Stochastic analysis of Marotzke and Stone climate model

by

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ABSTRACT

A deterministic analysis of an atmosphere-ocean coupled model is completed. A stochastic model is then created using the deterministic model with an imbedded Markov chain. An analysis of the stochastic model is then concluded.
CHAPTER 1. Introduction

Simple coupled ocean-atmospheric models are used frequently today by experts in the atmospheric and ocean sciences. The Marotzke and Stone model is a nonlinear system with temperature gradient and salinity gradient as state variables that describe the interaction between the ocean and atmosphere. The end result is a second order system as shown:

\[ \dot{T} = \lambda (T_E - T) - 2k|\alpha T - \beta S|T \]
\[ \dot{S} = \frac{2}{\epsilon} \frac{S_0}{D} \gamma T - 2k|\alpha T - \beta S|S \]

To study the stability of a deterministic nonlinear system many approaches can be taken. Generally, the first step is to find the fixed points and then study the local behavior around the fixed points. The most common way to study local behavior around the fixed points is by using the tool of linearization, which uses a linear approximation to a nonlinear system. The global stability of a system is then determined by combining the local solutions around the fixed points.

A stochastic process is a process that can be described using a probability distribution. A common stochastic process is a Markov chain, which is a discrete-time process with the Markov property. A stochastic model is a deterministic model with an aspect of randomness included and the stability of such a model is generally studied numerically. This paper describes an analysis of that nature and shows that the solution of the random system converges to an invariant set with invariant measure around the stable fixed points.

In chapter 2 a summary of the derivation of the Marotzke and Stone model is presented. The model stems from eleven basic equations describing ocean-atmosphere interaction and after certain assumptions are made such as heat and moisture being linear functions the system
above is yielded. The only nonlinear aspect of the model stems from interaction between ocean flow and density gradients.

A deterministic analysis of the model is completed in chapter 3. The first step is the computation of the fixed points and then a linearization of the model about those fixed points to describe local behavior. The global behavior of the system is determined using numerical tools.

Chapter 4 describes the stochastic Marotzke and Stone model. The first section offers an overview of Markov chains and then a description of the random Marotzke and Stone model is presented, which explains what the solutions are expected to be. The global behavior of the random model is presented with a description of the invariant sets around the stable fixed points. The last section is a numerical solution of the random Marotzke and Stone model.
CHAPTER 2. Marotzke and Stone Model

The following is a summary of Marotzke and Stone’s simple coupled model [7]. The Marotzke and Stone model is a simple coupled ocean-atmosphere model for the Atlantic ocean based on the work of Nakamura et al. The atmospheric transports have been simplified to yield an analytical solution in which heat and moisture are treated linearly. The nonlinear aspect of the model comes from the interaction between ocean flow and density gradients.

The model, shown in Figure 2.1, consists of two ocean boxes with depth $D$ where box 1 represents a high-latitude ocean and box 2 represents a low-latitude ocean and two atmospheric boxes. $H_1$ and $H_2$ are the heat gains through the surface of the ocean, $H_{01}$ and $H_{02}$ are the atmospheric energy gains, and $H_d$ is the meridional energy transport in the atmosphere. $E$ is the net evaporation at low latitudes and net precipitation at high latitudes and $F_w$ is the meridional atmospheric moisture transport.

The system consists of a set of eleven equations. The first four equations are the conservation equations for the ocean

\[
\begin{align*}
\dot{T}_1 &= H_1 + |q|(T_2 - T_1) \\
\dot{T}_2 &= H_2 - |q|(T_2 - T_1) \\
\dot{S}_1 &= -H_s + |q|(S_2 - S_1) \\
\dot{S}_2 &= H_s - |q|(S_2 - S_1),
\end{align*}
\]

where $H_s$ is the virtual surface salinity flux, $q$ is the flow strength, and $T_1$, $T_2$, $S_1$, and $S_2$ are the temperature and salinity of the respective oceans. The flow strength is represented by the following linear law

\[ q = k[\alpha(T_2 - T_1) - \beta(S_2 - S_1)], \]
where $\alpha$ is the thermal expansion coefficient and $\beta$ is the haline expansion coefficient. The virtual surface salinity flux, $H_s$, is connected to the surface freshwater flux via

$$H_s = S_0 \frac{E}{D},$$

where $S_0$ is a constant reference salinity. Multiplication of the heat fluxes by the heat capacity of the unit water column, $c\rho_0 D$, changes the quantities into physical heat fluxes represented by

$$\tilde{H}_1 = H_1 c\rho_0 D$$

$$\tilde{H}_2 = H_2 c\rho_0 D.$$

Using a time scale longer than a month, an assumption that the heat and moisture capacities of the atmosphere are negligible is allowed. The meridional fluxes of heat and moisture in the atmosphere are integrated over the 35 degree latitude circle to give

$$H_d = \tilde{\chi}(T_2 - T_1)$$

$$F_w = \tilde{\gamma}(T_2 - T_1),$$
with constants $\tilde{\chi}$ and $\tilde{\gamma}$. The final two equations are

\begin{align*}
H_{01} &= A_1 - BT_1 \\
H_{02} &= A_2 - BT_2,
\end{align*}

which represent the parameterization of radiation at the top of the atmosphere. $A_1$ and $A_2$ are the net incoming radiation for a surface temperature of $0^\circ C$ and $BT_1$ and $BT_2$ are the longwave fluxes for the respective oceans.

The heat supply for the high-latitude atmosphere is

$$\int_{\text{total}} H_{01} da - \int_{\text{ocean}} \tilde{H}_1 da + H_d = 0,$$

where model area is the quantity of integration. The first integral represents the negative energy gain at the top of the atmosphere integrated over the entire area north of the latitude circle that divides the low and high-altitude boxes. The negative heat loss to the ocean integrated over the ocean portion of the high-latitude box is represented by the second integral. Assuming that $H_{01}$, $H_1$ and hence $\tilde{H}_1$ are constant over the boxes gives the following result

$$F_{01} H_{01} - F_1 \tilde{H}_1 + H_d = 0.$$

where $F_{01}$ is the area north of the latitude circle that divides the boxes and $F_1$ is the area of the ocean part of the high-latitude box. This gives the physical heat flux of the high-latitude box as

$$\tilde{H}_1 = \frac{1}{\epsilon} H_{01} + \frac{1}{\epsilon F_{01}} H_d.$$

The relative ocean coverage of the high-latitude area is $\epsilon = \frac{F_1}{F_{01}}$. Assuming that $F_1 = F_2$ and $F_{01} = F_{02}$, the physical heat flux of the low-latitude box is

$$\tilde{H}_2 = \frac{1}{\epsilon} H_{02} - \frac{1}{\epsilon F_{01}} H_d.$$

The physical heat fluxes, as a function of ocean temperature, are

\begin{align*}
\tilde{H}_1 &= \frac{1}{\epsilon} (A_1 - BT_1) + \frac{\tilde{\chi}}{\epsilon F_{01}} (T_2 - T_1) \\
\tilde{H}_2 &= \frac{1}{\epsilon} (A_2 - BT_2) + \frac{\tilde{\chi}}{\epsilon F_{01}} (T_2 - T_1).
\end{align*}
The sum of heat fluxes is

\[ \tilde{H}_1 + \tilde{H}_2 = \frac{1}{\epsilon}(H_{01} + H_{02}). \]

Summing the first two conservation equations and the previous equation gives

\[ T_1 + T_2 = H_1 + H_2 = \frac{1}{\epsilon} \frac{H_{01} + H_{02}}{c\rho_0 D}. \]

As a function of temperature, the equation becomes

\[ \frac{1}{2}(\dot{T}_1 + \dot{T}_2) = \frac{1}{\epsilon} \frac{A_1 + A_2}{2c\rho_0 D} - \frac{1}{\epsilon} \frac{B}{c\rho_0 D} \frac{T_1 + T_2}{2}. \]

Defining the temperature gradient as \( T = T_2 - T_1 \), we obtain

\[ \dot{T} = H_2 - H_1 - 2|q|T = \frac{1}{\epsilon} \frac{A_2 - A_1}{c\rho_0 D} - \frac{1}{\epsilon} \frac{2\chi + B}{c\rho_0 D} T - 2|q|T, \]

where \( \chi = \tilde{\chi} F_0^{-1} \). Assume that the atmosphere and ocean have zero heat transport (\( \chi = q = 0 \)) and that the temperature gradient is determined by radiation alone, then

\[ T_R := T|_{\chi=0,q=0} = \frac{A_2 - A_1}{B}, \]

which is approximately \( 76^\circ C \), where \( C \) represents Celsius. The equilibrium temperature is found by assuming the atmosphere transfers heat horizontally while the ocean does not and balancing the dynamical and radioactive transports in the atmosphere. Thus,

\[ T_E = T|_{q=0} = \frac{A_2 - A_1}{2\chi + B}. \]

Assuming \( \chi \approx 1.3 W m^{-2} K^{-1} \) and \( F_0 \approx 1.25 \times 10^{14} m^2 \), where \( W \) stands for Watts, \( m \) is meters, and \( K \) is Kelvin, yields an equilibrium temperature of about \( 30^\circ C \). The Newton cooling law for ocean temperatures is used to find the surface heat fluxes that cause the meridional temperature gradient

\[ H_2 - H_1 = \lambda(T_E - T), \]

where

\[ \lambda = -\frac{\partial(H_2 - H_1)}{\partial T} = \frac{1}{\epsilon} \frac{2\chi + B}{C\rho_0 D}. \]

This leads to

\[ \dot{T} = \lambda(T_E - T) - 2|q|T. \]
The low-altitude and high-altitude net precipitation balance the meridional atmospheric moisture flux

\[ \int_{\text{total}} Eda + F_w = 0, \]

where it is assumed that \( E \) is constant along a latitude circle. The ratio of the ocean area to the catchment area of the ocean basin, \( \epsilon_w \), has a range from \( \epsilon \) to 1. When \( \epsilon_w = 1 \) the ocean only receives moisture from the atmosphere right above it and when it equals \( \epsilon \) it receives all the river runoff as well. Good values for the Atlantic are from .3 to .5. The net evaporation is

\[ E = \frac{1}{\epsilon_w} \frac{F_w}{F_{01}} \]

and using the sixth equation of the system we obtain

\[ H_s = \frac{1}{\epsilon_w} \frac{S_0}{D} \frac{F_w}{F_{01}}. \]

Letting \( \gamma = \frac{5}{F_{01}} \) with an approximate value of \( 1.2 \times 10^{-10} ms^{-1} K^{-1} \), where \( s \) stands for seconds, results in

\[ H_s = \frac{1}{\epsilon_w} S_0 \frac{F_w}{D} \gamma T. \]

Assuming the total salt content of the model ocean is constant and letting the salinity gradient be defined as \( S = S_2 - S_1 \) gives

\[ \dot{S} = \frac{2}{\epsilon_w} \frac{S_0}{D} \gamma T - 2|q|S. \]

The model equations are

\[ \dot{T} = \lambda(T_E - T) - 2k|\alpha T - \beta S|T \quad (2.1) \]

\[ \dot{S} = \frac{2}{\epsilon_w} \frac{S_0}{D} \gamma T - 2|q|S \quad (2.2) \]

with the values of the parameters shown in Table 2.1.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$2 \times 10^{-4} K^{-1}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$0.8 \times 10^{-3} psu^{-1}$</td>
</tr>
<tr>
<td>$T_E$</td>
<td>$303 K$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$2 \times 10^{-10} ms^{-1} K^{-1}$</td>
</tr>
<tr>
<td>$k$</td>
<td>$2 \times 10^{-8} s^{-1}$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>$\epsilon_w$</td>
<td>0.3</td>
</tr>
<tr>
<td>$S_0$</td>
<td>35 $psu$</td>
</tr>
<tr>
<td>$D$</td>
<td>$5 \times 10^4 m$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$1.26 \times 10^{-9} J m^{-2}$</td>
</tr>
</tbody>
</table>

Table 2.1 Model parameter values.
CHAPTER 3. Analysis of the Deterministic Model

This section presents the analysis of the deterministic Marotzke and Stone model, as described in (2.1), (2.2). The first step is to find the limit sets, which turn out to be fixed points for this model. We then linearize the system about those fixed points and analyze the linearized equations. A global simulation and local simulations around the fixed points are carried out to show the global behavior of the system. The following sections describe the results in detail.

3.1 Fixed Points

When discussing a system, we mean $\dot{x} = f(x)$ in $\mathbb{R}^n$. The system has a solution of the form $\varphi(t, x)$ for $t \geq 0$ with $\varphi(0, x) = x$ as initial condition.

**Definition 1.** Given a system a point $z \in \mathbb{R}^n$ is said to be a limit point of the system if the $\lim_{t \to \infty} \varphi(t, x) = z$. A **limit set** is the set of all limit points.

**Definition 2.** A **fixed point**, $x^*$, of a system is a point such that $\varphi(t, x^*) = x^* \forall t \geq 0$.

To find the fixed points, we set the right hand sides of both equations (2.1), (2.2) to zero, solve (2.1) for $T$, and substitute the value in the (2.2). The second equation (2.2) then reduces to a cubic polynomial, in which the roots are the salinity equilibrium values.

In case $\alpha T - \beta S > 0$ we obtain

$$T = \frac{-2k\beta S^2}{2 \epsilon_w \frac{S_0}{D} \gamma - 2k\alpha S},$$

$$(-\lambda 4k^2 \beta \alpha - 4k^2 \frac{2 \epsilon_w}{\epsilon_w} \frac{S_0}{D} \gamma k^2) s^3 + \lambda T E 4k^2 \alpha^2 + \lambda 2 \frac{2 \epsilon_w}{\epsilon_w} \frac{S_0}{D} \gamma k^2 s^2$$

$$- \lambda T E \frac{2 \epsilon_w}{\epsilon_w} \frac{S_0}{D} \gamma k \alpha s + \lambda T E \left( \frac{2 \epsilon_w}{\epsilon_w} \frac{S_0}{D} \gamma \right)^2 = 0.$$
Table 3.1  Fixed points of Marotzke and Stone model.

<table>
<thead>
<tr>
<th></th>
<th>S Value</th>
<th>T Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point 1</td>
<td>1.20341</td>
<td>157.644</td>
</tr>
<tr>
<td>Point 2</td>
<td>72.4569</td>
<td>294.571</td>
</tr>
<tr>
<td>Point 3</td>
<td>74.8554</td>
<td>294.827</td>
</tr>
</tbody>
</table>

In case \( \alpha_T - \beta_S < 0 \) the corresponding values are

\[
T = \frac{2k\beta S^2}{\epsilon_w D\gamma + 2k\alpha S},
\]

\[
(-\lambda 4k^2\beta \alpha - 4k^2 \times \frac{S_0}{\epsilon_w D\gamma \beta^2}) s^3 + (\lambda T_E 4k^2\alpha^2 - \lambda^2 \frac{2S_0}{\epsilon_w D\gamma k\beta}) s^2
\]

\[
+ (\lambda T_E^4 \frac{2S_0}{\epsilon_w D\gamma k\alpha}) s + \lambda T_E (\frac{2S_0}{\epsilon_w D\gamma})^2 = 0.
\]

Finding the roots of the cubic above is difficult because the coefficients vary by a factor of \(10^{12}\). The only way to get accurate values for the equilibrium points is by changing all the coefficients to rational numbers and then using the explicit solution of a cubic polynomial provided, e.g., in Mathematica.

All together the system (2.1), (2.2) has six fixed points, three with positive and three with negative \((S,T)\) values. The points with negative values are physically not feasible. Using the cubic equation solver in Mathematica results in the values \(S_1 = 1.20341 + 4.26326 \times 10^{-14} i\), \(T_1 = 157.644 - 1.71728 \times 10^{-10} i\) and \(S_2 = 72.4569 - 3.29597 \times 10^{-17} i\), \(T_2 = 294.571 - 1.31804 \times 10^{-16} i\) for \(\alpha_T - \beta_S > 0\), and \(S_3 = 74.8554\), \(T_3 = 294.827\) for \(\alpha_T - \beta_S < 0\). Looking at the global dynamics (see Figure 3.4), it appears that the real parts of \((S_1, T_1)\), \((S_2, T_2)\), and \((S_3, T_3)\) are the physically meaningful fixed points of the system. Table 3.1 lists the fixed points of (2.1), (2.2) that are analyzed further in the following sections.

### 3.2 Linearization at the Fixed Points

To obtain the local qualitative behavior at the fixed points, the system (2.1), (2.2) is linearized about the three hyperbolic fixed points.
\[
\begin{pmatrix}
-\lambda - 4k\alpha T + 2k\beta S & 2k\beta T \\
\frac{2}{\epsilon_w} \frac{S_0}{D} \gamma - 2k\alpha S & -2k\alpha T + 4k\beta S
\end{pmatrix}
\]

Figure 3.1 Jacobian of system for \(\alpha T - \beta S > 0\).

<table>
<thead>
<tr>
<th>Point</th>
<th>Eigenvalues</th>
<th>Eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_1 = 1.20341)</td>
<td>(-3.80923 \times 10^{-9})</td>
<td>[8929.83, 1]</td>
</tr>
<tr>
<td>(T_1 = 157.644)</td>
<td>(-1.1847 \times 10^{-9})</td>
<td>[1.92169, 1]</td>
</tr>
<tr>
<td>(S_2 = 72.4569)</td>
<td>(-2.62453 \times 10^{-9})</td>
<td>[8.60077, 1]</td>
</tr>
<tr>
<td>(T_2 = 294.571)</td>
<td>(1.1847 \times 10^{-9})</td>
<td>[1.92168, 1]</td>
</tr>
</tbody>
</table>

Table 3.2 Eigenvalues and eigenvectors of the stable and unstable fixed points.

**Definition 3.** A hyperbolic fixed point is a fixed point, \(x^*\), where the eigenvalues, \(\lambda_i\), of \(A(x^*)\) are such that \(\lambda_i \notin \text{Im}\), where \(A(x^*) = \frac{\partial f}{\partial x}(x^*)\), see Definition 5 below.

**Definition 4.** A homeomorphism is a mapping \(h : X \rightarrow Y\) where \(h\) is one-to-one, onto, continuous, and \(h^{-1}\) is continuous.

**Definition 5.** For a continuously differentiable function \(f : \mathbb{R}^n \rightarrow \mathbb{R}^m\), the Jacobian matrix, \(A, \left[\frac{\partial f}{\partial x}\right]\), is an \(m \times n\) matrix whose \(i^{th}\) row and \(j^{th}\) column entry is defined by \(\frac{\partial f_i}{\partial x_j}\).

**Theorem 1.** *Hartman-Grobman Theorem* If the linearization of the system, \(A(x^*)\) is hyperbolic, then there exists a homeomorphism from a neighborhood \(U\) of \(x^*\) onto \(\mathbb{R}^n\), \(h : U \rightarrow \mathbb{R}^n\), taking the trajectory of the nonlinear system and mapping it to a trajectory of the linear system [6].

Hence, the behavior of the linear system is the topologically conjugate to the behavior of the nonlinear system in a neighborhood of the fixed points.

In the case where \(\alpha T - \beta S > 0\) the Jacobian is shown in Figure 3.1. Table 3.2 shows the resulting eigenvalues and the eigenvectors of the Jacobian at the points \((S_1, T_1)\) and \((S_2, T_2)\).

**Definition 6.** Given a system, a fixed point, \(x^*\), is said to be

- stable if real parts of all the eigenvalues, \(\text{Re} \lambda_i\), of \(A(x^*)\) are less than zero or \(\text{Re} \lambda_i < 0\).
$$\begin{pmatrix} -\lambda + 4k\alpha T - 2k\beta S & -2k\beta T \\ \frac{2}{T} S^n \gamma + 2k\alpha S & 2k\alpha T - 4k\beta S \end{pmatrix}$$

Figure 3.2 Jacobian of system for $\alpha T - \beta S < 0$.

<table>
<thead>
<tr>
<th>Point</th>
<th>Eigenvalues</th>
<th>Eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_3 = 74.8554$</td>
<td>$-7.18141 \times 10^{-10} + 1.67333 \times 10^{-9}i$</td>
<td>$[2.81825 + 2.75139i, 1]$</td>
</tr>
<tr>
<td>$T_3 = 294.827$</td>
<td>$-7.18141 \times 10^{-10} - 1.67333 \times 10^{-9}i$</td>
<td>$[2.81825 - 2.75139i, 1]$</td>
</tr>
</tbody>
</table>

Table 3.3 Eigenvalues and eigenvectors of stable focus.

- unstable if $\text{Re}\lambda_i > 0$ [5].

As is shown in Table 3.1 the first point is a stable fixed point with two real negative eigenvalues. The second point has both a positive and negative real eigenvalue, hence the fixed point $(S_2, T_2)$ is unstable (hyperbolic saddle).

**Definition 7.** A **hyperbolic saddle point** is a hyperbolic fixed point with one $\text{Re}\lambda_i > 0$.

In the case where $\alpha T - \beta S < 0$ the Jacobian is shown in Figure 3.2. The eigenvalues and eigenvectors of the Jocabian evaluated at the fixed point $(S_3, T_3)$ are listed in Table 3.3. The fixed point $(S_3, T_3)$ is a stable point with a pair of complex conjugate eigenvalues with negative real parts (stable focus).

**Definition 8.** A **stable focus** is a fixed point with $\lambda_i \in \mathbb{C}$ where $\text{Re}\lambda_i < 0$.

Figure 3.3 shows the locations of the fixed points in the $ST-$plane. To obtain Figure 3.4, we numerically integrate the system equations (2.1), (2.2) for several initial values over the time interval $[0, 10^{10}]$. The resulting trajectories are plotted using Matlab. The figure shows convergence of trajectories to both stable points of the system.

**Definition 9.** A **manifold** is a topological space where every neighborhood in that space is Euclidean.

**Definition 10.** A **stable manifold**, $M^s$, is a positively invariant manifold such that $\lim_{t \to \infty} \varphi(t, x) = x^*$, for all $x \in M^s$.
Theorem 2. Given a system, assume \( x^* \) is a hyperbolic fixed point and \( \mathbb{R}^n = E^s \oplus E^u \) where \( E^s \) is the stable subspace of \( A(x^*) \) and \( E^u \) is the unstable subspace of \( A(x^*) \). Then there exists a \( \text{dim}(E^s) \) dimensional manifold, \( M^s \) tangent to \( E^s \) at \( x^* \) such that there exists a neighborhood \( N^+ \) of \( x^* \) in \( \mathbb{R}^n \) and for \( x \in M^s \cap N^+ \) we have \( \lim_{t \to \infty} \varphi(t, x) = x^* \).

The local area around the stable points illustrate the stable manifold theorem.

Figure 3.5 shows the behavior of the system around the fixed points \((S_2, T_2)\) and \((S_3, T_3)\). In the neighborhood of the stable focus \((S_3, T_3)\) the trajectories show the expected rotational behavior towards the equilibrium. In the neighborhood of the unstable fixed point \((S_2, T_2)\) the trajectories flow towards one of the stable points, except for the initial values on the stable manifold of \((S_2, T_2)\). This stable manifold is indicated by the black line. It was obtained by integrating the system equations backward in time from initial values close to the stable eigendirections.
Figure 3.4  Global dynamics of the system.

Figure 3.5  Global dynamics around the unstable fixed point.
CHAPTER 4. A Stochastic Version of the Marotzke and Stone Model

In this section, a stochastic approach is taken to study how the Marotzke and Stone model reacts to randomness in one of its parameters, namely $\lambda$. This is done by modeling $\lambda$ as a stochastic process (Markov chain) in the Marotzke and Stone model and analyzing the resulting system.

4.1 Review of Markov Chains

The following is based on chapter 4 in [8]. We consider a stochastic process \( \{X_n, n = 0, 1, 2, \ldots \} \) with \( X_n : \Omega \to S \) for all \( n \in \mathbb{N} \), where \( S \) is a finite set of values, denoted by

\[ S = \{1 \ldots m\}, \]

and \( m \in \mathbb{N} \).

**Definition 11.** While in state \( i \) there is a fixed probability, denoted \( P_{ij} \), that the process will next be in state \( j \):

\[ P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\} = P\{X_{n+1} = j \mid X_n = i\} = P_{ij}. \]

This process is a **Markov Chain** and according to the Markovian property, any future state is independent of past events and dependent only upon the present state. These probabilities have the following properties:

\[ P_{ij} \geq 0, \quad i, j \in S, \quad \sum_{j=1}^{m} P_{ij} = 1, \quad i = 1, \ldots, m. \]

**Definition 12.** The \( m \times m \) matrix of one step transition probabilities, called the **probability transition matrix**, is represented by
\[
P = \begin{pmatrix}
P_{11} & P_{12} & \ldots & P_{1m} \\
P_{21} & P_{22} & \ldots & P_{2m} \\
\vdots & \vdots & & \vdots \\
P_{m1} & P_{m2} & \ldots & P_{mm}
\end{pmatrix},
\]

**Definition 13.** The probability that the event in state \( i \) will after \( n \) steps be in state \( j \), denoted by \( P_{ij}^n \), is

\[
P_{ij}^n = P\{X_{n+p} = j \mid X_p = i\}, \quad n, p \in \mathbb{N}, \quad i, j \in S
\]

and is found using

\[
P_{ij}^{n+p} = \sum_{k=1}^{m} P_{ik}^n P_{kj}^p \quad \text{for all } n, p \in \mathbb{N}, \quad i, j \in S,
\]

which are known as the Chapman-Kolmogorov equations. Using transition matrix notation, the equations are

\[
P^{(n+p)} = P^{(n)} \cdot P^{(p)}
\]

where the dot represents matrix multiplication and \( P^{(n)} \) is found by multiplying the matrix \( P \) by itself \( n \) times.

A Markov chain \( \{X_n, n = 0, 1, 2, \ldots \} \) with values in \( S \) is therefore completely described by its transition probability matrix \( P \) and its initial distribution \( \pi_0 \). For convenience we will assume that the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is obtained via the Kolmogorov construction: \( \Omega \) consists of all paths with values in \( S \), the \( \sigma \)-algebra \( \mathcal{F} \) is generated by the cylinder sets, and the probability measure \( \mathbb{P} \) is the extension of the transition probabilities \( P^n \), \( n \in \mathbb{N} \), and the distribution \( \pi_0 \) to \( \mathcal{F} \).

**Definition 14.** A state \( j \in S \) is said to be **accessible** from state \( i \in S \) if the probability that the process \( X_n \) with initial value \( i \in S \) will be in state \( j \) after \( n \) steps is greater than zero or \( P_{ij}^n > 0 \), for some \( n \geq 0 \).

**Definition 15.** If \( i \) and \( j \) are (mutually) accessible, they are said to **communicate**, which is an equivalence relation. The equivalence classes of this relation are called communicating classes.
**Definition 16.** If all states in the Markov chain communicate with each other, then the Markov chain is called **irreducible**.

**Definition 17.** Within a class the greatest common divisor of the steps it takes to go from any state $i$ to any state $j$ is called the **period** and when the period is one, the Markov chain is called aperiodic.

To describe the long term behavior of a Markov chain, we introduce the first return time

$$R_i = \min\{n \geq 1, X_n = i\}.$$ 

The distribution of $R_i$ is denoted by

$$f_{ii}^{(n)} = \mathbb{P}\{R_i = n \mid X_0 = i\}$$

and its expectation is

$$\mu_{ii} = \mathbb{E}[R_i \mid X_0 = i] = \sum_{n=1}^{\infty} nf_{ii}^{(n)}.$$ 

**Definition 18.** A state $i \in S$ is called **recurrent**, if

$$\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$$

or equivalently, if

$$\sum_{n=1}^{\infty} P_{ii}^n = \infty.$$ 

Note that if $i$ is a recurrent state and $i$ communicates with $j$, then $j$ must be a recurrent state. On the other hand, a transient state is a state that is not recurrent, i.e. $j \in S$ is transient if

$$\sum_{n=1}^{\infty} P_{jj}^n < \infty.$$ 

In particular, a state is transient if it is visited only a finite number of times. For a finite state Markov chain, not all the states can be transient. Otherwise, after a finite amount of time the Markov chain would stop visiting all of the states and since at least one state must be visited at any time, this is impossible. A recurrent state $i \in S$ is called positive recurrent if $\mu_{ii} < \infty$, and null recurrent, if $\mu_{ii} = \infty$. Note that on a finite space $S$ all recurrent states are automatically positive recurrent.
The long term behavior of a Markov chain \(\{X_n, n = 0, 1, 2, \ldots\}\) on \(S\) is described by the following theorem [8]:

**Theorem 3.** We assume that \(S\) consists of one communicating class; in particular, all points in \(S\) are recurrent. We fix \(j \in S\), then for all \(i \in S\) it holds that

1. \(\mathbb{P}\{\lim_{t \to \infty} N_j(t)/t = 1/\mu_{jj} \mid X_0 = i\} = 1\), where \(N_j(t)\) is the number of transitions into state \(j\) by time \(t\).

2. \(\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^k = 1/\mu_{jj}\).

3. If \(j\) is aperiodic, then \(\lim_{n \to \infty} P_{ij}^n = 1/\mu_{jj}\).

4. If \(j\) has period \(d\), then \(\lim_{n \to \infty} P_{ij}^{nd} = d/\mu_{jj}\).

A Markov chain has a stationary distribution, \(\pi^*\), if

\[
\pi_j^* = \sum_{i=1}^{m} \pi_i^* P_{ij}, \quad \text{for all } j \in S.
\]

Markov chains that admit a unique stationary distribution are called ergodic. Note that for a Markov chain with initial distribution \(\pi_0 = \pi^*\) we have \(\pi_n = \pi^*\) for all \(n \in \mathbb{N}\). The following limit theorem describes the invariant distribution for aperiodic chains [8].

**Theorem 4.** Assume that \(S\) consists of one communicating class and that \(X_n\) is an aperiodic Markov chain on the finite space \(S\). Then all states \(j \in S\) are positive recurrent and

\[
\pi_j^* = \lim_{n \to \infty} P_{jj}^n = \lim_{n \to \infty} P_{ij}^n > 0
\]

for all \(i \in S\). In particular, the invariant distribution \(\pi^*\) is unique and \(\pi_j^* = 1/\mu_{jj}\) for all \(j \in S\).

In the following section we will consider the Marotzke and Stone model where the parameter \(\lambda\) is given by a finite state Markov chain that is, irreducible, with no assumption on the (a-)periodicity.
4.2 The Random Marotzke and Stone Model

In this section we introduce and analyze the Marotzke and Stone model (2.1), (2.2) with random parameter process $\lambda$. The behavior of $\lambda$ is modeled as a Markov chain $X = \{X_n, n \in \mathbb{N}\}$, i.e. as a sequence of random variables where $X_n : \Omega \rightarrow S$ in which $S$ is the set of all possible values of $\lambda$. Thus, the underlying Markov chain will be referred to as $\lambda$. For each $\omega \in \Omega$, $\{X_n(\omega), n \in \mathbb{N}\}$ is a trajectory of $X$. For a set $\Psi \subset \Omega$, $P(\Psi)$ describes the probability that this set occurs as a set of trajectories of $X$.

Let $\pi_n \in \mathbb{R}^m$ denote the distribution of $\lambda$ at each time step, i.e., the $i^{th}$ entry of the column vector $\pi_n$ is the probability that $X_n = i$, and $m$ is the number of states. $\pi_0$ is the initial distribution and each following distribution is found using

$$\pi_{n+1} = P\pi_n,$$

where $P \in \mathbb{R}^{m \times m}$ is the probability transition matrix. We assume that the Markov chain $X$ is periodic, possibly aperiodic, and irreducible. Hence, there exists a stationary distribution, $\pi^*$, where

$$\frac{1}{N} \sum_{n=1}^{N} \pi_n \rightarrow \pi^* \text{ as } N \rightarrow \infty,$$

see Section 4.1.

The Markov chain $X$ has the discrete time set $\mathbb{N}$. In order to use this chain as a perturbation model for the continuous time Marotzke and Stone model, we extend the trajectories $\{X_n(\omega), n \in \mathbb{N}\}$ to all of $\mathbb{R}^+$ as follows: Let $L \in \mathbb{N}$ denote the time scale of $X$, i.e. $\lambda_t(\omega) = X_n(\omega)$ for $t \in [nL, (n+1)L)$. The solutions of the combined system

$$\dot{T} = f(T, S, \lambda) \quad (4.1)$$

$$\dot{S} = g(T, S, \lambda) \quad (4.2)$$

$$\{X_n, n \in \mathbb{N}\} \text{ with } X_n : \Omega \rightarrow S, \lambda \in S \quad (4.3)$$

are interpreted the following way:

For $t \in [nL, (n+1)L)$ $n \geq 0$, the system dynamics are

$$\dot{T} = f(T, S, \lambda_t(\omega))$$
\[ \dot{S} = g(T, S, \lambda_t(\omega)) \]

with initial value \((T, S)(0) = (T_0, S_0), (T, S)(nL) = \lim_{t \to nL} (T, S)(t)\) for \(n \geq 1\), and \(\lambda_{nL}(\omega) = X_n(\omega)\). The limits exist because the system (4.1), (4.2) has unique solutions for all \(t \geq 0\), all \(\lambda \in S\). Hence the trajectories of the combined system are piecewise smooth and continuous in the first two components. The time scale \(L\) allows us to adjust the fluctuation clock to different time intervals in the Marotzke and Stone model, such as daily, seasonal, or long term fluctuations.

### 4.3 Global Behavior of the Random Marotzke and Stone Model

We consider the random Marotzke and Stone model (4.1, 4.2, 4.3) with \(S = \{\lambda_1, ..., \lambda_m\}\) and transition matrix \(P = (P_{ij})_{i,j=1,...,m}\). For sake of convenience we order the possible values of \(\lambda\) such that \(\lambda_1 < ... < \lambda_m\). Let us denote by \(\varphi(\cdot, (x, y), \lambda(\omega))\) the solutions of (4.1, 4.2) with initial value \((x, y) \in \mathbb{R}^+ \times \mathbb{R}^+\) under the random trajectory \(\lambda(\omega)\). For a fixed time scale \(L\), embedded in the system (4.1, 4.2, 4.3) is the discrete time system

\[ (T(nL), S(nL), X_n) \text{ for } n \geq 0. \]  

(4.4)

While the first two components \((T(nL), S(nL))\) are not a Markov chain, the complete system \((T(nL), S(nL), X_n)\) is a Markov chain with transition probability

\[ Q((x, y, i), (A, j)) = \begin{cases} 
P_{ij} & \text{if } \varphi(1, (x, y), i) \in A \\
0 & \text{otherwise}
\end{cases} \]

with \((x, y) \in \mathbb{R}^+ \times \mathbb{R}^+\) and \(A \subset \mathbb{R}^+ \times \mathbb{R}^+\) a Borel set.

**Definition 19.** A \(\sigma\)-algebra on a set \(X\) is a collection \(\mathcal{A}\) of subsets of \(X\) such that:

- \(\emptyset \in \mathcal{A}\)
- if \(A \in \mathcal{A}\), then \(A^c = X \setminus A \in \mathcal{A}\)
- if \(\{A_i|i \in \mathbb{N}\}\) is a countable family of sets in \(A\), then \(\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}\)

A measurable space \((X, \mathcal{A})\) is a set \(X\) and a \(\sigma\)-algebra \(\mathcal{A}\) on \(X\) [4].
Definition 20. The Borel $\sigma$-algebra $\mathcal{B}$ is the smallest $\sigma$-algebra generated by the open subsets. A Borel set is a set, which is an element of $\mathcal{B}$.

Definition 21. A forward invariant set, $F \subset \mathbb{R}^n$, is a set such that $\forall x \in F$, $\varphi(t, x) \in F$ for all $t \geq 0$.

Definition 22. An attracting set, $A$, is a set where given any neighborhood $U$ around $A$, $\forall x \in U$, $\lim_{t \to \infty} \varphi(t, x) \in A$.

Denote by $\lambda$ the nominal value of $\lambda$ from Table 2.1. As $\lambda$ varies around $\lambda$, the fixed points discussed in Chapter 3.1 will vary continuously, since the deterministic model (2.1), (2.2) is linear in $\lambda$. For the stable fixed points $(S_1, T_1)$ and $(S_3, T_3)$ from Table 3.1 we obtain compact, forward invariant, attracting sets $C_1$ and $C_3$ with $(S_1, T_1) \in C_1$ and $(S_3, T_3) \in C_3$ such that

- there exist $\varepsilon_i > 0$ with $(S_i(\lambda), T_i(\lambda)) \in C_i$ for all $\lambda \in [\bar{\lambda} - \varepsilon_i, \bar{\lambda} + \varepsilon_i]$, $i = 1, 3$, where $(S_i(\lambda), T_i(\lambda))$ denotes stable fixed points for $\lambda \in [\bar{\lambda} - \varepsilon_i, \bar{\lambda} + \varepsilon_i]$,
- $C_1 \cap C_3 = \emptyset$.

On the other hand, for the unstable (hyperbolic) fixed point $(S_2, T_2)$ from Table 3.1 we obtain a set $D_2$ with $(S_2, T_2) \in D_2$ such that

- there exists $\varepsilon_2 > 0$ with $(S_2(\lambda), T_2(\lambda)) \in D_2$ for all $\lambda \in [\bar{\lambda} - \varepsilon_2, \bar{\lambda} + \varepsilon_2]$, where $(S_2(\lambda), T_2(\lambda))$ denotes unstable fixed points for $\lambda \in [\bar{\lambda} - \varepsilon_2, \bar{\lambda} + \varepsilon_2]$,
- $C_1 \cap D_2 = \emptyset$ and $C_3 \cap D_2 = \emptyset$.

We refer to [1], Chapter 13 Corollary 13.1.5 for these results.

For the random system (4.4) this means that

1. there exists $\varepsilon_1 > 0$ such that for $S \subset [\bar{\lambda} - \varepsilon_1, \bar{\lambda} + \varepsilon_1]$ the random model (4.4) has an invariant probability distribution $\nu_1$ with $\text{supp} \nu_1 \subset C_1 \times S$, and the marginal distribution of $\nu_1$ in $S$ is $\pi^*$, the invariant distribution of the Markov chain $X$,
2. there exists $\varepsilon_3 > 0$ such that for $S \subset [\bar{\lambda} - \varepsilon_3, \bar{\lambda} + \varepsilon_3]$ the random model (4.4) has an invariant probability distribution $\nu_3$ with $\text{supp}\nu_3 \subset C_3 \times S$, and the marginal distribution of $\nu_3$ in $S$ is $\pi^*$, the invariant distribution of the Markov chain $X$,

3. there exists $\varepsilon_2 > 0$ such that for $S \subset [\bar{\lambda} - \varepsilon_2, \bar{\lambda} + \varepsilon_2]$ the random model (4.4) has a bistability region $B_2 \subset \mathbb{R}^+ \times \mathbb{R}^+$ with $(S_2(\lambda), T_2(\lambda)) \in B_2$ for all $\lambda \in [\bar{\lambda} - \varepsilon_2, \bar{\lambda} + \varepsilon_2]$, i.e., for $(x, y) \in B_2$ there exists $t_i > 0$ such that $\mathbb{P}\{\varphi(t, (x, y), \lambda(\omega)) \in C_i \text{ for all } t > t_i\} > 0$ for $i = 1$ and for $i = 3$,

4. for $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus B_2$ the solutions $\varphi(t, (x, y), \lambda(\omega))$ of (4.1, 4.2) will enter either $C_1$ with probability 1 or $C_3$ with probability 1, and they will remain in these sets for $t \to \infty$, since $C_1$ and $C_3$ are forward invariant.

We refer to [2] for these results. This paper considers continuous time random models, i.e. the random process is a Markov diffusion process, but compactness of $C_1$ and $C_3$ guarantees the same results for Markov chains, see [3] Theorem 1.2 Chapter 2.

Our discussions so far in this section have clarified the global behavior of the random model (4.1, 4.2, 4.3) for small values of the $\varepsilon_i > 0$, $i = 1, 2, 3$, i.e. for small variations of $\lambda$ around $\bar{\lambda}$. As variations in $\lambda$ increase, we may see global bifurcations of the sets $C_1$, $C_3$, and $B_2$.

### 4.4 Numerics of the Random Marotzke and Stone Model

#### 4.4.1 Computation of all Possible Trajectories

In this section with compute numerically the trajectories of the system (4.1, 4.2) for a set $S = \{\lambda_1, ..., \lambda_m\}$. We first create all possible trajectories of the system from an initial value $(T_0, S_0)$, resulting in a 'web' of paths that the random system will follow (with certain probabilities). Starting at time $t = 0$, the system

$$
\dot{T} = \lambda(T_E - T) - 2k|\alpha T - \beta S|T
$$

$$
\dot{S} = \frac{2}{\varepsilon_w D} \frac{S_0}{\gamma T - 2k|\alpha T - \beta S|S}
$$
Figure 4.1 Global dynamics using all possible solutions.

is integrated from an initial value using each possible value of \( \lambda \), which creates \( m \) different trajectories. At each time step \( nL, n \in \mathbb{N} \), where \( L \) is the time scale, \( m \) new directions are chosen, resulting in a 'web' depicting the possible behavior of the system.

An example is as follows: Let \( S = \{ \lambda_1, ..., \lambda_3 \} = \{0.3\bar{\lambda}, 0.9\bar{\lambda}, 1.1\bar{\lambda}\} \). The initial value is \((T_0, S_0) = (290, 71)\), the time scale is \( L = 10^9 \) and \( n = 1, \ldots, 7 \).

The following figures show how the system behaves traveling along all possible trajectories. Figure 4.1 shows the global 'web' of the system. Note that separation occurs at the initial value depending on the trajectory taken by the system: one group of trajectories moves towards the stable set \( C_1 \) around the fixed points \((S_1(\lambda), T_1(\lambda))\), another group of trajectories moves towards the stable set \( C_3 \) around the fixed points \((S_3(\lambda), T_3(\lambda))\). This shows that the initial value \((T_0, S_0) = (290, 71)\) is in the bistability area \( B_2 \) described in Section 4.3. The trajectories vary greatly as the simulation is running but as they approach the fixed points, the trajectories converge quickly to the sets \( C_1 \) or \( C_3 \).

Figures 4.2 and 4.3 show the trajectories around the two stable sets \( C_1 \) and \( C_3 \), respectively.
The trajectories converge to the areas around the stable fixed points determined by the different \( \lambda \) values in \( S \). Figure 4.3 shows clearly the bistability effect.

### 4.4.2 Computation of Random Trajectories

The next step is to create random trajectories of the system (4.1, 4.2, 4.3). This is accomplished in the following way:

Let \( U[0, 1] \) be the uniform distribution on \([0, 1]\), and let \( p \in \mathbb{R}^m \) be a probability vector, i.e. \( p_i \geq 0 \) for all \( i = 1, \ldots, m \) and \( \sum_{i=1}^{m} p_i = 1 \). We generate a random number \( \eta \) from the \( U[0, 1] \) distribution and pick \( i \in S \) if \( \eta \in [\sum_{j=0}^{i-1} p_j, \sum_{j=0}^{i} p_j) \), where we set \( p_0 = 0 \). This process is first used for \( t = 0 \) for the initial distribution \( \pi_0 \) of the Markov chain, and then for \( t = nL, \quad n \in \mathbb{N} \) using the appropriate row of the transition probability matrix \( P \), i.e. if \( X_{n-1} = i \), we use the \( i-th \) row of \( P \) to determine the next state \( j \) using the process above. In this way we generate a trajectory of the Markov process \( \{X_n, \quad n \in \mathbb{N}\} \). This trajectory is then used to integrate (4.1, 4.2) as described above.
As an example we use the system parameters as described in the previous section, with transition probability matrix

\[ P = \begin{pmatrix} .5 & .5 & 0 \\ 0 & 0 & 1 \\ .5 & .5 & 0 \end{pmatrix}. \]

Figure 4.4 shows a set of random trajectories of the system (4.1, 4.2, 4.3) with initial value \((T_0, S_0) = (160, 10)\).

This shows the system being integrated along the random trajectories generated using the steps above. As is shown, the random trajectories vary greatly but towards the end of the simulation they converge to the set \(C_1\). Therefore, the initial value \((T_0, S_0) = (160, 10)\) belongs to the domain of attraction of the compact, forward invariant set \(C_1\).
4.4.3 Computation of the Invariant Measure

As we have seen in Section 4.4.2, trajectories outside of the bistability set $B_2$ will converge to one of the compact, forward invariant sets $C_1$ or $C_3$. Once they enter one of these sets, they will build up an invariant measure of the random Marotzke and Stone model (4.1, 4.2, 4.3). The invariant measures $\nu_1$ and $\nu_3$ are best envisioned via a density plot. We generate a grid in the neighborhood of the stable fixed points and count for each cell the number of times the trajectory is in this cell at times $nL$, $n \in \mathbb{N}$. The relative frequencies of these counts form a density plot over the given grid.

As an example we have used the system parameter as in the previous section. Figures 4.5 and 4.7 show such density plots: The cells are color-coded from light blue to pink in increasing frequency. Figure 4.6 is a blow-up (closer view) of the area around the invariant set $C_1$. Both Figures 4.5 and 4.6 are generated from the initial value $(T_0, S_0) = (160, 10)$ and show substantial transient behavior of the system, since it converges slowly to the support of the invariant measure $\nu_1$. 

![Figure 4.4 System movement along random trajectories](image-url)
Figure 4.5  Density grid of system around the stable fixed point.

Figure 4.6  Zoom in of density grid.
CHAPTER 5. Conclusion

From this paper, one may conclude that when a system is made random it may still converge to an invariant set with invariant measure. The deterministic system is shown to be a stable system consisting of two stable fixed point and one saddle point that creates a separatrix. After creating the random system, the system is still stable with the main differences being that instead of convergence to a point, there is convergence to a set and the creation of a bistability region. This type of random system analysis can be applied to all sorts of systems to prove the capabilities of different models.
APPENDIX A. Matlab code for deterministic model

A.1 The function "thesis"

function dy=thesis(t,y)

alpha=2*10^(-4);
beta=0.8*10^(-3);
Te=303;
gamma=2*10^(-10);
k=2*10^(-8);
epsilon=1/6;
epsilonw=.3;
chi=1.3;
B=1.7;
Crho0D=2*10^(10);
S0=35;
D=5*10^3;
con=2/epsilonw*S0/D*gamma;
lambda=1/epsilon*(2*chi+B)/Crho0D;

dy=zeros(2,1);
dy(1)=lambda*(Te-y(1))-2*k*abs(alpha*y(1)-beta*y(2))*y(1);
dy(2)=con*y(1)-2*k*abs(alpha*y(1)-beta*y(2))*y(2);
A.2 Main program

clear all
format long

alpha=2*10^(-4);
beta=0.8*10^(-3);
Te=303;
gamma=2*10^(-10);
k=2*10^(-8);
epsilon=1/6;
epsilonw=.3;
chi=1.3;
B=1.7;
Crho0D=2*10^(10);
S0=35;
D=5*10^3;
con=2/epsilonw*S0/D*gamma;
lambda=1/epsilon*(2*chi+B)/Crho0D

S1=72.4569;
T1=294.571;

S2=1.20341;
T2=157.644;

S3=74.8554;
T3=294.827;

%Plots fixed points
figure(1)
plot(S1,T1,'*',S2,T2,'x',S3,T3,'o')
axis([0 100 1 400])

%Plots the global solution
[G1,Y1]=ode45(@thesis,[0 10000000000],[50 10]);
[G2,Y2]=ode45(@thesis,[0 10000000000],[150 10]);
[G3,Y3]=ode45(@thesis,[0 10000000000],[250 10]);
[G4,Y4]=ode45(@thesis,[0 10000000000],[350 10]);
[G5,Y5]=ode45(@thesis,[0 10000000000],[50 30]);
[G6,Y6]=ode45(@thesis,[0 10000000000],[150 30]);
[G7,Y7]=ode45(@thesis,[0 10000000000],[250 30]);
[G8,Y8]=ode45(@thesis,[0 10000000000],[350 30]);
[G9,Y9]=ode45(@thesis,[0 10000000000],[50 50]);
[G10,Y10]=ode45(@thesis,[0 10000000000],[150 50]);
[G11,Y11]=ode45(@thesis,[0 10000000000],[250 50]);
[G12,Y12]=ode45(@thesis,[0 10000000000],[350 50]);
[G13,Y13]=ode45(@thesis,[0 10000000000],[50 70]);
[G14,Y14]=ode45(@thesis,[0 10000000000],[150 70]);
[G15,Y15]=ode45(@thesis,[0 10000000000],[250 70]);
[G16,Y16]=ode45(@thesis,[0 10000000000],[350 70]);
[G17,Y17]=ode45(@thesis,[0 10000000000],[50 90]);
[G18,Y18]=ode45(@thesis,[0 10000000000],[150 90]);
[G19,Y19]=ode45(@thesis,[0 10000000000],[250 90]);
[G20,Y20]=ode45(@thesis,[0 10000000000],[350 90]);
```matlab
figure(2)
plot(S1,T1,'*',S2,T2,'x',S3,T3,'o')
axis([0 100 0 400])
hold on
plot(Y1(:,2),Y1(:,1),'y',Y2(:,2),Y2(:,1),'m',Y3(:,2),Y3(:,1),'c',Y4(:,2),
Y4(:,1),'r',Y5(:,2),Y5(:,1),'g',Y6(:,2),Y6(:,1),'b')
hold on
plot(Y7(:,2),Y7(:,1),'y',Y8(:,2),Y8(:,1),'m',Y9(:,2),Y9(:,1),'c',Y10(:,2),
Y10(:,1),'r',Y11(:,2),Y11(:,1),'g',Y12(:,2),Y12(:,1),'b')
hold on
plot(Y13(:,2),Y13(:,1),'y',Y14(:,2),Y14(:,1),'m',Y15(:,2),Y15(:,1),'c',Y16(:,2),
Y16(:,1),'r',Y17(:,2),Y17(:,1),'g',Y18(:,2),Y18(:,1),'b')
hold on
plot(Y19(:,2),Y19(:,1),'m',Y20(:,2),Y20(:,1),'g')%Plots unstable fixed point solution with unstable manifold
[G1,Y1]=ode45(@(thesis,[0 10000000000],[275 70]);
[G2,Y2]=ode45(@(thesis,[0 10000000000],[285 70]);
[G3,Y3]=ode45(@(thesis,[0 10000000000],[295 70]);
[G4,Y4]=ode45(@(thesis,[0 10000000000],[305 70]);
[G5,Y5]=ode45(@(thesis,[0 10000000000],[275 73]);
[G6,Y6]=ode45(@(thesis,[0 10000000000],[285 73]);
[G7,Y7]=ode45(@(thesis,[0 10000000000],[295 73]);
[G8,Y8]=ode45(@(thesis,[0 10000000000],[305 73]);
[G9,Y9]=ode45(@(thesis,[0 10000000000],[275 76]);
[G10,Y10]=ode45(@(thesis,[0 10000000000],[285 76]);
[G11,Y11]=ode45(@(thesis,[0 10000000000],[295 76]);
```
\[ G_{12}, Y_{12} = \text{ode45}(\text{@thesis}, [0 \ 10000000000], [305 \ 76]); \]
\[ G_{13}, Y_{13} = \text{ode45}(\text{@thesis}, [0 \ 10000000000], [275 \ 79]); \]
\[ G_{14}, Y_{14} = \text{ode45}(\text{@thesis}, [0 \ 10000000000], [285 \ 79]); \]
\[ G_{15}, Y_{15} = \text{ode45}(\text{@thesis}, [0 \ 10000000000], [295 \ 79]); \]
\[ G_{16}, Y_{16} = \text{ode45}(\text{@thesis}, [0 \ 10000000000], [305 \ 79]); \]

\text{figure(3)}
\text{plot(S1,T1,'o',S3,T3,'x')}  
\text{axis([65 85 270 310])} 
\text{hold on}
\text{plot(Y1(:,2),Y1(:,1),'y',Y2(:,2),Y2(:,1),'m',Y3(:,2),Y3(:,1),'c',Y4(:,2),Y4(:,1),'r',Y5(:,2),Y5(:,1),'g',Y6(:,2),Y6(:,1),'b')}  
\text{hold on}
\text{plot(Y7(:,2),Y7(:,1),'y',Y8(:,2),Y8(:,1),'m',Y9(:,2),Y9(:,1),'c',Y10(:,2),Y10(:,1),'r',Y11(:,2),Y11(:,1),'g',Y12(:,2),Y12(:,1),'b')}  
\text{hold on}

\%Stable Manifold
\[ [G_{G}, Y_{Y}] = \text{ode45}(\text{@thesis}, [0 \ -10^{10}], [T1+0.001 \ S1]); \]
\[ [G_{G2}, Y_{Y2}] = \text{ode45}(\text{@thesis}, [0 \ -10^{10}], [294 \ S1]); \]
\text{plot(YY(:,2),YY(:,1),'k',YY2(:,2),YY2(:,1),'k');}
APPENDIX B. Matlab code for stochastic model

B.1 Function file "thesismc"

```matlab
function dy=thesis_mc(t,y,L)

alpha=2*10^(-4);
beta=0.8*10^(-3);
Te=303;
gamma=2*10^(-10);
k=2*10^(-8);
epsilon=1/6;
epsilonw=.3;
chi=1.3;
B=1.7;
Crho0D=2*10^(-10);
S0=35;
D=5*10^(-3);
con=2/epsilonw*S0/D*gamma;

dy=zeros(2,1);
dy(1)=L*(Te-y(1))-2*k*abs(alpha*y(1)-beta*y(2))*y(1);
dy(2)=con*y(1)-2*k*abs(alpha*y(1)-beta*y(2))*y(2);
```
B.2 Main program for all possible solutions

```matlab
warning off
clear all

L=[.3*1.26*10^-9;.9*1.26*10^-9;1.1*1.26*10^-9];
ya=[290;71];

ic=ya;

options = odeset('RelTol',1e-20,'AbsTol',[1e-15 1e-15]);

timescale=10^10;
kicktime=10^9;
tau=kicktime;
tzau=[0 tau];
newic1(:,1) = ic;

for i=1:7
    for j=1:3^(i-1)
        ic = newic1(:,j) ;
        [T,Y1] = ODE45(@thesis_mc,tzau,ic,options,L(1));
        newic2(:,3*j-2) = Y1(end,:);
        [T,Y2] = ODE45(@thesis_mc,tzau,ic,options,L(2));
        newic2(:,3*j-1) = Y2(end,:);
        [T,Y3] = ODE45(@thesis_mc,tzau,ic,options,L(3));
        newic2(:,3*j) = Y3(end,:);
        newic2 ;
        plot(Y1(:,2),Y1(:,1),'m',Y2(:,2),Y2(:,1),'b',Y3(:,2),Y3(:,1),'g')
    end
end
```
hold on
end
newic1 = newic2;
tzau=[T(end) T(end)+tau];
end
xlabel('S')
ylabel('T')

B.3 Main program for random path and density grid

clear all
warning off

lambda=1.26*10^-9;
L=[.3*lambda;.9*lambda;1.1*lambda];

options = odeset('RelTol',1e-20,'AbsTol',[1e-15 1e-15]);
N=1000;
a=1;
b=3;
hx=(b-a)/N;
c=120;
d=160;
hy=(d-c)/N;
C=zeros(N,N);

P=[.5 .5 0 0 0 1;.5 .5 0];
pi0=[.2; .2; .6];
timescale=10^-10;
kicktime=1000;
tau=kicktime;
tzau=[0 tau];
ic=[157.64;1.2034];

for j=1:10
    Y=rand(100000,1);
    for i=1:length(Y)
        if (Y(i) <= pi0(1))
            T(i)=1;
            pi0=P(1,:);
        elseif ((Y(i) > pi0(1)) & (Y(i) <= pi0(1)+pi0(2)))
            T(i)=2;
            pi0=P(2,:);
        else
            T(i)=3;
            pi0=P(3,:);
        end
    end
    for i=1:length(T)
        [Time,YY] = ODE45(@thesis_mc,tzau,ic,options,L(T(i)));
        ic=YY(end,:);
        plot(YY(:,2),YY(:,1))
        hold on
    end
    if (i >= .20*length(Y))
if (a <= YY(end,2) <= b) & (c <= YY(end,1) <= d)
    n=floor((YY(end,2)-a)/hx+1);
    m=floor((YY(end,1)-c)/hy+1);
    C(N-m+1,n)=C(N-m+1,n)+1;
end

end

tzau=[Time(end) Time(end)+tau];

end

ic=[160;10];
pi0=[.2;.2;.6];
ic=[157.64;1.2034];
end

% density plot
conv=max(max(C))/255;
C=C./conv;
figure(2)
image(C)
colormap(cool)
BIBLIOGRAPHY


