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QUADRATIC PROGRAMMING COMPETITIVE EQUILIBRIUM MODELS FOR THE U. S. AGRICULTURAL SECTOR

by

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I. THE FARM POLICY PROBLEM

A. Introduction

A major socio-economic problem has confronted American agriculture for more than three decades and is still high on the agenda of national affairs. Large public investments have been and are made in continuous efforts to solve the problem, or at least to reduce some of its consequences, such as low farm income. Most of these efforts materialized in the form of ad-hoc programs, related to the immediate future, rather than to long range solutions.

The evaluation of these programs in terms of their quantitative effects, falls without the scope of this study. One thing is, however, clear: none of the heretofore tried programs have suceeded in bringing about a situation satisfactory to most groups concerned with agriculture.

In its broadest sense, the farm policy problem belongs, of course, to the problem of social choice. Our objective in this study is twofold: first, the farm problem is examined by using some theoretical tools provided by welfare economics, an examination which already serves to clarify some of the basic issues involved. This examination points to some possibilities of aiding the process of social choice, making it more systematic and explicit. Secondly, we develop the tools that may serve in carrying out the choice process in the manner indicated in the first stage.

The major question that has to be answered before any attempt at formulating farm programs is made, is what precisely does one mean by the Farm Problem? In general, a practical economic "problem" can be defined only relative to some desired state of affairs. Thus, for instance, a given employment situation will be regarded as problematic if the level of unemployment is higher than some specified level which is viewed as desired; a foreign trade situation will present a problem if the gap in the trade balance exceeds some maximal acceptable gap. Other examples may be found with respect to savings habits, wage rates etc. What then is the desired state of affairs to which the current situation in American agriculture is compared, and what is the problem observed on the basis of this comparison?

A recent definition of the problem is provided in a report of the National Agricultural Advisory Commission (1), hereafter referred to as the USDA report. Six policy goals are outlined (pp. 10-12) in this report. Two of these seem to imply the core of the problem. In the sequence in which they are mentioned, they are:

2. A level of farm income enabling efficient producers to earn returns on their labor and investment comparable with returns realized on similar resources outside of agriculture.

3. ... maximum freedom for individual farm operators within the limit of farm programs.

The first, profits which are too low for the efficient[1] farmer, is defined in comparison with profits that could be earned outside of agriculture. The observation is thus made that the productivity of at least part of the resources in agriculture is lower than their alternative costs. The second point concerns the degree of public intervention in the agricultural sector.

Accepting this definition of the problem, it is clear that what is sought is some lower bound on farm income and higher bound to public involvement. That is, while farm income is desired to exceed a certain level, public intervention should be kept under a certain level. Suppose now that one may define the set P of all levels of farm profits p such that $p \geq \bar{p}$, where $\bar{p}$ is the lower bound. In order to achieve any p of P, some level of public intervention g may be needed. We form the set G of all such levels g. Next, we define the set S of all levels of public involvement s such that $s \leq \bar{s}$ where $\bar{s}$ is the upper bound. Then one difficulty which may arise is $G \cap S = \emptyset$, where $\emptyset$ denotes the empty set. That is, none of the levels of public involvement which is

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[1]The term "efficient farmer" is not well defined. The definition rests primarily on the rate of profits presently earned, and hence involves some circularity.

acceptable, is sufficient to bring about an acceptable level of farm income. In that case, the policy goals are inconsistent or unattainable.

Suppose now that no such inconsistency exists i.e., $G \cap S \neq \emptyset$. We assume also that P and S are ordered by magnitude i.e., for every two points p' and p'' in P, p' is preferred to p'' if, and only if, $p' > p''$. Similarly, for every two points s_o and $\bar{s}$ in S, s_o is preferred to $\bar{s}$ if, and only if, $s_o < \bar{s}$. Three possibilities now evolve. First is the case where $G \cap S$ is a single point, say g_o. Then the "policymaker" is faced with no problem of choice. Only one acceptable state, (p_o, g_o), is attainable. For the other two cases, suppose that $G \cap S = G'$ and that the corresponding subset of P is P', where G' and P' contain each more than one point. The second case is now characterized by the existence of a pair $g_o \in G'$ and $p_o \in P'$ such that $g_o \leq g'$ for all $g' \in G'$ and $p_o \geq p'$ for all $p' \in P'$. That is, out of all acceptable states there is one which is best. As in the first case, no further choice is necessary. The only decision is made when P and S are defined.

It is the third situation which is most likely to occur and in which we are interested namely, when there is no best alternative. That is, for any two pairs (p', g') and (p'', g'') of P' and G', whenever $p' > p''$, $g' > g''$ and whenever $g' < g''$, $p' < p''$. Then one is confronted with a range of alternatives for which we did not establish an ordering criterion, yet from which a choice must be made. That is, a ranking of the alternatives is now required if a "rational" decision is desired.

The problem of ranking the alternatives involves a lot more than just ordering the elements of $P' x G'$ by some preference criterion which depends on P' and G' only. The choice of any particular pair $(\overline{p}, \overline{g})$ may have far reaching ramifications in terms of the number of farmers in business, the farm size, volume of output, prices of agricultural products etc. All these must be taken into consideration, since in practice they do influence decisions.

It is our task in this chapter to suggest a systematic way of going about the problem of choice. The terminology employed is the one provided by welfare economics.

B. The Problem of Choice

For the purpose of the ensuing discussion, we conceive of a social welfare function as consisting of two distinguishable parts: (a) the Bergson (2) economic welfare function; (b) a part which we lable the non-economic component of the social welfare function. This second part contains variables other than quantities of commodities and services consumed or delivered by individuals in the economy. It will contain variables representing society's institutional and ethical preferences, such as education, the beauty of the neighbor's lawn, space research, the form of municipal government etc. The construction of such a welfare function consists of determining which variables affect social welfare, by how much and in what direction. This is equivalent to the knowledge of the first partial derivatives of the social welfare function with respect to all variables. We shall refer to these partial derivatives as weights; in the case of a linear function, they will be constants.

Having outlined the general structure of the social welfare function, we must now concern ourselves with some specific variables pertinent to the analysis. As is well known, any attempt at dealing with the detailed structure of a welfare function involves value judgments. As a first step, we shall suppose that the following value loaded specifications are arrived at by general consent. As will be seen, these initial value judgments can be termed "minimal", at least as far as Western Democracies are concerned. First, we stipulate that the Bergson part of the function contains the utilities of farmers with positive weights. This means, that a ceteris paribus rise in farm income, increases social welfare. Secondly, we include in the second part of the social welfare function the following four variables with non-zero weights: (a) an income distribution variable which for our case will be taken to be the ratio of farm to non-farm income; (b) a variable representing the degree to which the farm sector is planned by the government, called the central planning variable and carrying a negative weight; (c) a variable representing public involvement in the form of price support, subsidies and the purchase of surpluses called the public intervention variable; (d) a variable representing public support to agricultural research and education. We assume that the last two variables are adequately represented in terms of public funds expended in the process of implementing the various schemes.

It will be easy to see now, that the above specifications are not sufficient as grounds for choice. One still must determine at the very least the relative weights of variables (a), (c) and (d). For if we consider the USDA report again, it seems that the weight it attaches to the public intervention variable is negative, but relatively small compared to the positive one it attaches to the income distribution variable[2]. But it is very easy to imagine an attitude reversing this relationship. It will be illustrative to picture the first attitude as resulting in a directive to the economist, asking him to find a program which minimizes public involvement subject to a minimum level of farm income, while viewing the second attitude as resulting in an effort

[2]The report does not seem to be concerned about variable (d) at all.

to maximize farm income subject to a ceiling on public intervention. Only an extraordinary coincidence would lead to identical programs in both cases.

It is thus clear, that the major problem we now confront is that for further specifications we cannot reasonably claim unanimity. This means, that we face at this point the Arrow (3) problem of constructing a welfare function. This is not, however, the purpose of our deliberation. Rather, we are interested here in a concrete problem which is faced in everyday practice. Facing the problem in everyday life is made possible, since some sort of a social welfare function is being used, or else no government would function. We shall therefore take the existing social welfare function, whatever it is, for granted and concentrate on the problem of its utilization in the process of choice.

Ideally, having accepted a social welfare function, we can simply maximize it over the possible alternatives, thereby identifying the optimal policy. Practically, there doesn't seem to be so far a reasonable way of doing it. First, there is no satisfactory way of obtaining an estimate to the welfare function. Even though some methods of measurement have been suggested, notably that by Frisch (4), their theoretical soundness and practical applicability are questionable. Finding the welfare maximizer by trial and error, even though possible, seems to have yielded up until now very little results, as evidenced by the continuous disenchantment with the farm situation.

The rest of the chapter is devoted to suggesting a method of choosing optimal policies by systematizing the trial-and-error procedure. The method is based on some fundamental results of economic theory.

C. The Economist's Contribution

The method which is suggested to overcome the encountered difficulties is based on explicit comparisons of various alternatives, the outcome of each of which can be computed if certain assumptions are met. We shall first indicate the theoretical foundations for the method. The next chapter establishes tools for carrying it out in practice. We shall see that in the course of developing the argument, considerable insight into the farm problem can already be gained.

For those parts of the analysis which warrant mathematical statements, the following notation is adopted: the k-dimensional linear space is denoted by E_k; points (vectors) in E_k are denoted by lower case letters; if x' and x'' are two such points in E_k, then $x' \geqq x''$ means $x'_i \geq x''_i$ for $i = 1,2,\ldots, k$, where i is the coordinate index; $x' \geq x''$ means $x' \geqq x''$ with $x'_i > x''_i$ for at least one i. Capital letters denote sets, and superscripts are used to discriminate vectors or sets.

We consider an economy with n desired (consumer) and m primary (non-producible) commodities. For simplicity, we omit intermediate commodities. Incorporating such goods would not alter the discussion in any significant way. The commodity space is thus given by $\bar{E}_{n+m}$. In that space, we define the set X^i to be the set of consumption choices of the i-th consumer and the set Y^j of all possible production choices of the j-th producer. The set X^i is characterized as follows: every point $x^i \in X^i$ is a list of quantities consumed of n + 1 commodities, the n + 1st being leisure. We must have $x^i \geqq 0$ which means, that X^i is confined to the non-negative orthant of E_{n+m}. As for Y^j, any point y^j of Y^j is a list of outputs, with positive signs, and inputs, with negative signs. We define the aggregate consumption and production sets by $X \equiv \sum_i X^i$ and $Y \equiv \sum_j Y^j$, respectively. We further define the set $Z = X \cap Y$ as the attainable set, and assume it is convex (non-increasing returns).

Starting now with the concept of saturation, we postulate the existence of a complete preference preordering on X^i for all i. That is, a preference relation which is reflexive, connected and transitive. The individual complete preorderings give rise to an aggregate preference preordering (reflexive and transitive) on X. We will say that for x' and x" in X, x' is preferred to x" if no consumer i prefers $x^{i''}$ to $x^{i'}$ and at least one consumer k prefers $x^{k'}$ to $x^{k''}$.

Let now $\bar{x}$ be a point in X and consider the set of all points x' in X such that

$$x^{k'} \geq \bar{x}^k \quad \text{for some } k \tag{1.1}$$

and $$x^{i'} \geqq \bar{x}^i \quad \text{for all } i \neq k \tag{1.2}$$

Then if $\bar{x}$ is preferred to x', $\bar{x}$ is a saturation point. We shall assume that no such $\bar{x}$ exists in Z. Whenever $\bar{x}$ and x' are two points of Z satisfying (1.1) and (1.2), we assume that x' is preferred to $\bar{x}$. We shall also assume that a movement from $\bar{x}$ to x' will not involve redistribution among the consumers of $\bar{x}$, so that (1.1) and (1.2) are always satisfied and hence x' always preferred to $\bar{x}$. "More" is always preferred to "less".

Turning now to the concept of efficiency, let y^o be a point in Z. Then we say that y^o is efficient if there is no y^1 in Z such that $y^1 \geq y^o$. This means, that if we cannot produce more of at least one commodity without reducing the output of another commodity, or cannot produce the same bundle by reducing at least one input without increasing the rate of another input, production is efficient. Viewing primary commodities as not desired themselves, we can simply state that any point on the production possibility frontier, is efficient.

Based on the assumptions made, it becomes now at once clear that the Bergson function, if based on the preference orderings on X^i, cannot reach its maximum at a non-efficient point. This is not, however, the only significance of efficient points. We have assumed above, that the social welfare function contains a variable whose weight reflects resentment of the introduction of central planning of the kind existing in Communist economies. That is, we assume that the American society much prefers an economy in which the (profit motivated) producers compete, to an economy in which they are told by a central authority what and how much to produce. As has been shown (5, pp. 87, 102),[3] production under competition is always efficient, regardless of how prices are formed. That is, non-efficiency will always indicate, to one degree or another, the lack of competition, and hence non-efficiency is not desirable from this standpoint as well.

I do not intend to infer from the two preceding findings, that it is impossible that the social welfare function reaches a maximum at a non-efficient point. Desires such as attaining a certain degree of self sufficiency and keeping a minimum percentage of the population in farming, may outweight the desirability for economic satisfaction and competition. But it seems equally likely, that the subset of efficient points is closer to the maximizer of the social welfare function ("closer" in terms of the value of the function)

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[3] The non-mathematically inclined reader is referred to an excellent discussion in Essay I of Koopmans (6).

than any other subset of points which could be specified on a priori grounds. The last is, of course, a value loaded postulate, expressing some belief concerning human behavior. It is this postulate which constitutes the logical basis for the decision making process to be introduced. For if the postulate is valid, it offers an important criterion that can be used to narrow the field from which final choice is to be made. We may think of the maximum value of the Bergson function as a numeraire, used to "price" goals whose realization requires non-efficiency. In linear production economies, this is a practically feasible procedure, since in such economies every efficient point can be computed by solving a linear programming problem (7, p. 88). Once this is done, goals that call for inefficiency, i.e. for a reduction in the value of the Bergson function and the introduction of central planning, can be judged by the amount of sacrifice in inefficiency and planning.

This is not enough. We still need some criterion for comparing efficient points to one another. This additional criterion is provided by the public intervention variable. Its role in this phase of the procedure is equivalent to the role played by the Bergson and central planning variables in the former phase. This will be facilitated by postulating, that the public intervention variable carries a negative weight. We would thus have some special interest at the point at which the value of the public intervention variable vanishes.

The point which is singled out is, of course, the point of market equilibrium. This is the point which will be (at least theoretically) realized, when not only central planning is absent but also prices are formed by Tatonnement, or what can be termed as the dollar-voting mechanism. That is, this is the efficient point which will be reached when no form of public intervention is present. It is therefore the point, where farm income is maximum when no market regulations are imposed.

We saw before, that we cannot be sure that efficiency is really desired, even though the case for efficiency was quite strong. The case for market equilibrium is considerably weaker but, as before, the mere knowledge of the point of market equilibrium, provides a criterion for an explicit elimination process, which may be beneficial. For it will be possible to see to what extent market freedom must be dispensed with and at what costs, in order to bring about what is essentially a change in income distribution. That is, we are again in a position to "price" alternatives.

As for the practicability of this phase, it is again true that linear programming may be used to carry it out. We shall examine this possibility towards the end of the next section, whose first part is devoted to a reconsideration of the farm problem, in the light of the preceding discussion and some of the major studies accomplished in this field heretofore. It will be the major purpose of Chapter II to develop more efficient tools for the process of comparisons in the efficient subset.

D. The Farm Problem Reconsidered

To facilitate the discussion from this section on, the following notational convention is adopted: lower case letters denote, as before, column vectors; prime denotes transposition, but will be used only when not absolutely clear from the context; capital letters denote matrices or, when so indicated, sets; superscripts are used as before.

The only empirical studies which, to my knowledge, attempted a major breakthrough along the lines of the above argument, are those by Heady and Egbert (8), Skold (9), Whittlesey (10) and Brokken (11). All of these studies applied a multi-regional formulation of a particular linear programming model to the field crops (feed grains, soybeans and cotton) and livestock (cattle and hogs) industries. The model used is the following: let there be K regions. Let x^k represent the output vector in region k $(k = 1,2,\ldots,K)$, c^k be the unit cost vector associated with x^k, A^k be the technology matrix, b^k the resource constraint vector and d be a vector of demanded quantities. Then the general formulation of the model is:

$$\text{to minimize} \quad \sum_k c^k x^k$$

$$\text{subject to} \quad A^k x^k \leqq b^k \qquad k = 1,\ldots,K$$

$$\sum_k x^k \geqq d$$

$$x^k \geqq 0$$

The objective in applying the model was to find the most efficient regional allocation of production for an array of final demands d. As stated, this is a limited practical objective, but we shall see that it falls in line with the above general procedure. First, note that the solution presents the result of competition among producers (regions). Once d is specified, competition among farmers would have led to the results obtained in solving the above problem, provided the farm sector meets the basic assumptions underlying linear production sectors. Secondly, and this is the chief feature of interest to us here, the solution to the above problem, as we shall see below, is probably not efficient. It may be worthwhile to say that this statement is not intended as a criticism, but rather as an important indication as to what we can learn from such a solution, in terms of the policy it represents.

Having indicated the merits of a regional formulation, it will be instructive to consider the above problem in an aggregative fashion. Theoretically, such an aggregation is possible, and we therefore let the triple (c, A, b) represent the aggregation over k of (c^k, A^k, b^k). The problem then becomes

$$\text{to minimize } f(x) = cx \tag{1.3}$$

$$\text{subject to } Ax \leqq b \tag{1.4}$$

$$x \geqq d \tag{1.5}$$

$$x \geqq 0 \tag{1.6}$$

Let us define the following (convex) sets: $X^1 \equiv \{x\colon\ Ax \leqq b,\ x \geqq 0\}$ and $X^2 \equiv \{x\colon\ x \geqq d\}$. Then our programming problem is to minimize f(x) over $X \equiv X^1 \cap X^2$. Fig. 1 portrays the sets X^1, X^2 and X for a two-dimensional example case. The set X^1 is given by the bounded plane OABC, X^2 is given by the unbounded shaded area and X by EDF. That is, only points of the closed bounded set EDF satisfy all the constraints. The enumerated parallel lines represent some of the "contours" of f(x), and the broken line is the normal to these lines through the origin, whose length can be shown to be proportional to the value of f(x). Obviously, the solution to the problem represented by Fig. 1 is not efficient; it will be at the point E. If it were to be efficient, the point E would have to lie on the efficiency frontier ABC, as shown in Fig. 2. The likelihood of a situation like the one represented by Fig. 1 is, of course, much greater than the one represented by Fig. 2, and hence we safely conclude

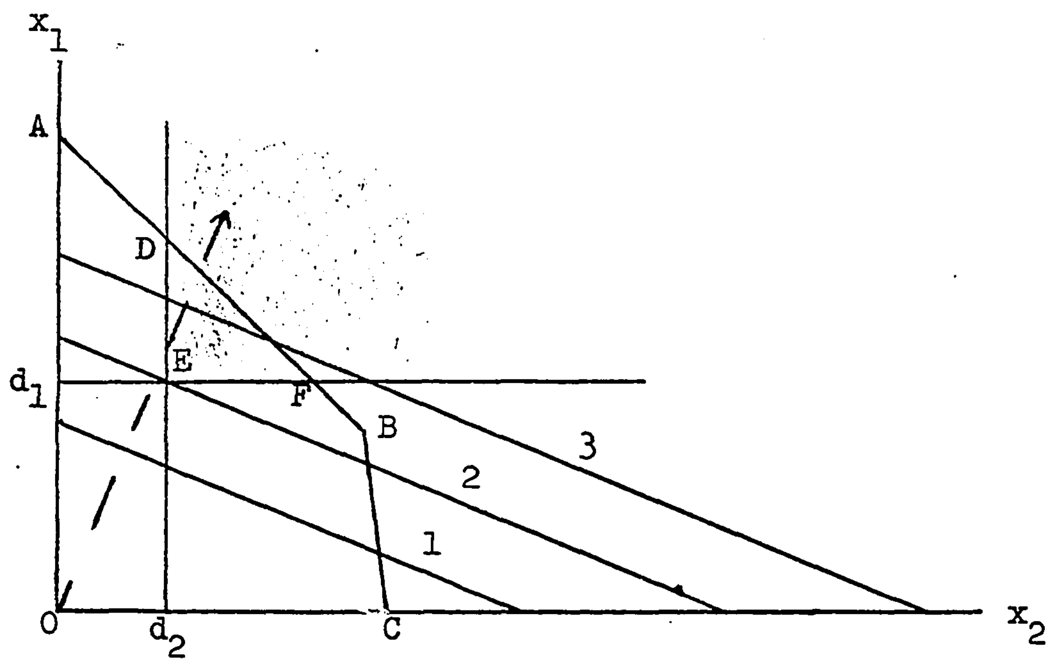

Fig. 1. Non-efficient solution

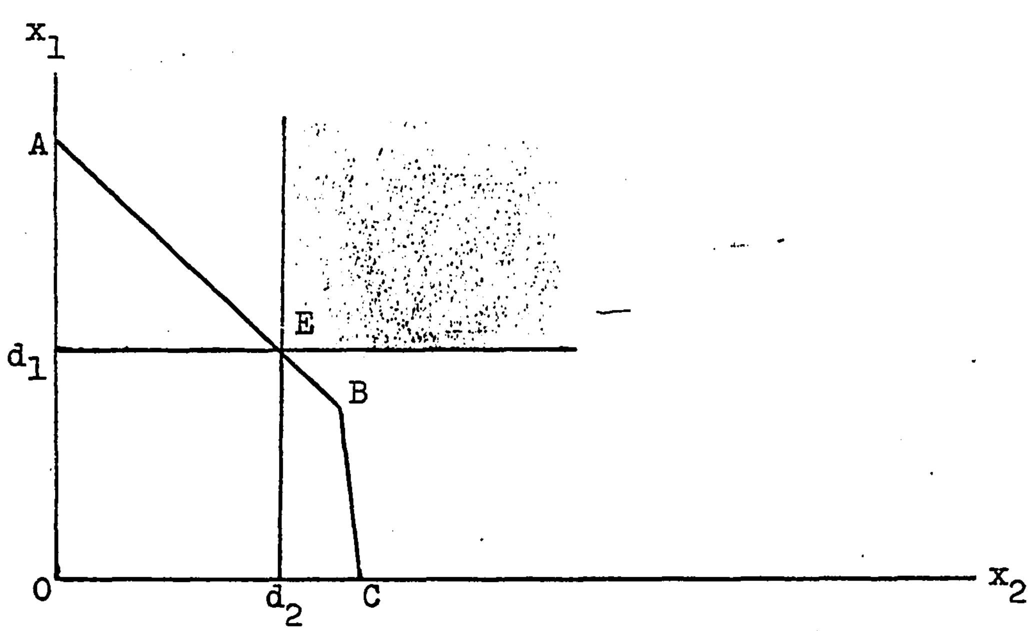

Fig. 2. Efficient solution

that the solutions in the above studies are not efficient. Yet there is little doubt that they constitute an improvement over the present state of affairs in the farm sector, at least as far as farm income and surpluses are concerned. It should be noted that, being non-efficient, practical implementation of such solutions would call for some degree of central planning. Still, the resulting pattern is considered superior to the present one. Why is this result important, and in what way does it serve us in the search for a systematic formulation of farm policy?

Returning to the present structure of the farm sector, it is obvious that, under current conditions, efficiency in the farm sector would, indeed, be disastrous for the farmer. Actually, the Land Retirement Program which is, to be sure, a relatively mild form of central planning, is aimed at preventing efficiency. The emphasis here is on "under current conditions". By this is meant chiefly the amount of primary resources presently committed to agriculture, or the vector b. The outstanding point concerning b is, that it probably is not a part of a price guided allocation pattern, neither in reality nor in the above studies. All that has been said above about efficient points and their characteristics, relates to efficient points when the economy as a whole is considered. These characteristics do not necessarily hold for an isolated sector, unless b belongs to a price guided allocation, and hence an efficient point, to begin with. When it does not, then profits will be, in general, no longer maximal at an efficient point of the isolated sector's possibility set.

It is thus clear, that the present ~~farm~~ policy is based not only on the variables that were explicitly included in the welfare function, but also on some of the variables mentioned in the previous section, such as the percentage of the population on farms, the cultural value of farm life etc. We have a situation in which the amount of resources, particularly labor, committed to farming is not just a means to production but an end in itself. Realizing this end necessitates foregoing efficiency and the introduction of public intervention of the forms specified. The solutions in the above studies indicate how the pursued policy can be achieved most efficiently in the sense that they minimize the "Bergson sacrifice" which must be made.

This takes us right back to the argument of the previous section. For the question to be asked now is, whether or not the present policy, when implemented in the fashion prescribed by the above studies, is optimal. Is the desire to prevent reallocation of resources i.e. migration from the farms, really "worth" the sacrifice in efficiency (hence central planning) and other forms of public intervention? If the desire to keep farm population in tact is not the motive, can this sacrifice be defended on other grounds? It seems, that the only other reason for keeping the present level of resources comitted to agriculture would be, that the degree of public intervention required in order to facilitate a transfer of labor force from the farm sector to other sectors, is considerably higher than the degree presently existing or the degree that is called for in conjunction with the above studies. No evidence for such reasoning exists. Moreover, I believe that the contrary is more likely. That is, in the long run, such shifting of resources would involve less public intervention.

The lesson is very clear: if the reluctance to engage in resource reallocation (because, say, of the values attached to cultural and other aspects of life on farms, the fear of over-urbanization, etc.) really outweighs all the other considerations, then the farm problem is not one of basic policy formulation, but rather a technical problem of how to best implement the existing policy. If it is the latter, the solution is given by the above studies.

It seems, however, that the farm problem is a more basic one. The USDA report is one indication of that. It is also well known that at least one of the most important farm organizations, namely the Farm Bureau, is diametrically opposed to most basics of the present farm policy. We shall thus assume that policy changes are needed.

As indicated in the concluding remarks of the last section, we proceed now to demonstrate how the above linear model can be utilized in carrying out the explicit decision making process. Such a demonstration seems worthwhile for history's sake as well as to further show how the applications initiated by Professor Heady are in line with the main stream of the present analysis. It will also aid in the understanding of the procedure.

Suppose, now, that the above model, in its regional or non-regional form, is applied to the economy as a whole. Further, suppose that the demand system for consumer goods is given by

$$d = d(p) \tag{1.7}$$

where p is the price vector, listing the prices of all final commodities. The process we shall carry out is as follows: first, the point of market equilibrium will be sought. After it is found, the decision must be made whether or not it is desirable. If, because of income distribution or other considerations it is not, other efficient points can be examined by, say, artificially fixing higher prices for farm products, with the understanding that the government will purchase surpluses. If no efficient point is satisfactory (because, say, farm population is too small or farm surpluses do not meet the needs of the Foreign Aid Plans), non-efficient points may be considered.

We thus start with finding the market solution. If we knew p^o, the market equilibrium price, we could simply compute $d^o = d(p^o)$ and insert d^o in (1.5). Such knowledge is not usually available, and hence p^o must be identified. To see how this can be done, consider the dual to our programming problem,

$$\text{to maximize } g(u,w) = bu - dw \tag{1.8}$$

$$\text{subject to } A'u - w \geqq c \tag{1.9}$$

$$u,w \geqq 0$$

where u and w are the (vectors of) "shadow prices" of the resources in b and the commodities in d, respectively. It can be easily shown, that if (u^o, w^o) solves the dual problem such that $w^o = p^o$, then $d = d^o$. The process under which such a solution is found is as follows: we select an initial price vector, say p^1, and solve for d^1 from (1.7). d^1 is then inserted in (1.8) and the dual problem is solved (given that a solution exists when $d = d^1$). We get a solution, say (u^1, w^1). If $w^1 \neq p^1$, we compute

$$d^2 = d(w^1),$$

insert d^2 in (1.8) and solve for a w^2. We continue in this fashion until, for some t, $w^{t+1} = w^t$. When this happens, $w^t = p^o$ and $d^t = d^o$. The method has, indeed, been used by Schrader and King (12).

Clearly, the described method is cumbersome, and would be infeasible for large scale problems, because of both computing limitations and budget constraints. However, at this stage we are satisfied with the principal results and may brush aside technical difficulties.[4]

Efficient points other than market equilibrium may have been already obtained during the above iterative procedure. The verification is easy, since any solution such that $u \geq 0$ is efficient. However, to systematically trace out other efficient points, one would use the following problem:

$$\begin{aligned} &\text{to maximize } (p - c)x \\ &\text{subject to } Ax \leqq b \\ &\qquad\qquad\quad x \geqq 0 \end{aligned}$$

where p now represents the (government) determined prices. If p is chosen, say, as p^t such that $p^t \geq p^o$, and x^t solves the problem, the public expenditures will be given by

$$p^t(x^t - d^t)$$

where $d^t = d(p^t)$

If none of the efficient solutions is satisfactory, non-efficient alternatives may be sought in two main directions. First, resources may be shifted from the non-farm to the farm sector (if the motive is farm population or the level of production). Secondly, additional restrictions may be imposed. For instance, if return to farm resources is

[4]Also, there is no proof that the iterative procedure described converges, but we shall neglect that as well.

deemed insufficient, restrictions on the relevant shadow prices may be imposed. For each new solution, the "sacrifice" in efficiency and public involvement can be computed. These are some of the courses that can be followed in quantifying and evaluating the results of as many policies as one wishes to consider. It should be noted that the variable representing public support to agricultural research and education can also be introduced into the process. One may represent the impact of such activities by changing A and resolving the array of problems. The new solutions can now be used to determine the "welfare value" of the public expenditures incurred in changing A.

Naturally, the choice of the particular policy to be followed, is not the function of the economist, at least not in his capacity as a "professional". This choice is to be made according to the rules by which the social welfare function is constructed. In the U.S. and other democracies, the choice is based on the relative majority rule.

What the economist can do, and this is, so I hope, what was accomplished in this chapter, is to improve the methodology of going about the process of choice. He can also contribute by supplying some operational tools, one of which was already discussed, more to be developed in the following chapter. Admittedly, the tools economics has to offer are far from being perfect. They are limited by assumptions part of which are known to be unrealistic, the credibility of others cannot yet be verified. However, within the limits of supporting tools, such as statistics, the techniques offered by the economist, so I believe, are liable to yield better results as compared with those obtained by some of the presently practiced procedures. I therefore believe, that if there is at all a chance of finding a modus vivendi with regard to American agriculture, the prescribed method must be employed.

II. MATHEMATICAL MODELS

A. Objectives

Before getting to the subject matter, a few more notational remarks are in order. Elements of matrices will be represented by the corresponding superscripted lower case letters if they are vectors, or by the corresponding subscripted lower case letters if they are scalars. Superscripts are generally used to discriminate matrices. ∂ denotes the gradiant operation. In particular, the following symbols will be kept throughout the chapter, unless otherwise specified.

x = an n-vector of outputs;

p = an n-vector of prices of the elements of x;

b = an m-vector of given amounts of restricted resources;

A = the $m \times n$ technology matrix relating x to b;

u = the m-vector of imputed prices of the limited resources;

c = the n-vector of pecuniary outlays, associated with the production of a unit of x.

As indicated at the conclusion of Chapter I, our prime objective in this chapter is to develop tools which will enable us to carry out the process of choice outlined above. In order that such tools be effective, they must enable us

1. To compute efficient points in general, and that efficient point which is associated with market equilibrium in particular.

2. To estimate the "sacrifices" associated with the various efficient or non-efficient alternatives. It is the first point which constitutes most of the burden since it must be proved, for any given model, that it is suitable for solving for a competitive equilibrium. That is, assuming non-saturation, we must show that a solution can be found which

a. Is efficient.

b. Guarantees maximum gross and net[1] profits for the sector, as well as for each firm in the sector.

c. Guarantees non-positive net profits for any firm in any activity.

d. Would be brought about by a free market which generates prices so as to equate supply and demand.

In the previous chapter, we saw already how linear programming may be utilized in obtaining competitive solutions. Some more elaboration in this direction will prove helpful in the development of quadratic programming models designed for the same purpose.

Consider, then, the linear programming Problem 1

$$\text{to maximize } f(x) = (p - c)x \tag{2.1}$$

$$\text{subject to } Ax \leqq b \tag{2.2}$$

$$x \geqq 0 \tag{2.3}$$

where p and c are given, and where A and c are supposed to

---

[1]Net profits are obtained after returns to fixed resources are subtracted from gross profits.

be representative of the sector as a whole. Will the solution to Problem 1 satisfy our conditions? We shall examine this with respect to each of them and, in doing so, base ourselves on some fundamental theorems in point set theory and linear programming.

With regard to efficiency, let $X = \{x\}$ be a closed bounded set and $g(x)$ be a linear function. Then if $x^o \epsilon X$ is such that $g(x^o) = \max\limits_{x \epsilon X} g(x)$, then x^o must be a boundary point of X. In Problem 1, $f(x) = (p - c)x$ is a linear function and $X = \{x: Ax \leqq b, x \geqq 0\}$ is a compact set. Suppose now that x^o maximizes $f(x)$. Then x^o is almost always efficient. Figs. 3 and 4 illustrate the theorem for $n = 2$, $m = 2$. The set X in each case is described by the plane bounded by OAB. The enumerated line segments are iso-gross-profit lines. They can be viewed as some of the contours of $f(x)$. The normals to these lines through the origins indicate, that as we go farther (from the origin) into the positive quadrants along these normals, $f(x)$ increases. Indeed, the length of these normals is proportional to the value of (2.1). In both cases it is then clear, that a maximum position cannot be reached in an interior point of X. In Fig. 3, the (unique) maximizer is x^o, and it is clearly efficient. In Fig. 4, both x^o and x^1 are maximizers and so is every point along the line segment between them i.e., any point x such that $x = tx^o + (1 - t)x^1$ for $t \epsilon (0,1)$. Yet out of these, only x^o

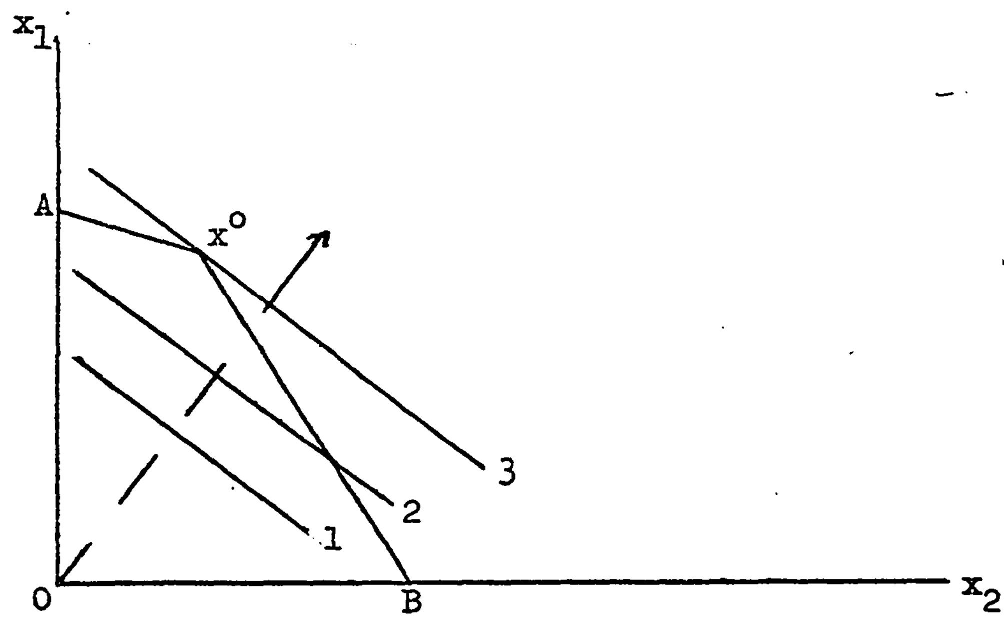

Fig. 3. Efficient solution

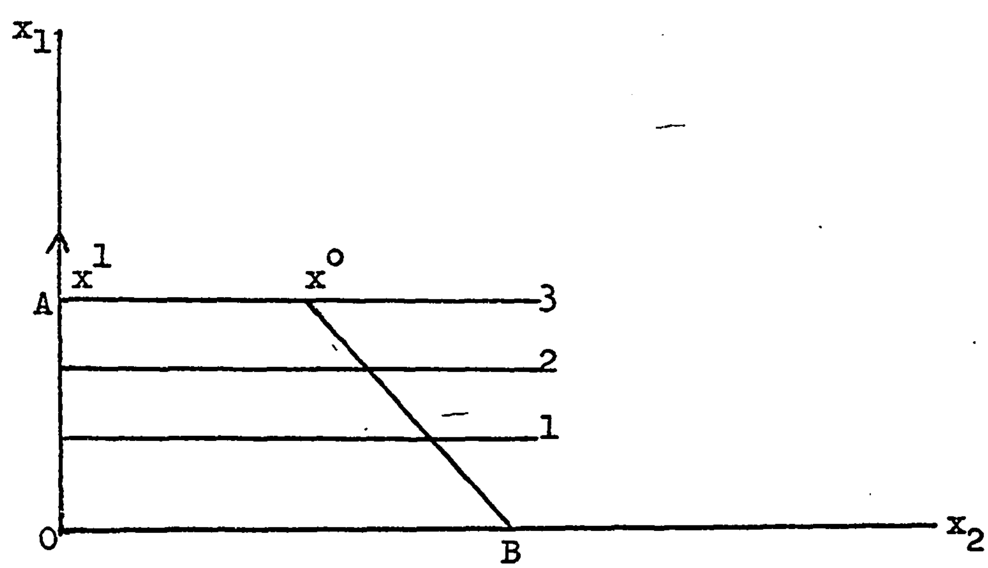

Fig. 4. Possibility of a non-efficient solution

is efficient. In reality, however, we need not be concerned with such situations. First, only x^0 and x^1, out of the uncountably infinite solutions, can be obtained by the simplex method, for the method considers only vertex solutions. Secondly, as Fig. 4 portrays, $(p_2 - c_2) = 0$ and when we have cases such as this, where $(p_i - c_i) = 0$ for some i, x_i can always be omitted before the problem is solved. Hence, we can generalize and say that Problem 1, if it has a solution i.e., if X is non-empty and compact, the solution will be efficient.[2] Thus, condition (a) is satisfied.

Our second condition is, that while f(x) is at its maximum, so are the gross and net profits for every firm in the sector. Suppose that there are K such independent firms and that $X^k \equiv \{x^k\colon A^k x^k \leqq b^k,\ x^k \geqq 0\} \subset R^n$ is the feasible set for $k = 1,2,\ldots,K$, and let $X = \sum_k X^k$. That is, $X = \left\{x\colon x = \sum_k x^k,\ x^k \in X^k\right\}$. Also, let $f(x^k)$ be the linear homogeneous gross profit function of the k-th firm. Then we have the following results;

(a) if $\max\limits_{x^k \in X^k} f(x^k) = f(\bar{x}^k) \qquad k = 1,\ldots,K$

and $\bar{x} \in X$ is such that $\bar{x} = \sum_k \bar{x}^k$,

---

[2]My analysis here draws heavily on Essary I of Koopmans (6) and on Dorfman, Samuelson and Solow (13), Chap. 14, both very comprehensive and relatively non-mathematical.

$$\text{then} \quad \max_{x \in X} f(x) = f(\bar{x}) .$$

(b) Conversely, if $\max_{x \in X} f(x) = f(\bar{x})$

and $\bar{x}^k$ are such that $\sum_k \bar{x}^k = \bar{x}$,

$$\text{then} \quad \max_{x^k \in X^k} f(x^k) = f(\bar{x}^k) \quad \text{for} \quad k = 1,\ldots,K .$$

This is called the decentralization property, and assures us that if x^k maximizes the profits for the k-th firm, then $\sum_k x^k$ maximizes profits for the sector and vice versa. Our Problem 1, then, does satisfy, theoretically, condition (b) with respect to gross profits. From the operational view point, however, it will be, in general, impossible to find the $\bar{x}$ such that $\bar{x} = \sum_k \bar{x}^k$, when solving the problem as formulated. The obstacle is the problem of aggregation. If the A^k are not identical and the c's are different for the different firms i.e., if the firms are not homogeneous, it will be impossible to define X and f(x). It may be noticed, that even as stated, the above results depend on the profits being the same function f for all firms. This is necessitated by the particular (which is the most usual) formulation of Problem 1.[3] Our concern at this point, however, is conceptual and we are satisfied that if technical difficulties are eliminated, Problem 1 does possess the decentralization property.

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[3]For the more general theorem, see Koopmans (6, p. 12) or Koopmans and Bausch (5, p. 81).

The third condition requires that net profits be zero at their maximum. To see that Problem 1 does satisfy it, consider the dual to Problem 1

$$\text{to minimize } g(u) = bu \tag{2.4}$$

$$\text{subject to } \quad A'u \geqq p - c \tag{2.5}$$

$$u \geqq 0 \tag{2.6}$$

The two problems can be treated as a primal-dual combination, Problem 2,

$$\text{to maximize } F(x,u) = f(x) - g(u) \tag{2.7}$$

subject to (2.2), (2.3), (2.5) and (2.6). $F(x,u)$ clearly represents net profits. If (x^o,u^o) solves Problem 2, then by the duality theorem[4] $F(x^o,u^o) = 0$. Thus, Problem 1 does satisfy the third condition, and it follows that condition (b) is also satisfied for net profits.

Finally, we require that if x^o maximizes (2.1) and $d = d(p)$ is the system of demand functions for the n commodities, with d being the vector of demanded quantities, $x^o = d(p^o)$, where p^o is the equilibrium price vector. This condition will not be satisfied, unless we are lucky and "guess" the right p, p^o. If we want to use linear programming as a systematic procedure to find the right p, we must return to the model presented in Chapter I. Consider the primal-dual combination of Problem 3,

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[4] See, for example, Charnes and Cooper (14, pp. 179-182).

$$\text{to maximize } G(x,u,w) = d^o w - cx - bu. \tag{2.8}$$

$$\text{subject to } Ax \leqq b \tag{2.9}$$

$$w - A'u \leqq c \tag{2.10}$$

$$-x \leqq -d^o \tag{2.11}$$

$$x,u,w \geqq 0$$

First, observe that Problem 3 is equivalent to Problem 2 in terms of a competitive equilibrium solution. To see it, suppose that (x^o,u^o,w^o) maximize (2.8) over the feasible set, and suppose that we are given ex ante that $w^o = p^o$ is the equilibrium price structure. Since

$$d^o = d(w^o) = d(p^o) \quad ,$$

d^o is the vector of demanded quantities under equilibrium, and we imposed the condition

$$x^o = d^o$$

So, Problem 3 can be reformulated as

$$\text{maximize} \quad (p^o - c)\,x$$

$$\text{subject to (2.9), and}$$

$$-A'u \leqq c - p^o$$

$$x,u \geqq 0$$

which is the same as Problem 2 when p is replaced in (2.7) by p^o. Hence, a solution to Problem 3, because of its identity with a solution to Problem 2 with $p = p^o$, satisfies all four basic conditions. The difficulty is, of course, operational, as I have indicated above. That is, the iterative procedure described in Section D of the previous chapter would be infeasible, both computationally and financially, for large scale problems. It is obvious, therefore, that we must resort to non-linear programming.

The task is to develop a model which, under the specified assumptions, will yield a solution which will satisfy the conditions we have imposed. The major difficulty arises from the fact that, unlike linear programming, non-linear models will not, in general, have efficient solutions. Figs. 5 and 6 depict the relevant situations for $n = 2$, $m = 2$. In both cases the feasible set is the plane bounded by OAB. The elipse-like curves represent the contours of the maximands, and the arrows denoted by 1 and 2 indicate the isotonic and antitonic parts of the concave maximands, respectively. In Fig. 5, it is clear that the point a, which is not efficient, is the maximizer of the objective function $f(x)$ over the feasible set X. If we consider only the subset of efficient points, $E \subset X$, then $\max_{X \in E} f(x) = f(x^o) = f(x^1) = f(x^2)$. But $f(x^o) < f(a)$ so that we cannot have both efficiency and maximum profits. In Fig. 6, on the other hand, x^o is the efficient maximizer of the objective function over the feasible set, since the point a does not belong to this set. If we are to accept a non-linear programming model as a framework for solving our problem, we must first show that it belongs to the class of models represented by Fig. 6.

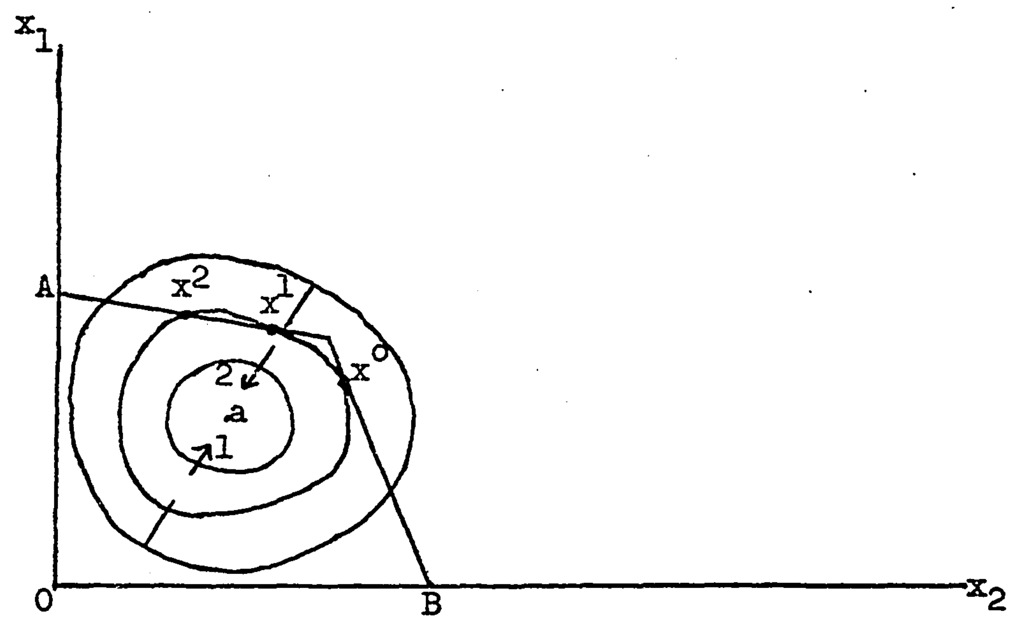

Fig. 5. Non-efficient maximum

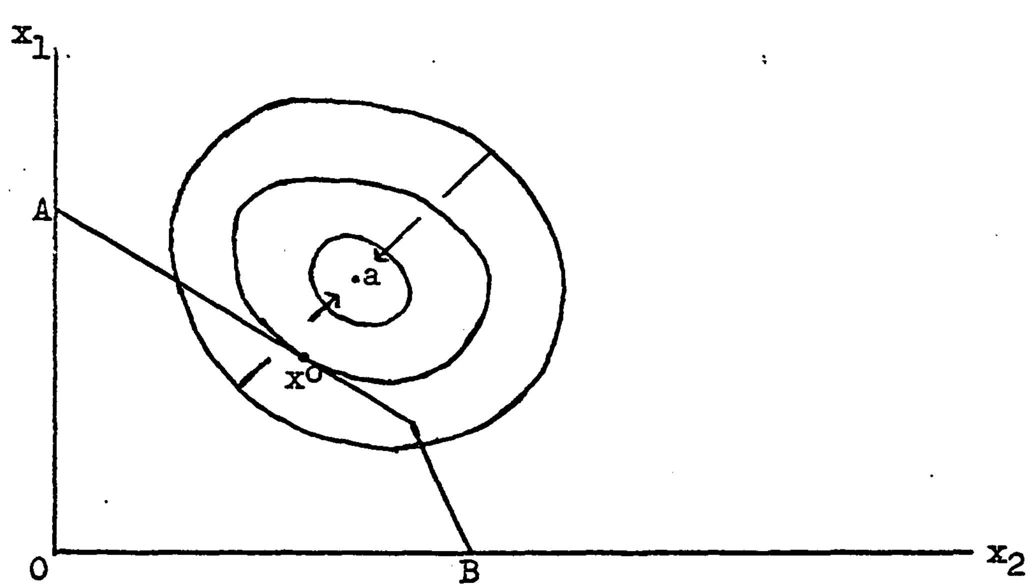

Fig. 6. Efficient maximum

B. The General Form

The general idea stems from Problem 3 above, which can be rewritten as follows:

$$\text{to maximize } G(x,u,w) = d^{o}w - cx - bu \tag{2.12}$$

subject to

$$\begin{bmatrix} 0 & A & 0 \\ -A' & 0 & I \\ 0 & -I & 0 \end{bmatrix} \begin{pmatrix} u \\ x \\ w \end{pmatrix} \leqq \begin{pmatrix} b \\ c \\ -d^{o} \end{pmatrix} \tag{2.13}$$

$$x,u,w \geqq 0$$

The coefficient matrix of (2.13), as is easily seen, is skew-symmetric i.e., equals the negative of its transpose. That is, the feasible space for Problem 3 is described by a self-dual system.[5] Furthermore, over this feasible space, max $G(x,u,w) = 0$, which means zero net profits. Note also, that the coefficients of the linear maximand (2.12) are the negatives of the right hand side of (2.13)[6]

Abstracting for a moment from economic interpretations of the quantities, consider Problem 4:

$$\text{to maximize } cx$$
$$\text{subject to } Ax \leqq b$$
$$x \geqq 0 \;.$$

---

[5]See Tucker (15) and Goldman and Tucker (16).

[6]See Dantzig (17).

We have the following result:

Lemma 1 Let x^o solve Problem 4. Then $b = -c$ and $A = -A'$ is a sufficient condition for $cx^o = 0$.

Proof Rewrite Problem 4, using $b = -c$, as

$$\max_{x \in X} \; cx$$

$$X \equiv \left\{x\colon \; Ax \leqq -c, \; x \geqq 0\right\}$$

The dual to this problem is

$$\min_{u \in U} \; -cu$$

$$U \equiv \left\{u\colon \; A'u \geqq c, \; u \geqq 0\right\}$$

But with $A' = -A$ the dual is

$$\min_{u \in U} \; -cu$$

$$U \equiv \left\{u\colon \; Au \leqq -c, \; u \geqq 0\right\}$$

which is equivalent to

$$\min_{x \in X} \; -cx$$

By the duality theorem

$$cx^o = -cx^o$$

which implies

$$cx^o = 0 \; .$$

The search for a non-linear model which will satisfy our conditions, was aimed at finding a model whose structure will be as close as possible to that of Problem 3, which, indeed, is the case, as will be seen, with the proposed quadratic programming model to be presented.

In preparation for the presentation, I shall state, without proof, Hanson's (18) duality theorem,[7] since it is essential for the argument. Again, we abstract from economic interpretations. Let $f\colon R^n \rightarrow R$ be convex and $h\colon R^n \rightarrow R^m$ be concave, both functions differentiable everywhere. Consider the problem

$$\min_{x \in X} f(x) \tag{2.14}$$

$$X \equiv \{x\colon h(x) \geqq 0,\ x \geqq 0.\}$$

Then, by Hanson's theorem, the problem

$$\max_{v \in V(x)} g(u,v) = f(u) - u'\partial f(u) - v'[h(u) - \partial h(u)\cdot u] \tag{2.15}$$

$$V(x) \equiv \{v\colon \partial h(u)\cdot v - \partial f(u) \leqq 0,\ v \geqq 0\}$$

is such that

$$f(x^o) \equiv \min_{x \in X} f(x) = \max_{v \in V(x)} g(u,v) \equiv g(u^o,v^o) \tag{2.16}$$

and $$u^o = x^o \tag{2.17}$$

The framework of the model to be discussed is exactly the same as that of Problem 3. That is, all the usual assumptions of activity analysis[8] hold. In addition, we assume:

(a) The supply functions for "unrestricted" inputs are perfectly elastic, so that prices of these inputs can be considered as given and constant.

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[7] See also Dorn (19).

[8] For a rigorous treatment, see Koopmans (7).

(b) The demand structure is given by the system

$$d = d^o + Dp \tag{2.18}$$

where D is negative semi-definite, d^o being a vector of constants.

The first assumption is not essential. I chose to impose it at this point for simplicity. It will be relaxed later.

The second assumption is essential, as will be seen later. Some elaboration on it seems desirable because of its economic implications. By negative semidefiniteness is meant, that for any $y \neq 0$ we must have $yDy \leq 0$. As far as I can see, the restriction on D cannot be clearly interpreted in economic terms, even though it is economically meaningful. According to Newman (20), it seems that the minimum restrictions on the elements of D, which are necessary to assure its negative semidefiniteness are, that all the diagonal elements of D be negative[9] and that D be strongly quasi-diagonalized. The second restriction is the one which does not lend itself to a clear interpretation. It means, that there must exist a vector $w > 0$ such that

$$|d_{ii}|\, w_i > \sum_{i \neq j = 1}^{n} |d_{ij}|\, w_j \quad i = 1,2,\ldots,n \tag{2.19}$$

and

---

[9] Economically, this means exclusion of Giffen goods.

$$|d_{jj}|\ w_j > \sum_{j \neq i = 1}^{n} |d_{ij}|\ w_i \quad j = 1,\ldots,n \qquad (2.20)$$

where d_{ij} are the elements of D, w_i those of w and the vertical bars indicate absolute values. A special case occurs when (2.18) and (2.19) hold for w = 1. Then D is strongly diagonalized which means, in economic terms, that the effect of p_i on d_i (the i-th element of d) is stronger than both the sum effects of all other prices on d_i, and the sum effects of p_i on all other commodities. All that is required by our restriction is $w > 0$, and hence we can say only, that there must exist at least one linear combination of the price effects such that p_i affects d_i more than does this linear combination, when applied both to all other prices in relation to d_i, and p_i in relation to all other commodities.

Another question is, whether or not negative semi-definiteness is actually met in reality or not. Again, there seems to be no basis for an a priori answer. The only, and very weak ground, for expecting D to be negative semidefinite (or negative definite), is related to the so called Hicksian stability conditions.[10] This can be shown as follows: let the supply structure be represented by

$$l = l^{o} + Lp \qquad (2.21)$$

[10] See, for instance, Allen (21, pp. 325-329).

where l is the n-vector of supplied quantities, l^o a vector and L a matrix of constants. Let $\bar{p}$ be such that

$$l^o + L\bar{p} = d^o + D\bar{p} ,$$

and define the excess demand vector

$$e = d - l .$$

Now, let there be an initial disturbance which causes p to deviate from $\bar{p}$ and let the adjustment process be described by

$$\dot{p} = \hat{K}e = \hat{K}(d^o - l^o) + \hat{K}(D - L)p$$

where $\hat{K}$ is a diagonal matrix of constants, representing the speed of adjustment, and $\dot{p}$ is the time derivative of p. The Hicksian conditions show, that if $\bar{p}$ is to be stable i.e., if after the initial disturbance $p \longrightarrow \bar{p}$, $(D - L)$ must be negative definite. Now, if D is negative semidefinite, and L is positive definite, then $(D - L)$ is negative definite. Thus, if there is a reason to believe that $\bar{p}$ is stable and that L is definite, then D is negative semidefinite. However, expecting stability does not imply that D is negative semidefinite, since for $h \geq 0$

$$h(D - L)h < 0 \not\Longrightarrow hDh \leq 0 \text{ and } hLh \geq 0 .$$

Finally, to test D for negative semidefiniteness, one investigates the signs of the principal minors of $(D + D')$. If these have the sign $(-1)^i$ for $i = 1,2,\ldots,n$, then $(D + D')$ is negative semidefinite and so is D. This can be shown as follows:

$h(D + D')h \leq 0 \Longrightarrow hDh + hD'h \leq 0$. Now, $hD'h = (hD'h)'$, since hD'h is a scalar. Hence,

$$hDh + (hD'h)' = 2hDh \leq 0 \Longrightarrow hDh \leq 0 .$$

We are now prepared to present the general form of the quadratic programming problem, called Problem I.1:

$$\text{to maximize } F_1(x,p,u) = d^o p + pDp - cx - bu \tag{2.22}$$

$$\text{subject to } Ax - b \leqq 0 \tag{2.23}$$

$$p - A'u - c \leqq 0 \tag{2.24}$$

$$d^o + Dp - x \leqq 0 \tag{2.25}$$

$$x,p,u \geqq 0$$

The resemblence to Problem 3 can be seen immediately, if we rewrite the system (2.23) through (2.25) as

$$\begin{bmatrix} 0 & -A' & I \\ A & 0 & 0 \\ -I & 0 & D \end{bmatrix} \begin{pmatrix} x \\ u \\ p \end{pmatrix} \leqq \begin{pmatrix} c \\ b \\ -d^o \end{pmatrix} \tag{2.26}$$

and realize that the coefficients of the linear part of (2.22) are the negatives of the right hand side of (2.26). We can also see now why assumption (b) is essential. It assures us of the concavity of (2.22), which, in turn, is required to assure that a maximum position will be a global one.

Our task now is to show that a solution to Problem I.1, if one exists, satisfies our conditions for efficiency and competitive equilibrium.

We first consider the dual to Problem I.1, which by Hanson's theorem is Problem I.2:

$$\text{to minimize } G(z,w,y,v) = -d^o y - zDz + cw + bv \quad (2.27)$$

$$\text{subject to } -Aw + b \geqq 0 \quad (2.28)$$

$$-y + A'v + c \geqq 0 \quad (2.29)$$

$$-d^o - Dz - D'(z - y) + w \geqq 0 \quad (2.30)$$

$$w,y,v \geqq 0$$

The vector z here represents that part of the vector u in Hanson's theorem, whose elements enter the dual objective function (2.15) with non-zero coefficients; v, w and y are the dual variables.

<u>Theorem 1</u> If Problem I.1 has a solution $(\bar{x}, \bar{p}, \bar{w})$, then $F_1(\bar{x}, \bar{p}, \bar{u}) = 0$

<u>Proof</u> First, rewrite (2.22) as

$$F_1(x,p,u) = -[(-d^o-Dp+x)'p] - [(-p+A'u+c)'x] - [(-Ax+b)'u] \quad (2.31)$$

Each of the three terms in (2.31) is an inner product of vectors which are restricted to be non-negative. Hence, over the feasible set for Problem I.1,

$$F_1(x,p,u) \leqq 0 \quad (2.32)$$

From the Kuhn-Tucker (22) necessary and sufficient conditions we have, if we let $(\bar{z},\bar{w},\bar{y},\bar{v})$ solve Problem I.2,

$$-[(-d^o-D\bar{p}+\bar{x})'\bar{y}] - [(-\bar{p}+A'\bar{u}+c)'\bar{w}] - [(-A\bar{x}+b)'\bar{v}] = 0, \quad (2.33)$$

since each of the three inner products of (2.33) vanishes. From (2.31), (2.32) and (2.33), $x = \bar{w}$, $p = \bar{y}$ and $u = \bar{v}$ is a solution for Problem I.1 if it is feasible. For feasibility, we must have

$$-A\bar{w}+b \geqq 0, \; -\bar{y}+A'\bar{v}+c \geqq 0 \text{ and } -d^o-D\bar{y}+\bar{w} \geqq 0.$$

But with $\bar{z} = \bar{p}$ by (2.17) and with $p = \bar{y}$, these are precisely the dual conditions, which are satisfied by $(\bar{w},\bar{y},\bar{v})$. Hence, $\bar{x} = \bar{w}$, $\bar{p} = \bar{y}$ and $\bar{u} = \bar{v}$, and $F_1(\bar{x}, \bar{p}, \bar{u}) = 0 \geqq F_1(x, p, u)$.

<u>Corollary</u> Whenever $\bar{p}_i > 0$, $x_i = d_i^o + \sum_{j=1}^{n} d_{ij} p_j$ i.e., $\bar{p}_i$ equates the supply and demand for the i-th commodity.

The corollary follows immediately from the first term in (2.33), since $\bar{y} = \bar{p}$.

<u>Theorem 2</u> If Problem I.1 has a solution $(\bar{x}, \bar{p}, \bar{u})$, then $\bar{x}$ is efficient i.e., is a boundary point of the set $X \equiv \{x\colon Ax \leqq b\}$

<u>Proof</u> Consider the linear programming problem

$$\begin{aligned} &\text{to maximize } F_1(x, \bar{p}, u) \\ &\text{subject to } \quad Ax \leqq b \\ &\qquad\qquad\quad -A'u \leqq -\bar{p} + c \\ &\qquad\qquad\quad -x \leqq -d_o - D\bar{p} \\ &\qquad\qquad\quad x, u \geqq 0 \end{aligned}$$

Clearly, this problem and Problem I.1 have the same solution in x and u. Since $\bar{p} > 0$ because of non-saturation, we can rewrite the linear problem using the corollary, as Problem I.3:

$$\text{to maximize } f_1(x,u) = (\bar{p} - c)x - bu \leqq F_1(\bar{x},\bar{p},\bar{u}) \qquad (2.34)$$

$$\text{subject to } \quad Ax \leqq b \qquad (2.35)$$

$$-A'u \leqq -\bar{p} + c \qquad (2.36)$$

$$x,u \geqq 0$$

Suppose now that $\bar{x}$ is not efficient i.e., $A\bar{x} < b$. Then from the third term in (2.33), since $\overline{v} = \overline{u}$,

$$\overline{u} = 0 \tag{2.37}$$

(2.36) and (2.37) imply $\overline{p} - c \leqq 0$. The case $\overline{p} - c = 0$ is trivial, since then $f_1(x,\overline{u}) = 0$ for all x. If $\overline{p} - c < 0$, then $\overline{x} \geq 0$ ($\overline{x} = 0$ is, again, trivial) implies

$$f_1(\overline{x},\overline{u}) = F_1(\overline{x},\overline{p},\overline{u}) < 0$$

which contradicts Theorem 1. Hence, $\overline{u} \geq 0$ and $\overline{x}$ is efficient.

Corollary The solution to Problem I.1 possesses the decentralization property, if $\overline{p}$ is universal.

This follows immediately from the fact that Problem I.3, whose solution in x is the same as for Problem I.1, is a self-dual linear programming problem. Its primal constituent is

$$\begin{aligned} &\text{to maximize } (\overline{p} - c)x \\ &\text{subject to } \quad Ax \leqq b \\ &\qquad\qquad\qquad\quad x \geqq 0, \end{aligned}$$

whose solution has been shown to satisfy decentralization.

The two theorems and their corollaries show, that the solution to problem I.1 satisfies all the desired conditions. That is, the solution is a competitive equilibrium. However, as has been already mentioned, the empirical results which can be obtained in the framework of Problem I.1 are of limited usefulness, for two reasons: first, there is, of course, the aggregation problem, which prevents us from formulating a representative technology matrix. Secondly, whether the model is applied to the entire economy or to an isolated sector, no information with regard to intersectoral allocation of resources can be obtained. The pattern of allocation is determined a priori, once b is determined, since no intersectoral movement of resources, such as labor, is allowed for in the model. The same is true with regard to information concerning firms: their optimal size and number.

For these reasons, it is desirable to reformulate the problem so as to make possible the derivation of results regarding interindustry and interfirm relationships. The next sections are devoted to this task. Models, having the same basic characteristics, will be constructed to allow flows of resources and/or final goods between any predetermined production units (firms, regions etc.).

C. Trade in Input Factors

In order to simplify terminology, I shall define a "producer" as a production unit representing a firm, industry, region or some aggregate thereof. This is done in order to avoid in each case the necessity of spelling out the exact variant of each model that can be applied to a particular production unit. In most cases it will be self evident and where not, I shall give the necessary specification.

We introduce a superscript k to discriminate between the producers, where always $k = 1,2,\ldots,K$. The notation introduced in Section A holds throughout, additional notation being defined when warranted.

We conceive of a market for the limited resources. Each potential producer, possessing no initial stock of these resources, can purchase non-negative amounts of each of the resources, the total purchases being limited by the availability of these resources. It is worthwhile to point out, that this device enables us to determine the optimal size of each producer, and the number of producers actually engaged in production. This is, particularly with respect to the agricultural sector, of utmost importance, as has been pointed out in Chapter I.

The following additional vectors are defined:

s^k = the purchases of the limited factors made by the k-th producer;

t^k = the given per unit transaction costs involved in these purchases;

u^k = the imputed factor prices of the k-th producer;

u = the "market" prices of the resources.

It should be noted, that even though u is obtained as an imputed price, it can be thought of as a market price, since it will be determined by the intersection of $\sum_k s^k$ and b, the demand and perfectly inelastic supply functions for the resources. t^k can accommodate a spatial setup, as it may include transportation costs from the market to the producer.

In addition to the above, we define the aggregates $x \equiv \sum_k x^k$ and $s = \sum_k s^k$. Also,

$$X \equiv (x^1, x^2, \ldots, x^K)'$$

$$U \equiv (u^1, u^2, \ldots, u^K)'$$

$$S \equiv (s^1, s^2, \ldots, s^K)'$$

Problem II.1 now is as follows:

maximize

$$F_2(X,p,U,u,S) = d^o p + pDp - \sum_k c^k x^k - bu - \sum_k t^k s^k \tag{2.38}$$

$$\text{subject to} \quad A^k x^k - s^k \leqq 0 \tag{2.39}$$

$$s \leqq b \tag{2.40}$$

$$p - A^{k\prime} u^k \leqq c^k \tag{2.41}$$

$$Dp - x \leqq - d^o \tag{2.42}$$

$$u^k - u \leqq t^k \tag{2.43}$$

$$X,p,U,u,S \geqq 0, \; k = 1, 2, \ldots, K.$$

The only restriction which deserves further interpretation, is (2.43). It can be recognized at once as the condition for equilibrium in trade. It requires that producer k purchase resources only up to the point where their value to him is equal to the costs of acquiring them.

A theorem equivalent to Theorem 1 could be proven here in exactly the same fashion. Instead, it would be useful to show, as a heuristic argument, that Problem II.1 has precisely the same structure as Problem I.1. Denoting by I_n and I_m

the n and m-dimensional identity matrices, and choosing for convenience K = 2, the coefficient matrix for Problem II.1 is

$$\begin{bmatrix} 0 & 0 & -A^{1'} & 0 & 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & -A^{2'} & 0 & 0 & 0 & I_n \\ A^1 & 0 & 0 & 0 & -I_m & 0 & 0 & 0 \\ 0 & A^2 & 0 & 0 & 0 & -I_m & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & 0 & I_m & I_m & 0 & 0 \\ -I_n & -I_n & 0 & 0 & 0 & 0 & 0 & D \end{bmatrix} \qquad (2.44)$$

It is immediately seen that (2.44) has exactly the same structure as the matrix in (2.26). This common structure seems to be a sufficient condition for the validity of Theorem 1. This can be seen when we recognize, that (2.31) is the corner stone of the proof of this theorem. For (abstracting again from any economic interpretation) consider the following problem:

$$\text{to maximize } (-b \;\; 0)\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + x^2Dx^2 \qquad (2.45)$$

$$\text{subject to } \begin{bmatrix} 0 & -A' \\ A & D \end{bmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} - \begin{pmatrix} -b \\ 0 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (2.46)$$

$$x \equiv (x^1 \; x^2)' \geqq 0$$

The problem has exactly the same general structure as Problem I.1. We now multiply (2.46) by x, an operation similar to the one which yielded (2.31). We get:

$$-x^{1'}A'x^2 - bx^1 + x^{2'}Ax^1 + x^2Dx^2 \geq 0$$

which is equivalent to (2.32), and requires (2.45) to be non-negative for all feasible x.

This demonstration of the connection between the structure of the problem and Theorem 1 may be taken formally as an equivalent to Lemma 1 in the case of quadratic programming. It makes clear that a theorem similar to Theorem 1 can be proved for Problem II.1. This being the case, we can conclude that a solution to Problem II.1 does satisfy conditions (c) and (d), and that part of condition (a) relative to the "economy" as a whole. We still have to show that the solution is decentralized and efficient.

We start out by assuming that Problem II.1 has a solution $(\bar{X}, \bar{p}, \bar{U}, \bar{u}, \bar{S})$ and that the competitively behaving producers face the prices $(\bar{p}, \bar{u})$. Then the profit maximization problem for the k-th producer is Problem II.2:

to maximize $g^k(x^k,u^k,s^k) = (\bar{p} - c^k)x^k - (\bar{u}+t^k)s^k$ (2.47)

subject to $A^kx^k - s^k \leqq 0$ (2.48)

$-A^{k'}u^k \leqq -(\bar{p} - c^k)$ (2.49)

$u^k \leqq (\bar{u} + t^k)$ (2.50)

$x^k, u^k, s^k \geqq 0$

for k = 1, 2,...,K. It is seen immediately, that II.2 is a self-dual linear programming problem to which Lemma 1 applies. Thus, if (x_o^k, u_o^k, s_o^k) solves it,

$$g^k(x_o^k, u_o^k, s_o^k) = 0 \qquad (2.51)$$

It is also obvious, because of the primal-dual structure of the problem, that the solution is efficient for each producer. Hence, the argument will be completed once we show $\sum_k x_o^k = \bar{x}$, $\sum_k s_o^k = \bar{s}$ and $U_o = \bar{U}$.

For this, consider the linear programming Problem II.3

$$\text{to maximize } F_2(X, \bar{p}, U, \bar{u}, S) \tag{2.52}$$

subject to (3.2), (3.3) and

$$-x \leqq - d^o - D\bar{p} \tag{2.53}$$

$$-A^{k'}u^k \leqq - (\bar{p} - c^k) \tag{2.54}$$

$$u^k \leqq (\bar{u} + t^k) \tag{2.55}$$

$$X, U, S \geqq 0, k = 1, 2, \ldots, K.$$

Clearly, Problems II.1 and II.3 have the same solution in (X, U, S) and hence, over the feasible set, (2.52) is zero at maximum.

<u>Theorem 3</u> If $(\bar{X}, \bar{p}, \bar{U}, \bar{u}, \bar{S})$ solves Problem II.1 and (x_o^k, u_o^k, s_o^k) solve Problem II.2 for k = 1, 2,...,K, then $\bar{X} = X_o$, $\bar{U} = U_o$ and $\bar{S} = S_o$.

<u>Proof</u> From the Kuhn-Tucker conditions and Theorem 1 (when properly applied to Problem II.1), we have for Problem II.3

$$(\bar{p}-A^{k'}\bar{u}^k-c^k)\bar{x}^k = (A^k\bar{x}^k-\bar{s}^k)\bar{u}^k = (\bar{u}^k-t^k-\bar{u})\bar{s}^k = 0 \tag{2.56}$$

The Kuhn-Tucker conditions for Problem II.2 read

$$(\bar{p}-A^{k'}u_o^k-c^k)x_o^k = (A^kx_o^k-s_o^k)u_o^k = (u_o^k-t^k-\bar{u})s_o^k = 0 \tag{2.57}$$

(2.56) and (2.57), being the same, and constituting, together with the relevant constraints, the necessary and sufficient conditions for Problems II.2 and II.3, lead to the required result, since Problem I.1 has the same solution in (X, U, S) as Problem II.3.

This completes the demonstration that the solution to Problem II.1 will comply with our imposed conditions. Note also, that the third inner product in (2.56) implies that if $s^k > 0$, $\bar{u}^k - \bar{u} = t^k$ and when $\bar{u}^k - \bar{u} < t^k$, $s^k = 0$. This means, that the condition for spatial equilibrium is also satisfied.

Clearly, Model II.1 does present a formulation which is applicable to the problems of intersectoral and interregional allocation. Unlike Problem 3 above, a solution to Problem II.1 will determine the efficiency frontier for each producer and guarantee an efficient production pattern. That is, we have here a way of solving not only for the optimal allocation of resources among products, but also among production units. Considering the U.S. agricultural sector, we have a way to determine not only the optimal commitment of resources (particularly labor) to the sector as a whole, but also the optimal size of farms under competitive equilibrium. As I have pointed out in Chapter I, these are, indeed, the basic problems facing American agriculture. In addition, in a situation where maintaining the family farm structure is deemed desirable,[11] it will be possible to determine whether

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[11] See (1, p. 12).

or not such an aspiration can be defended on economic grounds.

It is to be noted, that the formulation of Problem II.1 is flexible, in the sense that it can allow for the inclusion of immobile resources, and for initial quantities of mobile resources at the disposal of producers. In some cases, where spatial effects are important, such initial possessions may render the model more realistic. A less atomistic breakdown would be to conceive of a number of spatially separated markets for resources, a number smaller than the number of potential producers. This would be a typical regional setup.

Considering the case where each potential producer has an initial quantity of mobile resources, and letting $K = 2$, denote by s^{12} resource shipments from the first to the second producer and by s^{21} the reverse shipments. Assuming that per unit transportation rates are the same in both directions, and denoting these by t, Problem II.1 is reformulated as

to maximize $d^o p + pDp - c^1x^1 - c^2x^2 - b^1u^1 - b^2u^2 - t(s^{12}+s^{21})$

subject to

$$\begin{bmatrix} 0 & 0 & -A^{1\prime} & 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & -A^{2\prime} & 0 & 0 & I_n \\ A^1 & 0 & 0 & 0 & I_m & -I_m & 0 \\ 0 & A^2 & 0 & 0 & -I_m & I_m & 0 \\ 0 & 0 & -I_m & I_m & 0 & 0 & 0 \\ 0 & 0 & I_m & -I_m & 0 & 0 & 0 \\ -I_n & -I_n & 0 & 0 & 0 & 0 & D \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ u^1 \\ u^2 \\ s^{12} \\ s^{21} \\ p \end{bmatrix} \leqq \begin{bmatrix} c^1 \\ c^2 \\ b^1 \\ b^2 \\ t \\ t \\ -d^o \end{bmatrix}$$

This reformulation has exactly the same structure as II.1, and it can be verified that all that has been said with regard to II.1 applies here as well. The advantage of II.1 over the reformulation, concerns the size of the problem. The problems are of equal size only if K = 2. Otherwise, II.1 is smaller, by K - 2. (By size is meant the number of constraints, which is the same as the number of variables.)

D. Further Extensions

This section is devoted to a brief presentation of more applicable variations of Model I.1. In each case I shall present the model, point out its main advantages, assuming always that, since the structure of all these models is similar to I.1, their solutions always satisfy the conditions for efficiency and competitive equilibrium. That this is indeed the case, can be verified in each case by proving theorems equivalent to Theorems 1, 2 and 3.

We start out by introducing a spatial element to the commodity market. We conceive of L spatially separated markets for final goods, the demand system in each of which is given by

$$d = d_o^l + D^l p^l \qquad l = 1, 2, \ldots, L \tag{2.58}$$

where d_o^l is a vector of constants. No special restrictions must be imposed on D^l, since it can be thought of as given by

$$D^l = a_l D \qquad a_l \in (0,1), \quad \sum_l a_l = 1 \tag{2.59}$$

where a_l is a scalar. That is, we can think of D^l as obtained from D on a population basis. In other words, a_l is that portion of the total population which constitutes the l-th market. Then, since D is negative semidefinite, so is D^l. We assume, to avoid the necessity of using too many superscripts, that in each consumption region there are K producers. The transportation costs within each consumption region are neglected, primarily for convenience. s^{jl} and t^{jl} denote the shipments and per unit transportation costs from consumption region j to region l, respectively. We assume $t^{jl} = t^{lj}$ for l, j = 1, 2,...,L. Problem III.1 then is

$$\text{to maximize } \sum_l [(d_o^l + p^l D^l)p^l - \sum_k c^{kl} x^{kl} - \sum_k b^{kl} u^{kl}] - \sum_{l \neq j}\sum t^{jl} s^{jl} \tag{2.60}$$

$$\text{subject to } A^{kl} x^{kl} \leqq b^{kl} \tag{2.61}$$

$$p^l - A^{kl\prime} u^{kl} \leqq c^{kl} \tag{2.62}$$

$$D^l p^l + \sum_{l \neq j} (s^{lj} - s^{jl}) - \sum_k x^{kl} \leqq - d_o^l \tag{2.63}$$

$$p^l - p^j \leqq t^{jl} \tag{2.64}$$

$$p^j - p^l \leqq t^{lj} = t^{jl} \tag{2.65}$$

$$x^{kl}, p^l, u^{kl}, s^{jl} \geqq 0, \; k = 1, 2, \ldots, K. \tag{2.66}$$

It is worthwhile to point out, that (2.64) and (2.65) do not contradict each other. That is, the intersection of the sets defined by (2.64) and (2.65) is not empty. Fig. 7 depicts this intersection for L = 2, when restricted to the

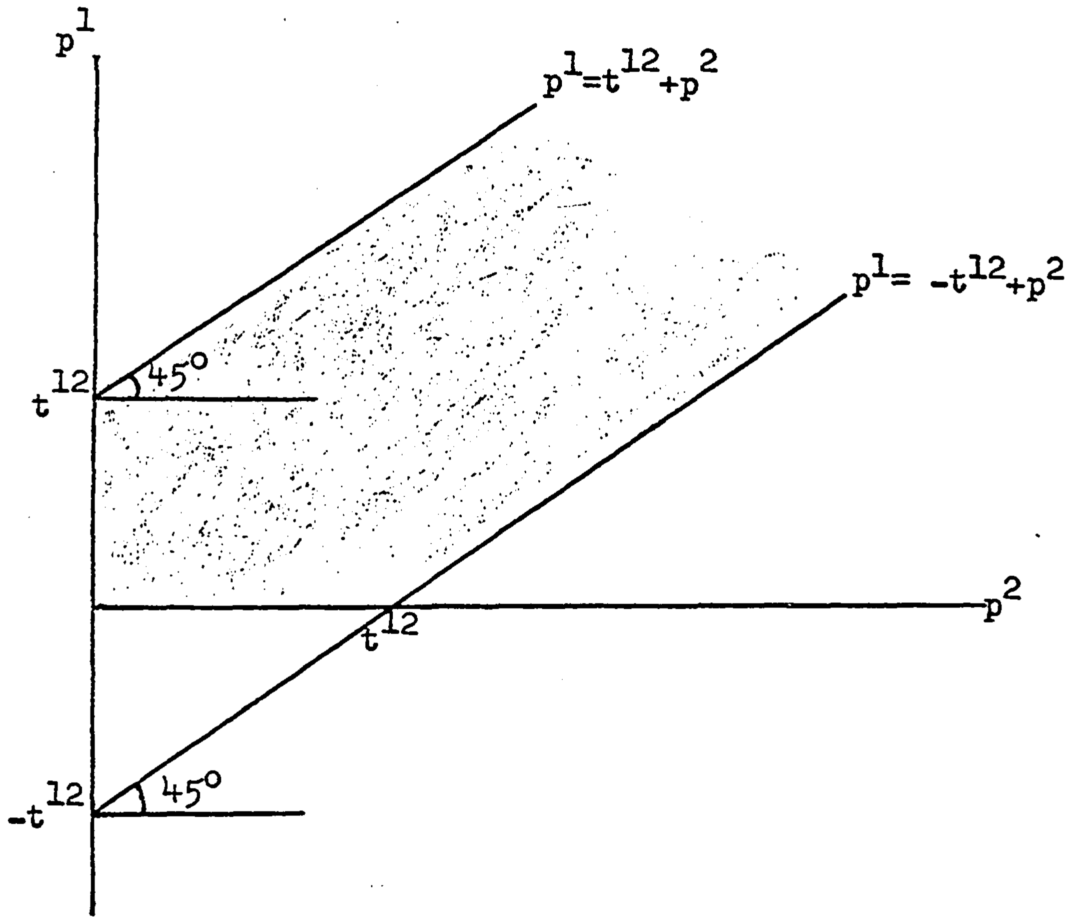

Fig. 7. The transportation restrictions

positive orthant, given by the shaded area. When $s^{12} > 0$, the point (p_1, p_2) will be somewhere along the line $p^1 = -t^{12} + p^2$ and if $s^{21} > 0$, along the line $p^1 = t^{12} + p^2$. If $s^{12} = s^{21} = 0$, (p_1, p_2) will be somewhere between these lines.

It should be understood, that a spatial setup of resource markets could be incorporated as well. I have not done so only for convenience. That is, Problems II.1 and III.1 can be combined into one problem. When this combination is used, the solution would indicate the following optimal patterns:

1. the allocation of resources among producers, and hence the optimal number of producers and their size;
2. the optimal rates of outputs of each commodity by each producer;
3. the optimal trade pattern in both resources and final goods.

It should be pointed out, that III.1 is, in a sense, a wasteful formulation. This results from the fact, that at the most half of the s^{jl} will assume non-zero values. This also means, that at the most half of the restrictions in (2.64) and (2.65) will become effective. I have not found a way to improve on this point, primarily because of the restrictions (2.66). Otherwise, one could define $s^{jl} = -s^{lj}$ and cut the restrictions by half.

We are now ready to dispense with assumption (a), postulated in Section B. It will be recalled, that this assumption stipulated that prices of "variable" inputs are fixed and given. I have also mentioned, that convenience was the main underlying reason. However, regarding empirical studies, we find almost invariably that the assumption is also made because of lack of reliable estimates of supply functions for such variable inputs. The following model is suggested for cases where such estimates are available.

We assume that the supply structure for the said inputs is given by

$$r = r^{o} + Rz \tag{2.67}$$

where r is the vector of supplied quantities of the Q variable inputs, r^{o} is a vector of constants, z is the vector of prices and R is a positive semidefinite matrix. The restriction on R is exactly equivalent to that on D. Thus, for instance, R must have positive diagonal elements.

We expand the technology matrix A, by introducing a matrix B whose elements are the input-output coefficients relating x to r. Problem III.2 then is

$$\text{to maximize } d^{o}p + pDp - r^{o}z - zRz - bu \tag{2.68}$$

$$\text{subject to } Ax \leqq b \tag{2.69}$$

$$Bx - Rz \leqq r^{o} \tag{2.70}$$

$$p - A'u - B'z \leqq 0 \tag{2.71}$$

$$Dp - x \leqq - d^{o} \tag{2.72}$$

$$x, p, u, z \geqq 0$$

Since both D and (-R) are negative semidefinite, (4.11) is a concave function. The coefficient matrix

$$\begin{bmatrix} 0 & -A' & -B' & I \\ A & 0 & 0 & 0 \\ B & 0 & -R & 0 \\ -I & 0 & 0 & D \end{bmatrix}$$

of the constraint set for Problem III.2, is seen to have, indeed, the same structure as in the previous cases. Model III.2 can be used in any combination with Models II.1 and III.1. Moreover, one could also introduce a spatial setup in the variable inputs market. When it is realized that (2.70) stands in the same relation to the input market as does (2.72) to the output market, we have here a model which will determine also the input market equilibrium prices. That is, the model allows for all possible variations, except for variations in the technical magnitudes. The model is, indeed, very general, a fact which can be recognized more fully by realizing that it is, in fact, a reformulation of the Walras-Cassel general equilibrium model.[12] It is a reformulation in two respects: first, it is explicitly presented as a constrained maximum problem. Secondly, it

[12] See, for instance, Dorfman, Samuelson and Solow (13, pp. 352-353). The systems (13-1), (13-2), (13-3) and (13-4) there, are equivalent to our (2.70), (2.18), (2.71) and (2.67), respectively.

specifies the functional form of the demand and supply relations. This specification is actually something more than formal, since it also specifies

$$\frac{\partial d}{\partial z} = \frac{\partial r}{\partial p} = 0$$

Another step in the direction of a completely generalized competitive equilibrium model is the introduction of intermediate commodities. That is, we may have a situation, where some, or all, of the inputs which in III.2 are supposed to be purchased, are actually produced within the "economy" under consideration.

To facilitate such a situation, we redefine our commodity space. Let a point in the commodity space be denoted by y. Then y can be partitioned as

$$y \equiv \begin{pmatrix} y^{d} \\ y^{int} \\ y^{pr} \end{pmatrix} \quad y^{d} \leq 0, \; y^{pr} \geq 0, \; y^{int} \gtrless 0 \qquad (2.73)$$

where y^{d} is a point in the subspace of desired commodities (final goods), y^{int} a point in the subspace of intermediate goods and y^{pr} a point in the subspace of primary commodities (resources). An activity is now defined as a vector in the commodity space, describing a production process. Such a vector has negative entries for commodities emerging as outputs from the process, positive entries for commodities going as inputs into the process, and zero entries for commodities not involved in the process. When A is so redefined, we can partition it in accordance with (2.73) to obtain

$$A \equiv \begin{bmatrix} A_d \\ A_{int} \\ A_{pr} \end{bmatrix} \tag{2.74}$$

where it is clear that $A_d \leq 0$, $A_{pr} \geq 0$. We also denote by w the imputed prices of the intermediate commodities; x denotes the level of use of the activities.

With these notations, Problem III.3 is formulated as follows:

$$\text{to maximize } G(x, p, u, w) = d^o p + pDp - bu \tag{2.75}$$

$$\text{subject to } \quad A_d x + Dp \leqq -d^o \tag{2.76}$$

$$A_{int}\, x \leqq 0 \tag{2.77}$$

$$A_{pr}\, x \leqq b \tag{2.78}$$

$$-A_d' p - A_{int}' w - A_{pr}' u \leqq 0 \tag{2.79}$$

$$x, p, u, w \geqq 0$$

Model III.3 can actually be looked upon as combining Koopmans' (7) linear production model with Walras' conception of the market, in a quadratic programming formulation. It should then suffice to remark, that III.3 can be combined with II.1, III.1 and III.2, in order to bring to bare the extent of the generality of the model.

I should like to demonstrate the fact that (2.75) is, indeed, the profit function, even though it does not contain the costs of non-primary inputs. First, we have again the formalistic assurance that Theorems 1 and 2 are satisfied by III.3, when we look at the coefficient matrix of the constraint set

$$\begin{bmatrix} 0 & -A'_{pr} & -A'_{int} & -A'_{d} \\ A_{pr} & 0 & 0 & 0 \\ A_{int} & 0 & 0 & 0 \\ A_{d} & 0 & 0 & D \end{bmatrix}, \tag{2.80}$$

which is of the same familiar form. Thus, if $(\bar{x}, \bar{p}, \bar{u}, \bar{w})$ solves Problem III.3,

$$G(\bar{x}, \bar{p}, \bar{u}, \bar{w}) = 0 \tag{2.81}$$

Next, we have, using (2.73), (2.74), and (2.76) to (2.78),

$$\begin{pmatrix} \bar{y}^{d} \\ \bar{y}^{int} \\ \bar{y}^{pr} \end{pmatrix} = \begin{bmatrix} A^{d} \\ A^{int} \\ A^{pr} \end{bmatrix} \bar{x} \leqq \begin{pmatrix} -d^{o}-D\bar{p} \\ 0 \\ b \end{pmatrix} \tag{2.82}$$

From Koopmans (7, pp. 63, 82, 89) we know that part of the necessary and sufficient conditions for efficient production is

$$(\bar{p}\ \bar{w}\ \bar{u}) \begin{pmatrix} \bar{y}^{d} \\ \bar{y}^{int} \\ \bar{y}^{pr} \end{pmatrix} = 0 \tag{2.83}$$

To avoid the necessity of introducing additional notation,[13]

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[13] See Koopmans (7, p. 82).

suppose that in (2.82) strict equality holds. Then from (2.82) and (2.83),

$$-d^o\overline{p}\ -\overline{p}D\overline{p} + \overline{wy}^{int} + \overline{u}b = 0 \tag{2.84}$$

But (2.81) and (2.84) imply

$$\overline{wy}^{int} = 0 \tag{2.85}$$

which means, that the net costs of intermediate commodities are zero, hence (2.75) is, indeed, the profit function. (2.76) is, again, the requirement for non-positive excess demand, (2.79) requires that no activity yield positive profits, and (2.77) requires that no intermediate commodity will have a net negative output. Again, (2.79) constitutes part of the necessary and sufficient conditions for efficient production.[14] It can be verified that the rest of these conditions are also satisfied.

E. Conclusion

As stated, the objective of this chapter was to devise tools which will enable us to follow the recommendations of Chapter I. That is, we had to provide the necessary means to carry out the process of formulating a choice. This objective was achieved for at least certain situations, namely those which meet the assumptions underlying the above models. True, these assumptions impose restrictions which, at least

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[14] See Koopmans, op. cit.

for the producing sector of the economy, are economically meaningful. Yet it seems to me, that for all practical purposes, these restrictions are not hindering. This is to say, such restrictions will not lead to "wrong" choices. Even if the economy under consideration does not meet exactly the assumptions of activity analysis and linear demand structure, the quantitative errors will not lead to qualitative changes. This is particularly true when one takes into account the imperfections of the statistical methods, usually used to estimate the necessary parameters and technical coefficients.

Apart from the more limited task of providing a feasible framework for empirical studies, it is very likely that the results of this chapter provide the opportunity for proving some competitive equilibrium existence theorems. Even though these would not be as general as some of the heretofore established theorems[15], they would have the advantage of being sensitive to empirical verification. Such existence theorems would be derived from the conditions under which the various models have a solution. These conditions are that the constraint set in each case be non-empty convex and compact. One would then have to investigate the economic

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[15]See, cf. Arrow and Debreu (23), Debreu (24, pp. 83-88) and Gale (25).

implications (if any) of these conditions. However, this falls beyond the scope of this study.

III. APPLICATION

A. The General Setting

The application as carried out for the purpose of this study, should not be considered as more than a first step illustrative example, convincing though it may be. As in most other instances, beginnings are modest, and ours is no exception. The restricted scope of our application is due largely to the fact that computational techniques and facilities, as well as the availability of data, are still very much a restricting factor. It might be added, however, that more elaborate applications will be possible in the not too distant future.

Keeping the notation of Chapter II, the model applied is the simplest possible, namely

$$\begin{aligned} &\text{to maximize } f(x,p,u) = -cx - bu + dp + pDp \\ &\text{subject to } -A'u + p \leqq c \\ &\qquad Ax \leqq b \\ &\qquad -x + Dp \leqq -d \\ &\qquad x,\ p,\ u \geqq 0. \end{aligned}$$

The "economy" considered is a sub-sector of agriculture, consisting of the following products:

1. Wheat, corn, oats and barley for food.

2. Feed grains, consisting of a mixture of corn, oats, barley and grain sorghum.

3. Soybeans and cottonseed oil meals.

4. Cotton lint.

The production set consisted, thus, of seven production activities: wheat, corn oats, barley, feed grains, soybeans and cotton.

This composition of the production set already presents a major difficulty, in that none of the products considered can be regarded as being strictly a final commodity. In particular, feed grains and oil meals are typical intermediate commodities, no portion of which is used for final consumption. The reason why this presents a difficulty is, that usually no demand functions for intermediate commodities are estimated, especially if such goods are durable, as are feed grains and oil meals. At best, estimations for intermediate demand functions are derived from the relevant final demands by employing some assumptions. Estimates obtained in this manner are not only questionable from the statistical point of view (biasedness, consistency, etc.) but, because of the assumptions employed, tend to affect the so derived price elasticities by an unknown percentage. For instance, a demand function for feed grains may be derived from demand functions for livestock products. However, to accomplish this, one would have to assume pre-determined fixed proportions between the various feeds, which will, no doubt, affect the obtained estimates for slope or elasticity coefficients.

Such difficulties do not arise if one is allowed to place products such as feed grains in their proper category. To do this, one would have to include all the livestock products and all types of feed in the model, using demand functions for the livestock products. The production and utilization of the intermediate goods, as well as their prices, will then be determined "within" the model, since the demand for these commodities will be generated on the basis of what is called for to facilitate the production of the final products.

Another difficulty arising from the treatment of intermediates as final goods occurs in cases of joint production. This can be illustrated by the example of cotton. Two of the products forthcoming from cotton are cotton lint and cottonseed oil meals. Demand functions will be used, say, for lint and oil meals, supply of oil meals coming also from soybeans. The ratio between lint and cottonseed oil meals is fixed. Then, depending on the coefficients of the demand functions, a situation may arise in which no combination of cotton and soybeans production exists which will equate the supply and demand for cotton lint and oilmeals simultaneously. As in the previous instance, had we treated oilmeals as an intermediate commodity, the difficulty could not have arisen.

The correct classification of the commodities involved was not possible for two reasons. First, a substantial amount of refinement in the data related to the livestock industry is needed before such data can be included in a sensitive model as ours. A tremendous effort concerning the livestock industry by Brokken (11), does provide a solid basis on which to proceed, but there is still a lot to be desired. Second, the inclusion of livestock products in the model at this stage, would have increased the size of the problem (in terms of the number of variables) beyond what can be presently handled by available computer programs which, as pointed out before, are still a limiting factor.

I have thus far discussed what has not been done and the underlying reasons. It is worthwhile to stress, however, that what has been done was not merely a forced choice. There are very good reasons for choosing the above crops even if there were no limiting factors. First of all, the crops chosen occupy the bulk of the intensively cultivated acreage which is devoted to field crops. That is, a sizeable portion of the agricultural economy is represented by the crops chosen. Secondly, part of the products included, are the ones which are considered to constitute a considerable part of what is referred to as the "farm problem". By that is meant primarily wheat, feed grains and cotton, which are produced in great "surpluses". In view of our analysis in the first chapter, it is of major interest to find out what would happen to these crops under competitive equilibrium.

That is, what would be the prices if the market were to be cleared. Third, there is the consideration of the reliability of the data. As pointed out in Chapter II, some very substantial studies were carried out, which treated the same group of commodities. Each of these studies, beginning with Heady and Egbert (8), involved a careful and very sophisticated assemblence of data. The result is a collection of data which probably presents the best that can be done in gathering information. Applying our type of model to such refined data is very likely to produce reasonable results, as, indeed, is the case.

These, then, are the positive and negative reasons which dictated the particular application attempted in this study.

B. The Data

Two are the major sources of data, both presenting them in a very detailed fashion. Hence, only a few comments will be needed here, the interested reader being referred to the sources.

On the supply side, the data used in this study are based solely on Skold (9) and Whittlesey (10). Both these works contain a collection of input-output and cost coefficients, land constraints and rotational weights for 144 regions of the U.S., projected to 1975 and 1965, respectively.

The data used here are 1965 projections, and the reference to Skold is made because some of the data are listed in more detail in his study.[1] The projections were made on the basis of time-argumented linear regressions. The sources of the crude data are listed in the bibliography list of the two mentioned studies, and I find it therefore superfluous to repeat them here. The rotational weights refer to feed grains. That is, it is assumed that out of every acre devoted to feed grains in a given region, a certain predetermined percentage will be under each of the four components of feed grains, including the possibility of 0% for some of them. Another way to do it would be to formulate explicitly constraints on rotation. This, however, would have increased substantially the size of the problems, and hence was not tried. For the purpose of our study, there is no meaning to talking about rotation restrictions on a non-regional basis, hence a predetermined composition of feed grains was assumed.

The actual input-output data for the present study were obtained by aggregating the regional data. Admittedly, the aggregation problem is one of the toughest in applied economics. It is not a theoretical problem, at least not within the realm of activity analysis. Theoretically, the

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[1] See Skold (9), Tables 12, 13, 14, 16, 65, 68 and 71.

aggregate production possibility set is the vector sum of the individual production possibility sets.[2] There is no way yet, however, to practically compute such an aggregated opportunity set. Any individual set, can be represented (spanned) by a finite number of meaningful activities (e.g. wheat, corn etc.). What is involved in aggregation would be to find the same type of meaningful activities spanning the aggregate possible cone. Specifically, given n possible cones representing n producers, each of these cones being spanned, say, by two activities labled "wheat" and "corn", what are the "wheat" and "corn" activities spanning the aggregate possible cone? No practical answer is available, and the only way open to the researcher at this point, is the use of common sense i.e., the exercise of prejudice.

This is, to a great extent, the procedure used here. The regional input-output coefficients were aggregated by way of using acreages as weights. These weights were a mixture of available regional acreages and the regional planted acreages as given in the various solutions of Skold. The resulting coefficients were then checked by comparing the total productive capacity under these coefficients with that under the regional coefficients.

A few words should be said concerning the place of labor

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[2] See Koopmans (6, pp. 9-10).

in the model. As pointed out in the first chapter, the portion of the labor force allocated to agriculture is probably one of the major, if not the major, factor in the U.S. farm problem. Coping with this problem would, therefore, require a very careful and explicit treatment of labor. What would be called for, is an explicit formulation of labor constraints and labor input-output coefficients, thereby facilitating the determination of the total labor power required to produce the emerging bill of goods, and the resulting reward to labor. The lack of this feature in the present study is one of the prime reasons for cautioning against regarding it as much more than an example.

As in other instances, the reason for omitting labor as an explicit factor in the model is the lack of data. As is well known, the gathering of reliable data concerning availability of labor is extremely difficult, particularly in agriculture, where a considerable portion of the labor force is not occupied consistently in farming. Some labor is hired periodically, especially in the major harvesting periods. Even some of the people living on farms engage in part time work outside their farms, either in other farms or in non-agricultural occupations. Some of the labor is supplied by part time work of farm youth, and the "availability" of such labor cannot virtually be determined. The task of gathering information on labor availability in the farm sector, is therefore a study in its own right, and could not be attempted for the purpose of our application.

The data for the consumer side of the market are taken almost entirely from Brandow (26). Only a few figures concerning foreign trade in livestock products were taken from statistical reports of the U.S.D.A. (27, 28). The basic procedure was as follows: the farm level linear demand functions, derived from the estimates of logarithmic-linear retail level demand functions, were taken as a basis.[3] A set of prices was then assumed for all the relevant commodities not in the model. These prices, assumed to remain at their 1955-57 means, were inserted into the included demand functions, thereby adjusting the intercepts (constant terms). The demand functions were then adjusted to reflect the 1965 per capita income, and population. The adjustment to per capita income was done by giving the appropriate values to the trend variables, whose coefficients were estimated by Brandow. Figures for the 1955-57 population were obtained from the U.S. Statistical Abstract (29). The latest population estimates available were then taken from Current Population Reports (30). The average annual population increase was then assumed to remain unchanged and used to project the 1965 population, a projection which yielded

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[3]The process of derivation is described in Brandow.

approximately 193,690,000. The ratio of this projection to the 1955-57 average was used to adjust the demand functions. Finally, demands were adjusted to foreign trade. The export figures for grains were taken from Skold, while the foreign trade data for livestock products were obtained from the above mentioned statistical reports (27, 28). These were used to adjust the relevant constant terms of the demand functions. The major assumption here is, that average grain exports of recent years (including those under Public Law 480) will remain unchanged in 1965.

A few words ought to be said with regard to the demand functions for feed grains and oil meals. It is, so it seems to me, very reasonable to believe, that in the short run, say within a time span of a year, the quantity demanded of feed grains is virtually independent of their price. The reason is, that any appreciable adjustments in the purchase of feed grains can be brought about only by adjustments in the number of livestock fed, and these are unlikely to take place in the short run. However, even if the above does not hold, there is certainly a considerable gap between short run and long run elasticities. Such differences exist in general, for any good, but are undoubtedly larger in cases such as ours.[4] Which demand function should one choose?

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[4] See e.g. Stigler (31, pp. 45-46).

As stated before, such a problem would not have occurred had we treated feed grains and oil meals as intermediate commodities. However, in our case a choice had to be made. The demand functions for feed grains and oil meals are based on those given in Table 16 of Brandow.[5] These are supposed to represent the long-run elasticities when prices are expressed as price indeces. These estimates were used only as a guideline. They could not be used without impairing the consistency of the model. The reason is, that the demand functions for livestock products are not included in the study. This being the case, prices of livestock products had to be assumed in advance as given, in order to adjust the demand functions for consumer products that are included in the model. But once the prices of livestock products are considered as fixed, so are the demanded quantities of these products. Hence, one cannot use, under these circumstances, long run demand functions for feeds. The result is, that we were virtually forced to use arbitrary coefficients for the demand functions for feed grains and oil meals. There is not much to be said in defense of the particular coefficients chosen from the theoretical standpoint. However, some consolation may be found in the fact that changing these coefficients over a considerable range would not have altered

[5]How they were derived is explained in Brandow, pp. 75-79.

the equilibrium prices. The reason for this is demonstrated in Fig. 8.

Let Q, P and C denote the quantity, price and costs of feed grains, respectively. Let S be the supply function, drawn after solutions for all other commodities in the model have been obtained and are kept constant. Then q_{max} is the maximum quantity that can be produced after the required land has been allocated to all other commodities. Let D, D' and D" be three demand functions for feed grains, having a common intercept d and different slopes. As can be seen from the diagram, the price per unit of feed grains under equilibrium will be equal to the costs per unit unless the demand function intersects the supply function above the line segment $P = C$. This occurs in the case of D". Then $P = C + U$, where U is the imputed value of a unit of land. That is, $P > C$ only if $U > 0$, which can happen only if the available land is exhausted.

Suppose now that D' represents the function actually used in the model. Then the question is whether or not, by changing the coefficients, we could have reached a situation like the one represented by D". That is: by how much do we have to decrease the slope of D', in order to bring about the exhaustion of the land available to feed grains?

The answer is that no such negative slope exists. The reason is, that under the actual solution, $d < q_{max}$. The

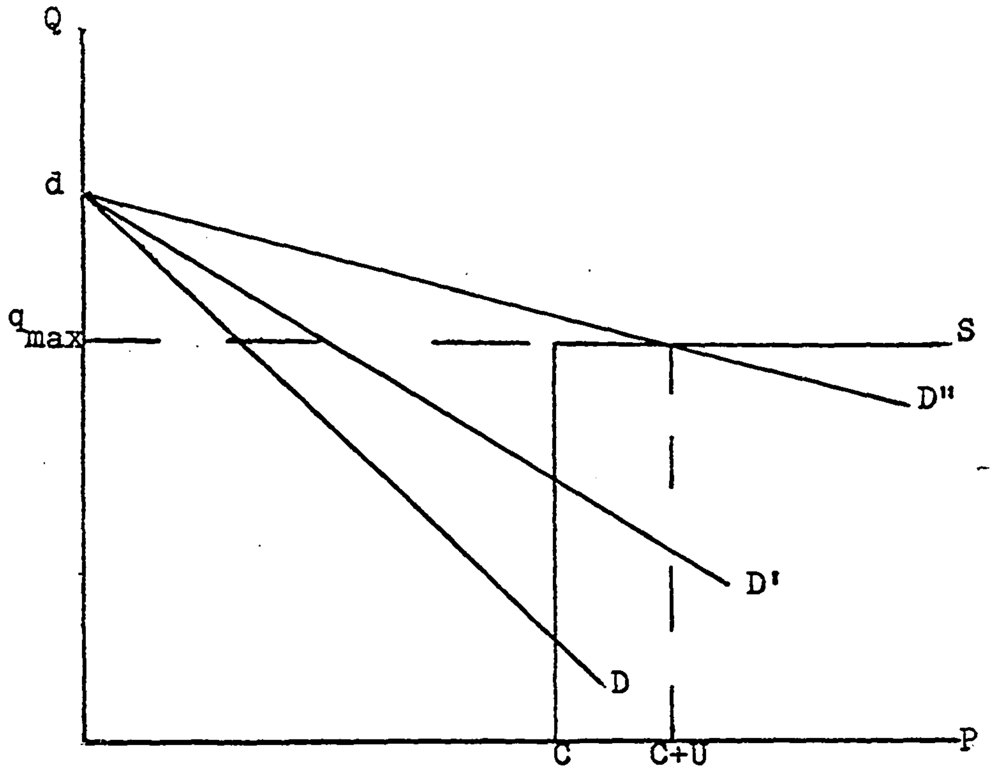

Fig. 8. Illustration of demand and supply functions

opposite possibility, of increasing the slope of D', does not affect the price as long as there exists a point of intersection between the supply and demand curves. An increase large enough to prevent such intersection, is clearly of no economic interest. The net result is, that we could have let the slope of D' vary from 0 to any reasonably large slope (say -2) without affecting equilibrium prices. The same holds true with regard to oil meals.

Table 1 displays the simplex tableau in its entirety. x_j for j = 1, 2,...,7 are production activities; u_j for j = 1, 2, 3 are imputed values in dollars per acre; p_j for j = 1, 2,...,7 are prices in dollars per ton. As for the column P_o (usually referred to as "the right hand side"), c_i for i = 1, 2,...,7 are production costs in dollars per ton; b_i for i = 1, 2, 3 are the different types of available land in tens of thousands of acres; d_i for i = 1, 2,...,7 are the intercepts of the demand functions, in tens of thousands of tons. The land input coefficients are in acres per ton. As can be seen, the ratio of cottonseed oil meals to cotton lint is taken to be .777 ton per ton. The elements of the lower right diagonal block are the slope coefficients of the demand functions. There are three groups distinguishable in that block:

1. food grains;
2. feed grains and oil meals;

Table 1. The simplex tableau

Item		P_o	Wheat x_1	Corn x_2	Oats x_3	Barley x_4	FG[a] x_5	Sb[b] x_6	Cotton x_7	Total land u_1	Sb land u_2
Wheat	c_1	26.53								-1.33	
Corn	c_2	23.57								- .558	
Oats	c_3	18.47								-1.39	
Barley	c_4	23.73								-1.21	
FG	c_5	25.83								- .785	
Sb	c_6	38.42								-1.548	-1.548
Cotton	c_7	363								-3.8	
Total land	b_1	24546	1.33	.558	1.39	1.21	.785	1.548	3.8		
Sb land	b_2	6634.9						1.548			
Cotton land	b_3	1689.6							3.8		
Wheat	$-d_1$	-2859.2	-1								
Corn	$-d_2$	-867.61		-1							
Oats	$-d_3$	-132.32			-1						
Barley	$-d_4$	-18.167				-1					
FG	$-d_5$	-14000					-1				
OM	$-d_6$	-1600						-1	- .777		
Cotton	$-d_7$	-463.66							-1		

[a] Feed grains.

[b] Soybeans.

Table 1. (Continued)

Item		Cotton land u_3	Wheat p_1	Corn p_2	Oats p_3	Barley p_4	FG p_5	OM[c] p_6	Cotton p_7
Wheat	c_1		1						
Corn	c_2			1					
Oats	c_3				1				
Barley	c_4					1			
FG	c_5						1		
Sb	c_6							1	
Cotton	c_7	-3.8						.777	1
Total land	b_1								
Sb land	b_2								
Cotton land	b_3								
Wheat	$-d_1$		-.6077	.05313	.00864	.0013225			
Corn	$-d_2$		.0532	-.51898	.004113				
Oats	$-d_3$		.008632	.0040923	-.1666				
Barley	$-d_4$		.0013953			-.018754			
FG	$-d_5$						-.233	.052	
OM	$-d_6$						.016	-.1	
Cotton	$-d_7$								-.20969

[c]Oil meals.

3. cotton lint.

The cross price elasticities between these three groups are assumed to be zero. The zero elements within the first group are not zero by assumption, but the coefficients which would have appeared there were so small that they were ignored.

Any and all additional explanations regarding the data can be found in the sources from which they were derived, and we shall therefore proceed to present and discuss the solution.

C. The Solution

In order to solve the quadratic programming problem, a computer program based on the Hartley-Hocking (32) algorithm for convex programming was used. The algorithm itself is based on successive implicit linear approximations to the non-linear functions in the system. Each non-linear function is approximated by a sequence of linear segments, thus converting the convex into a linear programming problem. The key feature of the algorithm is, that the linear approximations are not actually computed in advance. Instead, the "best" (most profitable) approximation is automatically chosen at each iteration and, using the product form of the inverse, computed for only one function at a time.

I would like to make clear at the outset, that the fact that the problem as presented had a solution at all, is merely a coincidence. As mentioned in the first section of this chapter, the equations relating to oil meals and cotton lint need not have a solution at all. They will not have a solution as soon as the slope coefficients in the demand function for oil meals are sufficiently changed. An infeasibility of this nature can be also brought about by changing c_6 and c_7 (Table 1), d_6 and d_7, or the slope of the demand function for cotton lint. All this is pointed out in order to make clear that the existence of a solution in our particular case, cannot be regarded as a proof for the validity of the formulation given in Table 1.

In Table 2, all the information derivable from the solution is presented. There is not much to be said about the quantities produced. Since the price elasticities for all products are small around the mean values, it could be expected that the x_j will be more or less close to the corresponding d_i, which is, indeed, what happened. There is more to be said about the prices.

As could be expected, the prices in the optimal solution are in most cases considerably lower than current market prices. In fact, since the values of land in all three land categories are zero, the demand functions for all the products intersect the corresponding supply functions at a point where P = C (see Fig. 8). Hence, unless the cost figures contain an "adequate" return for labor, competitive

Table 2. The solution

	Land use 000 acres	Production unit	Production 000 units	Prices $/unit
Wheat	37832.1	bu	948163.9	.796
Corn	4781.3	bu	306019.0	.660
Oats	1801.0	bu	80981.3	.230
Barley	214.9	bu	7399.9	.570
FG	109868.4	ton	139959.8	25.830
Sb	20053.6	---	---	---
Cotton	14726.5	ton	3875.4	363.000
OM	---	ton	15965.7	38.420
Unused land:				
total	56182.2	---		
Sb	46295.4	---		
Cotton	2169.5	---		

production under the current conditions in agriculture cannot provide the farmer with a satisfactory level of income. This is not surprising at all, and was pointed out in the first chapter as a distinct possibility. However, the results are still to be treated with caution, because of the oversimplified formulation of the problem. It is very possible that, once a multiregional setup is considered, land values will not be zero in all regions, resulting in higher prices for the products. It is impossible to speculate by how much this would change the prices in the present solution, but it still seems reasonable that any such change would not be sufficient to void some of the basic conclusions.

In order to see how some of these conclusions can be drawn, I choose to treat in detail the two major products: wheat and feed grains.

Suppose, that all prices and quantities except those of wheat remain constant at their solution level, and that the government imposes a minimum price of $1 per bushel of wheat. Even if this would not have changed production, buying the surplus wheat at the minimum price would have costed the government about 1.7 million dollars. This could not be considered an expense under Public Law 480, since all types of foreign aid are included in the demand function for wheat. To the mentioned expense one would then have to add shipment and storage expenses and overhead associated with storage capacity and administration.

Clearly, production would not have remained unchanged. In fact, for any price higher than $.796 per bushel one would have to expect that a good deal of the unused land would be drawn into wheat production. To speculate how much additional land would be planted with wheat is difficult, since this depends on regional technology. For purpose of illustration, suppose that the boost of 29% in the price of wheat would have brought about a similar increase in the wheat acreage. That would have added another 11 million acres of wheat. To prevent such change, the government would have to pay the difference between the price and the costs of a bushel of wheat, which is 20 cents. This would mean about another 56 million dollars in the form of a land retirement program. I think, that these simple calculations demonstrate very clearly the advantage to be gained from applications such as this one. As pointed out in our introductory discussion, consideration of a large number of alternatives in an explicit form is likely to improve the process of choice. The opportunity to do so is granted by our application, as the above calculations demonstrate.

Turning now to the case of feed grains, I would like to show first that any reasonable change in the slope coefficients of the demand function for feed grains would not have changed their price. As before, we suppose now that the outputs of all products other than feed grains are fixed at their solution level. As demonstrated in Fig. 8, the price of feed grains could have changed only if $u_1 > 0$ (see Table 1). To bring this situation about, the remainder of "total land" would have to be planted with feed grains. If this were done, the total output of feed grains would have been about 21 million tons, which is greater than d_5 by 5 million tons.

Calculations similar to those carried out for wheat can be performed with regard to feed grains. Taking as a basis, for instance, the 1955-57 average situation, total production of feed grains was about 124 million tons at a price of about $44 per ton.[6] This is a price which is higher than the solution price by about $14 per ton. If this price were to be maintained without any change in production, the costs of buying surpluses would have amounted to over 2.5 million dollars. Again, this is, of course, only a fraction of what the total expenses would have been.

Finally, a word is to be said about cotton and soybeans. Cotton is the only product which, within the framework of this application, carries what can be termed as "pure profits". This is again a result of the fact that oil meals are not treated as an intermediate commodity. We thus have a situation where the costs per ton of cotton lint are $363, but revenue is about $393, because the income from a ton of lint is $363 and there is an additional income of about $30 for the meals that emerge as a byproduct of lint. The same situation would have occurred with regard to soybeans, had we included the demand functions for oils in the model. From the theoretical viewpoint, the solution cannot, therefore, be regarded strictly as a competitive equilibrium, since there

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[6] Derived from data in USDA statistics (33, 34).

is some income not accounted for by expenses or rent. The only way to eliminate this difficulty, as well as the ones mentioned above, is to consider the various products in their proper categories.

D. Outline for a Proposed Application

As stated at the outset, the above application is of mere demonstrative value. It therefore seems desirable, in order to eliminate the possible impression of disproportion between theory and practice, to describe in detail a proposed application, to be carried out at Iowa State University in the near future.

The research will be executed in two stages. At first, the crops considered will be the same as in the above example. In the second stage, the field crops and livestock industries will be integrated in a single model. In both stages, the model used will be III.1 (see p. 55), where the K producers will be K (where K will be approximately 40) production regions. The number of consumption regions, L, will be about 10.

Suppose, then, that Model III.1 is applied, and that it yields the following competitive solution:

$$p^l = \bar{p}^l; \quad x^{kl} = \bar{x}^{kl}; \quad d^l = \bar{d}^l \; . \tag{3.1}$$

Since the vectors b^{lk} are most probably, as was observed in

Chapter I, not part of a price guided allocation, it is most reasonable to expect that the solution (3.1) will not be considered acceptable from the point of view of, say, farm income. In other words, since average profits

$$\sum_{l}\sum_{k} b^{kl}\, u^{kl} \,/\, K \tag{3.2}$$

or some individual terms $b^{kl}u^{kl}$ thereof will be "too small", some public intervention will be desired. We shall now see how a "farm program" can be obtained by using the results of (3.1).

Assuming that (3.2) is, indeed, below the desired minimum level, a price support system is called for. Suppose that the support prices are determined at $p_o^{\,l}$. Then there are three major possibilities:

(a) all prices are changed by the same factor in all consumption regions. That is,

$$p_{oi}^{\,j} \,/\, \overline{p}_i^{\,j} = m_o \text{ for all i and j;}$$

(b) the price for any given commodity is changed by the same factor in all regions i.e.,

$$p_{oi}^{\,j} \,/\, \overline{p}_i^{\,j} = m_i \text{ for all j;}$$

(c) prices are arbitrarily changed. Essentially, the alternative to be chosen will depend on the profit pattern associated with the solution to III.1, and on the supply level which the policymaker wishes to maintain. The latter will be based on what is considered to be "adequate" from the consumer's standpoint. It is easy to see that whereas (a) may not affect the level of production (and possibly not even profits) at all, (b) will preserve the interregional trade pattern i.e., relative distribution of every commodity between the consumption regions will be the same as in III.1, since the p_o^j's will satisfy (2.64) and (2.65) as do the $\bar{p}^j$'s for all consumption regions j. Alternative (c) may bring about a total change.

Since alternative (c) is the most general, we shall choose it for the continuation of the present outline. The reader will be readily able to see what simplifications can be made if (a) or (b) are chosen. It should be noted, that the choice of a general alternative is not a sufficient basis for the anticipation of a particular direction of change from the solution to III.1. As a matter of fact, even when the relation between $\bar{p}^j$ and p_o^j is exactly known, different directions of change are still possible, depending on the structures of D^j and A^{kj}. In order to be able to complete the description of the scheme, we will assume:

1. $p_o^j \geq \bar{p}^j$ for all j in such a way that

$$d_o^j + D^j p_o^j \leq \bar{d}^j \text{ for all j.}$$

2. A supply level of at least $\bar{d}^j$ is to be maintained in

all regions j.

As a first step, we shall now find the effect on profits and supply that the price support system, under Assumptions 1 and 2, exerts. This will be done by solving Problem A:

$$\text{to maximize} \quad \sum_j (p_o^j - \sum_k c^{kj}) x^{kj} - \sum_{l \neq j} \sum t^{jl} s^{jl}$$

$$\text{subject to} \quad A^{kj} x^{kj} \leqq b^{kj}$$

$$\sum_k x^{kl} + \sum_{l \neq j} (s^{lj} - s^{jl}) \geqq \bar{d}^l \qquad (3.3)$$

If (3.3) is not binding, the solution to Problem A will be a competitive equilibrium among the regions under the price regime p_o^j. In that case trade will also be at an equilibrium. If, however, (3.3) is binding for some l, the solution will merely be efficient. If the latter is the case, we face a difficult practical problem, since some administrative action is needed to secure the desired supply for those regions and commodities for which (3.3) is binding. The only other possibility is to change the prices in those regions. We shall therefore impose

3. p_o^j are determined in such a way that (3.3) is not binding.

Under this additional assumption, (3.3) is needed merely to solve for the new trade pattern.

We again find ourselves at a junction point. The avenue along which the investigation will proceed depends totally on the actual solution to Problem A and its relation to that of Problem III.1. First, the effect of the support system on profits in any one region cannot be anticipated. Second, we have to know exactly what the demanded quantities under p_o^j will be. Thirdly, we cannot anticipate the direction that the change in regional production may take. We shall therefore add assumptions as necessary to continue the presentation.

Suppose that the solution to Problem A is (x_o^{kl}, s_o^{jl}) and define r_o^j to be the regional supply vectors associated with this solution. Further, let

$$q_o^j = d_o^j + D^j p_o^j$$

be the regional demand vector under p_o^j. We add the following assumptions:

4. $q_o^j \leqq \bar{d}^j$ for all j.

5. Farm income as given by the solution to Problem A is deemed "satisfactory".

Since by Assumption 3 $r_o^j > \bar{d}^j$, it implies, together with Assumption 4, the existence of surpluses. The actual surpluses, if the consumers pay p_o^j, are $(r_o^j - q_o^j)$. However, we want the consumers to have $\bar{d}^j$. The first step is, therefore, to fix "consumer's prices" at $\bar{p}^j$. Once this is done, we face two problems: regional surpluses of $(r_o^j - \bar{d}^j)$ and price gaps of $(p_o^j - \bar{p}^j)$.

There are three courses of action that can be taken in dealing with these problems:

(a) pay the farmers the difference $p_o^j - \bar{p}^j$ for the quantities $\bar{d}^j$ and purchase the entire surpluses $(r_o^j - \bar{d}^j)$ at p_o^j;

(b) pay the farmers the difference as in (a), but eliminate the surpluses through acreage control measures, with or without compensating the farmers;

(c) using a combination of (a) and (b). As for the price gap, there is no way other than paying the farmer the difference of $p_o^j - \bar{p}^j$ for the quantity $\bar{d}^j$. The three methods differ, therefore, in the approach to the surplus problem. Our task becomes now to provide the information necessary for evaluating the alternatives in welfare terms.

For alternative (a), we compute

$$g = \sum_j p_o^j(r_o^j - \bar{d}^j).$$

The total public expenditures associated with (a) will be given by

$$g + \sum_j (p_o^j - \bar{p}^j)\ \bar{d}^j \tag{3.4}$$

It should be noted, that (a) does not involve any form of central planning, which may be very important in making the final policy choice.

As for alternative (b), we mentioned already that it may be implemented by law. This would be the purest form of government control and, keeping in line with the assumptions of Chapter I, we shall rule such a procedure out. The second method of implementation is a "land retirement program" which does not compel the farmer to subject himself to the central planning. The major difficulty it presents, as is well known from past and present experience, arises from the fact that the number of farmers participating in the program is not known in advance. This would lead us automatically to alternative (c). However, to complete the argument we shall pretend that the program will be fully effective. In order to realize its implications, we solve Problem B,

$$\text{to minimize} \quad \sum_{j}\sum_{k} c^{kj}x^{kj} + \sum\sum_{l \neq j} t^{jl}s^{jl}$$

$$\text{subject to} \quad A^{kj}x^{kj} \leqq b^{kj}$$

$$\sum_{k} x^{kl} + \sum_{l \neq j} (s^{lj} - s^{jl}) = \bar{d}^{\,l} \qquad (3.5)$$

The solution to Problem B will yield the most efficient interregional production pattern which will secure the supply of $\bar{d}^{l}$, the equilibrium quantity under $\bar{p}^{l}$. It is worthwhile to note, that Model B is the same as the one used by Heady <u>et al</u>. (8, 9, 10, 11), which reinforces our claim in Chapter

I, that it is a tool for finding the most efficient way of executing a particular policy, and is in line with the general prescribed process of choice. It is also worth mentioning that since by Assumption 3 $r_o^j \geq \overline{d}^j$, the solution to Problem B is not efficient, which reflects the element of central planning present in the land retirement scheme. This fact might play an important role when alternatives (a) and (b) are compared.

Based on the solutions to Problems A and B, we can now compute the payments that must be made to the farmers in order to bring about the necessary reduction in acreage. Using p_o^j, we compute the regional profits resulting in Model B and subtract them from those of Problem A. The difference is divided by the difference of regional acreages between the two solutions, and a system of payments per acre is thus established. This will be done, of course, for each crop separately.

Alternative (c) is comprised, of course, of an infinity of possibilities. We shall test a few by selecting some q^j such that

$$q_o^j \geq q^j \geq \overline{d}^j \quad \text{for all } \underline{j}.$$

The quantity vectors q^j will be evenly spaced in the relevant interval, each serving to resolve Problem B upon its substitution for $\overline{d}^j$ in (3.5). In each of these cases, regional surpluses of $(q^j - \overline{d}^j)$ will have to be purchased, and the surpluses $(r_o^j - q^j)$ eliminated through acreage reduction. The actual number of q^j tried will depend on the pattern of change emerging from trying the first few.

The results obtained in the entire process, when repeated for various price support systems, will enable the policy-maker to select the to him most acceptable policy, having considered the most efficient way of implementing each of the possible policies. This is precisely the method of choice recommended in Chapter I.

In concluding this section, it should be noted that, once sufficient data on farm labor are available, one could also consider the various policies on the grounds of farm population and returns to labor. It will also be very interesting to evaluate at that stage various policies of public support to agricultural research and education. In general, it will ultimately be possible to introduce any aspect which one may wish to consider.

E. Conclusion

We started out with a welfare analysis concerning the U.S. farm problem, which resulted in a suggested scheme of going about the choice of a farm policy. A series of models was then developed which make it possible to follow that scheme under various economic structures (regions etc.).

We proceeded to present a very limited application which demonstrated the working of one of the models. Finally, we presented an outline of an elaborate application, to be carried out in the near future.

There is still a long way to go before empirical results will be obtained that will be solid enough to serve as a basis for actual policy recommendations. In order to achieve such results, data have to be refined and the formulation of the model has to reflect reality in every possible way. Because of the aggregation problem, a regional approach is strongly recommended. Since man power in agriculture is probably a crucial ingredient in the farm problem, an attempt must be made to formulate labor restrictions. When satisfactory data concerning labor are available, the models can be used to determine the optimal farm size by regions, thereby contributing significantly towards the analysis of another important component of the farm problem.

What this study set out to do, is to propose methodological as well as technical improvements in dealing with the farm problem. These two rather ambitious goals were, so I hope, at least partially achieved.

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