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AN ALGORITHM FOR GEODETIC NAVIGATION FOR THE TRANSIT SATELLITE SYSTEM

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INTRODUCTION

In order to familiarize the reader a brief description of the Transit Satellite Navigation System as reported in the literature is given. Two previously known methods of data reduction are then described which a navigator may use to determine his position. The proposed method of data reduction presented herein is then discussed briefly and comparisons are made with the first two methods.

Description of the Transit Satellite System

The Transit Satellite Navigation System is being developed as an aid to high accuracy navigation on the surface of the earth by the Applied Physics Laboratory of The Johns Hopkins University (3, 4). The ultimate accuracy of the system is not yet known, but accuracies of one-half nautical mile and better are thought possible. The system does not allow for continuous navigation, but rather it is intended for correcting the navigator's position periodically. It is thought that ships will make primary use of the system, although it might prove feasible for aircraft navigation in the future (7).

The exact specifications for the system have not been published, but numerous magazine articles have described the basic concepts (2). The following is a simplified explanation of how navigation is accomplished. There will probably be four or more Transit satellites, each in a near circular, polar orbit approximately 500 nautical miles above the surface of the earth. The satellites will be positioned such that a navigator at any point on the earth's surface will be able to observe a satellite at least once every 90 minutes. The data taken by the navigator from observing one pass of any of the satellites will usually be sufficient to determine a position fix to the desired accuracy. Each satellite broadcasts a constant CW radio signal in the 100 to 500 megacycle region. When a satellite appears above the horizon, the navigator receives this CW radio signal and extracts the Doppler frequency shift caused by the relative motion of the satellite and the navigator. As the satellite crosses the sky, the navigator measures and records this Doppler frequency shift.

Each satellite has a memory that can be loaded with information from a ground injection station that is part of the Transit system. Ground tracking stations observe the orbit of each of the satellites and compute a set of parameters which will describe the orbit. Every twelve hours this orbital information is loaded into the satellite's memory. Once every minute the information is broadcast by the satellite by superimposing binary coded information on the CW signal. During the pass of the satellite the navigator must decode this orbital information and from it compute the position of the satellite as a function of time.

Using an assumed position for himself and the true position of the satellite, the navigator can compute a theoretical Doppler curve which he can compare with the Doppler curve he has observed. The true position of the navigator is found by adjusting his assumed position until the theoretical Doppler curve is a "best match" to the observed Doppler curve. For a high accuracy position fix, the curve matching computations are complex and must be done with a digital computer.

The navigator's position, determined in this manner, has one known ambiguity. There are two points on the earth's surface at which nearly identical Doppler curves will be observed. The false point is located symmetrically on the other side of the plane of the orbit to the navigator's position. The rotation of the earth causes dissimilarities between the two curves which usually make the distinction possible. If the satellite does not pass too close to the navigator, these points are relatively far apart and the more reasonable position is easily determined. If the satellite passes nearly overhead, the two points must be distinguished by determining which yields a "better match" between the theoretical and observed Doppler curves. However, when the satellite does pass nearly overhead, the effect of noise in the Doppler data prohibits an accurate position fix (3).

To be more exact, the satellite will broadcast two harmonically related CW signals so that a correction for ionospheric refraction is possible. The error in the Doppler frequency shift due to refraction is, to a first order approximation, inversely proportional to transmitter frequency. Hence, by suitably combining the Doppler shifts from two transmitter frequencies, the first order error terms can be made to cancel (3).

In obtaining a "best match" between the theoretical and observed Doppler frequency curves, three variables must be adjusted. Two variables relate to the navigator's position, his latitude and longitude. The third variable allows a constant bias on the observed Doppler curve to account for long term drift in the oscillators of the satellite and the navigator. These oscillators are assumed stable and constant during a given satellite pass (3).

Curve A of Figure 1 is a typical Doppler curve computed for a circular, polar orbit 500 nautical miles above the earth with the navigator on the

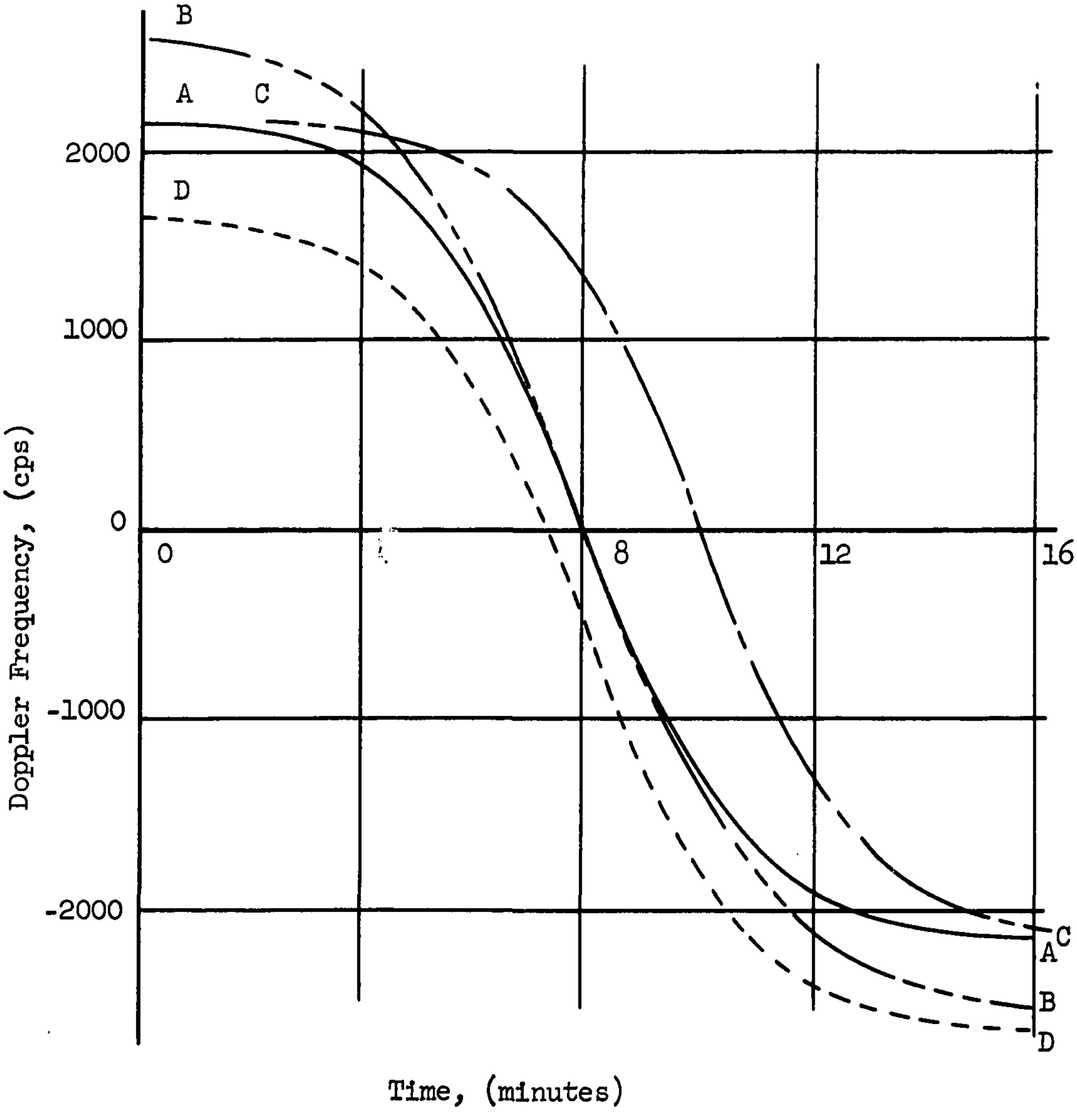


Figure 1. Typical Doppler curves

equator. The projection of the satellite's orbit on the surface of the earth lies about 400 nautical miles from the navigator; the transmitter frequency is 100 megacycles. Because of the rotation of the earth, the curve is not symmetrical with respect to the zero crossing. Curve B results from a change in the navigator's longitude. The major effect is to multiply the frequency by a constant. A change in the navigator's latitude essentially shifts the frequency curve in time as is shown by Curve C. A constant bias on the Doppler frequency shifts the curve in frequency by a constant amount as is shown by Curve D. Curves B and C are greatly exaggerated for clarity. If the navigator's position had been changed by only one nautical mile, the curves would not have been distinguishable on the graph. It therefore requires very precise frequency measurements to achieve high accuracy navigation (6).

Present Methods for Computing the Navigator's Position

The purpose of this research is to investigate a method for obtaining a "best match" between the theoretical and observed Doppler curves, and thus determining the navigator's position. A separate problem that will not be considered is that of determining the position of the satellite as a function of time from the orbital information broadcast by the satellite. For purposes here, the position of the satellite as a function of time will be assumed known.

Two methods for obtaining a "best match" have been reported in the literature. A least squares curve fit was suggested by the Applied Physics Laboratory (3) and the method of averages was proposed by K. C. Kochi of Autonetics Division of North American Aviation (6). The least squares fit is generally accepted to be more accurate and allows the navigator to estimate the accuracy of his position fix. The method of averages, however, requires much less computation time and considerably less storage of data.

Method of least squares

In the method of least squares, the observed Doppler curve is defined by a set of discrete data $\left\{f_{0_i}\right\}$, where f_{0_i} is the observed Doppler frequency shift at time t_i. The residual of a data point, μ_i, is defined to be the difference between the theoretical Doppler shift, f_{T_i}, and the observed Doppler shift at time t_i. The theoretical Doppler shift is a function of the navigator's latitude, Θ, and longitude, λ, and the bias, b.

$$\mu_i = f_{T_i}(\Theta, \lambda, b) - f_{0_i} \qquad 1$$

An error function, $F(\Theta, \lambda, b)$, is defined as the sum of the squared residuals.

$$F(\Theta, \lambda, b) = \sum_i \mu_i^2 \qquad 2$$

The "best match" occurs when the three variables are adjusted in such a way that the error function is a minimum, a condition that is found using the following three equations:

$$F_\Theta(\Theta, \lambda, b) = 0 \qquad 3$$

$$F_\lambda(\Theta, \lambda, b) = 0 \qquad 4$$

$$F_b(\Theta, \lambda, b) = 0 \qquad 5$$

The solution of these three equations, θ_c, λ_c, and b_c, is the navigator's computed position and bias. The minimum value of the error function, $F(\theta_c, \lambda_c, b_c)$, yields an estimate of the accuracy of the position fix.

Since Equations 3, 4, and 5 are nonlinear, an iterative technique is used to find their solution. The partial derivatives of the error function are expanded in first order Taylor series about the assumed position of the navigator, θ_a and λ_a, and the assumed bias, b_a.

$$F_\theta(\theta, \lambda, b) = F_\theta(\theta_a, \lambda_a, b_a) + F_{\theta\theta}(\theta_a, \lambda_a, b_a)\,\Delta\theta + F_{\theta\lambda}(\theta_a, \lambda_a, b_a)\Delta\lambda + F_{\theta b}(\theta_a, \lambda_a, b_a)\,\Delta b = 0 \quad 6$$

$$F_\lambda(\theta, \lambda, b) = F_\lambda(\theta_a,\lambda_a,b_a) + F_{\lambda\theta}(\theta_a,\lambda_a,b_a)\,\Delta\theta + F_{\lambda\lambda}(\theta_a,\lambda_a,b_a)\Delta\lambda + F_{\lambda b}(\theta_a,\lambda_a,b_a)\,\Delta b = 0 \quad 7$$

$$F_b(\theta, \lambda, b) = F_b(\theta_a,\lambda_a,b_a) + F_{b\theta}(\theta_a,\lambda_a,b_a)\,\Delta\theta + F_{b\lambda}(\theta_a,\lambda_a,b_a)\,\Delta\lambda + F_{bb}(\theta_a,\lambda_a,b_a)\,\Delta b = 0 \quad 8$$

where

$$\theta = \theta_a + \Delta\theta \quad 9$$

$$\lambda = \lambda_a + \Delta\lambda \quad 10$$

$$b = b_a + \Delta b \quad 11$$

Equations 6, 7 and 8 can be solved simultaneously for $\Delta\theta$, $\Delta\lambda$, and Δb. Using the newly computed assumed position, the iteration process is continued.

The Doppler frequency is proportional to the time derivative of the slant range from navigator to satellite, called the slant range rate. Hence, not only the satellite's position must be known to compute the theoretical Doppler curve, but also the satellite's velocity. In every iteration, at every data point the satellite's position and velocity and the navigator's newly computed position and velocity must be used to compute the partial derivatives of the error function. These computations are complex and require a great deal of computer capability, including both speed and storage.

Method of averages

The method of averages involves dividing the Doppler curve into three equal time periods. The integrals of the observed Doppler frequency, I_{O_1}, I_{O_2}, and I_{O_3}, are computed for each time period.

$$I_{O_1} = \int_{t_0}^{t_1} f_O(t)dt \qquad 12$$

$$I_{O_2} = \int_{t_1}^{t_2} f_O(t)dt \qquad 13$$

$$I_{O_3} = \int_{t_2}^{t_3} f_O(t)dt \qquad 14$$

When the theoretical values for these three integrals, I_{T_1}, I_{T_2}, and I_{T_3}, are equated to the observed values, three simultaneous equations in latitude, longitude and bias result.

$$I_{T_1}(\theta, \lambda, b) - I_{O_1} = 0 \qquad 15$$

$$I_{T_2}(\theta, \lambda, b) - I_{O_2} = 0 \qquad 16$$

$$I_{T_3}(\theta, \lambda, b) - I_{O_3} = 0 \qquad 17$$

The solution of these three equations is the navigator's position and bias. Since these equations are nonlinear, an iterative technique is used to find their solution. The theoretical integrals are expanded in first order Taylor series about the assumed position of the navigator, θ_a and λ_a, and the assumed bias, b_a.

$$I_{T_1}(\theta_a,\lambda_a,b_a) + I_{T_{1_\theta}}(\theta_a,\lambda_a,b_a)\Delta\theta$$
$$+ I_{T_{1_\lambda}}(\theta_a,\lambda_a,b_a)\Delta\lambda + I_{T_{1_b}}(\theta_a,\lambda_a,b_a)\Delta b = I_{O_1} \qquad 18$$

$$I_{T_2}(\theta_a,\lambda_a,b_a) + I_{T_{2_\theta}}(\theta_a,\lambda_a,b_a)\Delta\theta$$
$$+ I_{T_{2_\lambda}}(\theta_a,\lambda_a,b_a)\Delta\lambda + I_{T_{2_b}}(\theta_a,\lambda_a,b_a)\Delta b = I_{O_2} \qquad 19$$

$$I_{T_3}(\theta_a,\lambda_a,b_a) + I_{T_{3_\theta}}(\theta_a,\lambda_a,b_a)\Delta\theta$$
$$+ I_{T_{3_\lambda}}(\theta_a,\lambda_a,b_a)\Delta\lambda + I_{T_{3_b}}(\theta_a,\lambda_a,b_a)\Delta b = I_{O_3} \qquad 20$$

The computed position of the navigator and the computed bias are given by

$$\theta_c = \theta_a + \Delta\theta \qquad 21$$

$$\lambda_c = \lambda_a + \Delta\lambda \qquad 22$$

$$b_c = b_a + \Delta b \qquad 23$$

Equations 18, 19 and 20 can be solved simultaneously for $\Delta\theta$, $\Delta\lambda$, and Δb. Using the newly computed assumed position, the iteration process is continued.

The definite integral of the Doppler frequency is proportional to the slant range from navigator to satellite evaluated at the end points of the integration. So, in this method only the satellite's position must be evaluated, not its velocity. Also, there are only four points for which the position must be determined, the points defining the three time periods of integration. The computer requirements for this method are not nearly as stringent as for the least squares method. The author is not aware of any way for the navigator to estimate the accuracy of the position fix found using this method.

Proposed Truncation Method for Computing the Navigator's Position

The author proposes a method, arbitrarily called the truncation method, which is a combination of the method of least squares and the method of averages, and uses a modification of the iteration technique. In the truncation method, the Doppler curve is first divided into many time intervals. The integral of the Doppler curve is then computed for each interval and a least squares fit is made of the integrals of the

Doppler curve. The least squares fit is used to obtain the higher accuracies and to allow an estimate of the accuracy of the position fix. The fit is applied to integrals of the Doppler curve to allow a first order smoothing of the Doppler data by the integration and to eliminate the need for computing the satellite's velocity. The iteration technique is changed in the following way. In the two methods previously described, three first order Taylor series expansions were used and the coefficients of these series were recomputed with each iteration. If higher order expansions were used, say to the third or fourth order, then the solution should be sufficiently accurate without any iterations. That is, the coefficients need be computed only once using the assumed position of the ship and the true position of the satellite. These coefficients can be computed during the pass of the satellite, leaving only a small amount of computation after the pass to solve the Taylor series for the position of the navigator.

The truncation method lends itself to a real time method of solution. The coefficients of the Taylor series are summations which accumulate as each data point is encountered in time. At the end of each time interval, the coefficients form a complete set with which an attempt can be made to determine a position fix. After enough data has accumulated so that a fit can be made where the estimated accuracy is sufficiently good, the rest of the satellite's pass can be ignored.

Since the majority of the computations are made during the pass of the satellite, when several minutes are available, the computer speed requirements are not as stringent. Also, since no iterations are necessary, the observed Doppler data and the satellite's position for each data point need not be saved for future computations, making the storage requirements for the computer far less.

The major disadvantage of the truncation method is that the convergence of the Taylor series, that is, the number of terms required, depends on the original accuracy of the navigator's assumed position. Computer studies will be described which indicate that the third order terms are usually sufficient for one-tenth nautical mile accuracies, and fourth order terms are almost always sufficient.

DERIVATION OF THE TRUNCATION METHOD

The derivation of the truncation method, which has been broken down into several stages, will be discussed in this chapter.

Explanation of the Coordinate System

A geocentric coordinate system that is fixed in space, as shown in Figure 2, will be used. The geocentric latitude and right ascension of the satellite are θ_s and λ_s respectively. The corresponding coordinates of the navigator are θ and λ. Longitude and right ascension differ only by the angle of rotation of the earth. The radius of the earth at the position of the navigator is R, the radial distance to the satellite is r and the slant range from the navigator to the satellite is ρ. The geocentric angle between the navigator's radius vector and the satellite's radius vector is ϕ. The locus of points generated by the intersection of the satellite's radius vector and the surface of the earth is called the satellite's subtrack.

The slant range, ρ, is given by the following equation

$$\rho^2 = r^2 + R^2 - 2rR \cos \phi \qquad 24$$

where

$$\cos \phi = \sin \theta_s \sin \theta + \cos \theta_s \cos \theta \cos (\lambda_s - \lambda) \qquad 25$$

The latitude of the navigator is given by

$$\theta = \theta_o + \int_{t_o}^{t} \dot{\theta}(t)dt \qquad 26$$

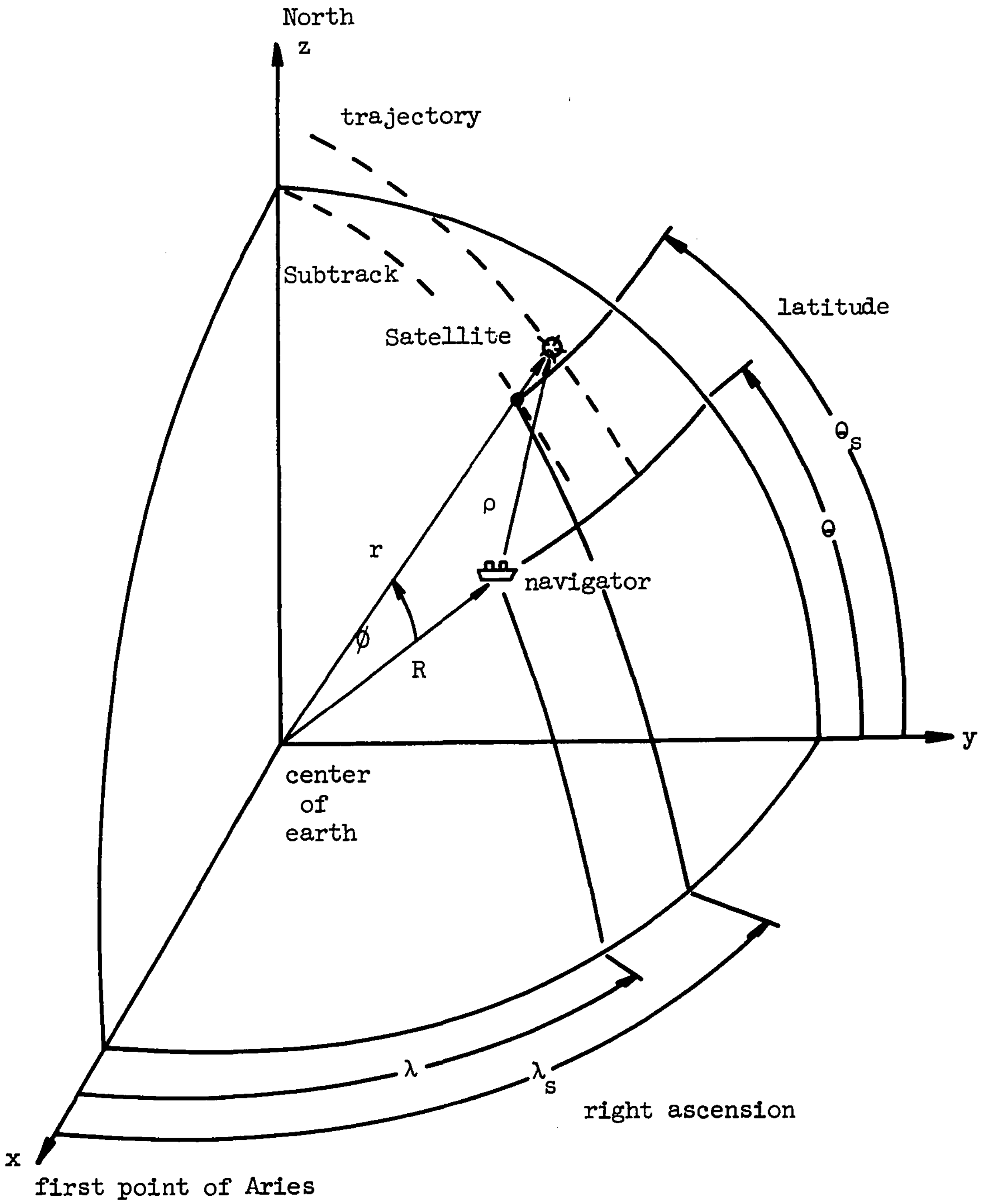


Figure 2. Coordinate system

where θ_0 is the latitude at some epoch t_0, and $\dot{\theta}(t)$ is the latitudinal angular velocity of the navigator. The right ascension of the navigator is given by

$$\lambda = \lambda_0 + \Omega(t - t_0) + \int_{t_0}^{t} \dot{\lambda}(t)dt \qquad 27$$

where λ_0 is the right ascension at t_0, Ω is the sidereal earth rate, and $\dot{\lambda}(t)$ is the longitudinal angular velocity with respect to a meridian fixed on the surface of the earth.

To identify the true, assumed and computed values of a variable, such as the navigator's latitude, the subscripts t, a and c are used respectively.

The correction terms for the navigator's position are η for latitude correction and ν for longitude correction.

$$\theta_t = \theta_a + \eta_t \qquad 28$$

$$\lambda_t = \lambda_a + \nu_t \qquad 29$$

If it is assumed that the navigator's angular velocities in latitude and longitude are accurately known, η and ν are constants. The errors in computing the navigator's position are δ_η for latitude error and δ_ν for longitude error.

$$\theta_t = \theta_c + \delta_\eta \qquad 30$$

$$\lambda_t = \lambda_c + \delta_\nu \qquad 31$$

Derivation of the Error Function

The true Doppler frequency, f_d, is proportional to the slant range rate, $\dot{\rho}$. The slant range rate is dependent on both the position and velocity of the satellite and the navigator. It will be assumed that the position and velocity of the satellite and the velocity of the navigator are known as functions of time. Therefore, the slant range rate can be expressed as a function of the navigator's position and time.

$$f_d = - \frac{f_0}{c} \dot{\rho}(\theta_t, \lambda_t, t) \qquad 32$$

The transmitter frequency of the satellite is f_0, and c is the velocity of light in free space.

In the navigator's receiver, the two received signals are appropriately combined and mixed with a local oscillator to yield a received Doppler frequency, f_{d_r}, that is void of first order refraction error. This received Doppler frequency is essentially the true Doppler frequency, but it also contains noise, f_n, and a constant bias, B.

$$f_{d_r} = f_d + f_n - B \qquad 33$$

The integral of the Doppler frequency between times t_{i-1} and t_i scaled to the dimension of velocity is called σ_i.

$$\sigma_i = - \frac{c}{f_0} \int_{t_{i-1}}^{t_i} f_{d_r} \, dt \qquad 34$$

The following definitions will be made.

$$N_i = \frac{c}{f_0} \int_{t_{i-1}}^{t_i} f_n \, dt \qquad 35$$

$$\rho_i(\theta_t, \lambda_t) = \rho(\theta_t, \lambda_t, t_i) \qquad 36$$

$$\Delta t_i = t_i - t_{i-1} \qquad 37$$

From the preceding equations,

$$\sigma_i = \rho_i(\theta_t, \lambda_t) - \rho_{i-1}(\theta_t, \lambda_t) - N_i + \frac{c}{f_0} B \, \Delta \, t_i \qquad 38$$

The Doppler data to be considered is the set $\{\sigma_i\}$ of integrals of the received Doppler frequency. It might be noted that the integration can be performed with no truncation error by counting the number of cycles in the received Doppler frequency signal.

The theoretical value corresponding to the integrated received Doppler frequency is σ_{T_i}, where

$$\sigma_{T_i}(\eta, \nu, b) = \rho_i(\theta_a + \eta, \lambda_a + \nu) - \rho_{i-1}(\theta_a + \eta, \lambda_a + \nu) + b \, \Delta \, t_i \qquad 39$$

The variable b is included to account for bias in the Doppler frequency and a possible constant term in the noise. The residual, μ_i, in the least squares fit is

$$\mu_i(\eta, \nu, b) = \sigma_{T_i}(\eta, \nu, b) - \sigma_i \qquad 40$$

The error function, $F(\eta, \nu, b)$, is defined as the weighted summation of the squared residuals over the number of intervals, M (5).

$$F(\eta,\nu,b) = \sum_{i=1}^{M} \omega_i \mu_i^2 \qquad 41$$

The weighting function, ω_i, is defined to be positive. The optimum value of the weighting function depends on the statistical properties of the noise in the received Doppler signal. The general weighting function will be used to derive the error function and an optimum value will be suggested later. In order to isolate the term for bias in the residual, a function $e_i(\eta,\nu)$ is defined as

$$e_i(\eta,\nu) = \rho_i(\theta_a + \eta,\ \lambda_a + \nu) - \rho_{i-1}(\theta_a + \eta,\ \lambda_a + \nu) - \sigma_i \qquad 42$$

Now, the error function can be written as

$$F(\eta,\nu,b) = \sum_{i=1}^{M} \omega_i \left[e_i(\eta,\nu) + b\ \Delta t_i \right]^2 \qquad 43$$

The minimum value of the error function is found using

$$F_\eta(\eta,\nu,b) = 0 \qquad 44$$

$$F_\nu(\eta,\nu,b) = 0 \qquad 45$$

$$F_b(\eta,\nu,b) = 0 \qquad 46$$

It is beneficial to consider the last equation first.

$$F_b(\eta,\nu,b)\ \frac{\partial}{\partial b} \sum_{i=1}^{M} \omega_i[e_i(\eta,\nu) + b\ \Delta t_i]^2 \qquad 47$$

$$= 2 \sum_{i=1}^{M} \omega_i[e_i(n,\nu) + b\ \Delta t_i]\ \Delta t_i = 0 \qquad 48$$

This equation can be solved for b in terms of η and ν.

$$b(\eta,\nu) = -\frac{1}{T} \sum_{i=1}^{M} \omega_i e_i(\eta,\nu)\,\Delta t_i \qquad 49$$

where

$$T = \sum_{i=1}^{M} \omega_i\,\Delta t_i^2 \qquad 50$$

The error function can now be expressed in two variables, η and ν.

$$F(\eta,\nu,b) = F(\eta,\nu,b(\eta,\nu)) = F(\eta,\nu) \qquad 51$$

From Equations 43, 49 and 50

$$F(\eta,\nu) = \sum_{i=1}^{M} \omega_i\,[e_i^2 + 2e_i b\Delta t_i + b^2\Delta t_i^2] \qquad 52$$

$$= \sum_{i=1}^{M} \omega_i e_i^2 + 2b \sum_{i=1}^{M} \omega_i e_i \Delta t_i + b^2 \sum_{i=1}^{M} \omega_i \Delta t_i^2 \qquad 53$$

$$= \sum_{i=1}^{M} \omega_i e_i^2(\eta,\nu) - Tb^2(\eta,\nu) \qquad 54$$

Notice that $F(\eta,\nu)$ contains a summation of terms which is the error function with no bias, and a correction term for bias.

Taylor Series Expansion of the Error Function

The solution of Equations 55 and 56 gives the computed position of the navigator, η_c and ν_c, where the error function is a minimum.

$$F_\eta(\eta,\nu) = 2 \sum_{i=1}^{M} \omega_i e_i e_{\eta_i} - 2Tbb_\eta = 0 \qquad 55$$

$$F_\nu(\eta,\nu) = 2 \sum_{i=1}^{M} \omega_i e_i e_{\nu_i} - 2Tbb_\nu = 0 \qquad 56$$

The partial derivatives of the error function are non-linear and explicit equations cannot be written for η and ν. The equations can be solved by an iteration process using first order Taylor series expansions for the partial derivatives of the error function about the assumed position of the navigator.

$$F_\eta(\eta,\nu) = F_\eta(\eta_a,\nu_a) + F_{\eta\eta}(\eta_a,\nu_a)\Delta\eta + F_{\eta\nu}(\eta_a,\nu_a)\Delta\nu = 0 \qquad 57$$

$$F_\nu(\eta,\nu) = F_\nu(\eta_a,\nu_a) + F_{\eta\nu}(\eta_a,\nu_a)\Delta\eta + F_{\nu\nu}(\eta_a,\nu_a)\Delta\nu = 0 \qquad 58$$

where

$$\eta = \eta_a + \Delta\eta \qquad 59$$

$$\nu = \nu_a + \Delta\nu \qquad 60$$

Equations 57 and 58 are solved for $\Delta\eta$ and $\Delta\nu$ using the assumed position of the navigator. Then a new assumed position is found from Equations 59 and 60, and the process is repeated until the desired accuracy is achieved. The five partial derivatives of the error function at the assumed position of the ship must be recomputed with each iteration.

The need for successive iterations is eliminated if more terms are carried in the Taylor series expansions for $F_\eta(\eta,\nu)$ and $F_\nu(\eta,\nu)$. These series form two polynomials in η and ν which can be solved simultaneously to find the navigator's position. For simplicity, the following notation will be used:

$$F_{jk} = \frac{\partial^{j+k}}{\partial\eta^j \partial\nu^k} F(\eta, \nu) \Bigg|_{\substack{\eta = 0 \\ \nu = 0}} \qquad 61$$

The general Taylor series in two variables for $F(\eta, \nu)$ expanded about $F(0,0)$ is

$$F(\eta,\nu) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{1}{(j-k)!k!} F_{j-k,k} \, \eta^{j-k} \nu^k \qquad 62$$

If F is replaced by F_η, and then by F_ν,

$$F_\eta(\eta, \nu) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{1}{(j-k)!k!} F_{j-k+1,k} \eta^{j-k} \nu^k \qquad 63$$

$$F_\nu(\eta, \nu) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{1}{(j-k)!k!} F_{j-k,k+1} \eta^{j-k} \nu^k \qquad 64$$

All three of the above series are based on the set $\left\{F_{jk}\right\}$ of partial derivatives of the error function evaluated at the assumed position of the navigator. This set of coefficients can be computed from Equation 54.

$$F_{jk} = \frac{\partial^{j+k}}{\partial\eta^j \partial\nu^k} \left[\sum_{i=1}^{M} \omega_i e_i^{\,2}(\eta, \nu) - Tb^2(\eta, \nu) \right] \qquad 65$$

The general form for the partial derivative of a squared function is given by

$$\frac{\partial^{j+k} f^2(x,y)}{\partial x^j \partial y^k} = \sum_{J=0}^{j} \sum_{K=0}^{k} \binom{j}{J}\binom{k}{K} f_{J,K} f_{j-J,k-K} \qquad 66$$

Equation 65 can be written as

$$F_{jk} = \sum_{J=0}^{j} \sum_{K=0}^{k} \binom{j}{J}\binom{k}{K} \Big[\sum_{i=1}^{M} \omega_i e_{J,K_i} e_{j-J,k-K_i}$$

$$- T\, b_{J,K} b_{j-J,k-K} \Big] \qquad 67$$

where

$$b_{jk} = -\frac{1}{T} \sum_{i=1}^{M} \omega_i e_{jk_i} \Delta t_i \qquad 68$$

and, for j and k not both equal to zero,

$$e_{jk_i} = \rho_{jk_i} - \rho_{jk_{i-1}} \qquad 69$$

The general expression for the partial derivatives of the slant range, ρ, is not known by the author, but the equations for some of the partial derivatives appear later in the derivation along with a summary of the other equations required to compute the error function coefficients.

Minimization of the Error Function

The truncated Taylor series expansion for a function f will be denoted by a superbar, $\overline{f}$. The error function $F(\eta,\nu)$ is approximated by its truncated Taylor series expansion.

$$F(\eta,\nu) \cong \overline{F(\eta,\nu)} \qquad 70$$

In order for the truncated error function to be a minimum,

$$\overline{F_\eta(\eta,\nu)} = 0 \qquad 71$$

$$\overline{F_\nu(\eta,\nu)} = 0 \qquad 72$$

Since $\overline{F_\eta(\eta,\nu)}$ and $\overline{F_\nu(\eta,\nu)}$ are each high order polynomials in two variables, an iterative technique is used to find their solution. The following first order approximations are made:

$$\overline{F_\eta(\eta,\nu)} = \overline{F_\eta(\eta_a,\nu_a)} + \overline{F_{\eta\eta}(\eta_a,\nu_a)}\Delta\eta + \overline{F_{\eta\nu}(\eta_a,\nu_a)}\ \Delta\nu \ = 0 \qquad 73$$

$$\overline{F_\nu(\eta,\nu)} = \overline{F_\nu(\eta_a,\nu_a)} + \overline{F_{\eta\nu}(\eta_a,\nu_a)}\Delta\eta + \overline{F_{\nu\nu}(\eta_a,\nu_a)}\ \Delta\nu \ = 0 \qquad 74$$

where

$$\eta = \eta_a + \Delta\eta \qquad 75$$

$$\nu = \nu_a + \Delta\nu \qquad 76$$

The solution to Equations 73 and 74 is given by

$$\Delta\eta = \frac{\overline{F_{\nu}(\eta_a, \nu_a)}\ \overline{F_{\eta\nu}(\eta_a, \nu_a)} - \overline{F_{\eta}(\eta_a, \nu_a)}\ \overline{F_{\nu\nu}(\eta_a, \nu_a)}}{\overline{F_{\eta\eta}(\eta_a, \nu_a)}\ \overline{F_{\nu\nu}(\eta_a, \nu_a)} - \overline{F_{\eta\nu}(\eta_a, \nu_a)}^2} \qquad 77$$

$$\Delta\nu = \frac{\overline{F_{\eta}(\eta_a, \nu_a)}\ \overline{F_{\eta\nu}(\eta_a, \nu_a)} - \overline{F_{\nu}(\eta_a, \nu_a)}\ \overline{F_{\eta\eta}(\eta_a, \nu_a)}}{\overline{F_{\eta\eta}(\eta_a, \nu_a)}\ \overline{F_{\nu\nu}(\eta_a, \nu_a)} - \overline{F_{\eta\nu}(\eta_a, \nu_a)}^2} \qquad 78$$

The partial derivatives of the truncated error function which appear in Equations 77 and 78 are each polynomials in the two variables η_a and ν_a whose coefficients are taken from the set $\{F_{jk}\}$. The set $\{F_{jk}\}$ is computed only once and is then held constant throughout the iteration process. The first iteration is performed with $\eta_a = \nu_a = 0$. Then a new η_a and ν_a are computed as the iteration continues.

Unfortunately, there are several solutions to Equations 71 and 72. As was mentioned in the introduction, there are two points on the surface of the earth for which the error function is a relative minimum. There may also be a third point between these two for which the error function is a relative maximum. The point which is found by the iteration process depends on where the first iteration is started. The best estimate of where to start the iteration is the assumed position of the navigator, where $\eta_a = \nu_a = 0$. This almost always leads to the correct position.

Optimization of the Weighting Function

The weighting function will be optimized in the following way. First certain statistical properties will be assumed for the noise function,

$f_n(t)$. Then the RMS position error due to this noise will be computed. The optimum weighting function is that which minimizes the RMS position error for the type of noise assumed. It will be assumed that $f_n(t)$ is a time stationary function with zero average value and has an autocorrelation function which decreases exponentially to zero in a time small with respect to Δt_i. The variance of the integral of such a function is approximately the product of the variance of the function and the integration time, Δt_i, and the average value is zero (1).

$$< N_i^2 > = N^2 \Delta t_i \qquad 79$$

where

$$N^2 = \frac{c^2}{f_o^2} < f_n^2 > \qquad 80$$

For simplicity, the weighting function will be optimized for an error function of one position variable, x. It can be shown that the same optimum value applies for a function of two position variables. The observed function, g_i, which is void of noise, is being matched by the computed function, $f(x)_i$, in the presence of noise, N_i. The perfect match occurs when $x = x_t$, because

$$g_i = f(x_t)_i \qquad 81$$

The residual, μ_i, is defined to be

$$\mu_i(x) = f(x)_i - g_i + N_i \qquad 82$$

The error function is

$$F(x) = \sum_{i=1}^{M} \omega_i \mu_i^2 \qquad 83$$

The error function is a minimum when $x = x_c$, where

$$F_x(x_c) = 2 \sum_{i=1}^{M} \omega_i [f(x_c)_i - g_i + N_i]\ f_x(x_c)_i = 0 \qquad 84$$

A first order approximation can be made for g_i using δ, the error in x.

$$g_i = f(x_c)_i + f_x(x_c)_i\ \delta \qquad 85$$

where

$$x_t = x_c + \delta \qquad 86$$

Equation 84 can be written as

$$\sum_{i=1}^{M} \omega_i\ [N_i - \delta f_{x_i}]\ f_{x_i} = 0 \qquad 87$$

The solution for δ is

$$\delta = \frac{\sum_{i=1}^{M} \omega_i N_i f_{x_i}}{\sum_{i=1}^{M} \omega_i f_{x_i}^2} \qquad 88$$

The average value of δ is zero since the average value of N_i is zero. The variance of the error in x is found by squaring both sides of Equation 88 and averaging over many positioning attempts.

$$\delta^2 = \frac{\sum_{i=1}^{M} \sum_{j=1}^{M} \omega_i \omega_j N_i N_j f_{x_i} f_{x_j}}{\left[\sum_{i=1}^{M} \omega_i f_{x_i}^{\ 2}\right]^2} \qquad 89$$

The average value of $N_i N_j$, where $i \neq j$, is zero; the average value of $N_i^{\ 2}$ is $N^2 \Delta t_i$.

$$< \delta^2 > = \frac{N^2 \sum_{i=1}^{M} \omega_i^{\ 2} \Delta t_i f_{x_i}^{\ 2}}{\left[\sum_{i=1}^{M} \omega_i f_{x_i}^{\ 2}\right]^2} \qquad 90$$

The optimum weighting function is that which minimizes the variance of the error in x.

$$\frac{\partial < \delta^2 >}{\partial \omega_j} = \frac{2N^2 \omega_j \Delta t_j f_{x_j}^{\ 2}}{\left[\sum_{i=1}^{M} \omega_i f_{x_i}^{\ 2}\right]^2} - \frac{2N^2 f_{x_j}^{\ 2} \sum_{i=1}^{M} \omega_i^{\ 2} \Delta t_i f_{x_i}^{\ 2}}{\left[\sum_{i=1}^{M} \omega_i f_{x_i}^{\ 2}\right]^3} = 0 \qquad 91$$

This is equivalent to

$$\sum_{i=1}^{M} \omega_i f_{x_i}^{\ 2} (\omega_j \Delta t_j - \omega_i \Delta t_i) = 0 \qquad 92$$

Let j be chosen such that for all i,

$$\omega_j \Delta t_j \geq \omega_i \Delta t_i \qquad 93$$

Since ω_i is positive, each term in Equation 92 must be zero, and for all i,

$$\omega_j \Delta t_j = \omega_i \Delta t_i \qquad 94$$

The solution to this is that

$$\omega_i = \frac{\text{constant}}{\Delta t_i} \qquad 95$$

It can be seen from Equation 88 that the constant is arbitrary, so it will be chosen to be unity.

$$\omega_i = \frac{1}{\Delta t_i} \qquad 96$$

It should be noted that this weighting function is optimum only for the time stationary, random noise that was assumed. In general, the noise function will not contain time stationary noise.

The optimum value of ω_i can be used to rewrite some of the previous equations. From Equation 50

$$T = \sum_{i=1}^{M} \Delta t_i = t_M - t_0 \qquad 97$$

From Equations 42 and 49

$$b(\eta,\nu) = -\frac{1}{T}(\rho_M - \rho_0 - \sum_{i=1}^{M} \sigma_i) \qquad 98$$

From Equation 68, for j and k not both equal to zero,

$$b_{jk}(\eta,\nu) = -\frac{1}{T}\left(\rho_{jk_M} - \rho_{jk_0}\right) \tag{99}$$

From Equation 54

$$F(\eta,\nu) = \sum_{i=1}^{M} \frac{e_i^{\,2}(\eta,\nu)}{\Delta t_i} - T\, b^2(\eta,\nu) \tag{100}$$

And from Equation 67

$$F_{jk} = \sum_{J=0}^{j} \sum_{K=0}^{k} \binom{j}{J}\binom{k}{K}\left[\sum_{i=1}^{M} \frac{1}{\Delta t_i}\, e_{J,K_i} e_{j-J,k-K_i} - T\, b_{J,K} b_{j-J,k-K}\right] \tag{101}$$

Estimation of the Computed Position Accuracy

In determining the navigator's computed position, Equations 71 and 72 are solved simultaneously for η_c and ν_c. If a sufficient number of terms are carried in the Taylor series expansions, η_c and ν_c should also satisfy Equations 55 and 56, which can be rewritten as

$$\sum_{i=1}^{M} \frac{e_i e_{\eta i}}{\Delta t_i} - \frac{1}{T} \sum_{i=1}^{M} e_i \sum_{i=1}^{M} e_{\eta_i} = 0 \tag{102}$$

$$\sum_{i=1}^{M} \frac{e_i e_{\nu_i}}{\Delta t_i} - \frac{1}{T} \sum_{i=1}^{M} e_i \sum_{i=1}^{M} e_{\nu_i} = 0 \tag{103}$$

For simplicity, the following notation will be adopted:

$$\Delta\rho_i = \rho_i(\theta_a + \eta_c, \lambda_a + \nu_c) - \rho_{i-1}(\theta_a + \eta_c, \lambda_a + \nu_c) \qquad 104$$

$$\Delta\rho_{t_i} = \rho_i(\theta_t, \lambda_t) - \rho_{i-1}(\theta_t, \lambda_t) \qquad 105$$

From Equations 38, 42, 104 and 105

$$e_i(\eta_c, \nu_c) = \Delta\rho_i - \Delta\rho_{t_i} + N_i - \frac{c}{f_0} B\Delta t_i \qquad 106$$

When Equation 106 is inserted in Equations 102 and 103, the terms containing B cancel leaving

$$\sum_{i=1}^{M} (\Delta\rho_i - \Delta\rho_{t_i} + N_i)\frac{e_{\eta_i}}{\Delta t_i} - \frac{1}{T} \sum_{i=1}^{M}(\Delta\rho_i - \Delta\rho_{t_i} + N_i) \sum_{i=1}^{M} e_{\eta_i} = 0 \qquad 107$$

$$\sum_{i=1}^{M} (\Delta\rho_i - \Delta\rho_{t_i} + N_i)\frac{e_{\nu_i}}{\Delta t_i} - \frac{1}{T} \sum_{i=1}^{M}(\Delta\rho_i - \Delta\rho_{t_i} + N_i) \sum_{i=1}^{M} e_{\nu_i} = 0 \qquad 108$$

The first order approximation is now made for $\Delta\rho_{t_i} - \Delta\rho_i$.

$$\Delta\rho_{t_i} - \Delta\rho_i = \delta_\eta e_{\eta_i} + \delta_\nu e_{\nu_i} \qquad 109$$

where

$$\eta_t = \eta_c + \delta_\eta \qquad 110$$

$$\nu_t = \nu_c + \delta_\nu \qquad 111$$

Now, Equations 107 and 108 can be written as

$$\delta_\eta \sum_{i=1}^{M} E_{\eta_i} \frac{e_{\eta_i}}{\Delta t_i} + \delta_\nu \sum_{i=1}^{M} E_{\eta_i} \frac{e_{\nu_i}}{\Delta t_i} = \sum_{i=1}^{M} E_{\eta_i} \frac{N_i}{\Delta t_i} \qquad 112$$

$$\delta_\eta \sum_{i=1}^{M} E_{\nu_i} \frac{e_{\eta_i}}{\Delta t_i} + \delta_\nu \sum_{i=1}^{M} E_{\nu_i} \frac{e_{\nu_i}}{\Delta t_i} = \sum_{i=1}^{M} E_{\nu_i} \frac{N_i}{\Delta t_i} \qquad 113$$

where

$$E_{\eta_i} = e_{\eta_i} - \frac{\Delta t_i}{T} \sum_{i=1}^{M} e_{\eta_i} \qquad 114$$

$$E_{\nu_i} = e_{\nu_i} - \frac{\Delta t_i}{T} \sum_{i=1}^{M} e_{\nu_i} \qquad 115$$

It is noted that

$$\sum_{i=1}^{M} E_{\eta_i} \frac{e_{\eta_i}}{\Delta t_i} = \sum_{i=1}^{M} \frac{E_{\eta_i}^{\ 2}}{\Delta t_i} \qquad 116$$

$$\sum_{i=1}^{M} E_{\eta_i} \frac{e_{\nu_i}}{\Delta t_i} = \sum_{i=1}^{M} E_{\nu_i} \frac{e_{\eta_i}}{\Delta t_i} = \sum_{i=1}^{M} \frac{E_{\eta_i} E_{\nu_i}}{\Delta t_i} \qquad 117$$

$$\sum_{i=1}^{M} E_{\nu_i} \frac{e_{\nu_i}}{\Delta t_i} = \sum_{i=1}^{M} \frac{E_{\nu_i}^{\ 2}}{\Delta t_i} \qquad 118$$

Equations 112 and 113 become

$$\delta_\eta \sum_{i=1}^{M} \frac{E_{\eta_i}^{2}}{\Delta t_i} + \delta_\nu \sum_{i=1}^{M} \frac{E_{\eta_i} E_{\nu_i}}{\Delta t_i} = \sum_{i=1}^{M} N_i \frac{E_{\eta_i}}{\Delta t_i} \qquad 119$$

$$\delta_\eta \sum_{i=1}^{M} \frac{E_{\eta_i} E_{\nu_i}}{\Delta t_i} + \delta_\nu \sum_{i=1}^{M} \frac{E_{\nu_i}^{2}}{\Delta t_i} = \sum_{i=1}^{M} N_i \frac{E_{\nu_i}}{\Delta t_i} \qquad 120$$

Equations 119 and 120 can be solved simultaneously for δ_η and δ_ν.

$$\delta_\eta = \frac{\sum_{i=1}^{M} N_i \frac{A_i}{t_i}}{D} \qquad 121$$

$$\delta_\nu = \frac{\sum_{i=1}^{M} N_i \frac{B_i}{\Delta t_i}}{D} \qquad 122$$

where

$$A_i = E_{\eta_i} \sum_{i=1}^{M} \frac{E_{\nu_i}^{2}}{\Delta t_i} - E_{\nu_i} \sum_{i=1}^{M} \frac{E_{\eta_i} E_{\nu_i}}{\Delta t_i} \qquad 123$$

$$B_i = E_{\nu_i} \sum_{i=1}^{M} \frac{E_{\eta_i}^{2}}{\Delta t_i} - E_{\eta_i} \sum_{i=1}^{M} \frac{E_{\eta_i} E_{\nu_i}}{\Delta t_i} \qquad 124$$

$$D = \sum_{i=1}^{M} \frac{E_{\eta_i}^{2}}{\Delta t_i} \sum_{i=1}^{M} \frac{E_{\nu_i}^{2}}{\Delta t_i} - \left[\sum_{i=1}^{M} \frac{E_{\eta_i} E_{\nu_i}}{\Delta t_i} \right]^{2} \qquad 125$$

The average value of δ_η and δ_ν is zero, since the average value of N_i is zero. The variance is found in the following way:

$$\delta_\eta^{\,2} = \frac{1}{D^2} \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{N_i N_j A_i A_j}{\Delta t_i \, \Delta t_j} \qquad 126$$

$$\delta_\nu^{\,2} = \frac{1}{D^2} \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{N_i N_j B_i B_j}{\Delta t_i \, \Delta t_j} \qquad 127$$

The average value of $N_i N_j$, for $i \neq j$, is zero; the average value of $N_i^{\,2}$ is $N^2 \Delta t_i$.

$$< \delta_\eta^{\,2} > \quad = \frac{N^2}{D^2} \sum_{i=1}^{M} \frac{A_i^{\,2}}{\Delta t_i} \qquad 128$$

$$< \delta_\nu^{\,2} > \quad = \frac{N^2}{D^2} \sum_{i=1}^{M} \frac{B_i^{\,2}}{\Delta t_i} \qquad 129$$

These equations can be reduced to

$$<\delta_\eta^{\,2}> \quad = \frac{N^2}{D} \sum_{i=1}^{M} \frac{E_{\nu_i}^{\,2}}{\Delta t_i} \qquad 130$$

$$< \delta_\nu^{\,2} > \quad = \frac{N^2}{D} \sum_{i=1}^{M} \frac{E_{\eta_i}^{\,2}}{\Delta t_i} \qquad 131$$

In order to estimate the variance of the positioning error, an estimate must be made for the variance of the noise, N^2. This estimate is obtained from the minimum value of the error function. The minimum value of the truncated error function evaluated at the computed position of the navigator is actually computed, but if a sufficient number of terms is carried in the expansion this is approximately equal to the error function given by Equation 100. This equation can be rewritten as

$$F(\eta_c, \nu_c) = \sum_{i=1}^{M} \frac{e_i^2}{\Delta t_i} - \frac{1}{T}\left(\sum_{i=1}^{M} e_i\right)^2 \qquad 132$$

When Equation 106 is inserted into Equation 132, the terms containing B cancel and the result is

$$F(\eta_c, \nu_c) = \sum_{i=1}^{M} (\Delta\rho_i - \Delta\rho_{t_i} + N_i)^2 \frac{1}{\Delta t_i} - \frac{1}{T}\left[\sum_{i=1}^{M} (\Delta\rho_i - \Delta\rho_{t_i} + N_i)\right]^2 \qquad 133$$

At this point it is necessary to assume that $\Delta\rho_i - \Delta\rho_{t_i}$ is small with respect to N_i. With this assumption made, the error function can be written as

$$F(\eta_c, \nu_c) = \sum_{i=1}^{M} \frac{N_i^2}{\Delta t_i} - \frac{1}{T}\left[\sum_{i=1}^{M} N_i\right]^2 \qquad 134$$

$$= \sum_{i=1}^{M} \frac{1}{\Delta t_i}\left[N_i - \frac{\Delta t_i}{T}\sum_{i=1}^{M} N_i\right]^2 \qquad 135$$

Since the noise signal is being considered for a finite amount of time, some average value will be observed.

$$< f_n(t) > = \frac{1}{T} \int_{t_0}^{t_M} f_n(t)dt = \frac{f_0}{Tc} \sum_{i=1}^{M} N_i \qquad 136$$

The integral of the AC component of the noise over the time interval Δt_i is

$$\int_{t_{i-1}}^{t_i} [\ f_n(t) - <f_n(t)>]\ \ dt = \frac{f_0}{c} [N_i - \frac{\Delta t_i}{T} \sum_{i=1}^{M} N_i] \qquad 137$$

The variance of this integral is equal to the product of the variance of the noise and the time interval, Δt_i.

$$< \frac{f_0^{\ 2}}{c^2} [N_i - \frac{\Delta t_i}{T}(\sum_{i=1}^{M} N_i)]^2 > = < f_n^{\ 2}(t)\Delta t_i > = \frac{f_0^{\ 2}}{c^2} N^2 \Delta t_i \qquad 138$$

This equation can be solved for N^2.

$$N^2 = < \frac{1}{\Delta t_i}[N_i - \frac{\Delta t_i}{T} \sum_{i=1}^{M} N_i]^2 > \qquad 139$$

$$= \frac{1}{M} \sum_{i=1}^{M} \frac{1}{\Delta t_i} [N_i - \frac{\Delta t_i}{T} \sum_{i=1}^{M} N_i\ ]^2 \qquad 140$$

$$= \frac{1}{M} F(\eta_c, \nu_c) \qquad 141$$

The RMS errors in the latitude and longitude predictions can now be written as

$$\delta_{\eta_{RMS}} = \sqrt{\frac{F(\eta_c, \nu_c)}{M \quad D} \sum_{i=1}^{M} \frac{E_{\nu_i}^{2}}{\Delta t_i}} \qquad 142$$

$$\delta_{\nu_{RMS}} = \sqrt{\frac{F(\eta_c, \nu_c)}{M \quad D} \sum_{i=1}^{M} \frac{E_{\eta_i}^{2}}{\Delta t_i}} \qquad 143$$

Summary of the Computations

At this point it seems appropriate to summarize the equations leading to the computation of a position fix. At each data point i, the spherical coordinates of the satellite and the navigator's assumed location are computed. The following terms are computed from these coordinates.

$$r_i R_{ss} = r_i R \sin \Theta_{s_i} \sin \Theta_{a_i} \qquad 144$$

$$r_i R_{sc} = r_i R \sin \Theta_{s_i} \cos \Theta_{a_i} \qquad 145$$

$$r_i R_{css} = r_i R \cos \Theta_{s_i} \sin \Theta_{a_i} \sin (\lambda_{s_i} - \lambda_{a_i}) \qquad 146$$

$$r_i R_{csc} = r_i R \cos \Theta_{s_i} \sin \Theta_{a_i} \cos (\lambda_{s_i} - \lambda_{a_i}) \qquad 147$$

$$r_i R_{ccs} = r_i R \cos \Theta_{s_i} \cos \Theta_{a_i} \sin (\lambda_{s_i} - \lambda_{a_i}) \qquad 148$$

$$r_i R_{ccc} = r_i R \cos \Theta_{s_i} \cos \Theta_{a_i} \cos (\lambda_{s_i} - \lambda_{a_i}) \qquad 149$$

The slant range and its partial derivatives with respect to η and ν are computed in the following order.

$$\rho_i = \sqrt{(r_i - R)^2 + 2\,[r_iR - r_iR_{ss} - r_iR_{ccc}]} \qquad 150$$

$$\rho_{10_i} = (-r_iR_{sc} + r_iR_{csc})/\rho_i \qquad 151$$

$$\rho_{01_i} = (-r_iR_{ccs})/\rho_i \qquad 152$$

$$\rho_{20_i} = (r_iR_{ss} + r_iR_{ccc} - \rho_{10_i}^{\;2})/\rho_i \qquad 153$$

$$\rho_{11_i} = (r_iR_{css} - \rho_{10_i}\rho_{01_i})/\rho_i \qquad 154$$

$$\rho_{02_i} = (r_iR_{ccc} - \rho_{01_i}^{\;2})/\rho_i \qquad 155$$

$$\rho_{30_i} = (r_iR_{sc} - r_iR_{csc} - 3\,\rho_{10_i}\rho_{20_i})/\rho_i \qquad 156$$

$$\rho_{21_i} = (r_iR_{ccs} - 2\,\rho_{10_i}\rho_{11_i} - \rho_{01_i}\rho_{20_i})/\rho_i \qquad 157$$

$$\rho_{12_i} = (-r_iR_{csc} - 2\,\rho_{01_i}\rho_{11_i} - \rho_{10_i}\rho_{02_i})/\rho_i \qquad 158$$

$$\rho_{03_i} = (r_iR_{ccs} - 3\,\rho_{01_i}\rho_{02_i})/\rho_i \qquad 159$$

These equations are sufficient if the error function is to be truncated after the third order terms for the third order method. The fourth order method requires the following additional equations.

$$\rho_{40_i} = (-r_iR_{ss} - r_iR_{ccc} - 3\,\rho_{20_i}^{\;2} - 4\,\rho_{10_i}\rho_{30_i})/\rho_i \qquad 160$$

$$\rho_{31_i} = (-r_iR_{css} - 3\,\rho_{20_i}\rho_{11_i} - 3\,\rho_{10_i}\rho_{21_i} - \rho_{30_i}\rho_{01_i})/\rho_i \qquad 161$$

$$\rho_{22_i} = (-r_i R_{ccc} - 2\rho_{11_i}{}^2 - 2\,\rho_{10_i}\rho_{12_i} - 2\,\rho_{01_i}\rho_{21_i} - \rho_{20_i}\rho_{02_i})/\rho_i \qquad 162$$

$$\rho_{13_i} = (-r_i R_{css} - 3\,\rho_{11_i}\rho_{02_i} - 3\,\rho_{01_i}\rho_{12_i} - \rho_{10_i}\rho_{03_i})/\rho_i \qquad 163$$

$$\rho_{04_i} = (-r_i R_{ccc} - 3\,\rho_{02_i}{}^2 - 4\,\rho_{01_i}\rho_{03_i})/\rho_i \qquad 164$$

The slant range and its partial derivatives for time t_0 are stored for use later. The set $\{e_{jk_i}\}$ is computed from the slant range and its partial derivatives at t_i and t_{i-1}.

$$e_i = \rho_i - \rho_{i-1} - \sigma_i \qquad 165$$

and, for j and k not both zero,

$$e_{jk_i} = \rho_{jk_i} - \rho_{jk_{i-1}} \qquad 166$$

The set of products $\left\{\frac{1}{\Delta t_i}\, e_{ab_i} e_{cd_i}\right\}$ is computed and the partial sum over i, which will be called S_{abcd_I}, is accumulated for each member.

$$S_{abcd_I} = \sum_{i=1}^{I} \frac{1}{\Delta t_i}\, e_{ab_i}\, e_{cd_i} \qquad 167$$

Table 1 indicates the numerical values of the subscripts for the products which must be computed for both the third order and fourth order methods. The partial sum over i of the Doppler data, σ_i, is also accumulated.

After the pass of the satellite, or more specifically, after any time t_M when an attempt is made to determine a position fix, the following computations are made. The total time interval T is

$$T = t_M - t_O \tag{168}$$

Table 1. Designation of subscripts for product terms $e_{ab_i} e_{cd_i}$

cd/ab	00	10	01	20	11	02
00	x		Third Order			
10	x	x				
01	x	x	x		Fourth Order	
20	x	x	x	x		
11	x	x	x	x	x	
02	x	x	x	x	x	x
30	x	x	x			
21	x	x	x			
12	x	x	x			
03	x	x	x			
40	x					
31	x					
22	x					
13	x					
04	x					

x: product term to be computed

The set $\left\{b_{jk}\right\}$ is computed from

$$b = -\frac{1}{T}\left(\rho_M - \rho_0 - \sum_{i=1}^{M} \sigma_i\right) \qquad 169$$

and, for i and j not both equal to zero,

$$b_{jk} = -\frac{1}{T}\left(\rho_{jk_M} - \rho_{jk_0}\right) \qquad 170$$

The set $\left\{F_{jk}\right\}$ is computed from:

$$F_{00} = S_{0000_M} - T\, b_{00}^{2} \qquad 171$$

$$F_{10} = 2(S_{0010_M} - T\, b_{00}b_{10}) \qquad 172$$

$$F_{01} = 2(S_{0001_M} - T\, b_{00}b_{01}) \qquad 173$$

$$F_{20} = 2(S_{1010_M} - T\, b_{10}^{2}) + 2(S_{0020_M} - T\, b_{00}b_{20}) \qquad 174$$

$$F_{11} = 2(S_{1001_M} - T\, b_{10}b_{01}) + 2(S_{0011_M} - T\, b_{00}b_{11}) \qquad 175$$

$$F_{02} = 2(S_{0101_M} - T\, b_{01}^{2}) + 2(S_{0002_M} - T\, b_{00}b_{02}) \qquad 176$$

$$F_{30} = 6(S_{1020_M} - T\, b_{10}b_{20}) + 2(S_{0030_M} - T\, b_{00}b_{30}) \qquad 177$$

$$F_{21} = 4(S_{1011_M} - T\, b_{10}b_{11}) + 2(S_{0120_M} - T\, b_{01}b_{20})$$
$$+ 2(S_{0021_M} - T\, b_{00}b_{21}) \qquad 178$$

$$F_{12} = 4(S_{0111_M} - T\, b_{01}b_{11}) + 2(S_{1002_M} - T\, b_{10}b_{02})$$
$$+ 2(S_{0012_M} - T\, b_{00}b_{12}) \qquad 179$$

$$F_{03} = 6(S_{0102_M} - T\, b_{01} b_{02}) + 2(S_{0003_M} - T\, b_{00} b_{03}) \qquad 180$$

The above equations are sufficient for the third order method. The following equations must be included for the fourth order method.

$$F_{40} = 6(S_{2020_M} - T\, b_{20}^{2}) + 8(S_{1030_M} - T\, b_{10} b_{30})$$
$$+ 2(S_{0040_M} - T\, b_{00} b_{40}) \qquad 181$$

$$F_{31} = 6(S_{2011_M} - T\, b_{20} b_{11}) + 6(S_{1021_M} - T\, b_{10} b_{21})$$
$$+ 2(S_{0130_M} - T\, b_{01} b_{30}) + 2(S_{0031_M} - T\, b_{00} b_{31}) \qquad 182$$

$$F_{22} = 2(S_{2002_M} - T\, b_{20} b_{02}) + 4(S_{1111_M} - T\, b_{11}^{2})$$
$$+ 4(S_{1012_M} - T\, b_{10} b_{12}) + 4(S_{0121_M} - T\, b_{01} b_{21})$$
$$+ 2(S_{0022_M} - T\, b_{00} b_{22}) \qquad 183$$

$$F_{13} = 6(S_{1102_M} - T\, b_{11} b_{02}) + 6(S_{0112_M} - T\, b_{01} b_{12})$$
$$+ 2(S_{1003_M} - T\, b_{10} b_{03}) + 2(S_{0013_M} - T\, b_{00} b_{13}) \qquad 184$$

$$F_{04} = 6(S_{0202_M} - T\, b_{02}^{2}) + 8(S_{0103_M} - T\, b_{01} b_{03})$$
$$+ 2(S_{0004_M} - T\, b_{00} b_{04}) \qquad 185$$

An iterative procedure is now started with $\eta_a = \nu_a = 0$. The set $\{F_{jk}\}$ is used to compute the following functions.

$$\overline{F_\eta(\eta_a, \nu_a)} = F_{10} + F_{20}\eta_a + F_{11}\nu_a$$

$$+ \frac{1}{2} F_{30}\eta_a^2 + f_{21}\eta_a\nu_a + \frac{1}{2} F_{12}\nu_a^2$$

$$+ \underline{\frac{1}{6} F_{40}\eta_a^3} + \underline{\frac{1}{2} F_{31}\eta_a^2\nu_a} + \underline{\frac{1}{2} F_{22}\eta_a\nu_a^2} + \underline{\frac{1}{6} F_{13}\nu_a^3} \qquad 186$$

$$\overline{F_\nu(\eta_a, \nu_a)} = F_{01} + F_{11}\eta_a + F_{02}\nu_a$$

$$+ \frac{1}{2} F_{21}\eta_a^2 + F_{12}\eta_a\nu_a + \frac{1}{2} F_{03}\nu_a^2$$

$$+ \underline{\frac{1}{6} F_{31}\eta_a^3} + \underline{\frac{1}{2} F_{22}\eta_a^2\nu_a} + \underline{\frac{1}{2} F_{13}\eta_a\nu_a^2} + \underline{\frac{1}{6} F_{04}\nu_a^3} \qquad 187$$

$$\overline{F_{\eta\eta}(\eta_a, \nu_a)} = F_{20} + F_{30}\eta_a + F_{21}\nu_a$$

$$+ \underline{\frac{1}{2} F_{40}\eta_a^2} + \underline{F_{31}\eta_a\nu_a} + \underline{\frac{1}{2} F_{22}\nu_a^2} \qquad 188$$

$$\overline{F_{\eta\nu}(\eta_a, \nu_a)} = F_{11} + F_{21}\eta_a + F_{12}\nu_a$$

$$+ \underline{\frac{1}{2} F_{31}\eta_a^2} + \underline{F_{22}\eta_a\nu_a} + \underline{\frac{1}{2} F_{13}\nu_a^2} \qquad 189$$

$$\overline{F_{\nu\nu}(\eta_a, \nu_a)} = F_{02} + F_{12}\eta_a + F_{03}\nu_a$$

$$+ \underline{\frac{1}{2} F_{22}\eta_a^2} + \underline{F_{13}\eta_a\nu_a} + \underline{F_{04}\nu_a^2} \qquad 190$$

All of the terms in the preceding equations are used in the fourth order method. The underscored terms are omitted for the third order method.

The following equations define the iteration procedure. For the first iteration, $\eta_a = \nu_a = 0$.

$$\Delta\eta = \frac{\overline{F_\nu(\eta_a,\nu_a)}\ \overline{F_{\eta\nu}(\eta_a,\nu_a)} - \overline{F_\eta(\eta_a,\nu_a)}\ \overline{F_{\nu\nu}(\eta_a,\nu_a)}}{\overline{F_{\eta\eta}(\eta_a,\nu_a)}\ \overline{F_{\nu\nu}(\eta_a,\nu_a)} - \overline{F_{\eta\nu}(\eta_a,\nu_a)}^2} \qquad 191$$

$$\Delta\nu = \frac{\overline{F_\eta(\eta_a,\nu_a)}\ \overline{F_{\eta\nu}(\eta_a,\nu_a)} - \overline{F_\nu(\eta_a,\nu_a)}\ \overline{F_{\eta\eta}(\eta_a,\nu_a)}}{\overline{F_{\eta\eta}(\eta_a,\nu_a)}\ \overline{F_{\nu\nu}(\eta_a,\nu_a)} - \overline{F_{\eta\nu}(\eta_a,\nu_a)}^2} \qquad 192$$

$$\eta_c = \eta_a + \Delta\eta \qquad 193$$

$$\nu_c = \nu_a + \Delta\nu \qquad 194$$

The computed position for any iteration becomes the assumed position for the next iteration.

When the magnitudes of $\Delta\eta$ and $\Delta\nu$ are both less than some limit (perhaps 10^{-7} radians), the iteration process is terminated and the accuracy of the position fix is estimated. First the error function is evaluated at the ship's computed position.

$$\overline{F(\eta_c,\nu_c)} = F_{00} + F_{10}\eta_c + F_{01}\nu_c$$

$$+ \frac{1}{2} F_{20}\eta_c^{\,2} + F_{11}\eta_c\nu_c + \frac{1}{2} F_{02}\nu_c^{\,2}$$

$$+ \frac{1}{6} F_{30}\eta_c^{\,3} + \frac{1}{2} F_{21}\eta_c^{\,2}\nu_c + \frac{1}{2} F_{12}\eta_c\nu_c^{\,2} + \frac{1}{6} F_{03}\nu_c^{\,3}$$

$$+ \underline{\frac{1}{24} F_{40}\eta_c^{\,4}} + \underline{\frac{1}{6} F_{31}\eta_c^{\,3}\nu_c} + \underline{\frac{1}{4} F_{22}\eta_c^{\,2}\nu_c^{\,2}} + \underline{\frac{1}{6} F_{13}\eta_c\nu_c^{\,3}} + \underline{\frac{1}{24} F_{04}\nu_c^{\,4}} \qquad 195$$

The underscored terms are the fourth order terms. The other functions needed to compute the estimated accuracy should be evaluated at the navigator's computed position. However, if the satellite subtrack does not come within 100 nautical miles of the navigator, these functions are closely approximated by their values at the original assumed position of the navigator. If these approximations are made, the estimated accuracies are given by

$$\delta_{\eta_{RMS}} = \sqrt{\frac{\overline{F(\eta_c,\nu_c)}}{M(S_{1010_M} - T\, b_{10}^{\,2})\, D'}} \qquad 196$$

$$\delta_{\nu_{RMS}} = \sqrt{\frac{\overline{F(\eta_c,\nu_c)}}{M(S_{0101_M} - T\, b_{01}^{\,2})\, D'}} \qquad 197$$

where

$$D' = 1 - \frac{(S_{1001_M} - T\,b_{10}b_{01})^2}{(S_{1010_M} - T\,b_{10}^2)\,(S_{0101_M} - T\,b_{01}^2)} \qquad 198$$

COMPUTER INVESTIGATION OF THE TRUNCATION METHOD

Two separate computer studies were used to analyze the truncation method. The first study investigated the effect of noise on the positioning error. The second study simulated the navigator making a position fix and investigated the effect of truncating the Taylor series of the error function.

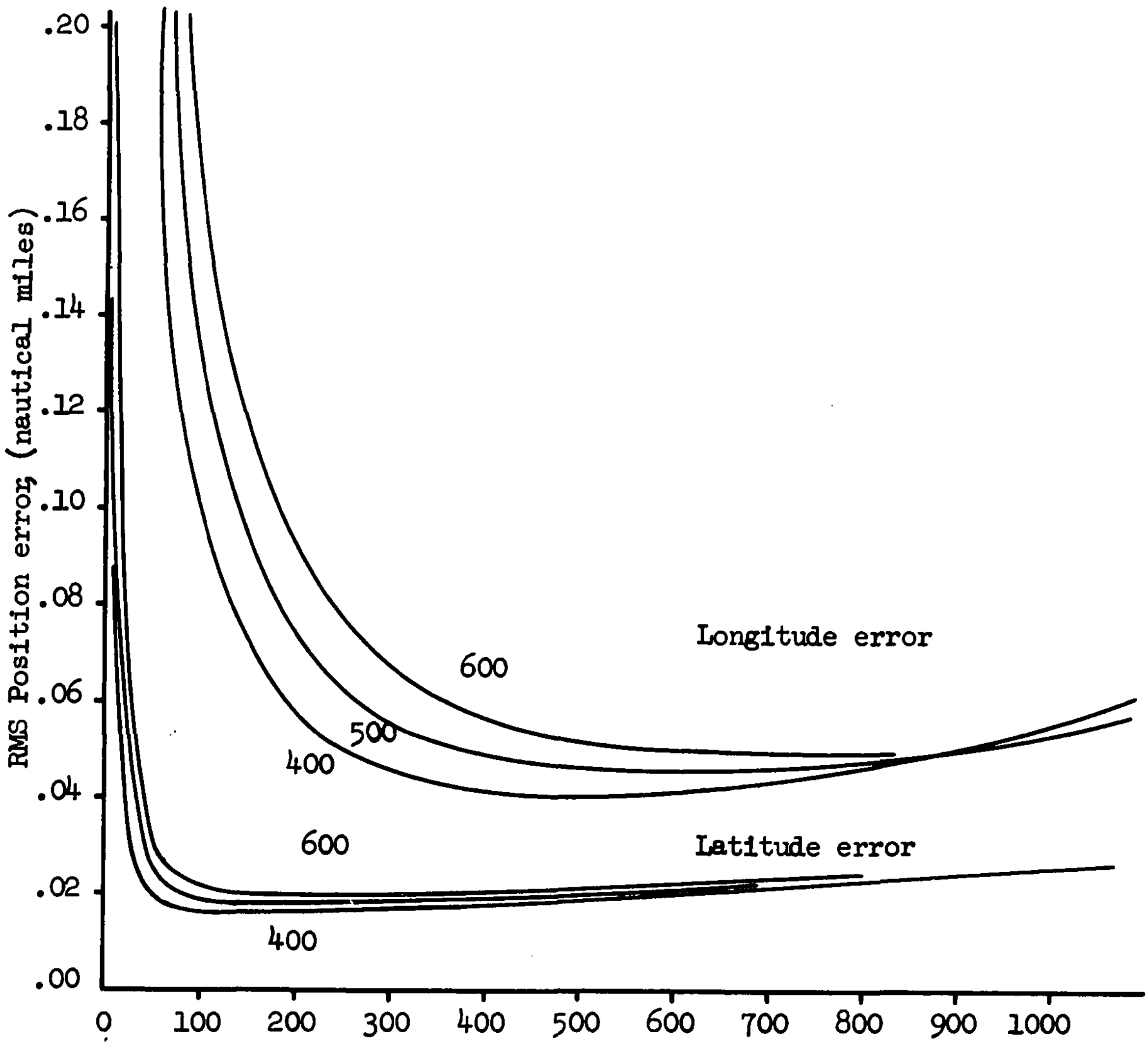
Noise Error Investigation

In the noise error investigation, Equations 130 and 131 were used to predict the RMS errors in latitude and longitude due to noise in the received Doppler frequency. The errors were computed for various positions of the navigator with respect to the plane of the satellite's orbit. The RMS position error is proportional to the RMS value of the noise in the received Doppler frequency. For convenience, a standard RMS noise figure was chosen to be a one cycle error in a one second count of the cycles in the Doppler signal for a transmitter frequency of 100 megacycles. A circular, polar orbit was chosen for the satellite. The rotation of the earth was included in the computations for the navigator's position. The full pass of the satellite, from horizon to horizon, was utilized in the computations.

From preliminary computations it became obvious that the length of the time interval Δt, and hence the number of intervals, did not greatly influence the errors. The RMS errors in latitude and longitude were computed for a 400 nautical mile high orbit with the navigator on the equator approximately 100 nautical miles from the satellite subtrack. The RMS errors improved by only 3% when the time interval Δt was changed from 60 seconds (13 intervals) to 5 seconds (173 intervals). This result should be anticipated since the full Doppler curve was utilized in both passes. For the study it was decided to use a representative value of 20 seconds for Δt.

The curves in Figure 3 illustrate the variation in the RMS error as a function of the distance of the navigator from the satellite subtrack for three different satellite heights. Notice that the errors are typically better than .06 nautical miles for the standard noise that was used. As the satellite orbital plane comes within 100 nautical miles of being overhead, the errors increase quite drastically. This is anticipated because the slant range is very insensitive to changes in longitude when the navigator is near the plane of the orbit.

The RMS position errors were also computed for various latitudes of the navigator. The results were almost identical with the curves in Figure 3 and would hardly be distinguishable if plotted. There is an exception to this which occurs for navigation near the poles of the earth which is explained in the following way. At the equator, longitude is measured perpendicular to the plane of the orbit and latitude is measured parallel to the plane. However, at the poles, latitude is measured perpendicular to the plane of the orbit and longitude is measured, more or less, parallel to the plane. Therefore, the latitude error curves near the pole approximate the longitude error curves near the equator. And the longitude error curves near the pole approximate the latitude error curves



Ground Range to Satellite Subtrack, (nautical miles)
Parameter: Satellite Height, (nautical miles)

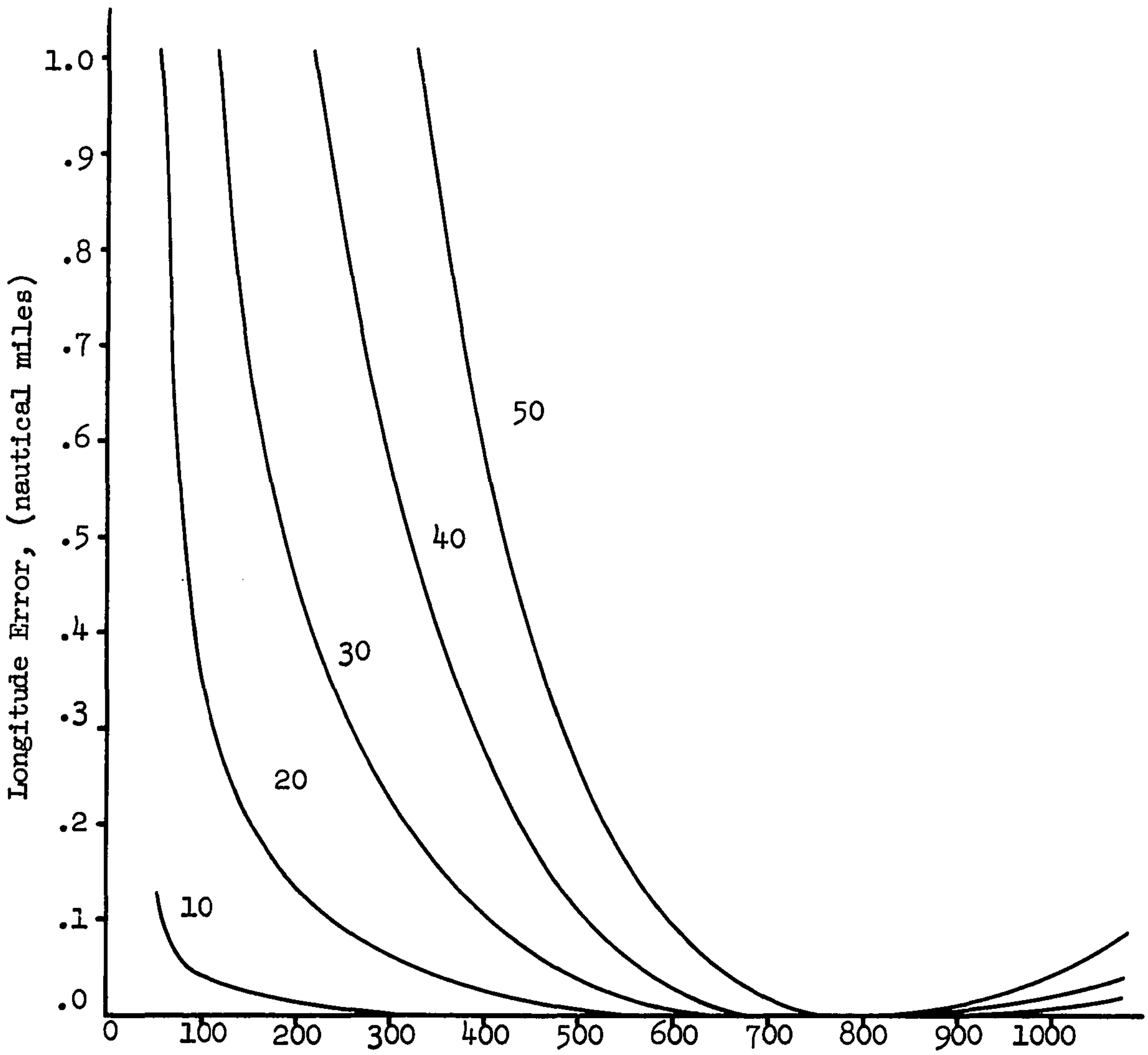
Figure 3. Position error due to noise

near the equator.

Truncation Error Investigation

To determine the effect on the position error of truncating the error function, the actual position fixes were simulated on the computer. The satellite was placed in a circular, polar orbit 400 nautical miles above the earth. The navigator was placed on the equator, but was assumed to be at a point nearby. The same error distance was assumed for both latitude and longitude. The coefficients for the Taylor series of the error function up through fourth order terms were computed using the full pass of the satellite. A position fix for the fourth order method was made using all of the terms. Then a position fix for the third order method was made using all but the fourth order terms. The curves in Figure 4 show the results for the third order method. The longitude error is plotted as a function of the ground range to the satellite subtrack with the original longitude error as a parameter. The errors in latitude were always substantially less than the errors in longitude. Notice that if the navigator's original longitude estimate was correct within 10 nautical miles, the error due to truncation after third order terms is less than the RMS error due to the standard value of noise.

The curves in Figure 5 show the results for the fourth order method. The longitude error is plotted as a function of the ground range to the satellite subtrack with the original longitude error as a parameter. The errors in latitude were always less than the errors in longitude. Notice that if the navigator's original longitude estimate was correct within 40 nautical miles, the error due to truncation after fourth order terms is

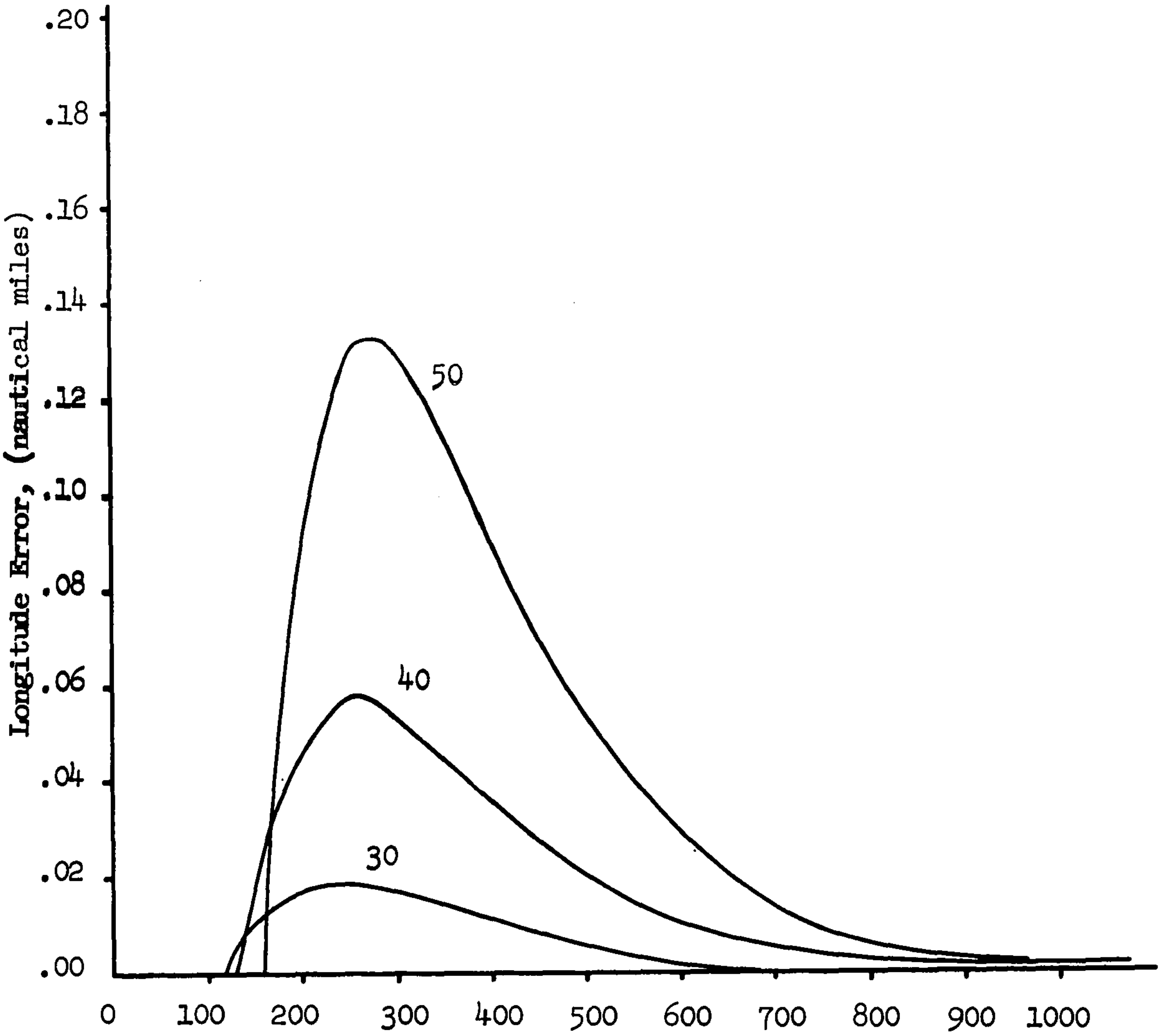


Ground Range to Satellite Subtrack, (nautical miles)

Parameter: Error in Assumed Longitude, (nautical miles)

Satellite Height: 400 nautical miles

Figure 4. Longitudinal truncation error for third order method



Ground Range to Satellite Subtrack, (nautical miles)

Parameter: Error in Assumed Longitude, (nautical miles)

Satellite Height: 400 nautical miles

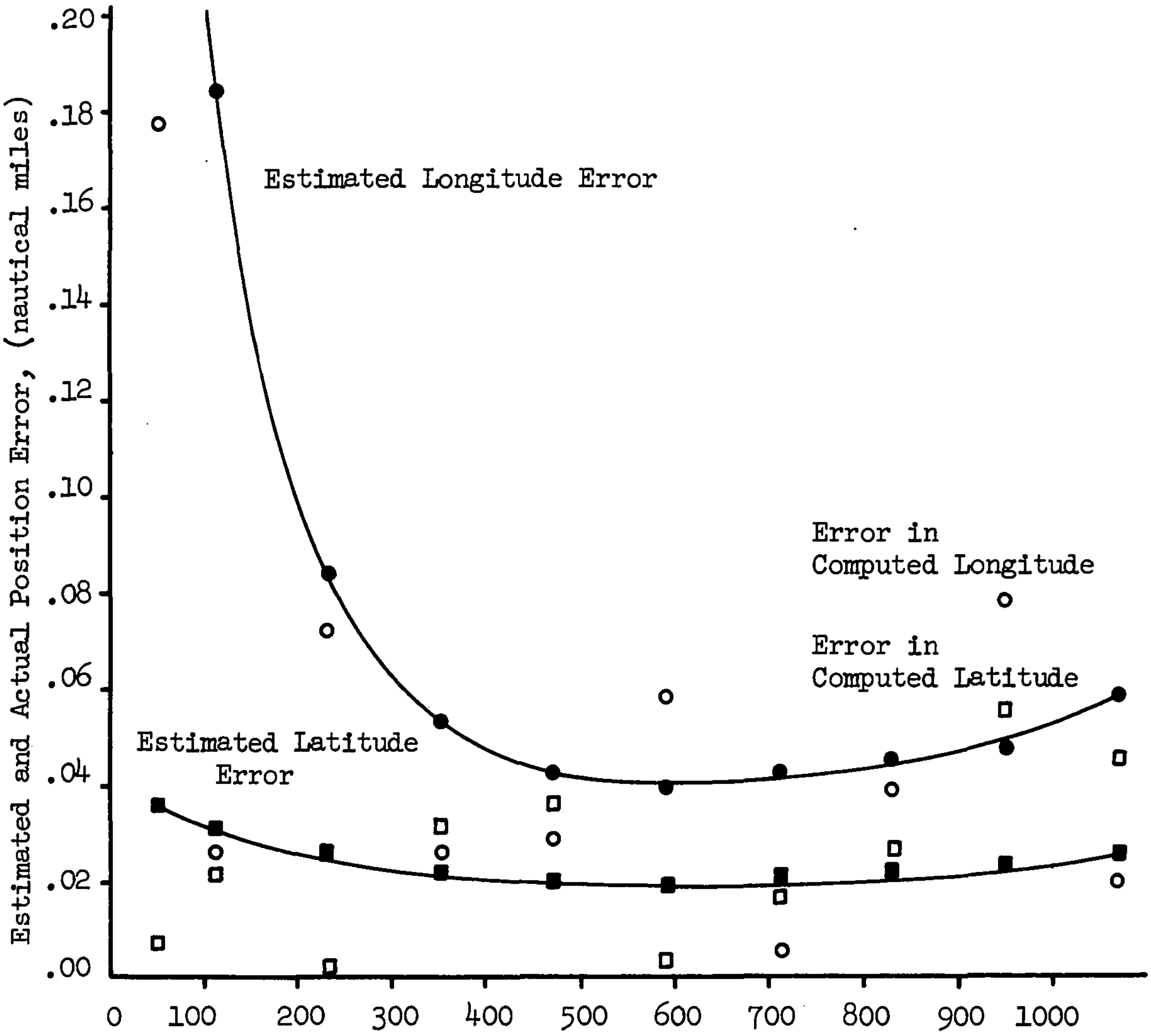
Figure 5. Longitudinal truncation error for fourth order method

on the order of the RMS error due to the standard value of noise.

The truncation error curves were computed for satellite heights of 500 and 600 nautical miles with the navigator's position in error by 10 nautical miles in both latitude and longitude. The curves were nearly identical with the 400 nautical mile high curve.

A bias of 10 cps was superimposed on the Doppler data for some sample satellite passes, and no noticeable effect was observed on the errors.

The curves in Figure 6 show the errors in the computed position for the third order method when the standard noise was introduced in the Doppler data during the satellite pass. The satellite height was 400 nautical miles and the navigator's position was assumed in error by 10 nautical miles in both latitude and longitude. The continuous curves indicate the estimated RMS position error. The discrete points indicate the actual error in the computed position.



Ground Range to Satellite Subtrack, (nautical miles)

Error in Assumed Latitude: 10 nautical miles
Error in Assumed Longitude: 10 nautical miles
Satellite Height: 400 nautical miles
Standard Noise: 1 cycle rms error in a 1 second count at 100 megacycles

Figure 6. Estimated and actual position error for third order method with standard noise introduced in Doppler data

CONCLUSIONS

The proposed truncation method is intended to allow high accuracy navigation using a more modest computer than is presently thought possible with the least squares method. No attempt has been made to justify this in detail. The basis for the contention is that the truncation method allows most of the computations to be made during the pass of the satellite, with few calculations following the pass. This permits a computed position soon after the pass is completed using a modest computer.

The major disadvantage of the method is that the accuracy of the position fix is dependent on the accuracy of the navigator's estimated position before the pass.

The computer study indicates that one-tenth nautical mile accuracies are possible if the navigator's estimated position is accurate to 10 nautical miles for the third order method and 40 nautical miles for the fourth order method. This assumes that the satellite's subtrack does not come within 100 nautical miles of the navigator. It should be noted that the accuracy of the position fix will always be at least as accurate as the first iteration in the least squares method. Therefore, a navigator who is lost can eventually find his position using the truncation method if he can find it with the least squares method.

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