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DEFICIENCY AND STABILITY IN INFINITE DIMENSIONAL LINEAR TOPOLOGY

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CHAPTER I. INTRODUCTION

R. D. Anderson(2, 3) has recently developed some very powerful methods in the topology of the Hilbert cube which, among other things, characterize sets of topological infinite deficiency, and give stable extensions of homeomorphisms between certain closed subsets. He is able to apply these methods to separable infinite-dimensional Fréchet spaces by considering the natural imbedding of s , the countable infinite product of lines, in I^∞ and by using the topological equivalence of all separable infinite-dimensional Fréchet spaces. This latter result follows from the combined work of Bessaga and Pelczynski, Kadec, and Anderson (see (1)).

Klee(10, 11) has previously established numerous topological properties of Hilbert space ℓ_2 through linear space methods, involving category, convexity, and deficiency. His results include the homogeneity of Hilbert space with respect to compact sets, the homogeneity of any infinite-dimensional Banach space with respect to finite-dimensional compact sets, and an extremely useful homeomorphism extension lemma for Fréchet spaces.

The methods presented here are of the linear space type, although Anderson's results figure prominently in Chapter III and part of Chapter V. The main topics of interest are the various deficiency conditions (finite, infinite, homotopy) and homogeneity, and isotopy and stability. Certain other results and applications appear in Chapter V. Various conditions on the spaces, such as Hausdorff, metrizability, local convexity, normability, inner-product, completeness, and separability, are used as needed.

In Chapter II it is shown that closed σ -compact (compact) subsets of infinite-dimensional Banach (Fréchet) spaces have finite deficiency

(which is defined as a topological property). Methods due to Klee and Cernavski are instrumental in the proof. Klee's extension lemma is used to obtain a homogeneity result for certain closed finite-dimensional subsets of infinite-dimensional Banach spaces, and at the same time the finite stability of such spaces is established.

The relationship of various deficiency conditions and their connection with homogeneity, in separable infinite-dimensional Fréchet spaces, is considered in Chapter III. In particular, it is shown that, under rather general conditions, topological infinite deficiency implies finite deficiency implies homotopy deficiency implies Property Z. Anderson's fundamental theorems, derived primarily by consideration of homeomorphisms in the Hilbert cube, establish the equivalence of Property Z and topological infinite deficiency, and homogeneity with respect to sets having Property Z, in separable infinite-dimensional Fréchet spaces. These have some rather interesting consequences: the equivalence of finite and topological infinite deficiency, the preservation of deficiency under closed countable unions, homogeneity with respect to closed σ -compact sets, the local character of homogeneity and deficiency.

Methods of Whittaker(15) are used in Chapter IV to show that, in a locally convex Hausdorff linear space, every stable homeomorphism is the product of two homeomorphisms each somewhere the identity. By Alexander's technique, applied to any topological linear space, every stable homeomorphism is isotopic to the identity. An application is made of Wong's isotopy methods(16) for infinite product spaces to certain separable infinite-dimensional inner-product spaces, and a condition for stability of homeomorphisms is given.

In Chapter V, Anderson's results are applied to show that, in separable infinite-dimensional Fréchet spaces, complements of deficient subsets are homeomorphic to the whole space. Certain results of Klee for infinite-dimensional normed linear spaces are noted, and an application is made to complements of countable locally compact sets. A non-homeomorphism theorem, dependent on the connectedness properties of complements of σ -compact sets in separable metric spaces, is given. The application is to the topological type represented by separable infinite-dimensional Fréchet space.

CHAPTER II. FINITE DEFICIENCY IN BANACH SPACES

The following lemmas will be useful at several points. For definitions and terminology involved, see Taylor(14).

Lemma 2.1. Let M be a finite-dimensional subspace of a locally convex Hausdorff linear space, and let P be a closed subspace with $M \cap P = \theta$. Then M has a closed complement N with $N \supset P$.

Proof: By induction on $\dim M$. Suppose $\dim M = 1$. Let f be a continuous linear functional on $M + P$, with $f^{-1}(0) = P$. By local convexity f may be extended to a continuous linear functional F on the whole space. The kernel of F is a closed hyperplane N , complementary to M . Now suppose $\dim M = k+1$, and let $M = M_1 \oplus M_k$, $\dim M_1 = 1$. Then $M_k \cap (M_1 + P) = \theta$, and by the inductive hypothesis M_k has a closed complement $N_k \supset M_1 + P$. By the above argument there exists a decomposition $N_k = M_1 \oplus N$ into closed subspaces with $N \supset P$, and $M \oplus N$ is the desired decomposition.

Lemma 2.2. Let $X = M \oplus N$ be a decomposition of a topological linear space, with M finite-dimensional and N closed. Then the projections onto M and N are continuous.

Proof: Since continuity of one projection implies continuity of the other (with the appropriate inclusion maps, the sum of the projections is the identity), it suffices to prove continuity of the projection onto M . The quotient map $Q: X \rightarrow X/N$ is continuous, and since N is closed, X/N is Hausdorff. Let $F: X/N \rightarrow M$ be the linear isomorphism defined by $F(m + N) = m$, $m \in M$. Since M is finite-dimensional, X/N is topologically isomorphic to some Euclidean space R^m , and M is topologically isomorphic to a product $T \times G$, where T has the trivial topology and G is Hausdorff, hence Euclidean (9, p. 41). Then obviously the topology of R^m refines that of

$T \times G$, and $F: X/N \rightarrow M$ is continuous. Therefore $FQ: X \rightarrow M$, the projection onto M , is continuous.

It is a consequence of the closed graph theorem that, for any decomposition of a complete metric linear space into closed subspaces, the projections are continuous (14, p. 242).

Lemma 2.3. In a locally convex metric linear space X the convex hull of a precompact set S is precompact.

Proof: Let $|x| = d(x, \theta)$, where d is an invariant metric for X . Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that the convex hull of $N_\delta(\theta)$ is contained in $N_{\varepsilon/2}(\theta)$. Choose $A = \{x_1, \dots, x_n\} \subset X$ such that $S \subset B = N_\delta(A)$. Then the convex hull $B_c \subset N_{\varepsilon/2}(A_c)$. Since A_c is compact (14, p. 134), this implies a finite number of ε -neighborhoods cover B_c , hence S_c . Thus S_c is precompact.

The following fundamental lemma is essentially due to Klee(10).

Lemma 2.4. Let A be a σ -precompact subset of an infinite-dimensional Fréchet space X . Then there is a line L in X , no translate of which intersects A_c more than once.

Proof: We may suppose $A = \bigcup_1^\infty A_i$, $A_i \subset A_{i+1}$, each A_i precompact. By the preceding lemma, $A_c = \bigcup_1^\infty (A_i)_c$ is σ -precompact. By completeness, the closure of a precompact set in X is compact, hence A_c , and therefore $A_c - A_c$, is contained in a σ -compact set B . Every compact set in X is nowhere dense, since otherwise X would be locally compact and therefore finite-dimensional (14, p. 129). For some $u \in X \setminus \theta$, the line $L = \{tu: -\infty < t < \infty\}$ intersects $A_c - A_c$ only at θ , for if not, $\bigcup_1^\infty nB \supset \bigcup_1^\infty n(A_c - A_c) = X$, and X would be first category. Thus no translate of L can intersect A_c more than once.

Definition. Let A be a closed subset of an infinite-dimensional

topological linear space X . If for each positive integer m , there is a decomposition $X = M \oplus N$ with $\dim M = m$ and N closed, and a homeomorphism h of X onto itself such that $hA \subset N$, then A has finite deficiency in X .

In (10, 11), Klee shows that compact sets in infinite-dimensional Banach spaces have finite deficiency.

Theorem 2.5. Let A be a closed σ -compact (compact) subset of an infinite-dimensional Banach (Fréchet) space X . Then A has finite deficiency in X .

Proof: Let $L = \{tu: -\infty < t < \infty\}$ be the line (through the origin) of Lemma 2.4. Let H be a closed complementary hyperplane (use Lemma 2.1). Let $\tau: X \rightarrow H$ be the (continuous) projection onto H . Then τ/A_c is 1-1. If A is compact, τ/A is a homeomorphism onto τA , which is closed in H . The homeomorphism $(\tau/A)^{-1}$ defines a map $m: \tau A \rightarrow (-\infty, \infty)$, which can be extended to a map $M: H \rightarrow (-\infty, \infty)$. Then $h + tu \rightarrow h + (t - M(h))u$ defines a homeomorphism of X , extending τ/A . By a simple inductive argument, A has finite deficiency.

Suppose X is a Banach space; assume $\|u\| = 1$. The geometric flavor of the following can be imputed to Cernavski(5), who uses similar constructions in Euclidean space. For $h \in H \setminus \theta$, define the 'semicircle' $S_h = \{t\|h\|u + rh: t^2 + r^2 = 1, r \geq 0\}$. Let $\mathcal{S} = \{S_h: h \in H \setminus \theta\}$. Each point of $X \setminus L$ lies on exactly one semicircle in \mathcal{S} . Assuming $\theta \in A$, we have $A \subset (X \setminus L) \cup \theta$, and no semicircle intersects A more than once (otherwise τ/A_c is not 1-1). For $y \in X \setminus L$, $y = t\|h\|u + rh$ is the canonical representation of y as a point on the appropriate semicircle S_h . Define $T = \{y \in X \setminus L: |t| \leq \sqrt{2}/2\} \cup \theta$. Note that $|t| \leq \sqrt{2}/2$ iff $|t| \leq r$ iff $r \geq \sqrt{2}/2$.

Lemma 2.6. There is a homeomorphism F of X such that $FA \subset T$, and

for $y \in FA \setminus \theta$, $t \rightarrow 0$ ($r \rightarrow 1$) as $\|y\| \rightarrow 0$.

Proof: Since A is closed and $L \setminus \theta$ does not intersect A , there is a map $\psi: (0, \infty) \rightarrow (0, 1)$ such that $\{y \in X \setminus L: r < \psi(\|h\|)\} \cap A = \emptyset$. Let $\xi: [0, \infty) \rightarrow [\sqrt{2}/2, 1]$ be a map such that $\xi(0) = 1$, $\xi(a) < 1$ for $a > 0$. Consider the collection $\{Q_h^+, Q_h^-\}: h \in H \setminus \theta\}$ of 'quartercircles' obtained by bisecting the semicircles $\{S_h\}$ at h . Define $F/L \cup H = \text{id}$; on each Q_h^+ (Q_h^-), F maps the point y with $r_y = \psi(\|h\|)$ onto the point z with $r_z = \xi(\|h\|)$, and is extended 'linearly' (with respect to r) on the entire quartercircle Q_h^+ (Q_h^-). Then F is the desired homeomorphism.

Thus we may assume, in proving the theorem, that $A \subset T$, and for $y \in A \setminus \theta$, $t \rightarrow 0$ as $\|y\| \rightarrow 0$. Let S_A denote the subfamily of S consisting of those semicircles which intersect A . Define a bijection $G_A: \bigcup S_A \cup \theta \rightarrow \bigcup S_A \cup \theta$ as follows: $G_A(\theta) = \theta$, and for $S_h \in S_A$, G_A/S_h is fixed on the endpoints (in L), takes $S_h \cap A$ onto $h = S_h \cap H$, and is extended linearly on S_h .

Lemma 2.7. G_A is a homeomorphism, and $G_A A$ is closed.

Proof: Obviously G_A/A is continuous. For the continuity of both G_A and G_A^{-1} it suffices to show continuity of $G_A^{-1}/G_A A$. Suppose $G_A^{-1}/G_A A$ is not continuous. Then there exists a sequence $G_A(a_n) \rightarrow G_A(a)$, $a_n, a \in A$, such that no subsequence of a_n converges to a . With $a_n = t_n \|h_n\| u + r_n h_n$, $a = t \|h\| u + r h$, we have $G_A(a_n) = h_n \rightarrow h = G_A(a)$. Since there is some subsequence $a_{n'}$ for which $t_{n'}$ and $r_{n'}$ converge, we have $a_{n'} \rightarrow a' \in A$, hence $G_A(a_{n'}) \rightarrow G_A(a')$, $G_A(a) = G_A(a')$, $a = a'$, and $a_{n'} \rightarrow a$, contrary to hypothesis.

To see that $G_A A$ is closed, let $G_A(a_n) \rightarrow h \in H$. Then, as before, $h_n \rightarrow h$, there is some subsequence $a_{n'}$ such that $t_{n'}$ and $r_{n'}$ converge, thus $a_{n'} \rightarrow a' \in A$, $G_A(a_{n'}) \rightarrow G_A(a') = h \in G_A A$, and $G_A A$ is closed.

The homeomorphism $G_A^{-1}/G_A A$ defines a map $m: G_A A \rightarrow [-\sqrt{2}/2, \sqrt{2}/2]$, where $m(G_A(t\|h\|u + rh)) = t$, $m(\theta) = 0$. Since $G_A A$ is closed, there is an extension of m to $M: H \rightarrow [-\sqrt{2}/2, \sqrt{2}/2]$, and this extension may be used in the obvious manner to define a homeomorphism $G: X \rightarrow X$, extending G_A . Again, a simple inductive argument completes the proof of the theorem.

The following homeomorphism extension lemma of Klee(11) is based on Dugundji's theorem that locally convex Hausdorff linear spaces are absolute extensors for the class of metrizable spaces.

Lemma 2.8. Let $X = L \oplus L_2$ be a decomposition of a Fréchet space into closed subspaces. Let A be a closed subset of L , and let h be a homeomorphism of A onto a closed subset of L_2 . Then there is a $*$ -isotopy \mathcal{J} of X onto X such that $\mathcal{J}_0 = \text{id}$ and $\mathcal{J}_1/A = h$. (An isotopy $\{H_t\}$ on X is called a $*$ -isotopy if its level map on $X \times I$, defined by $(x, t) \rightarrow (H_t(x), t)$, is a homeomorphism, i.e., if the family $\{H_t^{-1}\}$ of inverse homeomorphisms is also an isotopy).

Proof: By Dugundji's theorem, there are maps α of L , into L_2 and β of L_2 into L , such that $\alpha/A = h$ and $\beta/hA = h^{-1}$. By completeness, the projections of X onto L , and L_2 are continuous. For $y \in X$ and $t \in [0, 1]$, let $\xi_t y = y + t(\alpha y_1)$ and $\eta_t y = y - t(\beta y_2)$, where $y = y_1 + y_2$ with $y_i \in L_i$. Then ξ and η are $*$ -isotopies of X onto X . Now let $\mathcal{J}_t = \xi_{2t}$ for $t \in [0, 1/2]$ and $\mathcal{J}_t = \eta_{2t-1} \xi_1$, for $t \in [1/2, 1]$. Then if $y \in A$, $\mathcal{J}_1 y = \eta_1 \xi_1 y = \eta_1(y + \alpha y_1) = y + \alpha y_1 - \beta \alpha y_1 = y - h^{-1} h y_1 + \alpha y_1 = \alpha y_1$, so $\mathcal{J}_1/A = \alpha/A = h$, and \mathcal{J} is the desired $*$ -isotopy.

The proof of the following proposition is due to D. E. Sanderson.

Proposition 2.9. Every finite-dimensional locally compact separable metric space A can be imbedded as a closed subset of a Euclidean space.

Proof: The one-point compactification \tilde{A} is metrizable (7, p. 75) and finite-dimensional. Therefore \tilde{A} can be imbedded as a (closed) subset of S^n for some n , and deleting the compactifying point ω , we have $A = \tilde{A} \setminus \omega$ imbedded as a closed subset of $S^n \setminus \omega \sim R^n$.

Definition (Brown-Gluck(4)). A homeomorphism h on a space X is stable if h can be written as a composition of finitely many homeomorphisms on X each of which is the identity on some nonempty open set in X .

The stable homeomorphisms form a normal subgroup of the group of all homeomorphisms of X . From the proof of (2.5) it is clear that the homeomorphism constructed there can be chosen to be stable.

Klee(11) proves the following theorem (stressing isotopy rather than stability), for finite-dimensional compact sets in Banach spaces.

Theorem 2.10. Let X be an infinite-dimensional Banach (Fréchet) space, and let h be a homeomorphism between closed finite-dimensional separable locally compact (compact) subsets of X . Then h can be extended to a stable homeomorphism H of X .

Proof: Let $h: A \rightarrow h(A)$. Observe that $A \cup h(A)$ satisfies the hypotheses of Theorem 2.5. Thus there exists a decomposition $X = T \oplus E \oplus N$, where $\dim T = 1$, E is a finite-dimensional subspace containing a closed copy A' of A , and N is closed; and a homeomorphism F of X such that $F(A \cup h(A)) \subset N$. Applying (2.8) twice on $E \oplus N$ (first extending any homeomorphism of $F(A)$ onto A' , then extending the appropriate homeomorphism of A' onto $Fh(A)$), we obtain a $*$ -isotopy \mathcal{J} of $E \oplus N$, with $\mathcal{J}_i = \text{id}$ and such that $F^{-1} \mathcal{J}_0 F/A = h$. Define $G: X \rightarrow X$ as follows:

$$G(t + e + n) = \begin{cases} t + \mathcal{I}_2(e + n), & 0 \leq t \leq 1 \\ t + \mathcal{I}_0(e + n), & t \leq 0 \\ t + e + n, & t \geq 1. \end{cases}$$

Then $H = F^{-1}GF$ is the desired stable extension of h .

Definition. A topological linear space X is finitely stable if, for every homeomorphism f of X and every finite-dimensional subspace N , there is a stable homeomorphism g of X with $f/N = g/N$.

Corollary 2.11. Every infinite-dimensional Banach space is finitely stable.

Let $\{u_1, u_2, \dots\}$ be a complete orthonormal set in the Hilbert space ℓ_2 . Let $W = \text{sp} \{u_1, u_2, \dots\}$, and for each n , let $R^n = \text{sp} \{u_1, \dots, u_n\}$. From a consideration of Cernavski's methods(5) for finite-dimensional spaces, whereby he shows that every homeomorphism of R^n is k -stable (agrees with a stable homeomorphism on a k -dim subspace) for $k < \frac{2}{3}n - 1$, it seems likely that the following conjecture and its corollary are true.

Conjecture 2.12. Let $M \supset W$ be a linear subspace of ℓ_2 , and let h be a homeomorphism of M . Then, for each n , h agrees with a stable homeomorphism of M on R^n .

Corollary 2.13. Every separable infinite-dimensional inner-product space is finitely stable.

Proof: Let P be such a space, and let A be a finite-dimensional subspace, with basis $\{x_1, \dots, x_n\}$. Let $\{x_1, \dots, x_n, \dots\}$ be a countable dense subset of P . Let $y_1 = x_1$. Let y_{k+1} be the first x_j which is not in $\text{sp} \{y_1, \dots, y_k\}$. Then the linearly independent sequence $\{y_k\}$ generates the same linear subspace, hence the same closed linear subspace P , as the sequence $\{x_k\}$. Applying the Gram-Schmidt orthogonalization process to

$\{y_k\}$, we obtain an orthonormal set $\{v_1, v_2, \dots\}$ generating the closed linear subspace P , with $\text{sp}\{v_1, \dots, v_n\} = \text{sp}\{y_1, \dots, y_n\} = \text{sp}\{x_1, \dots, x_n\} = A$. Since P is congruent to a dense linear subspace of \mathcal{L}_2 , we may suppose $P = M \supset W$, $v_i = u_i$ for each i , and $A = \mathbb{R}^n$. The corollary follows.

CHAPTER III. DEFICIENCY AND HOMOGENEITY IN SEPARABLE FRÉCHET SPACES

Notation. E^{n+1} is an $(n+1)$ -cell, with $\text{Bd}(E^{n+1}) = S^n$.

Lemma 3.1. Let $X = L_1 \oplus L_2$ be a decomposition of a metric linear space with L_1 closed and $\dim L_2 = n + 2$, and let A be a closed subset of L_1 . Let $f: E^{n+1} \rightarrow X$ with $f(S^n) \cap A = \emptyset$, and let $\epsilon > 0$ be given. Then there exists $g: E^{n+1} \rightarrow X$ with $f/S^n = g/S^n$, $d(f, g) < \epsilon$, and $g(E^{n+1}) \cap A = \emptyset$.

Proof: Let \mathcal{T}_i be the projection onto L_i , and consider $f = (f_1, f_2)$, where $f_i = \mathcal{T}_i f$, $i = 1, 2$. We construct $g = (g_1, g_2)$ with $g_1 = f_1$, $g_2/S^n = f_2/S^n$, $d(f_2, g_2) < \epsilon$, and such that $g(E^{n+1}) \cap A = \emptyset$. Let $\|\cdot\|$ be a norm on L_2 , generating the same topology as the invariant metric d . Choose $\epsilon_1, \delta_1 > 0$ such that $d(y, \theta) < \delta_1 \Rightarrow \|y\| < \epsilon_1 \Rightarrow d(y, \theta) < \epsilon$. Since $f(S^n)$ is compact, there exists $0 < \delta < \delta_1$ with $S^n \cap f^{-1}(N_\delta(A)) = \emptyset$. Thus $S^n \cap f_1^{-1}(N_{\delta/2}(A)) \cap f_2^{-1}(N_{\delta/2}(\theta)) = \emptyset$. Set $C = f_1^{-1}(N_{\delta/4}(A)) \subset E^{n+1}$, and consider the restriction $f_2/C: C \rightarrow L_2$. If $\theta \notin f_2(C)$, take $g_2 = f_2$. Otherwise, since all values of f_2/C are unstable, θ is unstable, and for every neighborhood U of θ there exists a map $g'_2: C \rightarrow L_2$ satisfying $g'_2(x) = f_2(x)$ if $f_2(x) \notin U$, $g'_2(x) \in U$ if $f_2(x) \in U$, and $\theta \notin g'_2(C)$ (8, pp. 75-79). Take $U = N_{\delta/2}(\theta) \cap L_2$. Then $g'_2 = f_2/C \setminus f_2^{-1}(N_{\delta/2}(\theta))$, and $d(f_2/C, g'_2) < \delta$, hence $\sup_{x \in C} \|f_2(x) - g'_2(x)\| < \epsilon_1$. Define $k'(x) = f_2(x) - g'_2(x)$ for $x \in C$, $k'(x) = \theta$ for $x \in E^{n+1} \setminus f_1^{-1}(N_{\delta/2}(A)) \cap f_2^{-1}(N_{\delta/2}(\theta))$, and extend to $k: E^{n+1} \rightarrow L_2$ so that $\sup_{x \in E^{n+1}} \|k(x)\| < \epsilon_1$. Let $g_2 = f_2 - k$. Then $d(f_2, g_2) < \epsilon$, and $g = (f_1, g_2)$ is the desired map.

Definition. A closed subset A of a metric space X is homotopically deficient if, for each $n \geq -1$ and every map $f: E^{n+1} \rightarrow X$ with $f(S^n) \cap A = \emptyset$, and every $\epsilon > 0$, there exists a map $g: E^{n+1} \rightarrow X$ with $f/S^n = g/S^n$, $d(f, g) < \epsilon$, and $g(E^{n+1}) \cap A = \emptyset$.

Proposition 3.2. Let X be an infinite-dimensional metric linear space, and let $A \subset X$ be finitely deficient. Then A is homotopically deficient.

Proof: Let a map f , and $\varepsilon > 0$, be given. Let $X = L_1 \oplus L_2$ be a decomposition, with L_1 closed and $\dim L_2 = n + 2$, and let h be a homeomorphism such that $h(A) \subset L_1$. Consider $hf: E^{n+1} \rightarrow X$. Since $hf(E^{n+1})$ is compact, h^{-1} is uniformly continuous on it, and there exists $\varepsilon' > 0$ such that $r \in hf(E^{n+1}), s \in X, d(r, s) < \varepsilon' \Rightarrow d(h^{-1}(r), h^{-1}(s)) < \varepsilon$. By (3.1), there exists $hg: E^{n+1} \rightarrow X$ with $hg/S^n = hf/S^n, d(hg, hf) < \varepsilon'$, and $hg(E^{n+1}) \cap hA = \emptyset$. Then g is the desired map.

Proposition 3.3. Let X be a complete metric space, and let $L \subset X$ be a closed countable union of homotopically deficient sets. Then L is homotopically deficient.

Proof: Let $L = \bigcup_1^\infty A_i$. Let $f: E^{n+1} \rightarrow X$ with $f(S^n) \cap L = \emptyset$, and let $\mu > 0$ be given. Take $0 < \varepsilon_i < \mu/2$. There exists $g_i: E^{n+1} \rightarrow X$ with $g_i/S^n = f/S^n, d(f, g_i) < \varepsilon_i$, and $g_i(E^{n+1}) \cap A_i = \emptyset$. Since $g_i(E^{n+1})$ is compact, there exists $\delta_i > 0$ such that $d(g_i(E^{n+1}), A_i) > \delta_i$. Assume g_1, \dots, g_l have been defined, and $\delta_1, \dots, \delta_l$ have been chosen. Take $0 < \varepsilon_{l+1} < \min \{ \delta_1/2^l, \delta_2/2^{l-1}, \dots, \delta_l/2, \mu/2^{l+1} \}$. There exists $g_{l+1}: E^{n+1} \rightarrow X$ with $g_{l+1}/S^n = g_l/S^n = \dots = f/S^n, d(g_l, g_{l+1}) < \varepsilon_{l+1}$, and $g_{l+1}(E^{n+1}) \cap A_{l+1} = \emptyset$. Choose $\delta_{l+1} > 0$ so that $d(g_{l+1}(E^{n+1}), A_{l+1}) > \delta_{l+1}$. The sequence $\{g_i\}$ thus defined converges uniformly to a continuous map $g: E^{n+1} \rightarrow X$, and since $\varepsilon_i < \mu/2^i$ for each i , we have $d(f, g) < \mu$. Obviously $f/S^n = g/S^n$, and by construction $g(E^{n+1}) \cap A_i = \emptyset$ for each i .

Definition (Anderson(3)). A closed subset A of X has Property Z if for each nonempty homotopically trivial open set U in $X, U \setminus A$ is nonempty homotopically trivial.

If X is a metric space it is clear that every homotopically deficient subset has Property Z.

Definitions. A closed subset of a topological linear space has (strong) infinite deficiency if it is contained in a closed subspace with an infinite-dimensional (closed) complement.

A set has topological (strong) infinite deficiency if there is a space homeomorphism taking it onto a set with (strong) infinite deficiency.

The following proposition ensures the relevance of infinite deficiency for a large class of spaces.

Proposition 3.4. Every infinite-dimensional locally convex Hausdorff linear space admits a decomposition $X = L_1 \oplus L_2$ into infinite-dimensional subspaces, with L_1 closed.

Proof: A double sequence $\{x_i, f_i\}$ of points and continuous linear functionals is a biorthogonal sequence if $f_i x_j = \delta_{ij}$ for all i, j . There is a simple inductive method, used by Klee(13) and others, for the construction of such sequences. The Hausdorff condition and local convexity guarantee an adequate supply of continuous linear functionals. Choose x_1, f_1 with $f_1 x_1 = 1$. Suppose x_1, \dots, x_n and f_1, \dots, f_n have been chosen subject to the desired conditions. Set $F_n = \bigcap_1^n f_i^{-1}(0)$, $L_n = \text{sp}\{x_1, \dots, x_n\}$. Then $X = F_n \oplus L_n$. Take $x_{n+1} \in F_n \setminus \emptyset$, and choose f_{n+1} such that $f_{n+1}(x_{n+1}) = 1$, $f_{n+1}(L_n) = 0$. By induction we obtain a biorthogonal sequence $\{x_i, f_i\}$.

Now consider the infinite-dimensional subspaces $M = \text{sp}\{x_1, x_3, \dots\}$ and $N = \text{sp}\{x_2, x_4, \dots\}$. Let $v \in N \setminus \emptyset$. Then for some n , $f_{2n}(v) \neq 0$, and there is some neighborhood U of v such that f_{2n} does not vanish on U . Since $f_{2n}(M) = 0$, we have $M \cap U = \emptyset$, and $\bar{M} \cap N = \emptyset$. Thus $X = L_1 \oplus L_2$ with $L_1 = \bar{M}$ and $L_2 \supset N$.

Proposition 3.5. Let X be a locally convex Hausdorff linear space, and let $A \subset X$ have topological infinite deficiency. Then A has finite deficiency.

Proof: Let $X = L_1 \oplus L_2$ be a decomposition with L_1 closed, $\dim L_2 = \infty$, and let h be a homeomorphism of X such that $h(A) \subset L_1$. For any positive integer n , let $L_{2,n}$ be an n -dimensional subspace of L_2 . By (2.1), there exists a closed complementary subspace $L_{1,n}$ of $L_{2,n}$, such that $L_{1,n} \supset L_1$. Then $L = L_{1,n} \oplus L_{2,n}$, $h(A) \subset L_{1,n}$, and A has finite deficiency.

Thus in an infinite-dimensional locally convex metric linear space, topological infinite deficiency \Rightarrow finite deficiency \Rightarrow homotopy deficiency \Rightarrow Property Z. The following theorems and corollary of Anderson(3) give the converse, by establishing homogeneity with respect to subsets having Property Z, in separable infinite-dimensional Fréchet spaces. In what follows X denotes any such space. The Fréchet space s is the countable infinite product of lines, I^∞ the similar product of closed intervals.

Theorem 3.6. Let $A \subset s$ have Property Z. Then A has topological strong infinite deficiency.

The proof proceeds by imbedding s in I^∞ , establishing that the closure of A in I^∞ has Property Z (in I^∞), and constructing a β^* -homeomorphism of I^∞ which takes the closure onto a set of infinite deficiency in I^∞ (h is β if $h(s) \supset s$, β^* if $h(s) = s$).

Theorem 3.7. Each homeomorphism between two subsets of X with Property Z can be extended to a homeomorphism of X onto itself.

Proof: Let $h: A \rightarrow h(A)$, both A and $h(A)$ having Property Z. Since all separable infinite-dimensional Fréchet spaces are topologically equivalent, there is a homeomorphism k of X onto s . Since Property Z is

topological, $k(A)$ and $kh(A)$ have Property Z in s , and by (3.6) they both have topological strong infinite deficiency. Let $s = L_1 \oplus L_2 = M_1 \oplus M_2$ be decompositions into closed infinite-dimensional subspaces for which there exist homeomorphisms f, g of s with $fk(A) \subset L_1$, and $gkh(A) \subset M_1$. Let ρ be a homeomorphism of s such that $\rho(L_1) = M_2$ and $\rho(L_2) = M_1$. Then $\rho fk(A) \subset M_2$. By Klee's lemma (2.8) there is a homeomorphism η of s that extends $gkhk^{-1}f^{-1}\rho^{-1}$ from $\rho fk(A)$ onto $gkh(A)$. Then $k^{-1}g^{-1}\eta\rho fk$ is the desired extension of h from A onto $h(A)$.

Corollary 3.8. Let $A \subset X$ have Property Z. Then A has topological infinite deficiency.

Proof: Let $X = L_1 \oplus L_2$ be a decomposition into infinite-dimensional subspaces, with L_1 closed. Since L_1 is homeomorphic to X , it contains a closed copy A' of A , A' has infinite deficiency and therefore Property Z, and the homeomorphism $A \leftrightarrow A'$ can be extended to a homeomorphism of X .

Obviously, Property Z implies topological strong infinite deficiency wherever this is possible, i.e., in any separable infinite-dimensional Fréchet space admitting a decomposition $X = L_1 \oplus L_2$ into infinite-dimensional subspaces, with L_1 and L_2 closed. In particular, strong infinite deficiency is always obtainable in the Fréchet space s , and Hilbert space l_2 .

Anderson's results, together with the earlier propositions, lead to the following

Theorem 3.9. In a separable infinite-dimensional Fréchet space, topological infinite deficiency, finite deficiency, and homotopy deficiency are equivalent, and are preserved under closed countable unions. Furthermore, there is homogeneity with respect to deficient subsets.

Since every compact set in an infinite-dimensional Fréchet space is finitely deficient (2.5), we have the following homogeneity result:

Corollary 3.10. Each homeomorphism between two closed σ -compact subsets of X can be extended to a space homeomorphism.

Theorem 3.9 implies that, in a separable infinite-dimensional Fréchet space X , homogeneity and deficiency are local properties. In the following, \mathcal{C} is a class of closed subsets of X , containing all closed homeomorphs of its members (a closed topological class).

Definitions. X is homogeneous with respect to \mathcal{C} if each homeomorphism $f: A \rightarrow B$ between members of \mathcal{C} can be extended to a homeomorphism of X onto itself.

X is locally homogeneous with respect to \mathcal{C} if for each homeomorphism $f: A \rightarrow B$ and each $p \in A$, there is a neighborhood $N(p)$ in A such that $f|N(p)$ can be extended to a homeomorphism of X .

Clearly, X is (locally) homogeneous with respect to \mathcal{C} iff each set in \mathcal{C} is (locally) deficient in X . (A closed subset A is locally deficient if each point has a deficient neighborhood). Since a locally deficient set is σ -deficient and therefore deficient, local homogeneity with respect to \mathcal{C} implies homogeneity.

CHAPTER IV. STABILITY AND ISOTOPY

Let X be any topological space, and $H(X)$ the group of homeomorphisms of X . Let e denote the identity element of $H(X)$.

Definition (Whittaker(15)). $P(X)$ is the set of all $g \in H(X)$ satisfying the following condition: for each $x \in X$, there is a finite subset $\Delta(g, x) \subset X$ such that for each $y \in X \setminus \Delta(g, x)$, there exist neighborhoods U of x and V of y , and $f \in H(X)$, with $f = e/U$ and $f = g/V$.

Lemma 4.1 (Whittaker). $P(X)$ is a normal subgroup of $H(X)$.

Lemma 4.2. Let X be a topological linear space, and f a continuous linear functional. Let $x, y \in U = f^{-1}(a, b)$. Then there exists $h \in H(X)$ supported on U such that $h(x) = y$.

Proof: Immediate with the observation that the projection onto the closed subspace $f^{-1}(0)$ is continuous (Lemma 2.2).

Lemma 4.3. Let X be a locally convex Hausdorff linear space, and suppose $y \neq x \neq z$. Then there exists a continuous linear functional f and an interval (a, b) such that $x \in f^{-1}(a, b)$ and $\{y, z\} \cap f^{-1}[a, b] = \emptyset$.

Proof: There is a continuous linear functional f_0 on $\text{sp}\{x, y, z\}$ such that $f_0(y) \neq f_0(x) \neq f_0(z)$. By local convexity, f_0 can be extended to a continuous linear functional f on X , and the conclusion follows.

The method of proof of the following proposition is due to Whittaker.

Proposition 4.4. Let X be a locally convex Hausdorff linear space, $\dim X > 1$. Then the normal subgroup $S(X)$ of stable homeomorphisms of X coincides with the subgroup $P(X)$.

Proof: Since $P(X) \subset S(X)$ always, we need only show that if $f \in H(X)$ with $f = e/U$, for some nonempty open set U , then $f \in P(X)$. Let $x \in X$ be given. Take $\Delta(f, x) = \{x, f^{-1}(x)\}$. Let $y \in X \setminus \Delta(f, x)$. For each $z \in X \setminus$

$\{y, f(y)\}$, let U_z be a set of the form $f^{-1}(a, b)$, f a continuous linear functional, with $z \in f^{-1}(a, b)$ and $\{y, f(y)\} \cap f^{-1}[a, b] = \emptyset$. Since $X \setminus \{y, f(y)\}$ is connected, there is a chain U_1, \dots, U_n of such open sets between x and some $u \in U$. Put $A = \overline{U_1} \cup \dots \cup \overline{U_n}$. Then $A \cap \{y, f(y)\} = \emptyset$. For $1 \leq i \leq n$, let $h_i \in H(X)$, supported on U_i , be such that $h(x) = h_n \dots h_1(x) = u$. Set $V = h^{-1}(U)$ and $W = (X \setminus A) \cap (X \setminus f^{-1}(A))$. Then V and W are neighborhoods of x and y , respectively. We have $h^{-1}fh = e/V$, and $h^{-1}fh = f/W$. Thus $h^{-1}fh$ 'bridges' e and f , and $f \in P(X)$.

The restriction $\dim X > 1$ is necessary, since $P(R^1) = \{e\}$, while the stable homeomorphisms of R^1 are the order-preserving homeomorphisms. However, even for R^1 , we have the following property of stable homeomorphisms, given in the finite-dimensional case by Brown-Gluck(4).

Corollary 4.5. Every stable homeomorphism of a locally convex Hausdorff linear space is the product of two homeomorphisms each somewhere the identity.

Proposition 4.6. On a topological linear space X , every stable homeomorphism is $*$ -isotopic to the identity.

Proof: Let $h \in H(X)$ with $h = e/U$. Let $f \in H(X)$ be a translation taking \emptyset into U , and let $f^{-1}(U) = V$. Then $f^{-1}hf = e/V$. Define $G: X \times I \rightarrow X$ as follows: $G(x, t) = 1/t f^{-1}hf(tx)$ if $t \neq 0$, $G(x, 0) = x$. Then $\{G_t\}$ is a $*$ -isotopy between e and $f^{-1}hf$, and $\{fG_t f^{-1}\}$ is a $*$ -isotopy between e and h . The result follows, since the set $I(X)$ of homeomorphisms $*$ -isotopic to the identity is a (normal) subgroup.

Wong has shown (16) that for infinite product spaces $X^\infty = \prod_1^\infty X$ with X satisfying a certain Property $\overline{\Phi}$, every homeomorphism on X^∞ is isotopic to the identity. X satisfies Property $\overline{\Phi}$ if the homeomorphism g on X^∞

defined by $g(x_1, x_2, x_3, x_4, \dots) = (x_2, x_1, x_3, x_4, \dots)$ is isotopic to the identity. He then shows that both $I = [0, 1]$ and $I^\circ = (0, 1)$ satisfy Property $\bar{\Phi}$. Thus on the Hilbert cube I^∞ and on the Frechet space $s = (I^\circ)^\infty$ (which is homeomorphic to ℓ_2 by (1)), all homeomorphisms are isotopic to the identity. Using results proved in (2), Wong establishes, for s , the stronger result that every homeomorphism is stable. Anderson(3) shows that every homeomorphism on I^∞ is stable.

With merely an added emphasis on the norm, Wong's isotopy methods can be applied quite naturally to certain separable infinite-dimensional inner-product spaces. The relevant fact is that convergence in Hilbert space is convergence of coordinates and of norms. As in Chapter II, let $W \subset \ell_2$, where W is the span $\text{sp} \{u_1, u_2, \dots\}$ of a complete orthonormal set in ℓ_2 , and M is a linear subspace--any separable infinite-dimensional inner-product space is congruent to a space of this type. We have $\sum x_i u_i \in \ell_2$ iff $\sum x_i^2 < \infty$; denote $\sum x_i u_i$ by (x_1, x_2, \dots) . Let $e: M \rightarrow M$ denote the identity map, and define $g: M \rightarrow M$ by $g(x_1, x_2, x_3, x_4, \dots) = (x_2, x_1, x_3, x_4, \dots)$. An isotopy $\{h_t\}$ is called norm-preserving if $\|h_{t_1}(m)\| = \|h_{t_2}(m)\|$ for all $m \in M$ and every t_1, t_2 .

Lemma 4.7. There exists a norm-preserving *-isotopy between e and g .

Proof: For each n , define ω_n on $\text{sp} \{u_n, u_{n+1}\}$ by $\omega_n(x, y) = (-x, -y)$, and let $\{\Psi_{n,t}\}_{t \in [(n-1)/n, n/(n+1)]}$ be a norm-preserving *-isotopy between e and ω_n on $\text{sp} \{u_n, u_{n+1}\}$. Let $\bar{\omega}_n, \bar{\Psi}_{n,t}$ be the natural extensions to M (all other coordinates are fixed). Let $\bar{h}_n = \bar{\omega}_n \dots \bar{\omega}_2 \bar{\omega}_1, \bar{h}_n = e$. Define $h_{n,t} = \bar{\Psi}_{n,t} \bar{h}_n, t \in [(n-1)/n, n/(n+1)]$. Let $\sigma(x_1, x_2, \dots) = (-x_1, x_2, \dots)$. Then $M \times I \rightarrow M$, defined by $H/M \times t = h_{n,t}$ if $t \in [(n-1)/n, n/(n+1)]$ and $H/M \times 1 = \sigma$, is a norm-preserving *-isotopy between e and σ . Let F be

the rotation of $\text{sp} \{u_1, u_2\}$ counterclockwise through $\pi/4$ radians, and let \bar{F} be its natural extension to M . Then $g = \bar{F}'\sigma\bar{F}$, and since \bar{F} is norm-preserving, there is a norm-preserving $*$ -isotopy between e and g .

Define φ_n on $\text{sp} \{u_n, u_{n+1}\}$ by $\varphi_n(x, y) = (y, x)$, and let $\bar{\varphi}_n$ be the natural extension to M . Then there is, for each n , a norm-preserving $*$ -isotopy $\{\varphi_{n,t}\}_{t \in [(n-1)/n, n/(n+1)]}$ between e and $\bar{\varphi}_n$, fixed on the first $(n-1)$ coordinates.

Lemma 4.8. Let $P, P_i \in M$ with $P_i \rightarrow P$, and let $\{f_i\}$ be a sequence of functions with each $f_i = \varphi_{n,t}$ for some $t \in [(n-1)/n, n/(n+1)]$, such that $n \rightarrow \infty (t \rightarrow 1)$ as $i \rightarrow \infty$. Then $f_i(P_i) \rightarrow P$.

Proof: Convergence of coordinates is obvious. Since each f_i is norm-preserving, there is convergence of norms.

Definition. M is shift-complete with respect to $\{u_1, u_2, \dots\}$ if it satisfies the following condition: $\sum x_i u_i \in M$ iff $\sum x_i u_{i+1} \in M$.

W and \mathcal{L}_2 , among other subspaces, are shift-complete. Hereafter, we assume that M is shift-complete. For any $a \in R$ and any positive integer n , define maps $a^{(n)}$ and $\tilde{\pi}_n$ of M into itself by $a^{(n)}(x_1, x_2, \dots) = (x_1, \dots, x_{n-1}, a, x_n, \dots)$ and $\tilde{\pi}_n(x_1, x_2, \dots) = (x_1, \dots, x_{n-1}, x_{n+1}, \dots)$.

Lemma 4.9. Let $P, P_i \in M, a_i \in R$, with $P_i \rightarrow P$ and $a_i \rightarrow 0$, and suppose $n (=n_i) \rightarrow \infty$ as $i \rightarrow \infty$. Then $\tilde{\pi}_n(P_i) \rightarrow P$ and $a_n^{(n)}(P_i) \rightarrow P$.

Proof: Convergence of coordinates and of norms is easily verified.

Let $\pi_n: M \rightarrow R$ denote the n^{th} projection, and let $h: M \rightarrow M$ be a homeomorphism. Define $\tilde{h}_n: M \rightarrow M$ by $\tilde{h}_n(x) = (\pi_n(x))^{(n)} h \tilde{\pi}_n(x)$. For each n , \tilde{h}_n is a homeomorphism of M leaving the n^{th} coordinate fixed.

Lemma 4.10. Let $P, P_i \in M$ with $P_i \rightarrow P$, and suppose $n (=n_i) \rightarrow \infty$ as $i \rightarrow \infty$. Then $\tilde{h}_n(P_i) \rightarrow h(P)$.

Proof: We have $\widetilde{\pi}_n(P_i) \rightarrow P$, and $h\widetilde{\pi}_n(P_i) \rightarrow h(P)$. Since $\pi_n(P) \rightarrow 0$, $(\pi_n(P_i))^{(n)} h\widetilde{\pi}_n(P_i) \rightarrow h(P)$.

Lemma 4.11. \widetilde{h}_n is $*$ -isotopic to h .

Proof: Define $h_{n,t} = \varphi_{n,t} \widetilde{h}_n \varphi_{n,t}$ for $t \in [(n-1)/n, n/(n+1)]$. Note that $\{h_{n,t}\}$ is a $*$ -isotopy between \widetilde{h}_n and \widetilde{h}_{n+1} . Define $H: M \times I \rightarrow M$ by $H/M \times t = h_{n,t}$, $H/M \times 1 = h$. Suppose $(P_i, t_i) \rightarrow (P, 1)$, $t_i < 1$ for all i . We have $H(P_i, t_i) = h_{n,t_i}(P_i) = \varphi_{n,t_i} \widetilde{h}_n \varphi_{n,t_i}(P_i)$. Then $\varphi_{n,t_i}(P_i) \rightarrow P$ by (4.8), $\widetilde{h}_n \varphi_{n,t_i}(P_i) \rightarrow h(P)$ by (4.10), and $\varphi_{n,t_i} \widetilde{h}_n \varphi_{n,t_i}(P_i) \rightarrow h(P) = H(P, 1)$. Thus H is continuous, and $\{H_t\}$ is an isotopy. Similarly, $\{H_t^{-1}\}$ is an isotopy, therefore $\{H_t\}$ is a $*$ -isotopy.

Lemma 4.12. If $h(\theta) = \theta$, then h is $*$ -isotopic to e .

Proof: \widetilde{h}_n is the natural extension of a homeomorphism h , on the closed hyperplane $\pi_i^{-1}(0)$. We can repeat the same argument on $\pi_i^{-1}(0)$ and show that there exists a $*$ -isotopy, fixed on the first coordinate, between \widetilde{h}_n and a homeomorphism g , with the property that g is the natural extension of a homeomorphism h_2 on $\pi_i^{-1}(0) \cap \pi_2^{-1}(0)$. Iterating this process, we can define a function $H: M \times I \rightarrow M$ such that $H/M \times 1 = e$, $H/M \times 0 = h$, $H/M \times 1/2 = \widetilde{h}_n$, etc. Let $(P_i, t_i) \rightarrow (P, 1)$, $t_i < 1$ for all i . We need to show $H(P_i, t_i) \rightarrow P$. Certainly we have convergence of coordinates. Let $\epsilon > 0$ be given. Evidently $\|H(P_i, t_i)\| > \|P\| - \epsilon$ for i sufficiently large. Since $h(\theta) = \theta$, there exists $0 < \delta < \epsilon/2$ such that $\|x\| < \delta \Rightarrow \|h(x)\| < \epsilon/2$. Then, for any n , $\|x\| < \delta \Rightarrow \|\widetilde{h}_n(x)\| < \delta + \epsilon/2$. Thus, since $\varphi_{n,t}$ is norm-preserving, $\|x\| < \delta \Rightarrow \|h_{n,t}(x)\| < \delta + \epsilon/2$. Then the construction of H implies that $\|H(P_i, t_i)\| < \|P\| + \epsilon$ for i sufficiently large. Thus $\|H(P_i, t_i)\| \rightarrow \|P\|$, H is continuous, and $\{H_t\}$ is an isotopy. The same type of argument applies to $\{H_t^{-1}\}$, and therefore $\{H_t\}$ is a $*$ -isotopy.

Theorem 4.13. Every homeomorphism on M is $*$ -isotopic to the identity.

Proof: Every homeomorphism of a normed linear space is the product of a stable homeomorphism and one fixed at the origin. Since every stable homeomorphism is $*$ -isotopic to the identity (4.6), the theorem follows.

The following proposition resembles a lemma of Anderson (3, Lemma 3.1) for homeomorphisms on the Hilbert cube.

Proposition 4.14. Let f be a homeomorphism on M , and H a closed hyperplane such that $f(H) = H$. Then f is stable.

Proof: Klee has shown (see Chapter V) that for any closed hyperplane in an infinite-dimensional normed linear space, there is a space homeomorphism taking the hyperplane onto the unit sphere. Thus any two closed hyperplanes in M are equivalently imbedded, and f is the conjugate of a homeomorphism g such that $g(\pi_i^{-1}(0)) = \pi_i^{-1}(0)$. By normality, we may assume that $f(\pi_i^{-1}(0)) = \pi_i^{-1}(0)$. Let $\{f_t\}_{-1 \leq t \leq 0}$ be a $*$ -isotopy (on $\pi_i^{-1}(0)$) between e and $f/\pi_i^{-1}(0)$. By connectedness, either $f(\pi_i^{-1}(0, \infty)) = \pi_i^{-1}(0, \infty)$ or $f(\pi_i^{-1}(0, \infty)) = \pi_i^{-1}(-\infty, 0)$. Suppose the former. Define $F: M \rightarrow M$ as follows:

$$F(x_1, x_2, \dots) = \begin{cases} f(x_1, x_2, \dots) & \text{if } x_1 \geq 0 \\ (x_1, f_{x_1}(x_2, \dots)) & \text{if } -1 \leq x_1 \leq 0 \\ (x_1, x_2, \dots) & \text{if } x_1 \leq -1. \end{cases}$$

Then F is a homeomorphism which agrees with e and f on nonempty open sets, and f is stable. If $f(\pi_i^{-1}(0, \infty)) = \pi_i^{-1}(-\infty, 0)$, we make the obvious modifications in the definition of F , and the result follows, provided the homeomorphism σ defined by $\sigma(x_1, x_2, \dots) = (-x_1, x_2, \dots)$ is stable. But this is evident by the first part, considering the projection π_1 instead of π_i .

From the proof of (4.14) it is clear that the following proposition

holds for any infinite-dimensional normed linear space X for which every homeomorphism on the unit sphere S is weakly isotopic to the identity. ($H: S \times I \rightarrow S \times I$ is a weak isotopy if it is a homeomorphism and $H(S \times 0) = S \times 0$, $H(S \times 1) = S \times 1$). In particular, it holds for M , since M is homeomorphic to its closed hyperplanes and unit sphere, and we may apply (4.13).

Proposition 4.15. Let f be a homeomorphism of X such that $f(S) = g(S)$ for some stable homeomorphism g . Then f is stable.

CHAPTER V. APPLICATIONS

By pushing countable collections of weakly thin subsets of I^∞ to the pseudo-boundary $I^\infty \setminus \circ I^\infty$ through β -homeomorphisms, Anderson(2) obtains the following results for the space $\circ I^\infty (= s)$:

Theorem 5.1. Let $\{K_i\}_{i>0}$ be a countable collection of closed sets in $\circ I^\infty$ such that for each i , the projections of K_i onto infinitely many coordinate intervals are nondense. Then $\circ I^\infty \setminus \bigcup_{i>0} K_i$ is homeomorphic to $\circ I^\infty$.

Proof: The closure of each K_i in I^∞ is weakly thin.

Corollary 5.2. For any σ -compact subset K of $\circ I^\infty$, $\circ I^\infty \setminus K$ is homeomorphic to $\circ I^\infty$.

Theorem 5.1, for the case of a single closed subset, implies the following

Proposition 5.3. Let X be a separable infinite-dimensional Fréchet space, and let $A \subset X$ be deficient. Then $X \setminus A \sim X$.

Proof: Let $h: X \rightarrow \circ I^\infty$ be a homeomorphism. Then $h(A)$ has Property Z and has topological strong infinite deficiency in $\circ I^\infty$ (3.6). Let $\circ I^\infty = \circ I_\alpha \times \circ I_{\alpha'}$, where α and α' are complementary infinite subsets of positive integers. There is a homeomorphism g of $\circ I^\infty$ such that $gh(A) \subset \circ I_\alpha \times c$, $c \in \circ I_{\alpha'}$. By (5.1), $\circ I^\infty \setminus gh(A) \sim \circ I^\infty$.

In (10), Klee established the following

Theorem 5.4. Let Y be a compact set in the interior of the unit cell U of a nonreflexive normed linear space X . Then there is a homeomorphism of X onto $X \setminus Y$, supported on U .

The fundamental tool, due to Smulian, is the equivalence of nonreflexivity with the existence of a decreasing sequence of nonempty bounded

closed convex sets with empty intersection. Klee observes the theorem is applicable to any space which admits a unit cell, unit sphere-preserving homeomorphism with a nonreflexive normed linear space. In particular, it applies to any infinite-dimensional L^p space, hence to any infinite-dimensional Hilbert space, using Mazur's homeomorphism between the nonreflexive space L^1 and L^p .

In a later paper (12), Klee produced an alternative tool, for arbitrary infinite-dimensional normed linear spaces: every such space contains a decreasing sequence of nonempty linearly bounded closed convex sets with empty intersection. (A set is linearly bounded if its intersection with each line is bounded). With this, and evidently using the fact that, in a normed linear space, a closed convex body which is linearly bounded can be taken onto the unit cell, and its boundary onto the unit sphere, through a (stable) space homeomorphism, he is able to show that any infinite-dimensional normed linear space can lose a compact set without changing its topological character. Moreover, the full statement of Theorem 5.4 remains valid when Y is a single point, and thus every infinite-dimensional normed linear space can be 'inverted' across its unit sphere, i.e., there is a homeomorphism which is the identity on the unit sphere and takes the exterior of the unit cell onto the interior. It follows that every closed hyperplane can be thrown onto the unit sphere by a stable space homeomorphism, and there is a stable space homeomorphism taking the boundary of the unit cell onto itself, and the exterior onto the interior.

From Theorem 5.4 there can be deduced a similar result for countable locally compact sets:

Proposition 5.5. Let X be homeomorphic to a nonreflexive normed

linear space, and let $A \subset U \subset X$, where A is countable locally compact and U is open. Then there is a homeomorphism of X onto $X \setminus A$, supported on U .

Proof: Since the property is topological, we may assume X is a nonreflexive normed linear space. Let $A = \{a_i\}$. Choose a (closed) ball B_1 about a_1 , such that $B_1 \subset U$, $B_1 \cap A$ is compact, $\text{diam } B_1 < 1$, and $\text{Bd}(B_1) \cap A = \emptyset$. If $a_2 \in B_1$, let $B_2 = B_1$. Otherwise, choose a ball B_2 about a_2 , such that $B_2 \subset U$, $B_2 \cap A$ is compact, $\text{diam } B_2 < 1/2$, $\text{Bd}(B_2) \cap A = \emptyset$, and $B_1 \cap B_2 = \emptyset$. Continuing in this manner, we obtain a sequence $\{B_i\}$ of balls in U . By Klee's theorem, there exists, for each i , a homeomorphism $B_i \xrightarrow{\sim} B_i \setminus A$ which is the identity on $\text{Bd}(B_i)$. These define the desired homeomorphism $X \xrightarrow{\sim} X \setminus A$, supported on U .

Corollary 5.6. Let Z be any topological space, and let $A \subset U \subset \bar{U} \subset X \subset Z$, where A is countable locally compact, U is open, and X is homeomorphic to a nonreflexive normed linear space. Then there is a homeomorphism of Z onto $Z \setminus A$, supported on U .

Proof: By (5.5), there exists a homeomorphism of X onto $X \setminus A$, supported on U . Combined with the identity map on $Z \setminus X$, this gives the desired homeomorphism.

Proposition 5.7. There does not exist a subset $F \subset \mathbb{R}^n$ such that for every locally compact subset L of F , $F \setminus L$ is nonempty $(n-1)$ -connected.

Proof: Assume such an F exists. We shall construct an antipode-distinguishing map $f: S^n \rightarrow F \subset \mathbb{R}^n$, contradicting Borsuk's antipodal theorem (6, p. 349). Let $f_0: S^0 \rightarrow F$ be any map with $f_0(-1) \neq f_0(1)$. Assume there is defined an antipode-distinguishing map $f_{i-1}: S^{i-1} \rightarrow F$, $i \leq n$. Define $\xi(x) = d(f_{i-1}(x), f_{i-1}(-x))$ for all $x \in S^{i-1}$. By compactness, there exists $\epsilon > 0$ such that $\xi(x) > \epsilon$ for all x . Let $\mathcal{T} = \{T_j^a\}$ be a triangulation

of S^{i-1} , with $\dim T_j^a = a$, such that $\text{diam } f_{i-1}(T_j^{i-1}) < \epsilon/2$ for all T_j^{i-1} . Then $x \in T_j^{i-1}$, $-x \in T_k^{i-1}$ imply $f_{i-1}(T_j^{i-1}) \cap f_{i-1}(T_k^{i-1}) = \emptyset$. Choose $p \in F \setminus f_{i-1}(S^{i-1})$. We consider the cone cS^{i-1} over S^{i-1} , and its triangulation $c\mathcal{T}$ induced by \mathcal{T} . Define $f_i^p(c) = p$, $f_i^p/S = f_{i-1}$. Observe that for every face T_j^a of \mathcal{T} , $f_{i-1}(S^{i-1}) \setminus f_{i-1}(T_j^a)$ is locally compact. We extend f_{i-1} to $f_i^p: cS^{i-1} \rightarrow F$ by stages: first on the 1-faces cT_j^0 , then on the 2-faces, etc., and finally on the i -faces cT_j^{i-1} , such that, for each T_j^a , $f_i^p(cT_j^a) \cap f_{i-1}(S^{i-1}) = f_{i-1}(T_j^a)$. Now choose $q \in F \setminus f_i^p(cS^{i-1})$. Define $f_i^q(c) = q$, $f_i^q/S^{i-1} = f_{i-1}$, and as above, extend f_{i-1} to $f_i^q: cS^{i-1} \rightarrow F \setminus (f_i^p(cS^{i-1}) \setminus f_{i-1}(S^{i-1}))$ so that, for each T_j^a , $f_i^q(cT_j^a) \cap f_{i-1}(S^{i-1}) = f_{i-1}(T_j^a)$. Then f_i^p, f_i^q define a map $f_i: S^i \rightarrow F$ which distinguishes antipodes, and $f = f_n$ is the desired map.

Theorem 5.8. Let \mathcal{M} be a separable metric space such that the complement of every locally compact (σ -compact) subset is nonempty homotopically trivial. Then $\mathcal{M} \not\sim F \times C$, where F is finite-dimensional and C is locally compact (σ -compact).

Proof: Assume $\mathcal{M} \sim F \times C$, either case considered. Then $F \subset \mathbb{R}^n$ for some n , and for every locally compact subset L of F , $(F \setminus L) \times C$ is nonempty homotopically trivial, therefore $(n-1)$ -connected. Thus $F \setminus L$ is nonempty $(n-1)$ -connected, contradicting (5.7).

Corollary 5.9. Separable infinite-dimensional Fréchet space is not homeomorphic to a product $F \times C$, where F is finite-dimensional and C is σ -compact.

Proof: By the proof of (3.3), every σ -compact subset has homotopy deficiency, without the closure requirement, and therefore the complement is nonempty homotopically trivial. (Of course by Anderson's result (5.2), the complement is homeomorphic to the whole space).

CHAPTER VI. LITERATURE CITED

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CHAPTER VII. ACKNOWLEDGMENT

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