ if and only if } V^{EU}(H^1, \pi, \tilde{u}) \geq V^{EU}(H^2, \pi, \tilde{u}).$$

So, we show Observation O_1.

(2) For the second Observation, note that the value of information structure H is

$$V^{EU}(H, \pi, u) = \max_{\sigma \in (\Delta A)^S} U^{EU}(H, \pi, \sigma, u) = \max_{\sigma \in (\Delta A)^S} \int_\Omega \int_S \int_A u(a, \omega) d\sigma_s(a) dH(s|\omega) d\pi(\omega).$$

If $D^\pi u = D^{\pi'} u'$, then $u(a, \omega)\pi(\omega) = u'(a, \omega)\pi'(\omega)$ for any a, ω. From the above identity, it is clearly true that $V^{EU}(H, \pi, u) = V^{EU}(H, \pi', u')$, which shows Observation O_2. □

A.2 Proof of Lemma 1

Proof. Recall T is defined by $H(T(x, \omega)|\omega) = F(x|\omega)$. Pick any $x, y \in S$, define $\Delta^{x,y}(\omega) := F(x|\omega) - H(y|\omega) = H(T(x, \omega)|\omega) - H(y|\omega)$.

For all $x, y \in S$, $\Delta^{x,y}(\cdot)$ is single crossing in ω if and only if for any $\omega_1 > \omega_0$, $H(T(x, \omega_0)|\omega_0) \geq (>) H(y|\omega_0)$ implies $H(T(x, \omega_1)|\omega_1) \geq (>) H(y|\omega_1)$. The latter is equivalent to $T(x, \omega_1) \geq T(x, \omega_0)$, i.e., $T(x, \cdot)$ is increasing, which is exactly the definition of $H \succsim_L F$. □

A.3 Proof of Proposition 1

To distinguish two signals, we use X and Y to denote the signal space associated with experiment H and F, respectively. A typical element is denoted as $x \in X$, $y \in Y$.

Suppose H is not Lehmann more informative than F, by Lemma 1, there exists $\bar{x} \in X, \bar{y} \in Y$ and state $\omega_0 < \omega_1$ such that

$$H(\bar{x}|\omega_0) \leq F(\bar{y}|\omega_0), \quad H(\bar{x}|\omega_1) > F(\bar{y}|\omega_1). \tag{25}$$

Pick any $\hat{\omega} \in (\omega_0, \omega_1)$. Now we consider a special function $\psi(\omega) = \begin{cases} 0 & \text{if } \omega \leq \hat{\omega} \\ 1 & \text{otherwise} \end{cases}$.

Clearly ψ is monotone in ω.

Define

$$g(\omega) = \begin{cases} 1 & \text{if } \omega \in \{\omega_0, \omega_1\}, \\ \epsilon & \text{if } \omega \notin \{\omega_0, \omega_1\}, \end{cases} \tag{26}$$

for $\epsilon \in (0,1)$. Obviously g is positive for any $\epsilon > 0$, and it's uniformly bounded by 1.

Consider

$$h(\omega) = (\psi(\omega) - k)g(\omega),$$

which can be shown to be single crossing in ω.[34]

For such single-crossing $h(\omega)$, let $A = \{0,1\}$, define a utility function on $A \times \Omega$ as $u_\epsilon^\pi(a,\theta) = ah(\omega) = a(\psi(\omega) - k_\epsilon^\pi)g(\omega)$. Clearly, u satisfies the single-crossing property. By the MLR property, the function $\mathbb{E}[h(\omega)|X = x]$ is increasing in x following Quah and Strulovici (2009). And k_ϵ^π is chosen such that

$$\mathbb{E}[h(\omega)|X = \bar{x}] = \mathbb{E}[(\psi(\omega) - k)g(\omega))|X = \bar{x}] = 0,$$

where the conditional expectation is taken using H and the prior π. Such k always exists by setting $k_\epsilon^\pi = \frac{\mathbb{E}[\psi(\omega)g(\omega))|X=\bar{x}]}{\mathbb{E}[g(\omega))|X=\bar{x}]} \in [0,1]$.

An optimal decision rule for this $u := u_\epsilon^\pi$ under signal X is given by the following indicator function:

$$d^*(x) = \mathbf{1}_{\{\mathbb{E}[h(\omega)|X=x]\geq 0\}} = \begin{cases} 1 & \text{if } \mathbb{E}[h(\omega)|X = x] \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

with expected payoff

$$V^{EU}(H,\pi,u) := \int_\Omega \int_\mathcal{X} u(d^*(x),\omega)dH(x|\omega)d\pi(\omega) = \int_\Omega \int_\mathcal{X} \mathbf{1}_{\{\mathbb{E}[h(\omega)|X=x]\geq 0\}} h(\omega)dH(x|\omega)d\pi(\omega). \tag{27}$$

Similarly, for signal Y, we define the optimal strategy $d^*(y)$ in the similar way, we have:

$$\begin{aligned} V^{EU}(F,\pi,u) &:= \int_\Omega \int_\mathcal{Y} u(d^*(y),\omega)dF(y|\omega)d\pi(\omega) = \int_\Omega \int_\mathcal{Y} \mathbf{1}_{\{\mathbb{E}[h(\omega)|Y=y]\geq 0\}} h(\omega)dF(y|\omega)d\pi(\omega) \\ &\geq \int_\Omega \int_\mathcal{Y} u(d^0(y),\omega)dF(y|\omega)d\pi(\omega) = \int_\Omega \int_\mathcal{Y} \mathbf{1}_{\{y\geq y^0\}} h(\omega)dF(y|\omega)d\pi(\omega). \end{aligned} \tag{28}$$

for any $y^0 \in Y$. Here the inequality follows from the fact that the decision rule $d^0(y) = \mathbf{1}_{\{y\geq y^0\}}$ is clearly a feasible strategy, but may not be optimal.

[34] It means that, for all $\omega'' > \omega'$, $h(\omega') \geq (>) 0$ implies $h(\omega'') \geq (>) 0$.

From equation (27), we have:

$$V^{EU}(H,\pi,u) = \int_\Omega \int_\mathcal{X} \mathbf{1}_{\{x\geq \bar{x}\}} h(\omega) dH(x|\omega) d\pi(\omega), \tag{29}$$

as $\mathbf{1}_{\{\mathbb{E}[h(\omega)|X=x]\geq 0\}} = \mathbf{1}_{\{x\geq \bar{x}\}}$ from the construction of k.

By setting $y^0 = \bar{y}$ in (28), we have

$$V^{EU}(F,\pi,u) \geq \int_\Omega \int_\mathcal{Y} \mathbf{1}_{\{y\geq \bar{y}\}} h(\omega) dF(y|\omega) d\pi(\omega). \tag{30}$$

Consider the integration in (29), we can break the integration into two pieces.

$$\begin{aligned} & \int_\Omega \int_\mathcal{X} \mathbf{1}_{\{x\geq \bar{x}\}} h(\omega) dH(x|\omega) d\pi(\omega) \\ = & \int_{\omega\in\{\omega_0,\omega_1\}} \int_\mathcal{X} \mathbf{1}_{\{x\geq \bar{x}\}} h(\omega) dH(x|\omega) d\pi(\omega) + \int_{\omega\notin\{\omega_0,\omega_1\}} \int_\mathcal{X} \mathbf{1}_{\{x\geq \bar{x}\}} h(\omega) dH(x|\omega) d\pi(\omega). \end{aligned} \tag{31}$$

On the set $\omega \notin \{\omega_0, \omega_1\}$, the integrand is bounded by

$$|\mathbf{1}_{\{x\geq \bar{x}\}} h(\omega)| = |\mathbf{1}_{\{x\geq \bar{x}\}} (\psi(\omega) - k) g(\omega)| \leq \epsilon,$$

as $|\psi(\omega) - k| \leq 1$. Therefore,

$$\begin{aligned} |\int_{\omega\notin\{\omega_0,\omega_1\}} \int_\mathcal{X} \mathbf{1}_{\{x\geq \bar{x}\}} h(\omega) dH(x|\omega) d\pi(\omega)| &\leq \int_{\omega\notin\{\omega_0,\omega_1\}} \int_\mathcal{X} |\mathbf{1}_{\{x\geq \bar{x}\}} h(\omega)| dH(x|\omega) d\pi(\omega) \\ &\leq \int_{\omega\notin\{\omega_0,\omega_1\}} \int_\mathcal{X} \epsilon dH(x|\omega) d\pi(\omega) \leq \epsilon. \end{aligned} \tag{32}$$

Therefore,

$$\int_\Omega \int_\mathcal{X} \mathbf{1}_{\{x\geq \bar{x}\}} h(\omega) dH(x|\omega) d\pi(\omega) \leq \int_{\omega\in\{\omega_0,\omega_1\}} \int_\mathcal{X} \mathbf{1}_{\{x\geq \bar{x}\}} h(\omega) dH(x|\omega) d\pi(\omega) + \epsilon. \tag{33}$$

Similarly,

$$\int_\Omega \int_\mathcal{Y} \mathbf{1}_{\{y\geq \bar{y}\}} h(\omega) dF(y|\omega) d\pi(\omega) \geq \int_{\omega\in\{\omega_0,\omega_1\}} \int_\mathcal{Y} \mathbf{1}_{\{y\geq \bar{y}\}} h(\omega) dF(y|\omega) d\pi(\omega) - \epsilon. \tag{34}$$

As a result,

$$
\begin{aligned}
& V^{EU}(F,\pi,u) - V^{EU}(H,\pi,u) \\
\geq\ & -2\epsilon + \left(\int_{\omega\in\{\omega_0,\omega_1\}}\int_{\mathcal{Y}} \mathbf{1}_{\{y\geq\bar{y}\}} h(\omega) dF(y|\omega) d\pi(\omega)\right) \\
& - \left(\int_{\omega\in\{\omega_0,\omega_1\}}\int_{\mathcal{X}} \mathbf{1}_{\{x\geq\bar{x}\}} h(\omega) dH(x|\omega) d\pi(\omega)\right) \\
=\ & -2\epsilon + (h(\omega_0)[1-F(\bar{y}|\omega_0)]\pi(\omega_0) + h(\omega_1)[1-F(\bar{y}|\omega_1)]\pi(\omega_1)) \\
& - (h(\omega_0)[1-H(\bar{x}|\omega_0)]\pi(\omega_0) + h(\omega_1)[1-H(\bar{x}|\omega_1)]\pi(\omega_1)) \\
=\ & -2\epsilon + h(\omega_0)\pi(\omega_0)\underbrace{[H(\bar{x}|\omega_0) - F(\bar{y}|\omega_0)]}_{\leq 0} \quad + \quad h(\omega_1)\pi(\omega_1)\underbrace{[H(\bar{x}|\omega_1) - F(\bar{y}|\omega_1)]}_{>0}.
\end{aligned}
$$

Recall that $h(\omega_0) = (0-k)g(\omega_0) = -k_\epsilon^\pi \leq 0$ and $h(\omega_1) = (1-k_\epsilon^\pi)g(\omega_1) = (1-k_\epsilon^\pi) \geq 0$. Therefore,

$$
\begin{aligned}
& V^{EU}(F,\pi,u) - V^{EU}(H,\pi,u) \\
\geq\ & -2\epsilon - k_\epsilon^\pi\pi(\omega_0)\underbrace{[H(\bar{x}|\omega_0) - F(\bar{y}|\omega_0)]}_{\leq 0} + (1-k_\epsilon^\pi)\pi(\omega_1)\underbrace{[H(\bar{x}|\omega_1) - F(\bar{y}|\omega_1)]}_{>0} \\
\geq\ & -2\epsilon + (1-k_\epsilon^\pi)\pi(\omega_1)\underbrace{[H(\bar{x}|\omega_1) - F(\bar{y}|\omega_1)]}_{>0}. \qquad (35)
\end{aligned}
$$

Notice that in the above expression, the parameter k depends on the choice of ϵ. When $\epsilon \to 0$, k converges to some number in $(0,1)$.[35] By continuity, there exits a $\epsilon_1 > 0$ small enough , such that

$$V(F,\pi,u) - V(H,\pi,u) > 0, \text{for all } \epsilon \in (0,\epsilon_1]. \qquad (37)$$

This finishes the proof. □

A.4 Lemma 4

Define

$$Q(\epsilon,\pi) = -2\epsilon + (1-k_\epsilon^\pi)\pi(\omega_1)[H(\bar{x}|\omega_1) - F(\bar{y}|\omega_1)], \qquad (38)$$

[35] in the limit as $\epsilon \to 0$, the k_ϵ^π converges to

$$\Pr[(\omega > \bar{\omega})|X = \bar{x}, \omega \in \{\omega_0,\omega_1\}]. \qquad (36)$$

for $\epsilon \in [0,1]$, $\pi \in \Delta$.

Proposition 1 can be strengthened using the following lemma.

Lemma 4. *Fixing $\delta > 0$, there exists $\epsilon_1 > 0$ and $\underline{c} > 0$ such that for any $\epsilon \in [0, \epsilon_1]$ and $\pi \in \Delta^\delta$,*

$$Q(\epsilon, \pi) \geq \underline{c} > 0, \forall \epsilon \in [0, \epsilon_1], \ \forall \pi \in \Delta^\delta. \tag{39}$$

Proof of Lemma 4. Clearly Q is continuous as k_ϵ^π is continuous in both ϵ and π. Fixing $\delta > 0$, recall $\Delta^\delta := \{\pi \in \Delta\Omega : \pi(\omega) \geq \delta, \forall \omega\}$.

Define

$$m(\epsilon) := \inf_{\pi \in \Delta^\delta} Q(\epsilon, \pi) = \min_{\pi \in \Delta^\delta} Q(\epsilon, \pi).$$

(The min exists as Δ^δ is compact.) By the theorem of maximum, m is continuous in ϵ.

When $\epsilon = 0$, for any $\pi \in \Delta^\delta$, $\pi(\omega_1) \geq \delta$ and $\lim_{\epsilon \to 0} k_\epsilon^\pi < 1$. As a result, $Q(0, \pi) > 0$. Therefore, $m(0) > 0$. By continuity, the lemma follows. □

A.5 Proof of Lemma 2

To show Lemma 2, we need a general Minimax Theorem by Sion (1958).

Theorem 3 (Sion 1958, Theorem 3.4)**.** *Let M and N be convex, compact spaces, and f a function on $M \times N$, quasi-concave-convex and u.s.c.-l.s.c.*[36] *Then*

$$\sup_{\mu \in M} \inf_{\nu \in N} f(\mu, \nu) = \inf_{\nu \in N} \sup_{\mu \in M} f(\mu, \nu).$$

Proof. Fix u^π, we define a function $\tilde{G} : (\Delta A)^S \times \Delta\Omega \mapsto (-\infty, +\infty]$ as

$$\tilde{G}(\sigma, \tilde{\pi}) := G(U^{EU}(F, \tilde{\pi}, \sigma, u^\pi), \tilde{\pi}).$$

Note $M := \{\tilde{\pi} \in \Delta\Omega : dom G(\cdot, \tilde{\pi}) = \mathcal{X}\}$ is the subset of $\tilde{\pi} \in \Delta\Omega$ where $G(\cdot, \tilde{\pi})$ is not constant at $+\infty$. Recall $M := \{\tilde{\pi} \in \Delta\Omega : dom G(\cdot, \tilde{\pi}) = \mathcal{X}\}$. By assumption, it is without loss to restrict on $\tilde{G} : (\Delta A)^S \times M \mapsto \mathbb{R}$, where $(\Delta A)^S$ and $M \subseteq \Delta\Omega$ are convex and compact subspaces.

[36] A real-valued function f on $M \times N$ is *quasi-concave-convex* if it is quasi-concave in M and quasi-convex in N. A real-valued function f on $M \times N$ is *u.s.c.-l.s.c.* if $f(\cdot, \nu)$ is upper semi-continuous in μ for each $\nu \in N$ and $f(\mu, \cdot)$ is lower semi-continuous in ν for each $\mu \in M$.

For any $\tilde{\pi}$, $\tilde{G}(\cdot, \tilde{\pi}) := G(U^{EU}(F, \tilde{\pi}, \cdot, u^{\pi}), \tilde{\pi})$ is quasi-concave because the linear EU functional $U^{EU}(F, \tilde{\pi}, \cdot, u^{\pi})$ is quasi-concave and $G(\cdot, \tilde{\pi})$ is increasing. For any σ, $\tilde{G}(\sigma, \cdot) := G(U^{EU}(F, \cdot, \sigma, u^{\pi}), \cdot)$ is quasi-convex because $U^{EU}(F, \cdot, \sigma, u^{\pi})$ is linear in σ and G is quasi-convex. Thus $\tilde{G}$ is quasi-concave-convex.

Moreover, since $G(\cdot, \cdot)$ is lower-semi-continuous on $\mathcal{X} \times \Delta\Omega$ and $U(F, S, \cdot, \sigma)$ is linear in $\tilde{\pi}$, $\tilde{G}(\sigma, \cdot)$ is lower-semi-continuous on $\Delta\Omega$. Note that $\{G(\cdot, \tilde{\pi})\}_{\tilde{\pi} \in M}$ is uniformly equicontinous on $\mathcal{X}$ and $U(F, S, \tilde{\pi}, \cdot)$ is linear on $(\Delta A)^S$, hence $\tilde{G}(\cdot, \tilde{\pi})$ is continuous on $(\Delta A)^S$.

Hence all the conditions of Sion's Minimax Theorem are met. □

A.6 Lemma 5

In this appendix, we discuss an interesting property of the set of saddle points. Consider a general quasi-concave-convex function $f : X \times Y \mapsto \mathbb{R}$ that is upper semi-continuous (u.s.c.) in x and lower semi-continuous (l.s.c.) in y, where X and Y are some compact and convex spaces. The set of saddle points S is defined as

$$\{(x^*, y^*) : f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y) \quad \forall x \in X, y \in Y\}.$$

By Sion's Minimax Theorem, S is nonempty. Then, let X^* and Y^* be the projection of S onto X and Y, respectively. The next lemma says that any selection of $x^* \in X^*$ and $y^* \in Y^*$ forms a saddle point. That is, the set of saddle points has a product structure.

Lemma 5. *Let $f : X \times Y \mapsto \mathbb{R}$ be a quasi-concave-convex and u.s.c.-l.s.c. function defined on some compact and convex spaces X and Y. Let $S \subseteq X \times Y$ be the set of saddle points. Then there are some nonempty and convex subsets $X^* \subseteq X$ and $Y^* \subseteq Y$ such that $S = X^* \times Y^*$. Moreover, $f(x^*, y^*) = \bar{V}$ for all $(x^*, y^*) \in X^* \times Y^*$.*

Proof. Recall that S is defined as

$$\{(x^*, y^*) : f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y) \quad \forall x \in X, y \in Y\}.$$

Non-emptiness of S is given by Sion's Minimax Theorem. Pick any two saddle points (x_1^*, y_1^*) and (x_2^*, y_2^*) from S. Let X^* and Y^* be the projection of S onto X and Y, respectively. Then for all $x \in X, y \in Y$,

$$f(x, y_1^*) \leq f(x_1^*, y_1^*) \leq f(x_1^*, y), \tag{40}$$

$$f(x, y_2^*) \leq f(x_2^*, y_2^*) \leq f(x_2^*, y). \tag{41}$$

Then, by setting $y = y_2^*$ in the third term in (40) and $x = x_1^*$ in the first term in (41), and setting $x = x_2^*$ in the first term in (40) and $y = y_1^*$ in the third term in (41), we have

$$f(x_1^*, y_1^*) \leq f(x_1^*, y_2^*) \leq f(x_2^*, y_2^*) \leq f(x_2^*, y_1^*) \leq f(x_1^*, y_1^*). \tag{42}$$

Observe that f attains the same minimax value at the four pairs in (42). Let this value be $\bar{V}$.

Also, for all x, y, combining (40) and (42) yields

$$f(x_1^*, y_2^*) \leq f(x_1^*, y_1^*) \leq f(x_1^*, y),$$

and combining (41) and (42) yields

$$f(x, y_2^*) \leq f(x_2^*, y_2^*) \leq f(x_1^*, y_2^*).$$

Thus the "exchanged" pair (x_1^*, y_2^*) is also a saddle point. By the same argument, $(x_2^*, y_1^*) \in S$.

In light of this, it suffices to show that X^* and Y^* are convex. Pick any $\alpha \in (0, 1)$, let $x_\alpha^* = \alpha x_1^* + (1-\alpha) x_2^*$ and $y_\alpha^* = \alpha y_1^* + (1-\alpha) y_2^*$. For all x, y, by definition $f(x_1^*, y) \geq \bar{V}$ and $f(x_2^*, y) \geq \bar{V}$ and by quasi-concavity of $f(\cdot, y)$

$$f(x_\alpha^*, y) \geq \min\{f(x_1^*, y), f(x_2^*, y)\} \geq \bar{V} \quad \forall y \in Y.$$

Similarly $f(x, y_\alpha^*) \leq \bar{V}$ for all $x \in X$. Hence

$$f(x, y_\alpha^*) \leq f(x_\alpha^*, y_\alpha^*) = \bar{V} \leq f(x_\alpha^*, y).$$

So the convex combination (x_α^*, y_α^*) is also a saddle point. □

A.7 Proof of Lemma 3

Proof. For $\pi \in \Delta^\delta$, define

$$f_\pi(\sigma, \tilde{\pi}) := G(U^{EU}(F, \tilde{\pi}, \sigma, u^\pi), \tilde{\pi}).$$

Since Ω is finite, Δ^δ is a nonempty, compact, and convex subset of $\mathbb{R}^{|\Omega|}$ space. For all π, by Lemma 5, $S(\pi)$ is nonempty and convex.

Take $\pi^n \to \pi$ and $\tilde{\pi}^n \in \Sigma(\pi^n)$. Then there exists σ^n where $(\sigma^{n*}, \tilde{\pi}^{n*}) \in S(\pi^n)$ so that

$$f_{\pi^n}(\sigma, \tilde{\pi}^{n*}) \leq f_{\pi^n}(\sigma^{n*}, \tilde{\pi}^{n*}) \leq f_{\pi^n}(\sigma^{n*}, \tilde{\pi}), \quad \forall \tilde{\pi},\ \sigma. \tag{43}$$

Since $(\Delta A)^S$ and Δ^δ are compact, there exists some subsequence $\{(\sigma^{n_k*}, \tilde{\pi}^{n_k*})\}_{n_k}$ of $\{(\sigma^{n*}, \tilde{\pi}^{n*})\}_n$ that converges to some $(\sigma^*, \tilde{\pi}^*)$. Sending $n_k \to +\infty$, because u_π and hence $f_\pi(\sigma, \tilde{\pi})$ varies continuously with π, $f_\pi(\cdot, \tilde{\pi})$ is continuous, and $f_\pi(\sigma, \cdot)$ is continuous on M, (43) implies $\forall \tilde{\pi}, \sigma$:

$$f_\pi(\sigma, \tilde{\pi}^*) = \lim_{n_k} f_{\pi^{n_k}}(\sigma, \tilde{\pi}^{n_k*}) \leq \lim_{n_k} f_{\pi^{n_k}}(\sigma^{n_k*}, \tilde{\pi}^{n_k*}) \leq \lim_{n_k} f_{\pi^{n_k}}(\sigma^{n_k*}, \tilde{\pi}) = f_\pi(\sigma^*, \tilde{\pi}),$$

therefore $(\sigma^*, \tilde{\pi}^*) \in S(\pi)$. Hence $\tilde{\pi}^* \in \Sigma(\pi)$ and the correspondence Σ is upper-hemi-continuous. For any fixed π, repeat the above argument with $\pi^n = \pi$ implies $\Sigma(\pi)$ is closed and hence compact. By Kakutani's fixed point theorem, Σ has a fixed point. □

A.8 Proof of Proposition 2

Suppose (6) holds, then, since π^0 is of full support, using the result for the only if case for the EU case, we can construct a single-crossing utility index $v(a, \omega)$ such that

$$\max_\sigma U^{EU}(H, \pi^0, \sigma, v) < \max_\sigma U^{EU}(F, \pi^0, \sigma, v). \tag{44}$$

Define $\delta := \max_\sigma U^{EU}(F, \pi^0, \sigma, v) - \max_\sigma U^{EU}(H, \pi^0, \sigma, v)$. Clearly δ is positive by construction. For any $\epsilon > 0$, we define $u := \epsilon v$, which is also single crossing and satisfies (44). Next we show that for small enough ϵ, this u will do the job.

Assume $\phi(0) = 0$ and $\phi'(0) = 1$. This is WLOG because $\phi(\cdot)$ is unique up to a positive affine transformation.[37] Let $M_0 := \max_{a,\omega} |v(a, \omega)|$. We first claim that there exists a positive constant M_1 such that

$$|\phi(t) - t| \leq M_1 t^2, \qquad \forall t \in [-M_0, M_0].$$

(For example, pick $M_1 = \frac{1}{2} \max_{t \in [-M_0, M_0]} |\phi''(t)|$.)

For any strategy σ and any $\epsilon \in (0, 1)$, $|U^{EU}(H, \pi, \sigma, u)| \leq \max_{a,\omega} |u(a, \omega)| = \epsilon \max_{a,\omega} |v(a, \omega)| =$

[37] We assume ϕ is twice continuously differentiable around 0.

ϵM_0, therefore

$$
\begin{aligned}
& |\max_{\sigma} \int_{\Delta(\Omega)} \phi(U^{EU}(H,\pi,\sigma,u))d\mu(\pi) - \max_{\sigma} \int_{\Delta(\Omega)} U^{EU}(H,\pi,\sigma,u)d\mu(\pi)| && (45)\\
\leq\ & \max_{\sigma} \int_{\Delta(\Omega)} |\phi(U^{EU}(H,\pi,\sigma,u)) - U^{EU}(H,\pi,\sigma,u)|d\mu(\pi) && (46)\\
=\ & \max_{\sigma} \int_{\Delta(\Omega)} M_1(\epsilon M_0)^2 d\mu(\pi) = M_1(\epsilon M_0)^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& |\max_{\sigma} \int_{\Delta(\Omega)} \phi(U^{EU}(F,\pi,\sigma,u))d\mu(\pi) - \max_{\sigma} \int_{\Delta(\Omega)} U^{EU}(F,\pi,\sigma,u)d\mu(\pi)| && (47)\\
& \leq M_1(\epsilon M_0)^2. && (48)
\end{aligned}
$$

Moreover, by linearity of U^{EU} in π, we have

$$
\int_{\Delta(\Omega)} U^{EU}(H,\pi,\sigma,u)d\mu(\pi) = U^{EU}(H, \int_{\Delta(\Omega)} d\mu(\pi),\sigma,u) = U^{EU}(H,\pi^0,\sigma,u),
$$

for any σ. As a result,

$$
\max_{\sigma} \int_{\Delta(\Omega)} U^{EU}(H,\pi,\sigma,u)d\mu(\pi) = \max_{\sigma} U^{EU}(H,\pi^0,\sigma,u) = \epsilon \max_{\sigma} U^{EU}(H,\pi^0,\sigma,v). \tag{49}
$$

Similarly, we have

$$
\max_{\sigma} \int_{\Delta(\Omega)} U^{EU}(F,\pi,\sigma,u)d\mu(\pi) = \epsilon \max_{\sigma} U^{EU}(F,\pi^0,\sigma,v). \tag{50}
$$

Let

$$
\bar{\epsilon} := \min\left(1, \frac{\delta}{2M_1M_0^2}\right) > 0.
$$

Pick any ϵ satisfying $0 < \epsilon < \bar{\epsilon}$. Then by the triangular inequality and equations

(45)–(50), we have

$$
\begin{aligned}
& \max_{\sigma} \int_{\Delta(\Omega)} \phi(U^{EU}(H,\pi,\sigma,u))d\mu(\pi) - \max_{\sigma} \int_{\Delta(\Omega)} \phi(U^{EU}(F,\pi,\sigma,u))d\mu(\pi) \quad (51)\\
\leq \quad & \max_{\sigma} \int_{\Delta(\Omega)} U^{EU}(H,\pi,\sigma,u)d\mu(\pi) - \max_{\sigma} \int_{\Delta(\Omega)} U^{EU}(F,\pi,\sigma,u)d\mu(\pi)\\
& +|\max_{\sigma} \int_{\Delta(\Omega)} \phi(U^{EU}(H,\pi,\sigma,u))d\mu(\pi) - \max_{\sigma} \int_{\Delta(\Omega)} U^{EU}(H,\pi,\sigma,u)d\mu(\pi)|\\
& +|\max_{\sigma} \int_{\Delta(\Omega)} \phi(U^{EU}(F,\pi,\sigma,u))d\mu(\pi) - \max_{\sigma} \int_{\Delta(\Omega)} U^{EU}(F,\pi,\sigma,u)d\mu(\pi)|\\
\leq \quad & -\epsilon \underbrace{(\max_{\sigma} U^{EU}(F,\pi^0,\sigma,v) - \max_{\sigma} U^{EU}(H,\pi^0,\sigma,v))}_{=\delta} + M_1 M_0^2 \epsilon^2 + M_1 M_0^2 \epsilon^2\\
= \quad & -\epsilon(\delta - 2M_1 M_0^2 \epsilon) < 0. \quad (52)
\end{aligned}
$$

Then for $u = \epsilon v$, we have

$$
\max_{\sigma} \phi^{-1}\left(\int \phi(U^{EU}(H,\pi,\sigma,u))d\mu(\pi)\right) < \max_{\sigma} \phi^{-1}\left(\int \phi(U^{EU}(F,\pi,\sigma,u))d\mu(\pi)\right).
$$

□

Remark 9. The proof proceeds exactly as that of Theorem 1, except that here we construct a single crossing utility index using a different method. A similar method was used by Li and Zhou (2016).

A.9 Proof of Proposition 3

Suppose (6) holds, then, since π^0 is of full support, using the result for the only if case for the EU case, we can construct a single-crossing utility index $v(a,\omega)$ such that

$$
\max_{\sigma} U^{EU}(H,\pi^0,\sigma,v) < \max_{\sigma} U^{EU}(F,\pi^0,\sigma,v). \quad (53)
$$

Define $\delta := \max_{\sigma} U^{EU}(F,\pi^0,\sigma,v) - \max_{\sigma} U^{EU}(H,\pi^0,\sigma,v)$. Clearly δ is positive by construction. For any $\epsilon > 0$, we define $u := \epsilon v$, which is also single-crossing, and satisfies (53). Next we show that for small enough ϵ, this u will do the job.

Assume $\phi(0) = 0$ and $\phi'(0) = 1$. This is WLOG because $\phi(\cdot)$ is unique up to a positive affine transformation.[38] Let $M_0 := \max_{a,\omega} |v(a,\omega)|$. We first claim that there exists a

[38] We assume ϕ is twice continuously differentiable around 0.

positive constant M_1 such that

$$|\phi(t) - t| \leq M_1 t^2, \qquad \forall t \in [-M_0, M_0].$$

(For example, pick $M_1 = \frac{1}{2}\max_{t\in[-M_0,M_0]} |\phi''(t)|$.)

For any strategy σ and any $\epsilon \in (0,1)$, for any signal $s \in S$ and state $\omega \in \Omega$,

$$|\int_A u(a,\omega)d\sigma_s(a)| \leq \int_A \epsilon M_0 d\sigma_s(a) = \epsilon M_0,$$

therefore

$$|\phi(\int_A u(a,\omega)d\sigma_s(a)) - (\int_A u(a,\omega)d\sigma_s(a))| \leq M_1(\epsilon M_0)^2. \tag{54}$$

Therefore, for any strategy σ,

$$\begin{aligned}
&|\int_\Omega \int_S \phi\left(\int_A u(a,\omega)d\sigma_s(a)\right) dH(s|\omega)d\pi^0(\omega) - \int_\Omega \int_S \left(\int_A u(a,\omega)d\sigma_s(a)\right) dH(s|\omega)d\pi^0(\omega)| \\
&\leq \int_\Omega \int_S |\phi(\int_A u(a,\omega)d\sigma_s(a)) - (\int_A u(a,\omega)d\sigma_s(a))| dH(s|\omega)d\pi^0(\omega) \\
&\leq \int_\Omega \int_S M_1(\epsilon M_0)^2 dH(s|\omega)d\pi^0(\omega) = M_1(\epsilon M_0)^2.
\end{aligned} \tag{55}$$

Similarly, $\forall\sigma$, we have,

$$\begin{aligned}
&|\int_\Omega \int_S \phi\left(\int_A u(a,\omega)d\sigma_s(a)\right) dF(s|\omega)d\pi^0(\omega) - \int_\Omega \int_S \left(\int_A u(a,\omega)d\sigma_s(a)\right) dF(s|\omega)d\pi^0(\omega)| \\
&\leq M_1(\epsilon M_0)^2.
\end{aligned}$$

Let $\bar{\epsilon} := \min\left(1, \frac{\delta}{2M_1M_0^2}\right) > 0$. Then for any ϵ satisfying $0 < \epsilon < \bar{\epsilon}$, we have

$$\begin{aligned}
&\max_\sigma \int_\Omega \int_S \phi\left(\int_A u(a,\omega)d\sigma_s(a)\right) dF(s|\omega)d^0\pi(\omega) - \max_\sigma \int_\Omega \int_S \phi\left(\int_A u(a,\omega)d\sigma_s(a)\right) dH(s|\omega)d^0\pi(\omega) \\
= \;& \left(\max_\sigma \int_\Omega \int_S \phi\left(\int_A u(a,\omega)d\sigma_s(a)\right) dF(s|\omega)d\pi^0(\omega) - \max_\sigma \int_\Omega \int_S \left(\int_A u(a,\omega)d\sigma_s(a)\right) dF(s|\omega)d\pi^0(\omega)\right) \\
&+ \left(\max_\sigma \int_\Omega \int_S \left(\int_A u(a,\omega)d\sigma_s(a)\right) dH(s|\omega)d\pi^0(\omega) - \max_\sigma \int_\Omega \int_S \phi\left(\int_A u(a,\omega)d\sigma_s(a)\right) dH(s|\omega)d\pi^0(\omega)\right) \\
&+ \left(\max_\sigma \int_\Omega \int_S \left(\int_A u(a,\omega)d\sigma_s(a)\right) dF(s|\omega)d\pi^0(\omega) - \max_\sigma \int_\Omega \int_S \left(\int_A u(a,\omega)d\sigma_s(a)\right) dH(s|\omega)d\pi^0(\omega)\right)
\end{aligned}$$

By equation (56), the first term satisfies

$$\begin{aligned}
&\left(\max_{\sigma}\int_{\Omega}\int_{S}\phi\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dF(s|\omega)d\pi^0(\omega)-\max_{\sigma}\int_{\Omega}\int_{S}\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dF(s|\omega)d\pi^0(\omega)\right)\\
&\geq -\max_{\sigma}\left|\int_{\Omega}\int_{S}\phi\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dF(s|\omega)d\pi^0(\omega)-\int_{\Omega}\int_{S}\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dF(s|\omega)d\pi^0(\omega)\right|\\
&\geq -M_1M_0^2\epsilon^2.
\end{aligned}$$

Similarly, by equation (55), the second term satisfies

$$\begin{aligned}
&\left(\max_{\sigma}\int_{\Omega}\int_{S}\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dH(s|\omega)d\pi^0(\omega)-\max_{\sigma}\int_{\Omega}\int_{S}\phi\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dH(s|\omega)d\pi^0(\omega)\right)\\
&\geq -M_1M_0^2\epsilon^2.
\end{aligned}$$

And the third term is

$$\begin{aligned}
&\left(\max_{\sigma}\int_{\Omega}\int_{S}\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dF(s|\omega)d\pi^0(\omega)-\max_{\sigma}\int_{\Omega}\int_{S}\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dH(s|\omega)d\pi^0(\omega)\right)\\
&=\epsilon\max_{\sigma}U^{EU}(F,\pi^0,\sigma,v)-\epsilon\max_{\sigma}U^{EU}(H,\pi^0,\sigma,v)=\epsilon\delta.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&\max_{\sigma}\int_{\Omega}\int_{S}\phi\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dF(s|\omega)d\pi^0(\omega)\\
&\qquad-\max_{\sigma}\int_{\Omega}\int_{S}\phi\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dH(s|\omega)d\pi^0(\omega)\\
&\geq -M_1M_0^2\epsilon^2-M_1M_0^2\epsilon^2+\epsilon\delta=\epsilon(\delta-2M_1M_0^2\epsilon)>0.
\end{aligned}$$

Then for this $u=\epsilon v$, we have

$$\begin{aligned}
&\max_{\sigma}\phi^{-1}\left(\int_{\Omega}\int_{S}\phi\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dF(s|\omega)d\pi^0(\omega)\right)\\
>\ &\max_{\sigma}\phi^{-1}\left(\int_{\Omega}\int_{S}\phi\left(\int_{A}u(a,\omega)d\sigma_s(a)\right)dH(s|\omega)d\pi^0(\omega)\right),
\end{aligned}$$

which concludes the proof. □

A.10 Proof of Proposition 4

Given two interior priors π and π', define a positive function $\bar{g}$ as $\bar{g}(\omega)=\pi(\omega)/\pi'(\omega)$.

For any strategy σ, from the definition of value of information, the following holds:

$$U^{EU}(H, \pi, \sigma, u) = U^{EU}(H, \pi', \sigma, u^{\bar{g}}), \forall u. \tag{56}$$

Similarly, for information structure F,

$$U^{EU}(F, \pi, \sigma, u) = U^{EU}(F, \pi', \sigma, u^{\bar{g}}), \forall u.$$

Consider the mapping $\psi^{\bar{g}}$ by mapping each u to $u^{\bar{g}}$. Since $\mathbb{U}^*$ is closed under state-wise weighting, the image of this mapping on $\mathbb{U}^*$ is also in $\mathbb{U}^*$. In fact, this mapping is one to one and onto, and hence defines a bijection on $\mathbb{U}^*$. The inverse of this mapping is defined by $1/\bar{g}$. In particular, for any u, define $\tilde{u} = u^{1/\bar{g}}$, then it is easy to check that $\tilde{u}^{\bar{g}} = u$. By (56),

$$\max_{\sigma} U^{EU}(H, \pi, \sigma, u) \geq \max_{\sigma} U^{EU}(F, \pi, \sigma, u), \text{ for all u in } \mathbb{U}^*,$$

if and only if

$$\max_{\sigma} U^{EU}(H, \pi', \sigma, u^{\bar{g}}) \geq \max_{\sigma} U^{EU}(F, \pi', \sigma, u^{\bar{g}}), \text{ for all u in } \mathbb{U}^*,$$

if and only if

$$\max_{\sigma} U^{EU}(H, \pi', \sigma, u) \geq \max_{\sigma} U^{EU}(F, \pi', \sigma, u), \text{ for all u in } \mathbb{U}^*,$$

where the last step follows from the fact that $\{u : u \in \mathbb{U}^*\} = \{u^{\bar{g}}, u \in \mathbb{U}^*\}$. The result just follows. □

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