Development of an explicit time accurate scheme for incompressible flows

by

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NOMENCLATURE

\(a\) coefficient of discretised equations
\(A\) power law function
\(b\) integrated source term
\(D\) conductance
\(e, w, n, s\) east, west, north and south control volume faces
\(E, W, N, S, P\) east, west, north, south and central primary grids
\(J\) total flux
\(p\) pressure
\(R_i\) residual function
\(F\) mass flux
\(P\) peclet number
\(S\) source term
\(u, v\) Cartesian velocity components
\(x, y\) Cartesian coordinates
\(\alpha, \alpha'\) constants
\(\Gamma\) diffusion coefficient
\(\rho\) density
\(\phi\) scalar transport variable
\(\hat{u}, \hat{v}\) pseudovelocities
\(\mathcal{V}_i\) volume
Viscous incompressible flows play an important role in both engineering applications of hydrodynamics and aerodynamics and also in fundamental fluid dynamics. Numerical investigations of such flows are based on the solution of the Navier-Stokes equations. Computational studies have been performed for several decades and various methods for the solution of incompressible flows have been developed.

A significant difficulty for incompressible flow calculations occurs since the continuity equation is given not in a time evolution form, but in the form of a divergence free constraint. The pressure, which has no time term, is coupled implicitly with the divergence-free constraint on the velocity. This constraint, which is the continuity equation, prohibits time integration of the incompressible flow equations in a straightforward manner [1] [2].

Current techniques for the solution of incompressible viscous flows can be categorized as: (a) vorticity/stream-function methods, (b) projection methods, and (c) artificial compressibility methods [1]. These formulations generally lead to indirect solution of the discretized continuity equation. This is typically done either by adding ‘artificial compressibility’ to the continuity equation or by indirectly satisfying the continuity equation with the pressure Poisson equation. Pressure Poisson formulations work very well on staggered grids.

The vorticity/stream-function method [3] solves the vorticity transport equation which is constructed by taking the curl of the momentum equation. The terms containing the pressure may be eliminated by using the components of the equation of...
motion. The method requires the use of vorticity boundary conditions, which are difficult to implement, and an additional calculation is required if the pressure is desired.

The projection method is a fractional step method in which an intermediate velocity and pressure are calculated. The intermediate pressure and velocity are then corrected sequentially by the pressure gradient and the divergence of the intermediate velocity (continuity equation), respectively. New values for pressure and velocity are obtained until the divergence of the velocity vanishes. The SIMPLE and SIMPLER family of methods [4] fall in this class. The pressure correction equation plays a very important role in these methods, the derivation of which requires the use of the approximate forms of the momentum equations and the continuity equation.

An artificial equation of state $P = \delta \rho$, is the basis of the method of artificial compressibility [5]. This method differs from the projection method in that the continuity equation is not satisfied until a steady-state is reached. Chorin [5] uses central differences in space and time to arrive at a scheme designed for steady state solutions. Methods for unsteady solutions using artificial compressibility have also been developed by later researchers (like Soh and Goodrich [1]).

**Runge-Kutta Method**

An important family of explicit non-linear time-integration algorithms of higher order accuracy is provided by the Runge-Kutta methods. Compared with the linear multi-step methods, the Runge-Kutta schemes achieve higher order of accuracy by sacrificing the linearity of the method but retains the advantages of the one-step methods. In contrast the linear multi-step methods achieve higher accuracy by involving multiple time steps [6].

The use of explicit Runge-Kutta methods as time-stepping schemes for the solutions of the compressible Euler equations has become popular since the appearance of papers
by Jameson et al [7]. Implicit schemes have less restrictive stability requirements than explicit schemes but require more extensive calculations to such a degree that their use may compensate the gain in a larger stability limit. The severe restriction of the conventional explicit methods is relaxed by the enlarged stability region of Runge-Kutta methods [8].

**Current Work**

The present work uses the spatial discretization of the conservation equations based on the SIMPLER [4] algorithm in the context of a multi-stage explicit time accurate Runge-Kutta algorithm for incompressible flows in primitive variables. The spatially discretized governing equations are rewritten and made suitable for integrating explicitly in time using the four-stage Runge-Kutta algorithm [9]. The basic principle of the new algorithm stems from the fact that pressure is the dominant driving force behind incompressible flows and that implicit treatment is required only for the pressure equation if at all necessary. The discretized Poisson pressure equation is obtained by a direct substitution of the fully implicit steady discretized momentum equations into the fully implicit discretized incompressible continuity equation, following strictly the procedure outlined in the SIMPLER algorithm [4].

The SIMPLER algorithm develops, in addition to the fully implicit Poisson equation for the pressure and the implicit momentum equations, an additional equation for the perturbation of pressure called the pressure-correction equation [4]. This pressure-correction field is used to adjust the imperfect velocity field, obtained by solving the non-linear discretized momentum equations, to satisfy mass conservation. Thus, the SIMPLER algorithm solves a Poisson equation for the pressure, the momentum equations for the respective velocity components and a perturbation pressure field through the pressure correction field. In all a set of four non-linear equations for 2-D are solved.
iteratively.

The current algorithm uses the fully implicit pressure equation of the SIMPLER algorithm exactly but updates the velocities explicitly using the Runge-Kutta algorithm without iterations. In the process, the pressure correction equation is eliminated completely and so are the iterations required to solve the non-linear discretized momentum equations resulting in reduced computation time. Both the Crank-Nicolson scheme used for the SIMPLER algorithm and the Runge-Kutta scheme used in the new explicit algorithm presented here are second order accurate in time and spatial accuracy is a free choice that can be retained in both the algorithms. For this investigation, the power-law scheme [4] is used for spatial integration for both the algorithms.

The details of derivation of the formulation are discussed in Chapter 2 [4] [9]. The formulation of the SIMPLER algorithm in conjunction with the Crank-Nicolson time integration [4] [10] [11] scheme is also discussed in Chapter 2 for the sake of comparison. The algorithms and the corresponding flow-charts are presented in Chapter 3.

Chapter 4 presents a comparative study of the results produced by both of the algorithms. Three problems have been chosen to compare the results produced by both the schemes, one in the low Reynolds number regime (internal flow), one in the medium Reynolds number regime (internal flow) and one in the high Reynolds number regime (external flow), so that the validity of the new scheme could be established over a vast range of flows. The last chapter, Chapter 5, presents a comparison between the computer time required by both the schemes.
2 THEORY

Governing Equations for SIMPLER Crank-Nicholson Scheme

Time Integration

The method of time integration for several schemes including Crank-Nicolson can be conveniently stated as:

\[
\int_{t_0}^{t} \phi dt = [\alpha \phi + (1 - \alpha)\phi_0] \Delta t
\]  

(2.1)

where \(\phi\) is the quantity to be integrated, and the value of \(\alpha\) may be different depending on the scheme of choice. All time integrations in the following few sections are performed using the above stencil. The value of \(\alpha = 0.5\) in the above equation corresponds to the Crank-Nicolson time integration scheme. This scheme makes use of trapezoidal differencing to achieve second-order accuracy [12].

Continuity Equation

The continuity equation in two-dimensions can be stated as,

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0
\]  

(2.2)

Now we integrate this over the control volume shown in figure 2.1. Upon integration of the first term \(^1\) we get,

\[
\int_{t_0}^{t} \int_{x_w}^{x_e} \int_{y_s}^{y_n} \frac{\partial \rho}{\partial t} \, dx \, dy \approx \int_{t_0}^{t} \left( \frac{\partial \rho}{\partial t} \right) \Delta x \Delta y = [\rho - \rho_0] \Delta x \Delta y
\]  

(2.3)

\(^1\)This term can be dropped for incompressible flow but is retained here for algebraic convenience
Upon integrating the second term we get,

\[ \int_{t_0}^{t} \int_{x_w}^{x_e} \int_{y_s}^{y_n} \left( \frac{\partial (\rho u)}{\partial x} \right) dx \, dy \, dt \approx \int_{t_0}^{t} \Delta y \left[ \int \frac{\partial (\rho u)}{\partial x} dx \right] dt = \int_{t_0}^{t} [(\rho u)_e - (\rho u)_w] \Delta y dt \tag{2.4} \]

To simplify the above equation the following terms may be introduced:

\[
\begin{align*}
(\rho u)_e \Delta y &= F_e, \\
(\rho u)_w \Delta y &= F_w, \\
(\rho u)_n \Delta x &= F_n, \\
(\rho u)_s \Delta x &= F_s
\end{align*}
\tag{2.5}
\]

Therefore using eqns. (2.1) (2.5), eqn (2.4) becomes

\[
\int_{t_0}^{t} (F_e - F_w) = \left[ \alpha F_e + (1 - \alpha) F_e^o - \alpha F_w - (1 - \alpha) F_w^o \right] \Delta t \tag{2.6}
\]

\[
= \left[ \alpha (F_e - F_w) + (1 - \alpha) (F_e^o - F_w^o) \right] \Delta t \tag{2.7}
\]

Similarly integration of the third term of the continuity equation yields,

\[
\int_{t_0}^{t} \int_{x_w}^{x_e} \int_{y_s}^{y_n} \left( \frac{\partial (\rho v)}{\partial y} \right) dx \, dy \, dt = \left[ \alpha (F_n - F_s) + (1 - \alpha) (F_n^o - F_s^o) \right] \Delta t \tag{2.8}
\]
Upon collecting all the terms and rearranging, the continuity equation becomes

\[(\rho - \rho^0) \frac{\Delta x \Delta y}{\Delta t} + \alpha(F_e - F_w + F_n - F_s) + (1 - \alpha)(F_e^0 - F_w^0 + F_n^0 - F_s^0) = 0 \quad (2.9)\]

Here the value of \( \alpha \) defines the type of the scheme. For example

\[\begin{align*}
\alpha = 0 & : \text{ Explicit} \\
\alpha = 1 & : \text{ Fully Implicit} \\
\alpha = \frac{1}{2} & : \text{ Crank Nicholson}
\end{align*}\]

**X-momentum Equation**

The \(x\)-component of the Navier-Stokes equations in two-dimensions can be stated as

\[
\frac{\partial(\rho u)}{\partial t} + \frac{\partial J_{ux}}{\partial x} + \frac{\partial J_{uy}}{\partial y} = -\frac{\partial p}{\partial x} + S_u \quad (2.10)
\]

where \(J_{ux}\) and \(J_{uy}\) are the total (convection plus diffusion) fluxes defined by

\[
J_{ux} = \rho uu - \nu \frac{\partial u}{\partial x} \quad (2.11)
J_{uy} = \rho uv - \nu \frac{\partial u}{\partial y} \quad (2.12)
\]

Upon the integration of the momentum equation term by term (as was done for the continuity equation except for the pressure derivative term which is treated fully implicitly) we get

\[
[(\rho u) - (\rho u^0)] \frac{\Delta x \Delta y}{\Delta t} + \alpha(J_{u-e} - J_{u-w} + J_{u-n} - J_{u-s}) + (1 - \alpha)(J_{u-e}^0 - J_{u-w}^0 + J_{u-n}^0 - J_{u-s}^0) = (\alpha S_u + (1 - \alpha)S_u^0) \Delta x \Delta y
\]

\[+(p_w - p_e) \Delta y \quad (2.13)\]
Y-momentum Equation

The $y$-component of the Navier-Stokes equation in two-dimensions can be stated as

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial J_{vx}}{\partial x} + \frac{\partial J_{vy}}{\partial y} = -\frac{\partial p}{\partial y} + S_v$$

(2.14)

where $J_{vx}$ and $J_{vy}$ are the total (convection plus diffusion) fluxes defined by

$$J_{vx} = \rho vu - \Gamma \frac{\partial v}{\partial x}$$

(2.15)

$$J_{vy} = \rho vu - \Gamma \frac{\partial v}{\partial y}$$

(2.16)

In a similar fashion, integration of this equation gives

$$[\rho v - (\rho v)^o] \frac{\Delta x \Delta y}{\Delta t} + \alpha (J_{v-e} - J_{v-w} +$$

$$J_{v-n} - J_{v-s}) + (1 - \alpha) (J_{v-e}^o - J_{v-w}^o +$$

$$J_{v-n}^o - J_{v-s}^o) = (\alpha S_v + (1 - \alpha) S_v^o) \rho \Delta x \Delta y$$

$$+ (p_s - p_n) \Delta x$$

(2.17)

Assuming incompressible flow (constant density, $\rho$), the first term of eqn.(2.9) goes to zero. And since the choice of $\alpha$ is arbitrary, eqn.(2.9), which is the continuity equation, can be written as

$$F_e - F_w + F_n - F_s = 0$$

(2.18)

$$F_e^o - F_w^o + F_n^o - F_s^o = 0$$

(2.19)

Multiplying eqn (2.18) by $\alpha u_p$ and eqn (2.19) by $(1 - \alpha) u_p^o$, and subtracting both from eqn (2.13) we have

$$(u - u^o) \rho \frac{\Delta x \Delta y}{\Delta t} + \alpha [(J_{u-e} - F_e u_p)$$

$$-(J_{u-w} - F_w u_p) + (J_{u-n} - F_n u_p) - (J_{u-s} - F_s u_p)] + (1 - \alpha) [(J_{u-e}^o - F_e^o u_p^o)$$

$$-(J_{u-w}^o - F_w^o u_p^o) + (J_{u-n}^o - F_n^o u_p^o) - (J_{u-s}^o - F_s^o u_p^o)] = (p_w - p_e) \Delta y$$

(2.20)
Using the notation in [4] the following simplifications are used to help the algebraic manipulation of the conservation equations:

\[ J_{u-e} - F_e u_p = \bar{a}_{u-E} (u_p - u_E) \]
\[ J_{u-w} - F_w u_p = \bar{a}_{u-W} (u_p - u_W) \]
\[ J_{u-n} - F_n u_p = \bar{a}_{u-N} (u_p - u_N) \]
\[ J_{u-s} - F_s u_p = \bar{a}_{u-S} (u_p - u_S) \]

(2.21)

Similarly,

\[ J^0_{u-e} - F^0_e u^0_p = \bar{a}^0_{u-E} (u^0_p - u^0_E) \]
\[ J^0_{u-w} - F^0_w u^0_p = \bar{a}^0_{u-W} (u^0_p - u^0_W) \]
\[ J^0_{u-n} - F^0_n u^0_p = \bar{a}^0_{u-N} (u^0_p - u^0_N) \]
\[ J^0_{u-s} - F^0_s u^0_p = \bar{a}^0_{u-S} (u^0_p - u^0_S) \]

(2.22)

where,

\[ \bar{a}_{u-E} = D_e A(|P_e|) + [| - F_e, 0|] \]
\[ \bar{a}_{u-W} = D_w A(|P_w|) + [|F_w, 0|] \]
\[ \bar{a}_{u-N} = D_n A(|P_n|) + [| - F_n, 0|] \]
\[ \bar{a}_{u-S} = D_s A(|P_s|) + [|F_s, 0|] \]

(2.23) \hspace{1cm} (2.24) \hspace{1cm} (2.25) \hspace{1cm} (2.26)

Similar expressions can be written for the coefficients of the previous time-step (\(\bar{a}^0\)s).

Here

\[ P = \frac{F}{D} \]

(2.27)

and

\[ A(|P|) = \left[ |0, (1 - 0.1|P|)^5| \right] \]

(2.28)

Eqn (2.28) is a statement of the power-law scheme. It is important to mention here that power-law is used here for convenience and any spatial interpolation scheme could be as well used.
Upon substitution of the above quantities (eqns. (2.21) and (2.22)), the x-momentum equation becomes:

\[
\begin{align*}
    u_p \frac{\rho(\Delta x \Delta y)}{\Delta t} &+ \alpha(\bar{a}_{u-E} + \bar{a}_{u-W} + \bar{a}_{u-N} \\
    + \bar{a}_{u-S}) u_p = \alpha(\bar{a}_{u-E} u_E + \bar{a}_{u-W} u_W \\
    + \bar{a}_{u-N} u_N + \bar{a}_{u-S} u_S) + \frac{\rho(\Delta x \Delta y)}{\Delta t} u_p^o \\
    -(1 - \alpha)(\bar{a}_{u-E}^o + \bar{a}_{u-W}^o + \bar{a}_{u-N}^o + \bar{a}_{u-S}^o) u_p^o \\
    +(1 - \alpha)(\bar{a}_{u-E}^o u_E^o + \bar{a}_{u-W}^o u_W^o + \bar{a}_{u-N}^o u_N^o \\
    + \bar{a}_{u-S}^o u_S^o) + (p_w - p_e) \Delta y \\
    + [\alpha S_u + (1 - \alpha) S_u^o] \Delta x \Delta y \\
\end{align*}
\]  
(2.29)

Or in short,

\[
a_{u-p} u_p \equiv a_{u-E} u_E + a_{u-W} u_W + a_{u-N} u_N + a_{u-S} u_S + b_u + (p_w - p_e) \Delta y
\]  
(2.30)

where the coefficients \( \bar{a} \) can be further simplified as,

\[
a_{u-p} = \frac{\rho(\Delta x \Delta y)}{\Delta t} + \alpha(\bar{a}_{u-E} + \bar{a}_{u-W} + \bar{a}_{u-N} + \bar{a}_{u-S})
\]  
(2.31)

and

\[
a_{u-E} = \alpha(\bar{a}_{u-E})
\]  
(2.32)

\[
a_{u-W} = \alpha(\bar{a}_{u-W})
\]  
(2.33)

\[
a_{u-N} = \alpha(\bar{a}_{u-N})
\]  
(2.34)

\[
a_{u-S} = \alpha(\bar{a}_{u-S})
\]  
(2.35)

The term \( b_u \) is given by

\[
\begin{align*}
    b_u &= (1 - \alpha)[\bar{a}_{u-E}^o u_E^o + \bar{a}_{u-W}^o u_W^o + \bar{a}_{u-N}^o u_N^o \\
    + \bar{a}_{u-S}^o u_S^o] \Delta x \Delta y \\
    + [\alpha S_u + (1 - \alpha) S_u^o] \Delta x \Delta y
\end{align*}
\]  
(2.36)
A similar relation can be obtained for the y-momentum equation,

\[ a_{v-p} v_p = a_{v-E} v_E + a_{v-W} v_W + a_{v-N} v_N + a_{v-S} v_S + b_v + (p_a - p_n) \Delta x \]  

(2.37)

where,

\[ a_{v-p} = \frac{\rho(\Delta x \Delta y)}{\Delta t} + \alpha(a_{v-E} + a_{v-W} + a_{v-N} + a_{v-S}) \]  

(2.38)

\[ b_v = (1 - \alpha)[\bar{a}_{v-E}^o v_E^o + \bar{a}_{v-W}^o v_W^o + \bar{a}_{v-N}^o v_N^o + \bar{a}_{v-S}^o v_S^o] \]  

(2.39)

and the other coefficients can be defined in a manner similar to the coefficients in the x-momentum equation.

**Governing Equations for the New Explicit Runge-Kutta Scheme**

**Time Integration**

A system of ordinary differential equations (ODEs) can be integrated in time using the four-stage Runge-Kutta (RK-4) time stepping scheme to achieve a second-order accuracy in time. The spatial discretization, denoted by the residual vector \( R_i(\phi) \), is written as

\[ \frac{\partial \phi_i}{\partial t} v_i = R_i(\phi_i) \]  

(2.40)

The spatial discretization of the integral form of the equations transforms the system of partial differential equations (PDEs) into a coupled set of ODEs. The integration of these ODEs using the RK-4 explicit time integration [10] is given by

\[ \phi^{(0)} = \phi^n \]  

(2.41)

\[ \phi^{(1)} = \phi^{(0)} + \frac{1}{4} \Delta t \sum_i R_i(\phi^{(0)}) \]  

(2.42)
The continuity equation in two-dimensions in conservation form can be stated as

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \]  

(2.47)

After dropping the unsteady derivative \( \rho \) for incompressible flows, the continuity equation is integrated over the control volume in figure 2.1 to yield

\[ F_e - F_w + F_n - F_s = 0 \]  

(2.48)

where \( F_e, F_w, F_n \) and \( F_s \) are the mass flow rates through the faces of the control volume.

If \( \rho u \) at point \( e \) (figure 2.1 is taken to prevail over the whole interface \( e \), we can write

\[ F_e = (\rho u)_e \Delta y \]  

(2.49)

Similarly, considering all the interfaces we can write

\[ F_w = (\rho u)_w \Delta y \]  

(2.50)

\[ F_n = (\rho v)_n \Delta x \]  

(2.51)

\[ F_s = (\rho v)_s \Delta x \]  

(2.52)

**Momentum Equations**

The two dimensional form of the \( x \)-momentum equation can be written as

\[ \frac{\partial}{\partial t} (\rho \phi) + \frac{\partial J_{u-x}}{\partial x} + \frac{\partial J_{u-y}}{\partial y} = S_u - \frac{\partial p}{\partial x} \]  

(2.53)
where $J_{u-x}$ and $J_{u-y}$ are the total (convection plus diffusion) fluxes defined by

$$
J_{u-x} = \rho uu - \Gamma \frac{\partial u}{\partial x} \quad (2.54)
$$

$$
J_{u-y} = \rho uv - \Gamma \frac{\partial u}{\partial y} \quad (2.55)
$$

where $u$ and $v$ denote the velocity components in the $x$ and $y$ directions. The integration of eqn.(2.53) over the control volume shown in figure 2.1 would give

$$
\frac{\partial (\rho u)_p}{\partial t} \Delta x \Delta y + J_{u-e} - J_{u-w} + J_{u-n} - J_{u-s} = -(p_E - p_W) \Delta y + S_u \Delta x \Delta y \quad (2.56)
$$

If we now multiply eqn.(2.48) by $u_p$ and subtract it from eqn.(2.56) we obtain

$$
\frac{\partial (\rho u)_p}{\partial t} \Delta x \Delta y = -(J_e - F_e \phi_p) - (J_w - F_w \phi_p) + (J_n - F_n \phi_p) - (J_s - F_s \phi_p) - (p_E - p_W) \Delta y + S_u \Delta x \Delta y \quad (2.57)
$$

We can simplify the equation even further using the following equations,

$$
J_{u-e} - F_e u_p = a_{u-E}(u_p - u_E) \quad (2.58)
$$

$$
J_{u-w} - F_w u_p = a_{u-W}(u_W - u_p) \quad (2.59)
$$

$$
J_{u-n} - F_n u_p = a_{u-N}(u_p - u_N) \quad (2.60)
$$

$$
J_{u-s} - F_s u_p = a_{u-S}(u_S - u_p) \quad (2.61)
$$

Here the coefficients are

$$
a_{u-E} = D_e A(\| P_e \|) + [1 - F_e, 0] \quad (2.62)
$$

$$
a_{u-W} = D_w A(\| P_w \|) + [F_w, 0] \quad (2.63)
$$

$$
a_{u-N} = D_n A(\| P_n \|) + [1 - F_n, 0] \quad (2.64)
$$

$$
a_{u-S} = D_s A(\| P_s \|) + [F_s, 0] \quad (2.65)
$$

where $[\| p, q \|]$ implies the maximum of $p$ and $q$. The function $A$ is the power-law scheme which is defined in eqn.(2.28), and

$$
D_e = \frac{\Gamma_e \Delta y}{(\delta x)_e} \quad (2.66)
$$
\[ D_w = \frac{\Gamma_w \Delta y}{(\delta x)_w} \]  
\[ D_n = \frac{\Gamma_n \Delta x}{(\delta y)_n} \]  
\[ D_s = \frac{\Gamma_s \Delta x}{(\delta y)_s} \]

and the Ps are,

\[ P_e = \frac{F_e}{D_e} \]  
\[ P_w = \frac{F_w}{D_w} \]  
\[ P_n = \frac{F_n}{D_n} \]  
\[ P_s = \frac{F_s}{D_s} \]

Therefore, our original equation becomes,

\[ \frac{\partial (\rho u)}{\partial t} \Delta x \Delta y = -[a_{u-E}(u_p - u_E) - a_{u-W}(u_W - u_p) + a_{u-N}(u_p - u_N) - a_{u-S}(u_S - u_p)] \]

\[ - (p_E - p_W) \Delta y + S_u \Delta x \Delta y \]  
(2.74)

which can be written as,

\[ \frac{\partial (\rho u)}{\partial t} \Delta x \Delta y = [a_{u-E}u_E + a_{u-W}u_W + a_{u-N}u_N + a_{u-S}u_S - a_{u-p}u_p] + b_u \]  
(2.75)

where

\[ a_{u-p} = a_{u-E} + a_{u-W} + a_{u-N} + a_{u-S} \]  
(2.76)

\[ b_u = S_u \Delta x \Delta y + (p_W - p_E) \Delta y \]  
(2.77)

Now equation 2.75 can be written as

\[ \frac{\partial (u)}{\partial t} \Delta x \Delta y = R_i(u) \]  
(2.78)

\( \Delta x \Delta y \) is the volume \( \forall_i \). Therefore we have,

\[ \frac{\partial (u)}{\partial t} \forall_i = R_i(u) \]  
(2.79)
where,
\[ R_i(u) = \frac{[a_{u-E}u_E + a_{u-W}u_W + a_{u-N}u_N + a_{u-S}u_S - a_{u-p}u_p] + b_u}{\rho} \]  
\hspace{1cm} (2.80)

Applying Runge-Kutta time integration to this we get,

\[ u^{(0)} = u^n \] \hspace{1cm} (2.81)

\[ u^{(1)} = u^{(0)} + \frac{1}{4} \Delta t R_i(u^{(0)}) \] \hspace{1cm} (2.82)

\[ u^{(2)} = u^{(0)} + \frac{1}{3} \Delta t R_i(u^{(1)}) \] \hspace{1cm} (2.83)

\[ u^{(3)} = u^{(0)} + \frac{1}{2} \Delta t R_i(u^{(2)}) \] \hspace{1cm} (2.84)

\[ u^{(4)} = u^{(0)} + \Delta t R_i(u^{(3)}) \] \hspace{1cm} (2.85)

\[ u^{n+1} = u^{(4)} \] \hspace{1cm} (2.86)

Using these equations the velocities can be integrated in time.

A similar approach can be followed for the y-momentum equation to yield

\[ \frac{\partial (\rho v)}{\partial t} \Delta x \Delta y = [a_{v-E}v_E + a_{v-W}v_W + a_{v-N}v_N + a_{v-S}v_S - a_{v-p}v_p] + b_v \] \hspace{1cm} (2.87)

Here the coefficients are derived in a fashion similar to that of the x-momentum equation. In the above equation

\[ a_{v-p} = a_{v-E} + a_{v-W} + a_{v-N} + a_{v-S} \] \hspace{1cm} (2.88)

\[ b_v = S_v \Delta x \Delta y + (p_S - p_N) \Delta x \] \hspace{1cm} (2.89)

It should be noted that these equations have been arrived at from the Navier-Stokes equations and assuming incompressibility. Also, the spatial integration is done exactly using the stencil and technique followed in the formulation of SIMPLER.

**Computation of the Pressure Field**

Now we are in a position to solve the momentum equations to obtain the velocities for any given pressure field. The Crank-Nicolson or the Runge-Kutta time integration
schemes could be used for this purpose. The merits of each will be discussed in the following chapters. The question now is how to obtain the pressure field.

![Figure 2.2 Control volume for u](image)

The discretized equation for the calculation of the diffusion coefficient and the mass flow rate at the faces of the $u$ control volume as shown in figure 2.2 can be written as

$$a_e u_e = \sum a_{u-nb} u_{nb} + b_u + (p_P - p_E) A_e$$  \hspace{1cm} (2.90)

Here $nb$ is the number of neighbors, which is four in case of a 2D grid. Similarly the momentum equation for $v$ can be discretized as

$$a_n v_n = \sum a_{v-nb} v_{nb} + b_v + (p_P - p_N) A_n$$  \hspace{1cm} (2.91)

where $(p_P - p_N) A_n$ is the appropriate pressure force on a control volume face. The momentum equations can be solved only when the pressure field is given or somehow estimated. Unless the correct pressure field is employed, the resulting velocity field will not satisfy the continuity equation. Such an imperfect velocity field based on the guessed pressure $p^*$ will be denoted by $u^*$ and $v^*$. The “starred” velocity field will result from the solution of the following discretization equations.

$$a_e u_e^* = \sum a_{u-nb} u_{nb}^* + b_u + (p_P^* - p_E^*) A_e$$  \hspace{1cm} (2.92)

$$a_n v_n^* = \sum a_{v-nb} v_{nb}^* + b_v + (p_P^* - p_N^*) A_n$$  \hspace{1cm} (2.93)
The correct pressure $p$ is obtained from the following equation

$$p = p^* + p'$$  \hspace{1cm} (2.94)

where $p'$ is the pressure correction. The corresponding velocity corrections $u'$ and $v'$ can be introduced in a similar manner.

$$u = u^* + u'$$  \hspace{1cm} (2.95)

$$v = v^* + v'$$  \hspace{1cm} (2.96)

Subtracting eqn.(2.92) from eqn.(2.90), we have

$$a_e u'_e = \sum a_{u-nb}u'_n + (p'_p - p'_E)A_e$$  \hspace{1cm} (2.97)

For algebraic and computational convenience we can now drop the term $\sum a_{nb}u'_{nb}$ form the above equation. It so happens that the converged solution given by SIMPLER does not contain any error resulting from the omission of $\sum a_{nb}u'_{nb}$. This omission would of course be unacceptable if it altered the final solution. After the omission of the above mentioned terms, the following equation results

$$a_e u'_e = (p'_p - p'_E)A_e$$  \hspace{1cm} (2.98)

$$\Rightarrow u'_e = d_e(p'_p - p'_E)$$  \hspace{1cm} (2.99)

where

$$d_e \equiv \frac{A_e}{a_e}$$  \hspace{1cm} (2.100)

Now we can write the velocity correction formula which is same as eqn.(2.95),

$$u_e = u_e^* + d_e(p'_p - p'_E)$$  \hspace{1cm} (2.101)

Similarly for $v$ we have,

$$v_n = v_n^* + d_n(p'_p - p'_N)$$  \hspace{1cm} (2.102)

where

$$d_n = \frac{A_n}{a_n}$$  \hspace{1cm} (2.103)
The Pressure-Correction Equation

Integrating the continuity equation (eqn.(2.47)) over the control volume (figure 2.1), and assuming incompressible flow (constant density, ρ) the first term of the integral would vanish, and hence we get

\[(\rho u)_e - (\rho u)_w] \Delta y + [(\rho v)_n - (\rho v)_s] \Delta x = 0 \quad (2.104)\]

If we substitute the expressions of the velocity correction formulas, that is eqns.(2.101) and (2.102), for the velocities we get,

\[a_p p'_p = a_E p'_E + a_W p'_W + a_N p'_N + a_S p'_S + \bar{b} \quad (2.105)\]

where

\[a_E = \rho_e d_e \Delta y \quad (2.106)\]
\[a_W = \rho_w d_w \Delta y \quad (2.107)\]
\[a_N = \rho_n d_n \Delta x \quad (2.108)\]
\[a_S = \rho_s d_s \Delta x \quad (2.109)\]
\[a_P = a_E + a_W + a_N + a_S \quad (2.110)\]
\[\bar{b} = [(\rho u^*)_w - (\rho u^*)_e] \Delta y + [(\rho v^*)_n - (\rho v^*)_s] \Delta x \quad (2.111)\]

The Pressure Equation

An equation for obtaining the pressure field can be derived as follows: The x-momentum equation, regardless of the time integration scheme, is first written as

\[u_e = \frac{\sum a_{u-nb} u_{nb} + b}{a_{u-e}} + d_e(p_p - p_E) \quad (2.112)\]

where

\[d_e \equiv \frac{A_e}{a_{u-e}} \quad (2.113)\]
Now we define,

\[ \hat{u}_e = \frac{\sum a_{u-nb} u_{n_b} + b}{a_{u-e}} \]  

(2.114)

Now eqn.(2.112) becomes,

\[ u_e = \hat{u}_e + d_e(p_p - p_E) \]  

(2.115)

Similarly we can write

\[ \hat{v}_n = \frac{\sum a_{v-nb} v_{n_b} + b}{a_{v-n}} \]  

(2.116)

\[ v_n = \hat{v}_n + d_n(p_p - p_N) \]  

(2.117)

Integrating the continuity equation (eqn.(2.47)) over the control volume (figure 2.1), and assuming \( \rho \) to be constant (which means the first term goes to zero), we get

\[ [(\rho u)_e - (\rho u)_w] \Delta y + [(\rho v)_n - (\rho v)_s] \Delta x = 0 \]  

(2.118)

Substituting the velocities from eqns.(2.115) and (2.116), and rearranging we get,

\[ a_p p_p = a_E p_E + a_W p_W + a_N p_N + a_S p_S + b \]  

(2.119)

where

\[ a_E = \rho_e d_e \Delta y \]  

(2.120)

\[ a_W = \rho_w d_w \Delta y \]  

(2.121)

\[ a_N = \rho_n d_n \Delta x \]  

(2.122)

\[ a_S = \rho_s d_s \Delta x \]  

(2.123)

\[ a_p = a_E + a_W + a_N + a_S \]  

(2.124)

and

\[ b = [(\rho \hat{u})_w - (\rho \hat{u})_e] \Delta y + [(\rho \hat{v})_s - (\rho \hat{v})_n] \Delta x \]  

(2.125)

It can be seen that the only difference between the pressure equation and the pressure-correction equation is the expression for \( b \). Also, the meaning of the coefficients \( a_p, a_E, a_W, a_N, a_S \) and \( b \) depend on the time integration scheme followed.
3 NUMERICAL ALGORITHM

Both the schemes presented in the previous chapters use a displaced or "staggered" grid for the velocity components. In a staggered grid, the velocity components are calculated for the points that lie on the faces of the control volumes. Thus, the $x$-direction velocity $u$ is calculated at the faces that are normal to the $x$ direction. The locations for $u$ and $v$ are shown in figure 3.1 by short arrows, while the grid points are shown by small circles. The solid lines depict the control volumes. As can be seen in the figure, the $u$ locations are staggered only in the $x$ direction, and $v$ locations in the $y$ direction respectively. Both the algorithms presented in this work store the variables in a staggered grid, that is, the $u$ and $v$ velocities are not stored at the grid points, though the pressure is stored at the grid point.

Figure 3.1 Staggered locations for $u \rightarrow$, $v \uparrow$ and other $\circ$ variables
As mentioned earlier, no approximations are introduced in the derivation of the pressure equation (2.119). Thus, if the pseudo-velocities could be calculated precisely, the pressure equation would at once give the precise pressure.

**The Implicit Algorithm based on Crank-Nicolson (CN) Scheme:**

The Crank-Nicolson time integration scheme is a semi-implicit method, which is given by eqn.(2.1) where the value of $\alpha$ is 0.5. The SIMPLER algorithm using Crank-Nicolson consists of solving the pressure equation to obtain the pressure field and solving the pressure-correction equation only to correct the velocities. The sequence of operations can be stated as

1. Start with a guessed velocity field.
2. Calculate the Crank-Nicolson unsteady source terms using eqns.(2.36) and (2.39).
3. Calculate the coefficients of the $x$-momentum equation using equations (2.23) to (2.28) and (2.31) to (2.36) and using a similar expression for the $y$-momentum equation.
4. Calculate $\hat{u}$ and $\hat{v}$ from eqns.(2.114) and (2.116) respectively by substituting the values of the neighbor velocities $u_{nb}$.
5. Calculate the coefficients for the pressure equation (2.119), and solve it to obtain the pressure field.
6. Using this pressure field, solve the momentum equations (eqn.(2.30) for the $x$-component of the Navier-Stokes equations and eqn.(2.37) for the $y$-component) to obtain $u^*$ and $v^*$, since the pressure was from a guessed velocity field.
7. Calculate the mass source term $\bar{b}$ from eqn.(2.111), and hence solve the $p'$ equation, eqn.(2.105).
8. Correct the velocity field using the corrections obtained from the pressure correction equation, using equations (2.101)-(2.102).

9. Return to step 3 and repeat until convergence.

10. Start with a new time level.

The algorithm is also presented in the form of a flow-chart in figure 3.2.

Figure 3.2 Flow-chart for SIMPLER based on Crank-Nicolson scheme
The Explicit Algorithm based on Runge-Kutta (RK) Scheme

This algorithm is based on the fact that, if the pressure field is known the momentum equations can be solved to obtain the velocities correctly. Therefore we solve the pressure equation first and then update the velocities using the Runge-Kutta time integration scheme. The sequence of operations can be stated as

1. Start with a guessed velocity field.
2. Calculate the coefficients of the $x$-momentum equation using eqns.(2.62) - (2.65) and eqns.(2.76) - (2.77) and similarly for the $y$-momentum equation using eqns.(2.88) - (2.89).
3. Calculate $\hat{u}$ and $\hat{v}$ from eqns.(2.114) and (2.116) respectively by substituting the values of the neighbor velocities $u_{nb}$. 
4. Calculate the coefficients for the pressure equation (2.119), and solve it to obtain the pressure field.
5. Update the velocities using four-stage Runge-Kutta equations (2.41) - (2.46).
6. Start with a new time level.

The flow-chart of this algorithm is shown in figure 3.3. It should be noted that we do not need to iterate over the momentum equations to get the solution at a new time level. The intermediate velocities are updated over the entire domain after each Runge-Kutta stage. This is done by creating a temporary variable which stores the intermediate velocities after every Runge-Kutta stage and the actual velocities are updated once the intermediate velocities are calculated over the entire domain. This ensures that at each stage the neighboring velocities are completely updated from the previous stage, so that there are no approximations involved. The coefficients remain constant over all
the Runge-Kutta stages even though the velocities change. The coefficients are updated at every time-step. Such a procedure (no iterations) implies that the difference in the two time levels has to be very small, which means that the time-step has to be small. As will be seen in the following chapters, the time-step required by the Runge-Kutta time integration scheme is indeed much smaller than that of the Crank-Nicolson time integration scheme. The only iterations involved in the Runge-Kutta time integration scheme, is the one for the pressure equation.

Figure 3.3 Flow-chart for the new explicit Runge-Kutta scheme

Implementation Details

To implement the Runge-Kutta time integration scheme discussed in the previous section, an existing FORTRAN program using the Crank-Nicolson time integration scheme was chosen. The following changes were made to the program:

- The spatial discretization subroutines were changed according to the differences between equations (2.30) and (2.75), which are the discretized momentum equa-
tions for the Crank-Nicolson and the Runge-Kutta formulations, respectively.

- The source term $b$ in the discretized momentum equations also changes, therefore the subroutine calculating that term was also changed (equations (2.36) and (2.77)).

- The subroutine calculating the source term in the pressure equation was also changed (equation (2.125)).

- The time integration subroutines were rewritten for the Runge-Kutta time integration scheme.
Table 3.1 Comparison: Crank-Nicolson and Runge-Kutta schemes

<table>
<thead>
<tr>
<th>Step</th>
<th>Crank-Nicolson</th>
<th>Runge-Kutta</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Start with a guessed velocity field</td>
<td>Start with a guessed velocity field</td>
</tr>
<tr>
<td>2</td>
<td>Calculate the Crank-Nicolson unsteady source terms, eqn.(2.36) and eqn.(2.39)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Calculate the Coeff. of momentum eqns eqn.(2.31) to eqn.(2.35) and eqn.(2.38)</td>
<td>Calculate the coeff. of momentum eqns eqn.(2.76) and eqn.(2.77) and eqn.(2.88) to eqn.(2.89)</td>
</tr>
<tr>
<td>4</td>
<td>Calculate ( \hat{u} ) and ( \hat{v} ) from eqn.(2.114) and eqn.(2.116)</td>
<td>Calculate ( \hat{u} ) and ( \hat{v} ) from eqn.(2.114) and eqn.(2.116)</td>
</tr>
<tr>
<td>5</td>
<td>Calculate the coefficients for the pressure equation (2.119), and solve it to obtain the pressure field.</td>
<td>Calculate the coefficients for the pressure equation (2.119), and solve it to obtain the pressure field.</td>
</tr>
<tr>
<td>6</td>
<td>Solve the momentum equations (2.30) and (2.37) to obtain ( u^* ) and ( v^* ).</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Calculate the mass source term from eqn.(2.111) and solve the ( p' ) equation, (2.105)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Correct the velocity field using equations(2.101)-(2.102)</td>
<td>Update the velocities using four-stage Runge-Kutta equations (2.41)-(2.46) (Note: velocities are updated after every stage but the coefficients are held constant)</td>
</tr>
<tr>
<td>9</td>
<td>Return to step 3 and repeat until mass convergence is satisfied</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Advance in time and return to step 2</td>
<td>Advance in time and return to step 2</td>
</tr>
</tbody>
</table>
4 COMPARATIVE STUDY OF THE TWO SCHEMES

All the test cases presented in this chapter use a Cartesian grid and solve the unsteady incompressible Navier-Stokes equations in conjunction with the SIMPLER algorithm. The test problems chosen to validate the new formulation are the well known:

- two-dimensional lid-driven cavity (internal flow)
- two-dimensional laminar backward-facing step flow (internal flow)
- flow over a flat-plate at 90° incidence (external flow).

These are important benchmark problems, as they cover a wide range of Reynolds numbers, to test the relative accuracy of a newly formulated scheme. All the test problems, though simple in geometry, consist of complex flow structures and do not lack important interaction between the various flow structures present. These problems as well as the results obtained are discussed in the following sections.

To validate the accuracy of the algorithms, it needs to be tested over a wide range of Reynolds numbers. This led to the choice of the problems mentioned above. The two-dimensional lid-driven cavity problem is an internal flow with simple boundary conditions for testing the schemes at low Reynolds numbers. The two-dimensional laminar backward-facing step problem ranges from low to moderately high Reynolds numbers with inlet and outlet boundary conditions. And the external flow over a flat-plate at 90° incidence is a very high Reynolds number flow with significant unsteady phenomena.
In spite of its idealization, the flow inside a lid-driven cavity has been widely acknowledged as an excellent test case for evaluating numerical schemes. This is due to the presence of large streamline to grid skewness over most of the flow domain (on a rectangular grid). Moreover, there exist several relatively large recirculating regions, where convection and diffusion are of comparable magnitude. The standard well-known laminar steady two dimensional lid-driven square cavity is shown in figure 4.1 will be used as our first test problem. The Reynolds number is defined by $Re = \frac{\rho U_{lid} L}{\mu}$, where $\rho$ is the constant density of the fluid, $U_{lid}$ is the speed of the sliding wall, $L$ is the length of the square enclosure side wall, and $\mu$ is the fluid viscosity. The cavity length is chosen to be unity, and a $42 \times 42$ grid is used which is shown in figure 4.2.

All calculations start with a unit velocity field in the $x$-direction and zero in the $y$-direction and a unit pressure field. Since we are interested in a steady state solution, each calculation is terminated when the residual $\epsilon$ becomes smaller than $10^{-9}$, where
\( \epsilon \) is defined as the maximum value of the residuals for the continuity equation, and the difference between the \( u \) and \( v \)-velocities between two successive iterations. No-slip boundary conditions are used on all the four walls. All computations are performed using a constant time-step of different value for the two algorithms.

A moderately high Reynolds number, \( Re = 400 \), is chosen for the purpose of testing. The results of both the schemes are given for the purpose of comparison. The results obtained for \( Re = 1000 \) are also presented in figs. (4.6) and (4.7).

Along the vertical centerline \( (x = 0.5) \), the \( u \)-velocity is the dominant component, and along the horizontal centerline \( (y = 0.5) \), the \( v \)-velocity is the dominant component. Therefore, the accuracy with which the formulation can model these dominant velocity components will be analyzed.

Observing the \( u \)- and \( v \)-velocity components along the vertical and horizontal center-
Figure 4.3  Vertical centerline $u$-profile for driven-cavity at $Re = 400$

lines respectively (figures 4.3 and 4.4), we can see that the two schemes predict almost the same velocities. The difference in the velocities from the two schemes is observed in the fourth decimal place, which cannot be observed in the graphs. Figure 4.5 shows the convergence histories (maximum mass residuals of the continuity equation versus the iterations) on a log-log scale for both the schemes.

Furthermore it can be observed from the plots that the solution predicted from the two schemes is not in perfect agreement with the FCM (Flux Corrected Method) solution [10]. The FCM [10] solutions are used as a reference for comparison with the present schemes. The differences with the FCM solution can be attributed to the significantly diffusive nature of the Power-Law scheme for high Reynolds number recirculating flows. These conclusions are clearly stated by previous researchers, Wirogo [10] and Leonard [13]. The differences with the FCM solution can be observed for both, $Re = 400$ as well as $Re = 1000$. It is clear that the Power-Law based solutions would require considerable
grid refinement in order to match the reference centerline $u$-velocity. The results of such a study can be seen in the $102 \times 102$ grid solution, where the present schemes agree with the reference solution closely. The finer grid is used for $Re = 400$, and the centerline $u$- and $v$-velocities can be seen in figure 4.8 and figure 4.9, respectively. Also shown is a vector plot (figure 4.10) of the driven cavity as predicted by the Runge-Kutta scheme.

For a Reynolds number of 400 the maximum permissible time-step that could be used for the Crank-Nicolson time integration scheme was approximately $2 \times 10^{-2}$, and that for the Runge-Kutta time integration scheme was approximately $6 \times 10^{-3}$. These values are for the $42 \times 42$ grid. Hence it can be seen that the Runge-Kutta scheme requires a 3.33 times smaller time-step than the Crank-Nicolson scheme for the same grid size. The time-steps for $Re = 1000$ are smaller for both the algorithms. The Runge-Kutta scheme required a time-step of $9 \times 10^{-4}$ and that of the Crank-Nicolson was $3 \times 10^{-3}$. 

![Driven Cavity (Re = 400)](image.png)

Figure 4.4 Horizontal centerline $v$-profile for driven-cavity at $Re = 400$
Figure 4.5 Convergence history for driven cavity at $Re = 400$

Figure 4.6 Vertical centerline $u$-profile for driven-cavity at $Re = 1000$
Driven Cavity (Re = 1000)

Figure 4.7  Horizontal centerline $v$-profile for driven-cavity at $Re = 1000$

Driven Cavity (Re=400)

Vertical centerline $u$-profile

Figure 4.8  Vertical centerline $u$-profile for driven-cavity at $Re = 400$ using $102 \times 102$ grid
Figure 4.9  Horizontal centerline $v$-profile for driven-cavity at $Re = 400$ using $102 \times 102$ grid

Figure 4.10  Vector plot for driven-cavity at $Re = 400$
Flow Over a Backward-Facing Step

The simulation of viscous incompressible flow over a backward-facing step has been recognized as another excellent test problem for evaluating various schemes. This particular flow problem represents a simplification of the well known internal flow over a sudden expansion found in many important industrial applications. A special characteristic of this flow is the interaction between two different flow structures, which are the primary reverse flow just behind the step and the shear layer emanating form the step edge. It has been verified that, both experimentally and numerically, that for a given expansion ratio and assumed inlet profile, the re-attachment point of the separated region is dependent only on the Reynolds number. The schematic diagram of this flow problem is shown in figure 4.11. The dimensions of the channel and the three possible recirculating regions are also shown in the figure. The Reynolds number is given by:

\[ Re = \frac{U_{avg} \rho (2L)}{\mu} \]  

where \( U_{avg} \) is the average inlet velocity and \( \mu \) is the viscosity of the fluid.

Over the past two decades, both experimental and numerical investigations have been performed for this test case. The most notable experimental investigation is the
Laser-Doppler measurements of Armaly et. al. [14] which investigated in detail the characteristics of this flow for Reynolds number range of $70 \leq Re \leq 8000$, covering the laminar, transitional and turbulent flows. In Ref [14], Armaly reported that for a certain range of Reynolds numbers, there are two other possible secondary recirculating regions (II and III) of vastly different strength besides the primary recirculating region just behind the step (I).

The boundary conditions for the step geometry included the usual no-slip velocity specification for all solid surfaces, see figure 4.11. The inlet velocity field was specified as a flow with a parabolic profile given by $u(y) = 12(2y - 1)(1 - y)$ for $0.5 \leq y \leq 1.0$. This produces a maximum inflow velocity of $U_{max} = 1.5$ and an average inflow velocity of $U_{avg} = 1.0$.

In this case too, all calculations start with a unit velocity field in the $x$-direction and zero in the $y$-direction and a unit pressure field. Each calculation is terminated when the residual $\epsilon$ becomes smaller than $10^{-9}$, where $\epsilon$ is defined as the maximum value of the residuals for the continuity equation, and the difference between the $u$ and $v$-velocities between two successive iterations. The flow is started impulsively and the time-step is held constant during the computations.

Fig. (4.12) shows the plot between primary separation lengths versus the Reynolds number. The primary separation length is defined as $x$ shown in figure 4.11. Figure 4.12 is a comparison between experimental values by Armaly et al. [14] and Gartling [15] and computational values generated by the schemes presented in this work, by Armaly et al. and the Power-Law scheme results by S. Wirogo [10] for three different Reynolds numbers (namely $Re = 100$, $Re = 400$ and $Re = 800$). As is evident from the figure, the Runge-Kutta and the Crank-Nicolson time integration schemes are in good agreement with the other Power-law computational results available.

Figure 4.13 and figure 4.14 give the streamline plots of the transient development of flow using the Crank-Nicolson algorithm at $Re = 800$. The secondary recirculation
regions (II and III) can be seen in these plots. The grid used was a $172 \times 172$ grid. Following those figures are the streamline plots of the transient development of flow using the Runge-Kutta algorithm at $Re = 800$ at the same time intervals as the Crank-Nicolson scheme. The plots are shown in figure 4.15 and figure 4.16. As can be observed from the plots, both the schemes give approximately the same results for any given time, although the Runge-Kutta algorithm uses a much smaller time step.

The maximum permissible time-step for Crank-Nicolson in this case was approximately $1.0 \times 10^{-2}$ and that for Runge-Kutta was approximately $5.0 \times 10^{-4}$, for the $102 \times 42$ grid and the $Re = 100$ case.
Figure 4.13 Crank-Nicolson unsteady development part 1 for Re = 800

Figure 4.14 Crank-Nicolson unsteady development part 2 for Re = 800
Figure 4.15  Runge-Kutta unsteady development part 1 for $Re = 800$

Figure 4.16  Runge-Kutta unsteady development part 2 for $Re = 800$
Flat-Plate at $90^\circ$ Incidence

The last case is the simulation of laminar flow at high Reynolds numbers. A flat-plate in a flow normal to its surface at a very high Reynolds number is simulated. It is observed that when bluff bodies are held in a stream of fluid at large Reynolds numbers, the streamlines passing near the forward face of a rigid body break away at the sides of the body and enclose the fluid in slow and unsteady motion, which is an example of boundary-layer separation. Breakaway of the surface streamlines may occur even at Reynolds numbers near 10, but it takes particular importance at large Reynolds numbers because the streamlines leaving the surface then carry vorticity of large magnitude away from the surface [16]. Moreover, for bodies with sharp edges, such as disks and flat-plates, the pressure gradient would have to be extremely high for a flow to remain attached to the rear of the plate. No boundary layer, whether laminar or turbulent, can follow the way around the edges of such plates [17]. The flow separates behind the plates and there is a periodic shedding of the vortices which causes the $C_d$ to oscillate in a smooth sine-wave pattern.

The total drag on a body placed in a stream of fluid consists of skin friction (equal to the integral of all shearing stresses taken over the surface of the body) and of pressure drag (integral of normal forces). The sum of the two is called the profile drag. The profile drag does not exist in frictionless flow. In a real fluid drag is finite due to the fact that the presence of the boundary layer which modifies the pressure distribution on the body as compared with ideal flow [18]. At very high Reynolds numbers the contribution of skin-friction drag to the total drag is very small particularly for bluff body flows. Therefore the pressure drag will give us a reasonable estimate of the total drag. Hence the coefficient of drag ($C_d$) for the flat-plate at $90^\circ$ incidence can be predicted approximately by considering only the normal pressure drag.

Extensive experimental research in the past has shown that the flow around bluff
bodies is inherently unsteady as mentioned above. Periodic shedding of vortices has also been observed for a certain range of Reynolds numbers. However, in the present work no attempt has been made to investigate the instabilities which cause the flow to oscillate.

The Reynolds number for this flow is defined to be $Re = (\rho V_i L) / \mu$ and is varied by changing the velocity $V_i$. Other parameters chosen for the problem are $\rho = 1.0$, $L = 0.03m$ with a domain size of $1m \times 1.2m$. Here $\rho$ is the density, $V_i$ is the incident velocity, and $\mu$ is the fluid viscosity. The Strouhal number is defined as

$$Sr = \frac{Lf}{V_i}$$

where $f$ is the frequency of shedding in Hz. The test case was simulated on a $172 \times 172$ stretched cartesian grid shown in figure 4.18 with 10 points on the plate in the streamwise direction (that is along $d$) and 20 points along $L$.

Figure 4.17 Schematic of the flat-plate at 90° incidence

**Results of the Crank-Nicolson Time Integration**

Two different values of Reynolds numbers are chosen ($Re = 1.78 \times 10^6$ and $Re = 1 \times 10^5$) for a total of two cases. The grid used is stretched in the region in the vicinity of the plate to capture the complex flow field near it. For simplicity, impulsively started uniform flow is assumed at the inlet boundary. At the outlet boundary, in addition to
assuming the diffusion to be free, the normal velocity profile is adjusted for overall mass conservation. All computations are performed using a constant time step.

The time history of drag is shown in figure 4.19 for $Re = 1.78 \times 10^6$. It is evident from the figure that after the impulsive start the flow is in a state of development. However, after about 0.025 seconds oscillations begin to appear. The flow is developed and becomes periodic after about 0.03 seconds. The mean drag coefficient is about 2.35. In order to understand the nature of the flow and vortex dynamics, two cycles of the drag history are discussed.

The drag cycles discussed for the Crank-Nicolson(CN) scheme for $Re = 1.78 \times 10^6$ start from $t = 0.11579$ secs to $t = 0.11919$ secs for an elapsed time of 0.0034 secs. The period of 0.0034 secs marks two complete cycles of drag coefficient and as shown in figure 4.20, points A through I are important instances of time to observe the flow. The two complete cycles of drag history represent only one cycle of vortex shedding. Points
Figure 4.19  Time history of flat-plate drag coefficient convergence (CN)

A, E and I denote the instances when vortex shedding happens. The Strouhal number calculated for this case is $Sr = 6.724$.

Another dimensionless parameter to estimate the variation pressure is the coefficient of pressure. It is defined as $c_p = \frac{P-P_{\infty}}{\frac{1}{2} \rho V^2}$. The variation of $c_p$ along the plate is shown in figure 4.21. It is obvious from the figure that the pressure in front of the plate varies negligibly through the entire two cycles of the drag coefficient variation, whereas figure 4.22 manifests the dramatic variation of the back pressure of the plate which undergoes two cycles in synchronization with the vortex shedding cycle. Also, obvious from figure 4.22 is the mirror image type of distribution of the pressure in time instances A, E and I as well as in instances C and G.

Figures 4.23 to 4.31 depict the streamlines of the flow for points A through I sequentially. The complete cycle of vortex shedding can be seen in the figures. As stated before, these correspond to two drag history cycles.
Figure 4.20 Two Cycles of Drag Coefficient (CN)

Figure 4.21 Pressure distribution on the front side of the plate (CN)
Cp at Back of Plate .vs. Location

Mach = 0.17 and Re = 1.78E6

Crank-Nicolson 172X172

Figure 4.22 Pressure distribution on the back side of the plate (CN)

Figure 4.23 Condition A: Top-shed, beginning of eddy pressure cycle (CN)
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Figure 4.24 Condition B: Growth of the bottom vortex (CN)

Figure 4.25 Condition C: First minimum drag point (CN)
Condition D: Before bottom vortex shed (CN)

Condition E: Bottom-shed (CN)
Figure 4.28  Condition F: Growth of top vortex (CN)

Figure 4.29  Condition G: Second minimum drag point (CN)
Figure 4.30  Condition H: Before top vortex shed (CN)

Figure 4.31  Condition I: Top-shed, end of eddy pressure cycle (CN)
The second value of Reynolds number used was $Re = 1 \times 10^5$. As mentioned before, the Reynolds number is varied by varying the inlet velocity. All the other boundary conditions and initial conditions are the same. The time-step is also changed, since it varies inversely with the Reynolds number. A higher time-step could be used for this purpose since we have a lower Reynolds number. The time-step used for the computations was fixed.

The time history of drag is shown in figure 4.32 for $Re = 1 \times 10^5$. After an impulsive start the flow becomes periodic after about 0.6 seconds. The mean drag coefficient is still about 2.35. It should be noted that by decreasing the order of the Reynolds number we get an increase in the period of the oscillations by the same order. Figure 4.33 shows two cycles of the drag coefficient.

The variation of $c_p$ along the front of the plate is shown in figure 4.34. It can be seen in the figure that the pressure in the front of the plate varies negligible through the
entire two cycles of the drag coefficient variation. Figure 4.35 shows the variation of $c_p$ along the back of the plate. The mirror image type of variation for instances A, E and I and of instances C and G can be seen. These are in synchronization with the vortex shedding cycle.

These plots show that the pressure distribution and the drag history predicted by the Crank-Nicolson time integration scheme are essentially the same for both values of the Reynolds numbers, except for the fact that the velocities and hence the pressures and the frequency of vortex shedding are higher for the higher Reynolds number case.

**Results of the Explicit Runge-Kutta Time Integration**

The two different values of Reynolds number used for the Crank-Nicolson time integration scheme were also used for the Runge-Kutta time integration scheme namely $Re = 1.78 \times 10^6$ and $Re = 1 \times 10^5$. The grid was the same as the Crank-Nicolson case
Figure 4.34 Pressure distribution on the front side of the plate (CN)

(figure 4.18). The boundary conditions and the initial conditions were also the same. Impulsively started uniform flow is assumed at the inlet. The outlet boundary is assumed to be diffusion free and the normal velocity profile is adjusted for overall mass conservation. All computations are performed at a constant time step (though a much smaller time-step is needed for Runge-Kutta, the details of which shall be addressed later).

The time history of drag is shown in figure 4.36 for $Re = 1.78 \times 10^6$. In this case too the flow is in a development stage after the impulsive start. The flow becomes periodic after about 0.03 seconds. The mean drag coefficient is about 2.35.

The drag cycles discussed for the Runge-Kutta(RK) scheme for $Re = 1.78 \times 10^6$ start from $t = 0.12309$ secs to $t = 0.12649$ secs for an elapsed time of 0.0034 secs. The period of 0.0034 secs marks two complete cycles of drag coefficient and as marked in figure 4.37, points A through I are important instances of time to observe the flow. Points A, E
and I denote the instances when vortex shedding happens. The Strouhal number in this case too is the same $Sr = 6.724$. As was discussed earlier, these two cycles of the drag coefficient actually represent only one cycle of of the vortex shedding.

Figure 4.38 shows the variation of $c_p$ along the plate. It can be seen that the pressure varies negligibly on the front of the plate. Figure 4.39 shows the dramatic variation of the back pressure of the plate which undergoes two cycles in synchronization with the vortex shedding cycle. The mirror image type of distribution of the pressure in time instances A, E and I as well as in instances C and G can be seen in figure 4.39. It is clear that both the schemes give approximately the same variation.

Figures 4.40 to 4.48 depict the streamlines of the flow for points A through I sequentially for the Runge-Kutta time integration scheme. One complete cycle of vortex shedding can be seen very clearly in these figures similar to the Crank-Nicolson scheme.

For the Runge-Kutta time integration scheme the second value of Reynolds number
used was $Re = 1 \times 10^5$, which is the same as the one used for the Crank-Nicolson time integration scheme. All the other boundary conditions and initial conditions are the same. The time-step is also changed, since it varies inversely with the Reynolds number.

The time history of drag is shown in figure 4.49 for $Re = 1 \times 10^5$. After an impulsive start the flow becomes periodic after about 0.6 seconds. The mean drag coefficient is still about 2.35. Here too by decreasing the order of the Reynolds number we get an increase in the period of the oscillations by the same order.

The variation of $c_p$ along the front of the plate is shown in figure 4.51. It can be seen in the figure that the pressure in the front of the plate varies negligibly through the entire two cycles of the drag coefficient variation. Figure 4.52 shows the variation of $c_p$ along the back of the plate. The mirror image type of variation for instances A, E and I and for instances C and G can be seen very clearly. These are in synchronization with the vortex shedding cycle.

Figure 4.36  Time history of flat-plate drag coefficient convergence (RK)
Figure 4.37 Two Cycles of Drag Coefficient (RK)

Figure 4.38 Pressure distribution on the front side of the plate (RK)
Figure 4.39 Pressure distribution on the back side of the plate (RK)

Figure 4.40 Condition A: Top-shed, beginning of eddy pressure cycle (RK)
Flat-plate at 90° incident angle at Mach=0.17
$R_e = 1.78 \times 10^6$, time = 0.12349, $\Delta t = 0.00001$
Condition B : $C_d = 2.3491$

Figure 4.41 Condition B: Growth of the bottom vortex (RK)

Flat-plate at 90° incident angle at Mach=0.17
$R_e = 1.78 \times 10^6$, time = 0.12389, $\Delta t = 0.00001$
Condition C : $C_d = 2.1889$

Figure 4.42 Condition C: First minimum drag point (RK)
Figure 4.43  Condition D: Before bottom vortex shed (RK)

Figure 4.44  Condition E: Bottom-shed (RK)
Figure 4.45  Condition F: Growth of top vortex (RK)

Figure 4.46  Condition G: Second minimum drag point (RK)
Figure 4.47  Condition H: Before top vortex shed (RK)

Figure 4.48  Condition I: Top-shed, end of eddy pressure cycle (RK)
Figure 4.49 Time history of flat-plate drag coefficient convergence (RK)

Figure 4.50 Two cycles of drag coefficient (RK)
Cp at Front of Plate .vs. Location

Mach = 0.0096 (Re = 100000)

Runge-Kutta 172X172

Pressure Coefficient, Cp

Plate Distance, y

Figure 4.51 Pressure distribution on the front side of the plate (RK)

Cp at Back of Plate .vs. Location

Mach = 0.0096 (Re = 100000)

Runge-Kutta 172X172

Pressure Coefficient, Cp

Plate Distance, y

Figure 4.52 Pressure distribution on the back side of the plate (RK)
Comparison of the Results Predicted by the Two Schemes

As we saw from the plots before, the two time integration schemes are in good agreement with each other. Figure 4.53 is a comparison of the $C_d$ versus time for both the schemes for $Re = 1.78 \times 10^6$. The largest time-step that could be used for the Crank-Nicolson time integration scheme for $Re = 1.78 \times 10^6$ was approximately $1.0 \times 10^{-4}$ and that for the Runge-Kutta time integration scheme was approximately $1.0 \times 10^{-5}$.

![Flat Plate Drag vs. Time](image)

Figure 4.53  Flat-plate drag coefficient for the two schemes ($Re = 1.78 \times 10^6$)

Figure 4.54 shows a comparison between the coefficient of pressures predicted by Runge-Kutta and Crank-Nicolson time integration schemes for instances A and E for $Re = 1.78 \times 10^6$. Instance A is the point in the vortex-shedding cycle where we have a top-vortex shed. On the contrary instance E corresponds to a bottom-vortex shed. The mirror image type distribution can be seen in the graphs.

Figure 4.55 shows the comparison of the drag history between the Crank-Nicolson and the Runge-Kutta time integration schemes for $Re = 1.0 \times 10^5$. It should be noted
that the maximum permissible time-step for the Crank-Nicolson time integration scheme was approximately $2.0 \times 10^{-4}$ and for the Runge-Kutta time integration scheme was approximately $2.0 \times 10^{-5}$. Table (4.1) tabulates the time-steps used for both the schemes.

Figure 4.56 shows a comparison between the coefficient of pressures given by Runge-Kutta and Crank-Nicolson time integration schemes for instances A and E for $Re = 1.0 \times 10^5$. In this case too instances A and E correspond to the top-vortex shed and the bottom-vortex shed respectively. The mirror image type distribution is predicted here as well.
Figure 4.55 Flat-plate drag coefficient for the two schemes \((Re = 1.0 \times 10^5)\)

Figure 4.56 Comparison of the \(C_p\) at the back of the plate for the two schemes
Figure 4.57  Time history of the flat-plate drag coefficient convergence (CN $Re = 1.78 \times 10^6$)

Figure 4.58  Time history of the flat-plate drag coefficient convergence (RK $Re = 1.78 \times 10^6$)
It can be observed that there is a period of time in the beginning where there are no oscillations in the drag (figure 4.57 and figure 4.58). In this period of time there is no vortex shedding, but two symmetric vortices are attached to the plate until one of them breaks off and the process of vortex shedding starts. One may falsely assume convergence during this period and the actual physics of the flow may still remain to be captured. It is also interesting to note that the minimum drag point occurs during this period.
5 CONCLUDING REMARKS AND RECOMMENDATIONS

In this study, a new algorithm for solving the unsteady incompressible Navier-Stokes equations using the Runge-Kutta time integration was formulated and implemented. The central idea was to treat the pressure equation implicitly similar to the Crank-Nicolson based SIMPLER algorithm but use the explicit multi-stage Runge-Kutta time stepping to advance the velocities in time. The Navier-Stokes momentum and continuity equations were rewritten and brought to a form where the four-stage Runge-Kutta time integration algorithm could be applied to it. The spatial discretization followed the same procedure in both cases but the momentum equations were not solved in the Runge-Kutta scheme.

One important criterion which needs attention is the time of execution for both the schemes - Runge-Kutta and Crank-Nicolson. The following figure and table give a comparison of the computations for the driven-cavity problem. Figure 5.1 shows the number of iterations versus the time required for a 42 x 42 grid on a log-log scale. The following table 5.1 presents the same data.

It can be inferred from the table and the graph that the Runge-Kutta scheme is faster than the Crank-Nicolson scheme. But it should also be kept in mind that Crank-Nicolson is an implicit scheme and is capable of a much larger time step than that of Runge-Kutta scheme which is an explicit scheme. It was observed that the Crank-Nicolson could operate at a time step 10 - 20 times larger than Runge-Kutta. But as we can see from the figure and the table, the Runge-Kutta time integration algorithm is approximately 27.5714 times faster than the Crank-Nicolson time integration algorithm. So it turns out
that eventually the Runge-Kutta time integration algorithm would be more appropriate for steady as well as unsteady flows. Figure 5.2 shows the plot between time of execution versus the simulated time for both the schemes. Table 5.2 gives the data. The above conclusions are reinforced by the figure and the table. The results provided in the above tables and figures (figure 5.1 and table 5.1) were generated using the 42 x 42 grid for the $Re = 400$ driven cavity problem. As mentioned in the previous chapter, the maximum time-step that could be used for the Crank-Nicolson time integration scheme was $2.0 \times 10^{-2}$ and that for the Runge-Kutta time integration scheme was $6.0 \times 10^{-3}$. This implies that the Runge-Kutta time integration scheme was using a 3.333 times smaller time-step than the Crank-Nicolson time integration scheme. As can be inferred from figure 5.2 and table 5.2, the Runge-Kutta time integration scheme reaches a given simulated time 8.5888 times faster than the Crank-Nicolson time integration scheme for the driven cavity 42 x 42 grid. This conclusion can also be derived from the figure 5.1 and
Table 5.1 Execution time for both the schemes

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<thead>
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<th>Number of iterations</th>
<th>Runge-Kutta (sec)</th>
<th>Crank-Nicolson (sec)</th>
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<tr>
<td>convergence</td>
<td>100.27</td>
<td>1204.31</td>
</tr>
</tbody>
</table>

Table 5.2 Simulation time versus execution time for driven-cavity

<table>
<thead>
<tr>
<th>Simulation time (sec)</th>
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<th>Crank-Nicolson (sec)</th>
</tr>
</thead>
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</tr>
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<tr>
<td>convergence</td>
<td>100.27</td>
<td>1204.31</td>
</tr>
</tbody>
</table>

table 5.1. Since the time required to execute a Runge-Kutta time-step is 27.5714 times faster than that of Crank-Nicolson, and the time-step taken is 3.333 times smaller, the execution time required to reach a certain simulated time will be $27.5714/3.333 = 8.2715$ times faster for the Runge-Kutta time integration scheme. This is nearly the same value we got from figure 5.2 and the corresponding table 5.2.

Furthermore, the systematic study conducted in the present work results in the following conclusions.

- The new explicit Runge-Kutta scheme is faster than the previously existing Crank-Nicolson scheme even though the latter is capable of a much larger time steps.

- The Power-Law scheme used in both the formulations is highly diffusive and as a result requires a much finer grid to capture the flow characteristics properly.

Finally, it has been shown that the new explicit algorithm based on the four-stage
explicit Runge-Kutta time integration in conjunction with the fully implicit pressure formulation is a highly viable technique for modeling complex flow structures. But, there are also a few limitations to it which could be researched in the future. Following are the possible areas of future research or improvements to the formulation discussed.

- The highly diffusive Power-Law scheme could be replaced by a more accurate method such as the second order QUICK scheme [19] [20] or FCM [10]. This would enable the use of coarser grids and hence save valuable computer time.

- The present formulation could be extended to three-dimensional flows as well as other grid systems including unstructured grids.
BIBLIOGRAPHY


