

**A formalization of logic in diagonal-free cylindric algebras**

by

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## CHAPTER 1. BACKGROUND MATERIAL

### 1.1 Introduction

In a 1915 paper, Leopold Löwenheim first conjectured that “all important questions of mathematics” may be reducible to equations of the Peirce-Schröder calculus of binary relations [Löw15]. Taking up this task, in 1942-1943 Alfred Tarski proved that almost all axiomatized set theories can be formalized as sets of relation algebraic (RA) equations [Tar43, Theorem 5.22]. He developed the Translation Mapping from first-order logic (FOL) to the equational theory of RA which preserves meaning and provability for sufficiently strong theories (including most axiomatized set theories). The Translation Mapping theorem was first announced in the abstract [Tar53a], and eventually published in the monograph [TG87, 4.4(xxxiv)] (the original manuscript [Tar43] was never published in its original form).

Tarski also showed that the equational theory of RA is, in fact, rich enough to capture the whole of first-order logic, and not just set theories formalized therein [Tar43, Theorem 5.31]. This theorem was not included in [TG87] as its primary consequence (the undecidability of RA) is obtained there by other means. Consequently, it was published independently by Maddux as [Mad89a, Theorem 23]. For every first-order sentence  $\varphi$  (in symbols, every  $\varphi \in \Sigma[\mathcal{L}]$ ), Tarski and Maddux give a recursive construction of an RA equation  $E(\varphi)$  such that  $\varphi$  is logically valid iff  $\text{RA} \models E(\varphi)$ .

Tarski’s proofs in [Tar43, TG87] are syntactical and metamathematical. From an algebraic point of view, the proof of the Translation Mapping theorem can roughly be described as showing that every RA with quasi-projections is representable (more concisely stated  $\text{QRA} \subset \text{RRA}$ ). Indeed, this purely algebraic result (known as the QRA theorem) can be derived from the Translation Mapping theorem, and vice versa [TG87, p. 242]. Tarski thus posed the problem of finding a purely algebraic proof of the QRA theorem which does not rely upon the Translation Mapping theorem. Roger Maddux subsequently accomplished this in [Mad78b].

Tarski’s formalization of set theory has some remarkable properties, effectively eliminating many of the idiosyncrasies associated with traditional first-order logic. First and foremost, it uses no variables (thus the title of the monograph, *A formalization of set theory without variables*). This is remarkable in

and of itself, especially considering that all of the variable-free formalizations (also called propositional logics) which had been previously known were proven to be decidable (therefore insufficient for the study of set theory and arithmetic). It also circumvents the need to distinguish between free and bound variables in a formula, and the (somewhat tediously defined) process of respelling bound variables. Equally remarkable from our point of view, is that the only rule of inference employed in Tarski's formalization is the "high school algebra" rule of replacing equals with equals (as opposed to the typical first-order rules, modus ponens and generalization). This second property is common to the target formalization of the present work. Indeed this is true of all algebraizations of logical formalisms, owing to the replacement of logical connectives and quantifiers with algebraic operations, which allows us to speak of equality between terms in place of the equivalence of sentences.

Relation algebras are closely related to cylindric algebras of finite dimension ( $CA_\alpha$  for  $\alpha \in \omega$ ) and thus to the corresponding finite-variable logics. Of the ten equational axioms which define RA, all but two can be stated and proved in restricted 3-variable first-order logic ( $L_3$ ). The problematic axioms ( $R_1$  and  $R_4$  from [Definition 12](#) below) assert that the relative product of binary relations is associative and that the inverse of the inverse of a relation is the same relation, both requiring 4 variables to prove. The necessity of a 4<sup>th</sup> variable for proof is algebraically reflected in the fact that when  $\alpha \geq 4$ , the natural RA reduct (in symbols,  $\mathfrak{Ra}$ — [HMT85, 5.3.7]) of any  $CA_\alpha$  is indeed a relation algebra [HMT85, 5.3.8], however there are algebras  $\mathfrak{A} \in CA_3$  for which  $\mathfrak{Ra}\mathfrak{A} \not\models R_1$  [HT61, p. 103], and for which  $\mathfrak{A}\mathfrak{a}\mathfrak{A} \not\models R_4$ . Such algebras are illustrated in [HMT85] as Constructions 3.2.71 and 3.2.69 respectively. We summarize these results by saying that  $CA_3$  is weaker than RA, which is in turn weaker than  $CA_4$ .

In 1985 István Németi constructed an RA reduct,  $\mathfrak{Or}\mathfrak{a}$  (which we call the Németi reduct) of the formula algebra of  $L_3$ , by making use of projection functions which satisfy a finite set of assumptions (similar to, but stronger than Tarski's quasi-projections) [Ném85, Proposition 2.10]. Utilizing the fact that  $CA_3$  is an algebraization of  $L_3$ , he also showed the existence of an RA reduct of any  $\mathfrak{A} \in CA_3$  having sufficiently strong projection parameters [Ném85, Remark 3.13]. Németi's idea was to use projections to store information from the first two dimensions (where binary relations of the natural RA reduct are stored) in the first dimension only. Effectively, he coded binary relations as a sets (of "ordered pairs"). In this manner, he gave himself two spare variables for quantification instead of one, which turned out to be sufficient to recover both  $R_1$  and  $R_4$  (as it is in  $\mathfrak{A}\mathfrak{a}\mathfrak{A}$  for  $\mathfrak{A} \in CA_4$ ). Németi's technique did create a new issue with the identity law ( $R_3$ ), however he was able to get around this by adding the assumption that the "ordered pairs" induced by the projections were unique. Later, in [Ném86, Theorem 9], he abandoned this assumption in favor of another technique for recovering the identity law (c.f. [Remark 33](#)).

Using his reduct, Némethi constructed a recursively defined translation function from  $L_\omega$  to  $L_3$  which preserved meaning and provability (relative to theories proving the existence of such projection parameters) [Ném85, Proposition 3.3], thus solving a problem posed in [TG87, p. 143]. He also conjectured that these results could be extended to restricted 3-variable first-order logic without equality ( $L_3^\neq$ ), a problem which also appears in [SA04, p. 12], [SA05, p. 476], and [Sim07, p. 300]. Just as  $CA_3$  is an algebraization of  $L_3$ , diagonal-free cylindric algebras of dimension 3 ( $Df_3$ ) constitute an algebraization of  $L_3^\neq$ . In [Theorem 34](#), we show that every  $Df_3$  with an equality parameter and projection parameter satisfying a finite set of assumptions has an RA reduct, thus generalizing Némethi's construction from  $CA_3$  to the weaker class  $Df_3$ , and confirming his conjecture.

In [Theorem 56](#) we employ some algebraic properties of the variety  $Df_3$ , along with the Némethi reduct, to construct an analog of the Tarski-Maddux function translating the sentences of first-order logic to the equational theory of  $Df_3$  which: (1) preserves validity, and (2) is not relative to any particular theory. This strengthens aforementioned Tarski-Maddux result since, as we have previously mentioned and shall illustrate quite thoroughly,  $Df_3$  is a strictly weaker variety than RA.

**Remark 1** Hajnal Andréka and Némethi have independently described modifications to [Ném86, p. 41-64] (in Hungarian) which yield an RA reduct of the formula algebra of  $L_3^\neq$  [AN11, Theorem 4.3] and a formalization of set theory in  $L_3^\neq$  [AN11, Theorem 2.1]. Somewhat similar to our own, their construction uses 4 parameters (2 for equality and 2 for projection functions), and is carried out entirely in  $L_3^\neq$  as opposed to  $Df_3$ . □

**Remark 2 (Notational conventions)** We use Fraktur typeface (such as  $\mathfrak{A}$ ) to denote an algebra or an operator which outputs an algebra, and sans-serif typeface (such as  $BA$ ) to denote a class of algebras or an operator which outputs a class of algebras. The Greek letters  $\alpha, \beta, \kappa, \lambda, \mu, \omega$  denote ordinals (with  $\omega$  being the first infinite ordinal); and  $\rho$  is a rank function or sequence of ranks. We frequently use the fact that each ordinal may be defined as the set of all lesser ordinals (0 being identified with the empty set in this manner). When  $A$  and  $B$  are sets, we write  ${}^B A$  to denote the set of all functions from  $B$  to  $A$ . In particular for any ordinal  $\alpha$  and set  $A$ ,  ${}^\alpha A$  denotes the set of  $A$ -sequences of length  $\alpha$ . If  $x \in {}^\alpha A$  and  $\beta \in \alpha$ , then we understand  $x_\beta$  to be the image of  $\beta$  under the function  $x$ . □

## 1.2 First-order logic

Let  $\mathcal{L}$  be a first-order language with a countable set of relation symbols  $\{\mathbf{R}_\kappa : \kappa \in \beta \in \omega + 1\}$ , but no function symbols or constants<sup>1</sup>. Let  $\rho \in {}^\beta(\omega + 1)$  such that for every  $\kappa \in \beta$ , the relation symbol  $\mathbf{R}_\kappa$  has rank  $\rho_\kappa$ . The variables of  $\mathcal{L}$  are the countable set  $\{\mathbf{v}_\kappa : \kappa \in \alpha \in \omega + 1\}$ . The connectives of  $\mathcal{L}$  are disjunction ( $\vee$ ) and negation ( $\neg$ ). The only quantifier of  $\mathcal{L}$  is existential ( $\exists$ ), and there is an equality symbol ( $=$ ). All other logical connectives, and the universal quantifier are introduced as abbreviations (for example,  $\forall_{\mathbf{v}_\kappa} \varphi := \neg \exists_{\mathbf{v}_\kappa} \neg \varphi$  and  $\varphi \rightarrow \psi := \neg \varphi \vee \psi$ ). If  $\alpha \geq \omega$  then we say  $\mathcal{L}$  is an **ordinary language**. When  $\rho_\kappa = \alpha$  for each  $\kappa \in \beta$  we say  $\mathcal{L}$  is a **full language**.

The **atomic formulas** of  $\mathcal{L}$  are those of the form  $\mathbf{v}_\kappa = \mathbf{v}_\lambda$  for  $\kappa, \lambda \in \alpha$  (an **equation**) and  $\mathbf{R}_\mu(\mathbf{v}_{\kappa_0}, \dots, \mathbf{v}_{\kappa_\eta}, \dots)_{\eta \in \rho_\mu}$  (called a **relational atomic formula**), where  $\mu \in \beta$  and  $\kappa \in {}^{(\rho_\mu)}\alpha$ . We denote by  $\Phi[\mathcal{L}]$  the set of all **formulas** of  $\mathcal{L}$ —the closure of the set of atomic formulas under the connectives and quantifiers of  $\mathcal{L}$ . For  $\varphi \in \Phi[\mathcal{L}]$  let  $\text{Var}(\varphi)$ ,  $\text{In}(\varphi)$ , and  $\text{Rel}(\varphi)$  be the set of indices of variables, free variables, and relation symbols occurring in  $\varphi$  respectively—recursively defined according to Table 1.1.

$\varphi$	$\text{Var}(\varphi)$	$\text{In}(\varphi)$	$\text{Rel}(\varphi)$
$\mathbf{R}_\mu(\mathbf{v}_{\kappa_0}, \dots, \mathbf{v}_{\kappa_\eta}, \dots)_{\eta \in \rho_\mu}$	$\{\kappa_\eta : \eta \in \rho_\mu\}$	$\{\kappa_\eta : \eta \in \rho_\mu\}$	$\{\mu\}$
$\mathbf{v}_\kappa = \mathbf{v}_\lambda$	$\{\kappa, \lambda\}$	$\{\kappa, \lambda\}$	$\emptyset$
$\neg \varphi$	$\text{Var}(\varphi)$	$\text{In}(\varphi)$	$\text{Rel}(\varphi)$
$\exists_{\mathbf{v}_\kappa} \varphi$	$\text{Var}(\varphi)$	$\text{In}(\varphi) \setminus \{\kappa\}$	$\text{Rel}(\varphi)$
$\varphi \vee \psi$	$\text{Var}(\varphi) \cup \text{Var}(\psi)$	$\text{In}(\varphi) \cup \text{In}(\psi)$	$\text{Rel}(\varphi) \cup \text{Rel}(\psi)$

Table 1.1 Definitions of  $\text{Var}(\varphi)$ ,  $\text{In}(\varphi)$ , and  $\text{Rel}(\varphi)$

The set of **sentences** of  $\mathcal{L}$  is  $\Sigma[\mathcal{L}] = \{\varphi \in \Phi[\mathcal{L}] : \text{In}(\varphi) = \emptyset\}$ . The set of **subformulas** of  $\varphi \in \Phi[\mathcal{L}]$  is defined to be the smallest set  $\Gamma \subset \Phi[\mathcal{L}]$  such that  $\varphi \in \Gamma$ , and  $\psi \in \Gamma$  whenever:  $\exists_{\mathbf{v}_\kappa} \psi \in \Gamma$ , or  $\neg \psi \in \Gamma$ , or  $\psi \vee \chi \in \Gamma$  for some  $\chi \in \Phi[\mathcal{L}]$ . We say  $\varphi$  is a **restricted formula** when all of the relational atomic subformulas have the form  $\mathbf{R}_\mu(\mathbf{v}_0, \dots, \mathbf{v}_\eta, \dots)_{\eta \in \rho_\mu}$ , and set  $\Phi_r[\mathcal{L}]$  to be the set of all such formulas. We say  $\varphi \in \Phi[\mathcal{L}]$  is an **equality-free formula** when all of the atomic subformulas of  $\varphi$  are relational and set  $\Phi_e[\mathcal{L}]$  to be the set of all such formulas. The sets  $\Phi_{r,e}[\mathcal{L}]$ ,  $\Sigma_r[\mathcal{L}]$ ,  $\Sigma_e[\mathcal{L}]$ , and  $\Sigma_{r,e}[\mathcal{L}]$  are named and defined analogously.

<sup>1</sup>The elimination of function symbols and constants does not really restrict our choice of language since every  $n$ -ary function can be equivalently looked upon as an  $(n + 1)$ -ary relation, and similarly every constant as a unary relation containing only a single element.



**Definition 3** A **realization** for  $\mathcal{L}$  is a structure of the form  $\mathfrak{U} = \langle U, \mathbf{R} \rangle$  where  $U$  is a nonempty set and  $\mathbf{R} = \langle R_\mu : \mu \in \beta \rangle$  is a sequence of relations on  $U$  with  $R_\mu \subseteq {}^{(\rho_\mu)}U$  for each  $\mu \in \beta$ .  $\text{RE}[\mathcal{L}]$  is the class of all realizations for  $\mathcal{L}$ . Given  $\mathbf{a} \in {}^\omega U$ , and  $\varphi, \psi \in \Phi[\mathcal{L}]$ , we say:

1.  $\mathfrak{U} \models v_\kappa = v_\lambda[\mathbf{a}]$  if and only if  $a_\kappa = a_\lambda$ ,
2.  $\mathfrak{U} \models R_\mu v_{\kappa_0} \dots v_{\kappa_{\rho_\mu-1}}[\mathbf{a}]$  if and only if  $\langle a_{\kappa_0}, \dots, a_{\kappa_{\rho_\mu-1}} \rangle \in R_\mu$ ,
3.  $\mathfrak{U} \models \varphi \vee \psi[\mathbf{a}]$  if and only if  $\mathfrak{U} \models \varphi[\mathbf{a}]$  or  $\mathfrak{U} \models \psi[\mathbf{a}]$
4.  $\mathfrak{U} \models \neg\varphi[\mathbf{a}]$  if and only if it is not the case that  $\mathfrak{U} \models \varphi[\mathbf{a}]$ ,
5.  $\mathfrak{U} \models \exists_{v_\kappa} \varphi[\mathbf{a}]$  if and only if there is some  $\mathbf{b} \in {}^\omega U$  such that  $b_\lambda = a_\lambda$  for each  $\lambda \in \text{In}(\exists_{v_\kappa} \varphi)$  and  $\mathfrak{U} \models \varphi[\mathbf{b}]$ .

This recursively defines  $\mathfrak{U} \models \varphi[\mathbf{a}]$  for every formula  $\varphi$ . We say  $\mathfrak{U} \models \varphi$  when  $\mathfrak{U} \models \varphi[\mathbf{a}]$  for every  $\mathbf{a} \in {}^\omega U$ . In particular, for every  $\varphi \in \Sigma[\mathcal{L}]$  and  $\mathbf{a} \in {}^\omega U$ ,  $\mathfrak{U} \models \varphi[\mathbf{a}]$  iff  $\mathfrak{U} \models \varphi$ . When  $\mathfrak{U} \models \varphi$  for every  $\mathfrak{U} \in \text{RE}[\mathcal{L}]$  we write  $\models \varphi$ , and say that  $\varphi$  is **logically valid**. For a set of formulas  $\Gamma \subset \Phi[\mathcal{L}]$  we say  $\Gamma \models \varphi$  if and only if  $\mathfrak{U} \models \varphi$  whenever  $\mathfrak{U} \models \psi$  for each  $\psi \in \Gamma$ . When  $\models \varphi \leftrightarrow \psi$ , we say that  $\varphi$  and  $\psi$  are **logically equivalent**. □

In an ordinary language, every formula  $\varphi$  is logically equivalent to some restricted formula  $\varphi_r$  [HMT85, 4.3.6]. Similarly, if  $\Gamma \subseteq \Phi[\mathcal{L}]$  then there is some  $\Gamma_r \in \Phi_r[\mathcal{L}]$  such that  $\Gamma \models \varphi$  if and only if  $\Gamma_r \models \varphi_r$  [HMT85, 4.3.15].

Next we introduce a proof system suitable for  $\mathcal{L}$  and several of its subformalisms. A set of formulas  $\Gamma$  is said to be closed under **detachment** (also called modus ponens) iff  $\psi \in \Gamma$  whenever  $\{\varphi \rightarrow \psi, \varphi\} \subset \Gamma$ . The set  $\Gamma$  is said to be closed under **generalization** if  $\forall_{v_\kappa} \varphi \in \Gamma$  whenever  $\varphi \in \Gamma$ .

**Definition 4** The set of **logical axioms** for  $\mathcal{L}$  is denoted by  $\Lambda[\mathcal{L}]$ . The sets  $\Lambda_r[\mathcal{L}]$ ,  $\Lambda_e[\mathcal{L}]$ , and  $\Lambda_{r,e}[\mathcal{L}]$  are the corresponding restricted, equality-free, and restricted equality-free logical axioms.  $\Lambda[\mathcal{L}]$  consists of all formulas of the following kind, where  $\varphi, \psi \in \Phi[\mathcal{L}]$  and  $\kappa, \lambda, \mu \in \alpha$  are arbitrary:

- (1)  $\varphi$ , a propositional tautology,
- (2)  $\forall_{v_\kappa} (\varphi \rightarrow \psi) \rightarrow (\forall_{v_\kappa} \varphi \rightarrow \forall_{v_\kappa} \psi)$ ,
- (3)  $\forall_{v_\kappa} \varphi \rightarrow \varphi$ ,
- (4)  $\varphi \rightarrow \forall_{v_\kappa} \varphi$ , if  $v_\kappa$  does not occur free in  $\varphi$ ,
- (5)  $v_\kappa = v_\kappa$ ,

$$(6) \exists_{v_\kappa} (v_\kappa = v_\lambda),$$

$$(7) v_\kappa = v_\lambda \rightarrow (v_\kappa = v_\mu \rightarrow v_\lambda = v_\mu),$$

$$(8) v_\kappa = v_\lambda \rightarrow [\varphi \rightarrow \forall_{v_\kappa} (v_\kappa = v_\lambda \rightarrow \varphi)] \text{ if } \kappa \neq \lambda.$$

For  $\Gamma \cup \{\varphi\} \subseteq \Phi[\mathcal{L}]$ , define  $\Gamma \vdash \varphi$  if  $\varphi \in \Omega$  whenever  $\Omega \supset \Gamma \cup \Lambda$  is closed under detachment and generalization. Analogously define  $\overline{\vdash}_r$ ,  $\overline{\vdash}_e$ , and  $\overline{\vdash}_{r,e}$  for  $\Gamma \cup \{\varphi\}$  contained in  $\Phi_r$ ,  $\Phi_e$ , and  $\Phi_{r,e}$  respectively.

Restrictions of  $\mathcal{L}$  which use these proof systems are summarized in Table 1.2.  $\square$

Language	Variables	Formulas	Proof system
$L_\alpha$	$\{v_\kappa : \kappa \in \alpha\}$	$\Phi[L_\alpha] = \Phi_r[\mathcal{L}]$	$\overline{\vdash}_r$
$L_\alpha^\neq$	$\{v_\kappa : \kappa \in \alpha\}$	$\Phi[L_\alpha^\neq] = \Phi_{r,e}[\mathcal{L}]$	$\overline{\vdash}_{r,e}$

Table 1.2 Restrictions of  $\mathcal{L}$

By Gödel's completeness theorem, we have  $\models \varphi$  if and only if  $\vdash \varphi$  when  $\alpha \geq \omega$ . As we shall see, this is not the case for  $\alpha < \omega$ .

**Remark 5** As a historical note, we mention that the formalism which might best be described in our notation as  $\mathcal{L}_3$  is also of interest in the field of algebraic logic. It uses only 3 variables, but admits relational atomic formulas which are not restricted. As a consequence of this, the identity law (R<sub>4</sub>) becomes  $\mathcal{L}_3$ -provable, leaving only R<sub>1</sub> without proof. It was this that motivated Maddux's invention of so-called semi-associative relation algebras (SA). In this broader class, the axiom R<sub>1</sub> asserting the associativity of relative product is replaced with the special case:

$$(x;1);1 = x;(1;1),$$

which is provable using only 3 variables [Mad78b]. Since  $L_3$  is strictly weaker than  $\mathcal{L}_3$ , we are justified in concluding that the corresponding variety of algebras, CA<sub>3</sub> is strictly weaker than SA. As with CA<sub>3</sub>, however, the QRA theorem does not extend to SA. While representable SA's coincide with representable RA's, Németi has shown in [Ném85] that there are quasi-projectional SA's which are not even relation algebras.  $\square$

### 1.3 General algebra and Boolean algebras with operators

We begin our investigation by defining some general algebra (also called universal algebra) notions. These provide the framework upon which our target formalism shall be built. For a more detailed and

comprehensive exposition of results and definitions concerning general algebra, we refer the reader to e.g. [HMT71, §0].

**Definition 6** A sequence  $\mathfrak{A} = \langle A, \sigma_i \rangle_{i \in I}$  is an **algebra** if and only if  $A$  is a set and for each  $i \in I$ ,  $\sigma_i$  is an operation on  $A$  to  $A$  of rank  $\rho_i$ . In case  $\rho_i = 0$  we call  $\sigma_i$  a **distinguished element** of  $\mathfrak{A}$ . We call the sequence  $\langle \rho_i : i \in I \rangle$  the **similarity type** of  $\mathfrak{A}$ , and the set  $A$  the **universe** of  $\mathfrak{A}$ . Now let  $\mathbf{K}$  be a class of similar algebras and  $\mathfrak{A} = \langle A, \sigma_i \rangle_{i \in I} \in \mathbf{K}$ .

1. A **subalgebra** of  $\mathfrak{A}$  is an algebra  $\mathfrak{B} = \langle B, \sigma_i \rangle_{i \in I}$  such that  $B \subset A$ . Symbolically, we write  $\mathfrak{B} \leq \mathfrak{A}$ . That  $\mathfrak{B}$  is an algebra with the same operations as  $\mathfrak{A}$  entails that each distinguished element of  $\mathfrak{A}$  is also an element of  $B$ , and that  $B$  is closed under all of the operations of  $\mathfrak{A}$ . The class of all subalgebras of algebras in  $\mathbf{K}$  is **SK**.
2. A **homomorphic image** of  $\mathfrak{A}$  is an algebra  $\mathfrak{B} = \langle B, \tau_i \rangle_{i \in I}$  of the same similarity type as  $\mathfrak{A}$  such that there is a function  $f$  from  $A$  onto  $B$  which respects the operations of  $\mathfrak{A}$  (such an  $f$  is called a **homomorphism**). That is to say, for each  $i \in I$  and  $\mathbf{x} \in {}^{\rho_i}A$ ,  $f\sigma_i\mathbf{x} = \tau_i f\mathbf{x}$  (where  $f\mathbf{x}$  denotes the sequence  $\langle fx_0, \dots, fx_{\rho_i-1} \rangle \in {}^{\rho_i}B$ ). The class of all homomorphic images of algebras in  $\mathbf{K}$  is **HK**. The operator **I** (the class of all **isomorphic images**) is defined just as **H**, but with the additional requirement that  $f$  be bijective.
3. A **direct product** of an indexed system of algebras  $\mathfrak{A} \in {}^J\mathbf{K}$  is the algebra  $\mathfrak{B} = \prod \mathfrak{A} = \prod_{j \in J} \mathfrak{A}_j$  (with, say  $\mathfrak{A}_j = \langle A_j, \sigma_i^{\mathfrak{A}_j} \rangle_{i \in I}$  for each  $j \in J$ ) whose universe is the (not necessarily finite) Cartesian product  $B = \prod_{j \in J} A_j$  and whose operations  $\sigma_i^{\mathfrak{B}}$  are defined component-wise. That is to say, if  $\mathbf{x} \in {}^{\rho_i}B$  (so that for  $\lambda \in \rho_i$  and  $\kappa \in \alpha$ , we have  $x_\lambda \in B$  and  $(x_\lambda)_\kappa$  or simply  $x_{\lambda, \kappa}$  is an element of  $A_\kappa$ ), then  $\sigma_i^{\mathfrak{B}}\mathbf{x} := \langle \sigma_i^{\mathfrak{A}_\kappa} x_{0, \kappa} \dots x_{\rho_i-1, \kappa} : j \in J \rangle \in B$ . We define **PK** to be the class  $\mathbf{I}\{\prod_{j \in J} \mathfrak{A}_j : J \text{ is a set and } \mathfrak{A} \in {}^J\mathbf{K}\}$ .

We say  $\mathfrak{A}$  is a **subdirect product** of algebras in  $\mathbf{K}$  when  $\mathfrak{A} \in \mathbf{SPK}$ , i.e.  $\mathfrak{A}$  is a subalgebra of a direct product of algebras in  $\mathbf{K}$  (up to isomorphism). We say that  $\mathbf{K}$  is a **variety** just in case  $\mathbf{HSPK} = \mathbf{K}$ . **HSPK** is called the **variety generated by K**. □

The reason for closing the set of direct products under **I** in the definition of **P** is that this ensures that if  $\mathbf{K}$  is closed under isomorphisms, then **PK** will be as well. It is easy to see that **SK** inherits this closure from  $\mathbf{K}$  and that **HK** is such for every  $\mathbf{K}$ . If  $\mathbf{K}$  is a variety then it is not difficult to show that  $\mathbf{K} = \mathbf{HK} = \mathbf{SK} = \mathbf{PK}$  [HMT71, 0.3.13]. Garret Birkhoff's famous theorem [Bir35, Theorem 10] (sometimes called Birkhoff's **HSP** theorem) says that a class of similar algebras  $\mathbf{K}$  is a variety iff  $\mathbf{K}$  is an equational class, i.e.  $\mathbf{K}$  is precisely the class of all algebras satisfying a certain set of equations.

**Definition 7** An algebra  $\mathfrak{A}$  is said to be **simple** if and only if the only homomorphic images of  $\mathfrak{A}$  are itself and the trivial 1-element algebra. The simplicity of  $\mathfrak{A}$  can be symbolically expressed as:  $\mathfrak{B} \in \mathbf{H}\{\mathfrak{A}\}$  implies either  $\mathfrak{A} \cong \mathfrak{B}$  or  $|B| = 1$ . We say  $\mathfrak{A}$  is **semisimple** if it is isomorphic to a subdirect product of simple algebras.  $\square$

Simple algebras often have more convenient properties than the varieties that they generate. Moreover they form a sufficiently large class to generate the varieties we are concerned with. These facts will be stated more precisely after we introduce CA and RA (c.f. Theorems 22-24 and 29-31). We do note here, however, that the algebraic systems first used by Tarski (beginning with [Tar41]) coincide with what we now call simple relation algebras [Mad06, p. ix]. A definition equivalent to the modern variety RA was first published in the abstract [JT48].

**Definition 8** Fix any similarity type  $\rho = \langle \rho_i : i \in I \rangle$  and let  $\langle \sigma_i : i \in I \rangle$  be an arbitrary sequence of operation symbols. Let  $\mathfrak{A} = \langle A^{\mathfrak{A}}, \sigma_i^{\mathfrak{A}} \rangle_{i \in I}$  be an algebra and  $\mathbf{K}$  be a class of similar algebras, both having similarity type  $\rho$ .

1. The set of  $\mathfrak{A}$  **terms** (in symbols  $\text{Tm}[\mathfrak{A}]$ ), is the closure of the variables  $\{w_\kappa : \kappa \in \omega\}$  under the operation symbols  $\{\sigma_i\}_{i \in I}$ . Thus  $\{w_\kappa : \kappa \in \omega\} \subset \text{Tm}[\mathfrak{A}]$  and  $\sigma_i t_0 \cdots t_{\rho_i-1} \in \text{Tm}[\mathfrak{A}]$  whenever  $i \in I$  and  $t_0, \dots, t_{\rho_i-1} \in \text{Tm}[\mathfrak{A}]$ . For each  $t_0, t_1 \in \text{Tm}[\mathfrak{A}]$ , we call  $t_0 \doteq t_1$  an  $\mathfrak{A}$  **equation** and let  $\text{Eq}[\mathfrak{A}]$  be the set of all  $\mathfrak{A}$  equations. Neither  $\text{Tm}[\mathfrak{A}]$  nor  $\text{Eq}[\mathfrak{A}]$  depend on anything about  $\mathfrak{A}$  except its similarity type,  $\rho$ . Thus we analogously define  $\text{Tm}[\mathbf{K}]$  and  $\text{Eq}[\mathbf{K}]$ , or even  $\text{Tm}[\rho]$  and  $\text{Eq}[\rho]$ .
2. By an **assignment** to  $A^{\mathfrak{A}}$  we mean a sequence  $\mathbf{s} \in \omega(A^{\mathfrak{A}})$ . If  $t \in \text{Tm}[\mathfrak{A}]$  and  $\mathbf{s}$  is an assignment to  $A^{\mathfrak{A}}$ , then the **interpretation of  $t$  under  $\mathbf{s}$**  is defined to be the element  $t[\mathbf{s}] \in A^{\mathfrak{A}}$  obtained by replacing each variable  $w_\kappa$  with  $s_\kappa \in A^{\mathfrak{A}}$  and each operation symbol  $\sigma_i$  with the function  $\sigma_i^{\mathfrak{A}} : {}^{(\rho_i)}(A^{\mathfrak{A}}) \rightarrow A^{\mathfrak{A}}$ .
3. Let  $E = t_0 \doteq t_1 \in \text{Eq}[\rho]$  and  $\mathbf{s}$  be an assignment to  $A^{\mathfrak{A}}$ .
  - (a)  $\mathfrak{A} \models E[\mathbf{s}]$  (in words  $\mathfrak{A}$  **models  $E$  under  $\mathbf{s}$** ) if and only if  $t_0[\mathbf{s}] = t_1[\mathbf{s}]$ .
  - (b)  $\mathfrak{A} \models E$  if and only if  $\mathfrak{A} \models E[\mathbf{s}]$  for every  $\mathbf{s} \in \omega A$ .
  - (c)  $\mathbf{K} \models E$  if and only if  $\mathfrak{A} \models E$  for every  $\mathfrak{A} \in \mathbf{K}$ .  $\square$

It is easy to see that whenever  $\mathbf{K} \models E$ ,  $\mathbf{HSPK} \models E$  as well.

Next we turn our attention to a special class of algebras which contains all three of RA, CA, and Df; namely Boolean algebras with operators. Named for George Boole, who pioneered their study in the mid-nineteenth century, Boolean algebras are an algebraization of propositional (truth table) logic.

As suggested by the name, Boolean algebras with operators extend Boolean algebras. Thus we can extend all Boolean definitions, notations, and abbreviations to this much larger class. For more results concerning Boolean algebras with operators, we refer the reader to [JT51, JT52].

**Definition 9** The algebra  $\mathfrak{A} = \langle A, +, - \rangle$  of similarity type  $\langle 2, 1 \rangle$  is a **Boolean algebra** whenever it satisfies all of the identities in Table 1.3 for each  $x, y, z \in A$ .

1. If  $\mathfrak{A}$  is a Boolean algebra and  $\sigma : {}^\rho A \rightarrow A$ , then we say that  $\sigma$  is **additive** if and only if  $\sigma$  distributes over Boolean addition. Precisely, when  $\mathbf{x}, \mathbf{y} \in {}^\rho A$  and  $\mathbf{x} + \mathbf{y} := \langle x_\kappa + y_\kappa : \kappa \in \rho \rangle \in {}^\rho A$ , we have  $\sigma(\mathbf{x} + \mathbf{y}) = \sigma\mathbf{x} + \sigma\mathbf{y}$ .
2.  $\mathfrak{A} = \langle A, +, -, \sigma_i \rangle_{i \in I}$  is called a **Boolean algebra with operators** when  $\mathfrak{A}$  is an algebra, the **Boolean reduct**  $\mathfrak{B}\mathfrak{A} := \langle A, +, - \rangle$  is a Boolean algebra, and each  $\sigma_i$  is additive.

We let  $\mathbf{BA}$  and  $\mathbf{Bo}$  be the classes of all Boolean algebras and algebras with operators, respectively.  $\square$

$B_1$	$x + y = y + x$	+ - commutativity
$B_2$	$x + (y + z) = (x + y) + z$	+ - associativity
$B_3$	$(\bar{x} + \bar{y})^- + (\bar{x} + \bar{y})^- = x$	Huntington's axiom

Table 1.3 BA axioms

We use any of  $-x$ ,  $\bar{x}$ , and  $x^-$  interchangeably for the Boolean complement of  $x$ . Since  $x + \bar{x} = y + \bar{y}$  holds for every  $\mathfrak{A} \in \mathbf{BA}$  and  $x, y \in A$ , we may unambiguously define  $1 := x + \bar{x}$  and  $0 := \bar{1}$ . We also make use of the defined operations  $x \cdot y := (\bar{x} + \bar{y})^-$  and  $x \oplus y := x \cdot \bar{y} + y \cdot \bar{x}$ . As one last abbreviation we let  $x \leq y$  indicate  $x + y = y$ .

**Remark 10** The axiomatization given here is due to Edward V. Huntington [Hun33b, Hun33a]<sup>2</sup>. A beautiful feature of Huntington's axiomatization is that axioms  $B_1$ - $B_2$  together with the Boolean dual of  $B_3$ :

$$((x + \bar{y})^- + (x + \bar{y})^-)^- = x,$$

also axiomatize  $\mathbf{BA}$ —a surprisingly non-trivial fact. Soon after Huntington's axiomatization was published, Herbert Robbins conjectured that this was the case. The dual of  $B_3$  became known as the Robbins axiom, and the class of algebras satisfying  $B_1$ - $B_2$  plus Robbins axiom were called Robbins algebras. The problem gained much notoriety over the years, and remained open until 1996 when W.

<sup>2</sup>The idempotence axiom  $x + x = x$  was included in the axiomatization appearing in [Hun33b], along with an incorrect proof of its independence of the remaining three. This was corrected in [Hun33a].

McCune was able to construct a proof of  $B_3$  from  $B_1$ ,  $B_2$ , and Robbins's axiom using his automated theorem prover, EQP [McC97] (for a simplified derivation, see [Dah98]). It turns out that it is even possible to axiomatize BA using only one operation (the Sheffer stroke,  $x \uparrow y = \bar{x} + \bar{y}$ , from which  $x + y$  and  $\bar{x}$  can both be defined) and a single axiom (also found with computer assistance, this is known to be the shortest possible axiomatization) [Wol02].  $\square$

In addition to the three operators described in Definition 6, Boolean algebras with operators yield a fourth way of creating new algebras from old ones (which may or may not lead out of the variety generated by the parent algebra).

**Definition 11** Let  $\mathfrak{A} = \langle A, +, -, \sigma_i \rangle_{i \in I}$  be a Boolean algebra with operators and  $a \in A$ . We define  $\text{Rl}_a A := \{x \cdot a : x \in A\}$ ,  $\bar{x}^a := a \cdot \bar{x}$ , and for each  $i \in I$ ,  $x \in {}^{\rho_i} \text{Rl}_a A$  we let  $\sigma_i^a(\mathbf{x}) := a \cdot \sigma_i(\mathbf{x})$ . Then  $\mathfrak{Rl}_a \mathfrak{A} := \langle \text{Rl}_a A, +, -^a, \sigma_i^a \rangle_{i \in I}$  is the **relativization to  $a$  of  $\mathfrak{A}$** .  $\square$

It is well known and easy to show that  $\mathfrak{Rl}_a \mathfrak{A} \in \text{BA}$  when  $\mathfrak{A} \in \text{BA}$  and  $a \in A$ . This does not generally extend to other varieties contained in the class  $\text{Bo}$ . If  $\mathbf{K} \subset \text{Bo}$  is a variety and  $\mathfrak{A} \in \mathbf{K}$  with  $a \in A$ , then it is possible that  $\mathfrak{Rl}_a \mathfrak{A} \notin \mathbf{K}$  (though the structure is, by definition, guaranteed to be an algebra with the same similarity type as  $\mathbf{K}$ ). Examples of such algebras (for  $\mathbf{K} = \text{RA}$ ) can be found in [Mad06, p. 379] (along with some sufficient conditions for  $\mathfrak{Rl}_a \mathfrak{A} \in \text{RA}$ ). Relativizations of  $\text{CA}_\alpha$ 's are discussed in [HMT71, §2.2].

When studying the algebraization of a particular logic, those algebras which correspond to models of the logic are especially interesting. We call these algebras **representable**. The class of representable Boolean algebras consists of those which are isomorphic to an algebra  $\mathfrak{A} = \langle A, +, - \rangle$  is such that  $A$  is a family of subsets of some largest set  $1^{\mathfrak{A}}$ ,  $+$  is union, and  $-$  is complementation with respect to  $1^{\mathfrak{A}}$ . Since propositional logic is complete, every BA is representable. Conversely, by Stone's representation theorem for Boolean algebras [Sto36], every BA is representable, and thus propositional logic is seen to be complete. Just as  $\text{Bo}$  extends BA, the notion of representability for the Boolean operations extends, for example, to RA. Thus a representable relation algebra will be isomorphic to some algebra where the interpretations of  $+$  and  $-$  are union and complementation. Of course, we shall impose requirements on the additional operators of RA as well, making the problem of classifying representable algebras much more difficult.

## 1.4 Relation algebras

Next we investigate relation algebras. There is a natural correspondence between the equations which hold in every RA and the formulas of Tarski's  $\mathcal{L}^\times$  which are provable. We shall not discuss  $\mathcal{L}^\times$  in any detail here, but mention in passing that: (1) the works [Tar43, Tar53a, Tar53b, TG87] are primarily formulated in  $\mathcal{L}^\times$  rather than what we would call relation algebras, (2) the formulas of  $\mathcal{L}^\times$  involve no variables and are generated from a single binary predicate symbol, and (2) the provability relation in  $\mathcal{L}^\times$  falls somewhere between that of  $L_3$  and  $L_4$  (which is reflected in our discussion of  $\mathfrak{R}aCA_\alpha$  for  $3 \leq \alpha \leq 4$  on page 2).

**Definition 12** A **relation algebra** is any algebra

$$\mathfrak{A} = \langle A, +, -, ;, \checkmark, 1' \rangle$$

with similarity type  $\langle 2, 1, 2, 1, 0 \rangle$  which satisfies axioms  $R_0$ - $R_7$  in Table 1.4 (due to Tarski). The binary operation  $;$  is called **relative product** and the unary  $\checkmark$  is called **converse**. The variety consisting of all relation algebras is denoted RA. [Mad06, 6.0.1]  $\square$

$R_0$	$\langle A, +, - \rangle \in \mathbf{BA}$	
$R_1$	$x; (y; z) = (x; y); z$	$;$ - associativity
$R_2$	$(x + y); z = x; z + y; z$	$;$ - distributivity
$R_3$	$x; 1' = x$	identity law
$R_4$	$\check{\check{x}} = x$	$\checkmark$ - involution
$R_5$	$(x + y)^\checkmark = \check{x} + \check{y}$	$\checkmark$ - distributivity
$R_6$	$(x; y)^\checkmark = \check{y}; \check{x}$	$\checkmark$ - involutive distributivity
$R_7$	$\check{x}; (x; y)^- \leq \bar{y}$	Tarski's formulation of De Morgan's Theorem K

Table 1.4 RA axioms

We also define the additional relational constant  $0' := \bar{1}'$ , called the **diversity** relation. Similarly to the unary  $-$ , we use  $\check{x}$  and  $x^\checkmark$  interchangeably (dissimilarly, we do not use any prefix notation for the operation). In everything to follow, we let unary operations bind closer than binary operations and relational operations bind closer than Boolean operations. For example, we unambiguously read  $x; y \cdot z$  as  $(x; y) \cdot z$ . When  $\mathfrak{A} \in \mathbf{RA}$  and  $x, y, z \in A$ , all of the theorems in Table 1.5 hold. For reference, we have included theorem numbers from [Mad06] where proofs of each can be found.

$x \leq y \Leftrightarrow \check{x} \leq \check{y}$	240	$x; (y + z) = x; y + x; z$	261
$\check{0} = 0$	242	$x \leq y \Rightarrow z; x \leq z; y$	262
$\check{1} = 1$	243	$\check{1}' = 1'$	269
$\check{\bar{x}} = \bar{\check{x}}$	246	$1'; x = x = x; 1'$	270
$(x \cdot y)^\check{\vee} = \check{x} \cdot \check{y}$	247	$\check{0}' = 0'$	272
$(x + y)^\check{\vee} = \check{x} + \check{y}$	249	$x; 0 = 0 = 0; x$	287
$x \leq y \Rightarrow x; z \leq y; z$	253	$1; x = x \Rightarrow 1; \bar{x} = \bar{x}$	305(ii)

Table 1.5 Important RA theorems

**Definition 13** When  $U$  is any set and  $R, S \subseteq U \times U$ , we let:

$$\begin{aligned}
R \cup S &:= \{\langle x, y \rangle : \langle x, y \rangle \in R \text{ or } \langle x, y \rangle \in S\}, & R|S &:= \{\langle x, y \rangle : \exists z[\langle x, z \rangle \in R \text{ and } \langle z, y \rangle \in S]\}, \\
\sim R &:= \{\langle x, y \rangle : \langle x, y \rangle \notin R\}, & R^{-1} &:= \{\langle x, y \rangle : \langle y, x \rangle \in R\}, \\
\text{Re}U &:= \text{Sb}(U \times U), & \text{Id}_U &:= \{\langle x, x \rangle : \langle x, x \rangle \in 1\}, \\
\mathfrak{Rc}U &:= \langle \text{Re}(U), \cup, \sim, |, ^{-1}, \text{Id}_U \rangle.
\end{aligned}$$

We call  $\mathfrak{Rc}U$  the **full relation algebra on  $U$**  and any subalgebra of  $\mathfrak{Rc}U$  is called a **proper relation algebra**. We denote the class of all full relation algebras by **Rs**. A **representable relation algebra** is an algebra which is isomorphic to a subdirect product of algebras of the form  $\mathfrak{Rc}U$ . The class of representable relation algebras is denoted **RRA**. [Mad06, 6.0.3]  $\square$

Every representable relation algebra is indeed a relation algebra [Mad06, 229]. The containment  $\text{RRA} \subset \text{RA}$  was first shown to be proper by Roger Lyndon [Lyn50]. Lyndon's non-representable relation algebra is discussed in [Mad06, p. 358-62] (along with several other constructions). It turns out that **RRA** is a variety, but is not finitely based. Tarski first showed that **RRA** is closed under homomorphic images in [Tar55], thereby establishing that **RRA** is an equational class via Birkhoff's theorem (this is also proved as [Mad06, 120]). J. Donald Monk first showed that **RRA** has no finite axiomatization in [Mon64] (he also gave an infinite axiomatization in [Mon69], where he also established that  $\text{RCA}_\alpha$  is not finitely axiomatizable for  $3 \leq \alpha \in \omega$ ).

The provable sentences of  $\mathcal{L}^\times$  correspond to those equations which are true in the free relation algebra on one generator. The existence of non-representable relation algebras which are generated by a single element establishes that  $\mathcal{L}^\times$  is incomplete. Maddux has given a construction of an infinite sequence of one-generated non-representable relation algebras with a representable ultraproduct [Mad89b]. This shows that  $\mathcal{L}^\times$  is essentially incomplete in the sense that the addition of finitely many logical axiom schemes will not suffice for completion of the provability relation. Despite this inherent deficiency of **RA**,



we *can* impose some simple (though obviously not equational) conditions on a relation algebra which are sufficient to guarantee representability.

**Definition 14** A **quasi-projectional relation algebra** is an algebra  $\mathfrak{A} \in \text{RA}$  which has a pair of (**conjugated**) **quasi-projections**  $p_0, p_1 \in A$  satisfying the following equations.

$$\check{p}_0; p_0 + \check{p}_1; p_1 \leq 1' \tag{1.1}$$

$$\check{p}_0; p_1 = 1 \tag{1.2}$$

The class of all quasi-projectional relation algebras is denoted by **QRA**. □

**Theorem 15** *If  $\mathfrak{A} \in \text{QRA}$  then  $\mathfrak{A}$  is representable. Equivalently,  $\text{QRA} \subseteq \text{RRA}$ .* □

The above is famously known as Tarski's **QRA** theorem. It was first announced in the abstract [Tar53b, VII], and it appears as Theorem 8.4(iii) in [TG87] with a metamathematical proof. In [Mad78b] Maddux generalized the theorem to the broader class of relation algebras satisfying:

$$\sum_{p, q \in \text{Fn } A} \check{p}; q = 1,$$

where  $\text{Fn } A$  is defined to be the set of all  $p \in A$  such that  $\check{p}; p \leq 1'$  (the **functional** elements). Algebras satisfying this condition are called **tabular** relation algebras. It is quite easy to see that every **QRA** is tabular since the quasi-projections  $p_0$  and  $p_1$  are functional by (1.1). Maddux's purely algebraic proof of this generalization (and respectively the **QRA** corollary) can also be found as [Mad78a, Theorem 7, Corollary 8] and [Mad06, 423, 427].

Condition (1.2) of **Definition 14** asserts that for each  $x, y$  there is some  $z$  such that  $p_0 z = x$  and  $p_1 z = y$ , and thus  $p_0$  and  $p_1$  act as projection functions on an abstract pairing. In case  $\mathfrak{A} = \mathfrak{Rc}U$ , this "pairing" consists of a one-to-one correspondence between  $U \times U$  and  $U$ . The existence of such a correspondence implies that the cardinality of  $U$  is either 0 or infinite. Therein lies the reason that the formalization of [TG87] is restricted to set theory. Since **RRA** corresponds to the models of  $\mathcal{L}^\times$  and every nontrivial  $\mathfrak{A} \in \text{QRA} \subset \text{RRA}$  is infinite, the class **QRA** is naturally suited for those theories whose models are infinite. However, as Tarski was aware, Maddux later rediscovered, and we shall soon see; **QRA** is not *exclusively* suited for such theories.

A slight idiosyncrasy associated with **QRA** is that the quasi-projections are not included in the similarity type of the algebra, i.e. any given **QRA** has multiple choices of conjugated quasi-projections. It is easy to see, for example, that interchanging the roles of  $p_0$  and  $p_1$  yields another pair of quasi-projections. The reason for this convention is that the inclusion of the quasi-projections would mean

that the natural representation of a QRA would be a TPA (see [Definition 19](#) below) instead of an RRA. This would be especially disadvantageous from a general algebra perspective in light of the fact that **HSPTPA** is not axiomatizable by any decidable system of equations [[MSS92](#), Theorem 2].

We can actually use this minor quirk to our advantage by strengthening the conditions on quasi-projections in various ways. It is not generally necessary to do this (c.f. [Remark 33](#)), but we find it more convenient and as long as we keep the number of assumptions finite, we find that it creates no difficulties (c.f. commentary following [Theorem 31](#)). Our first such assumption (which can be made without loss of generality) is that the domains of  $p_0$  and  $p_1$  coincide (see e.g. [[Sim07](#), Lemma 2.6] for proof).

**Proposition 16** *Let  $\mathfrak{A} \in \text{QRA}$  with conjugated quasi-projections  $p_0$  and  $p_1$ . Then  $p'_0 = p_0 \cdot p_1;1$  and  $p'_1 = p_1 \cdot p_0;1$  also satisfy (1.1)-(1.2), and additionally satisfy:*

$$p'_0;1 = p'_1;1. \quad (1.3)$$

*In other words,  $p'_0$  and  $p'_1$  are conjugated quasi-projections which have a common domain.* □

Next we introduce the assumption that “pairs” formed by  $p_0$  and  $p_1$  are unique. This becomes particularly useful in proving that the Némethi reduct of a  $\text{CA}_\alpha$  or  $\text{Df}_\alpha$  obeys the identity law. In contrast to (1.3), we cannot assume that *every* QRA has such quasi-projections, and so are forced to define a proper subclass.

**Definition 17**  $\text{Q}^+\text{RA}$  is the subclass of QRA where the conjugated quasi-projections  $p_0$  and  $p_1$  additionally satisfy the following equation.

$$(p_0; \check{p}_0) \cdot (p_1; \check{p}_1) \leq 1' \quad (1.4)$$

In words, the pairs formed by  $p_0$  and  $p_1$  are unique. □

If  $p_0$  and  $p_1$  satisfy (1.4), then  $p'_0$  and  $p'_1$  from [Proposition 16](#) must as well. This follows immediately from the monotonicity of conversion and relative product [[Mad06](#), 240, 253, 262]. In view of this we can equivalently define  $\text{Q}^+\text{RA}$  as the class of all relation algebras having elements  $p_0$  and  $p_1$  satisfying (1.1)-(1.4); and similarly QRA satisfying (1.1)-(1.3). We note that it can also be assumed without loss of generality that the functions  $p_0$  and  $p_1$  are entire by simply defining  $p_i(x) = x$  for each  $x$  outside the domain of  $p_i$ . However, from our point of view this would be quite disadvantageous (c.f. [Proposition 20](#) and the commentary following).

We have not yet addressed the question of whether or not QRA and  $\text{Q}^+\text{RA}$  are varieties. It turns out that neither are closed under subalgebras, for whenever  $U$  is infinite,  $\mathfrak{Rc}U \in \text{Q}^+\text{RA}$ , but the subalgebra

of  $\mathfrak{Rc}U$  with universe  $\{0, 1, 0', 1'\}$  is clearly not a QRA. It is straightforward to see that QRA and  $Q^+RA$  (indeed, every class defined by extending the conditions on quasi-projections by equations without variables) are closed under **H** and **P**. By a result of Maddux the operators **H** and **S** commute on RA and classes contained in RA, so **SQRA** and **SQ<sup>+</sup>RA** can be seen to be varieties [Mad78a, Lemma 10] (also found as [Mad06, 112]). Partially because of this observation, but more so because we wish to be able to discuss  $Q^+RA$  equations which involve the projection functions, we have the following definition.

**Definition 18** An algebra  $\mathfrak{A} = \langle A, +, -, ;, \smile, 1', p_0, p_1 \rangle$  is called a **pairing relation algebra** whenever  $\langle A, +, -, ;, \smile, 1' \rangle \in RA$  and  $p_0, p_1$  satisfy Equations (1.1)-(1.3). We call the class of all such algebras PRA. The class  $P^+RA$  consists of those PRA's which also satisfy (1.4).  $\square$

Clearly every QRA can be extended to a PRA and every PRA has a QRA reduct; and similarly for  $Q^+RA$  and  $P^+RA$ . Since PRA and  $P^+RA$  are equationally defined, they form varieties by Birkhoff's theorem. The difference here is that since the quasi-projections are included in the similarity type, they must be preserved by subalgebras. Thus in the counterexample we gave previously,  $\{0, 1, 0', 1'\}$  fails to be the universe of a subalgebra.

**Definition 19** Let  $U$  be any set.

1.  $\ddot{U} = \bigcap \{W : U \cup (W \times W) \subset W\}$
2.  $\mathfrak{p}_0^U = \{\langle \langle x, y \rangle, x \rangle : x, y \in \ddot{U}\}$
3.  $\mathfrak{p}_1^U = \{\langle \langle x, y \rangle, y \rangle : x, y \in \ddot{U}\}$
4.  $\mathfrak{B}(U) = \langle \text{Re}\ddot{U}, \cup, \sim, |, ^{-1}, Id_{\ddot{U}}, \mathfrak{p}_0^U, \mathfrak{p}_1^U \rangle$

$\mathfrak{B}(U)$  is the **true pairing algebra** on  $U$ . The class of true pairing algebras is denoted  $TPA := \{\mathfrak{B}(U) : \ddot{U} \times \ddot{U} = \ddot{U} \setminus U \neq \emptyset\}$ .  $\square$

In the previous definition (taken from [Mad89a, Definitions 18, 20]),  $\ddot{U}$  is seen to be the closure of  $U$  under ordered pairs. The requirement  $\ddot{U} \times \ddot{U} = \ddot{U} \setminus U$  for membership of  $\mathfrak{B}(U)$  in TPA ensures that  $U$  is free of ordered pairs of its own elements. Whenever  $U$  is nonempty,  $\ddot{U}$  is necessarily infinite. Next we shall see that  $U$  ( $Id_U$  to be precise) can be expressed in terms of the projection functions in TPA. This is the key to expressing theories with finite models. Even though  $\ddot{U}$  is infinite (as it must be, since every TPA has a natural  $Q^+RA$  reduct), we can refer back to the original set  $U$  which does not have such a restriction. For more results concerning TPA, see [Sai00, MSS92].

**Proposition 20** If  $\mathfrak{B}(U) \in TPA$  then  $u := 1' \cdot (p_0; 1 \cdot p_1; 1)^{-}$  denotes  $Id_U$ .  $\square$

PROOF By definition,  $\mathfrak{p}_k^U;1 = \{\langle\langle x, y \rangle, z \rangle : x, y, z \in \ddot{U}\}$ . Since  $\mathfrak{B}(U) \in \text{TPA}$  we have  $\ddot{U} \times \ddot{U} = \ddot{U} \setminus U$ , so we may write  $\mathfrak{p}_k^U;1 = \{\langle x, y \rangle \in \ddot{U} \times \ddot{U} : x \notin U\}$ . Then the interpretation of  $\mathfrak{u}$  is seen to be  $Id_{\ddot{U}} \cap \{\langle x, y \rangle \in \ddot{U} \times \ddot{U} : x \in U\} = Id_U$ . ■

**Remark 21** In [Mad89a], Maddux originally used the simpler condition  $U \cap (U \times U) = \emptyset$  to define TPA. This does guarantee that  $U$  is free of ordered pairs of its own elements, however it was pointed out in [Mad93] that this condition is insufficient for our purposes. In particular, [Proposition 20](#) fails when  $U = \{x, \langle\langle x, x \rangle, x \rangle\}$  because while  $U \cap (U \times U) = \emptyset$ , the interpretation of  $\mathfrak{u}$  becomes  $\{\langle x, x \rangle\} \neq Id_U$ . □

We conclude this section with some useful general algebraic results concerning RA. That each of the next three theorems has a counterpart for (diagonal-free) cylindric algebras (Theorems [29-31](#)) further illustrates the connections between these varieties. Theorems [22-23](#) are found as Theorems 4.10,11,15 of [JT52]. [Theorem 24](#) is a corollary which was originally observed by Ernst Schröder in 1895 [Mad91, Theorem 24].

**Theorem 22** *Every RA is semisimple.* □

**Theorem 23** *For every  $\mathfrak{A} \in \text{RA}$ , the following are equivalent:*

- (i)  $\mathfrak{A}$  is simple,
- (ii) every subalgebra of  $\mathfrak{A}$  is simple,
- (iii) for every  $x \in A$ ,  $x \neq 0$  iff  $1; x; 1 = 1$ . □

**Theorem 24** *For every Boolean combination  $\varphi$  of relation algebraic equations, there is a correlated term  $\varphi^*$  such that  $\varphi$  and  $\varphi^* \doteq 0$  are equivalent in every simple RA. The mapping  $*$  is defined thusly:*

- (i)  $(t_0 \doteq t_1)^* = t_0 \oplus t_1$ ,
- (ii)  $(\varphi \wedge \psi)^* = \varphi^* + \psi^*$ ,
- (iii)  $(\neg\varphi)^* = (1; \varphi^*; 1)^-$ . □

In order to apply Theorems [23](#) and [24](#) to PRA and  $\text{P}^+\text{RA}$ , we require the following lemma.

**Lemma 25** *Every PRA is semisimple.* □

PROOF Let  $\mathfrak{A} \in \text{PRA}$  and  $\mathfrak{A}'$  be the RA reduct of  $\mathfrak{A}$ . Clearly since the universes of  $\mathfrak{A}$  and  $\mathfrak{A}'$  coincide, any homomorphism on  $\mathfrak{A}$  is also a homomorphism on  $\mathfrak{A}'$  and vice versa. Thus  $\mathfrak{A}$  is simple iff  $\mathfrak{A}'$  is

simple. We have that  $\mathfrak{A}'$  is simple iff  $\mathfrak{A}'$  is subdirectly indecomposable iff  $\mathfrak{A}'$  is directly indecomposable by [TG87, Theorem 8.2(v)]. We show that if  $\mathfrak{A}'$  is directly decomposable, then so is  $\mathfrak{A}$ . Then if  $\mathfrak{A}$  is subdirectly indecomposable (hence also directly),  $\mathfrak{A}'$  must be as well so that  $\mathfrak{A}'$  is simple and we may conclude that  $\mathfrak{A}$  is simple. This is sufficient because every general algebra can be written as a subdirect product of subdirectly indecomposable algebras by [Bir44].

We use  $p$  and  $q$  instead of  $p_0$  and  $p_1$  to avoid confusion with subscripts. Suppose  $\mathfrak{A}' \cong \mathfrak{B} \times \mathfrak{C}$  by a homomorphism  $h$ . Let  $h(p) = \langle p_B, p_C \rangle$  and  $h(q) = \langle q_B, q_C \rangle$ . We claim that  $\mathfrak{A} \cong \langle \mathfrak{B}, p_B, q_B \rangle \times \langle \mathfrak{C}, p_C, q_C \rangle$  and that  $\langle \mathfrak{B}, p_B, q_B \rangle, \langle \mathfrak{C}, p_C, q_C \rangle \in \text{PRA}$ . Again, since the universes of  $\mathfrak{A}$  and  $\mathfrak{A}'$  are identical,  $h$  is an isomorphism satisfying the first condition. By symmetry we only need show that  $\langle \mathfrak{B}, p_B, q_B \rangle \in \text{PRA}$ . First, note that since  $\check{p};q = 1$  in  $\mathfrak{A}'$  we have  $h(p)^\check{;}h(q) = \langle 1_B, 1_C \rangle$ , so:

$$\begin{aligned} \langle 1_B, 1_C \rangle &= \langle p_B, p_C \rangle^\check{;} \langle q_B, q_C \rangle \\ &= \langle \check{p}_B, \check{p}_C \rangle ; \langle q_B, q_C \rangle \\ &= \langle (\check{p}_B; q_B), (\check{p}_C; q_C) \rangle. \end{aligned}$$

Thus  $\check{p}_B; q_B = 1_B$  as required. Similarly  $h(p)^\check{;}h(p) = \langle 1_B, 1_C \rangle$ , so:

$$\begin{aligned} \langle 1_B, 1_C \rangle &= \langle p_B, p_C \rangle^\check{;} \langle p_B, p_C \rangle \\ &= \langle \check{p}_B, \check{p}_C \rangle ; \langle p_B, p_C \rangle \\ &= \langle (\check{p}_B; p_B), (\check{p}_C; p_C) \rangle. \end{aligned}$$

Thus  $\check{p}_B; p_B = 1_B$  as required. A symmetric argument establishes the same for  $q_B$  so  $\mathfrak{B} \in \text{QRA}$  implying  $\langle \mathfrak{B}, p_B, q_B \rangle \in \text{PRA}$ . In fact, if  $p$  and  $q$  satisfy the uniqueness of pairs (1.4), then  $p_B$  and  $q_B$  must as well. We conclude that every  $\mathfrak{A} \in \text{PRA}$  is isomorphic to a subdirect product of subdirectly indecomposable (hence simple) algebras in PRA, as desired. ■

## 1.5 Cylindric algebras

We now turn our attention to the classes of cylindric and diagonal-free cylindric algebras. Included throughout this section in `typewriter` font, are references to [HMT71, HMT85]<sup>3</sup> where definitions and proofs of theorems can be found. We maintain this practice for annotating proofs in the chapters to come.

**Definition 26** (1.1.1, 1.1.2) Let  $\alpha$  be any ordinal.

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<sup>3</sup>The chapters in [HMT71] and [HMT85] are respectively numbered 0-2 and 3-5, so we do not risk ambiguity with this referencing system.

1. A **cylindric algebra of dimension  $\alpha$**  is any algebra,

$$\mathfrak{A} = \langle A, +, -, \mathbf{c}_\kappa, \mathbf{d}_{\kappa\lambda} \rangle_{\kappa, \lambda \in \alpha}$$

with similarity type  $\langle 2, 1, \mathbf{1}, \mathbf{0} \rangle$  which satisfies axioms C<sub>0</sub>-C<sub>7</sub> in Table 1.6.

2. A **diagonal-free cylindric algebra of dimension  $\alpha$**  is any algebra,

$$\mathfrak{A} = \langle A, +, -, \mathbf{c}_\kappa \rangle_{\kappa \in \alpha}$$

with similarity type  $\langle 2, 1, \mathbf{1} \rangle$  which satisfies axioms C<sub>0</sub>-C<sub>4</sub> in Table 1.6.

$\mathbf{CA}_\alpha$  and  $\mathbf{Df}_\alpha$  respectively denote the classes of all such algebras.  $\mathbf{CA}$  and  $\mathbf{Df}$  are the classes of cylindric and diagonal-free cylindric algebras of arbitrary dimension. The unary operations  $\mathbf{c}_\kappa$  are called **cylindrifications** and the distinguished elements  $\mathbf{d}_{\kappa\lambda}$  are called **diagonal elements**.  $\square$

C <sub>0</sub>	$\langle A, +, - \rangle \in \mathbf{BA}$	
C <sub>1</sub>	$\mathbf{c}_\kappa 0 = 0$	
C <sub>2</sub>	$x \leq \mathbf{c}_\kappa x$	
C <sub>3</sub>	$\mathbf{c}_\kappa (x \cdot \mathbf{c}_\kappa y) = \mathbf{c}_\kappa x \cdot \mathbf{c}_\kappa y$	
C <sub>4</sub>	$\mathbf{c}_\kappa \mathbf{c}_\lambda x = \mathbf{c}_\lambda \mathbf{c}_\kappa x$	
C <sub>5</sub>	$\mathbf{d}_{\kappa\kappa} = 1$	
C <sub>6</sub>	$\mathbf{d}_{\lambda\mu} = \mathbf{c}_\kappa (\mathbf{d}_{\lambda\kappa} \cdot \mathbf{d}_{\kappa\mu})$	$\kappa \neq \lambda, \mu$
C <sub>7</sub>	$\mathbf{c}_\kappa (\mathbf{d}_{\kappa\lambda} \cdot x) \cdot \mathbf{c}_\kappa (\mathbf{d}_{\kappa\lambda} \cdot \bar{x}) = 0$	$\kappa \neq \lambda$

Table 1.6  $\mathbf{CA}_\alpha$  and  $\mathbf{Df}_\alpha$  axioms

Although C<sub>0</sub> and C<sub>2</sub> are not equations, C<sub>0</sub> could be replaced by B<sub>1</sub>-B<sub>3</sub> and C<sub>2</sub> abbreviates  $x + \mathbf{c}_\kappa x = \mathbf{c}_\kappa x$  by definition. Thus Birkhoff's theorem applies to  $\mathbf{CA}_\alpha$  and  $\mathbf{Df}_\alpha$ , so that both can be seen to be varieties. If  $\mathfrak{A} = \langle A, +, -, \mathbf{c}_\kappa, \mathbf{d}_{\kappa\lambda} \rangle_{\kappa, \lambda \in \alpha} \in \mathbf{CA}_\alpha$  then clearly

$$\mathfrak{Df}\mathfrak{A} := \langle A, +, -, \mathbf{c}_\kappa \rangle_{\kappa \in \alpha} \in \mathbf{Df}_\alpha$$

[HMT71, 1.1.3]. One might ask whether *every*  $\mathfrak{A} \in \mathbf{Df}_\alpha$  is equal to (or at least isomorphic to)  $\mathfrak{Df}\mathfrak{B}$  for some  $\mathfrak{B} \in \mathbf{CA}_\alpha$ , and whether  $\mathfrak{Df}\mathfrak{B} \cong \mathfrak{Df}\mathfrak{C}$  implies  $\mathfrak{B} \cong \mathfrak{C}$  (i.e. whether the function  $\mathfrak{Df}$  is surjective and injective, respectively). Except in the somewhat trivial case  $\alpha \leq 1$ , the answer to both is negative [HMT85, 5.1.4, 5.1.9]. Moreover, when  $\alpha \geq 4$  there is an equation not involving diagonal elements which hold in every  $\mathbf{CA}_\alpha$  but not in every  $\mathbf{Df}_\alpha$  [HMT85, 5.1.6] (originally appearing with a metalogical formulation in [Hen67]). Whether or not there is such an equation for  $\alpha = 3$  is an open question. Next we turn to defining the notion of representability for cylindric algebras.

**Definition 27** (1.1.5, 5.1.33) The **full cylindric set algebra of dimension  $\alpha$  with base  $U$** , in symbols  $\mathfrak{Cs}_\alpha U$ , is the algebra  $\langle \text{Sb } ^\alpha U, \cup, \sim, \mathbf{c}_\kappa, \mathbf{d}_{\kappa\lambda} \rangle_{\kappa \in \alpha}$  where  $U$  is any set and:

$$\begin{aligned} \text{Sb } ^\alpha U &:= \{X : X \subset ^\alpha U\}, \\ \mathbf{c}_\kappa X &:= \{\mathbf{y} \in ^\alpha U : \exists \mathbf{x} \in X [x_\lambda = y_\lambda \text{ for each } \lambda \neq \kappa]\}, \\ \mathbf{d}_{\kappa\lambda} &:= \{\mathbf{x} \in ^\alpha U : x_\kappa = x_\lambda\}, \end{aligned}$$

and  $\cup$  and  $\sim$  are union and complementation with respect to  $^\alpha U$ . The class of all such algebras is  $\text{Cs}$ . An algebra  $\mathfrak{A}$  is called a **representable cylindric algebra of dimension  $\alpha$**  if it is isomorphic to a subdirect product of algebras in  $\text{Cs}$ .  $\text{RCA}_\alpha$  denotes the class of all representable cylindric algebras of dimension  $\alpha$ . A **representable diagonal-free cylindric algebra** (and the corresponding class  $\text{RDf}_\alpha$ ) are defined analogously by omitting any reference to  $\mathbf{d}_{\kappa\lambda}$ .<sup>4</sup>  $\square$

As we would expect, for any set  $U$  we have that  $\mathfrak{Cs}_\alpha U \in \text{CA}_\alpha$  [HMT71, 1.1.6]. Since  $\text{CA}_\alpha$  is a variety, this also implies that  $\text{RCA}_\alpha \subseteq \text{CA}_\alpha$ ; and similarly  $\text{RDf}_\alpha \subseteq \text{Df}_\alpha$ . The proper containment of representable algebras greatly on the value of  $\alpha$ . We briefly summarize results concerning the different cases—what follows reflects comments from [HMT71, p. 171-3] as well as items 3.1.108, 3.2.65, 4.1.3-7, and 5.1.57, 62, 63 from [HMT71, HMT85].<sup>5</sup> Firstly, for every  $\alpha$  all four classes are axiomatizable. When  $\alpha \geq \omega$ , none of the four are finitely axiomatizable. When  $\alpha \leq 1$  all four varieties coincide, and when  $\alpha = 0$  they coincide with BA. When  $\alpha = 2$ ,  $\text{RDf}_\alpha = \text{Df}_\alpha \neq \text{CA}_\alpha \neq \text{RCA}_\alpha$ , and  $\text{RCA}_\alpha$  is finitely axiomatizable. When  $3 \leq \alpha < \omega$ , neither of  $\text{RCA}_\alpha$  and  $\text{RDf}_\alpha$  are finitely axiomatizable, while  $\text{CA}_\alpha$  and  $\text{Df}_\alpha$  are so by definition.

**Theorem 28** (4.3.57, 4.3.59) *Let  $L_\alpha$  be a full language and  $f : \text{Tm}[\text{CA}_\alpha] \rightarrow \Phi[L_\alpha]$  be recursively defined as follows, where  $s, t \in \text{Tm}[\text{CA}_\alpha]$  are arbitrary:*

$$\begin{aligned} f(\mathbf{w}_\kappa) &:= \mathbf{R}_\kappa \mathbf{v}_0 \dots \mathbf{v}_{\alpha-1}, & f(\mathbf{d}_{\kappa\lambda}) &:= \mathbf{v}_\kappa = \mathbf{v}_\lambda, & f(s+t) &:= f(s) \vee f(t), \\ f(\mathbf{c}_\kappa s) &:= \exists_{\mathbf{v}_\kappa} f(s), & f(\bar{s}) &:= \neg f(s). \end{aligned}$$

*Then for any  $s, t \in \text{Tm}[\text{CA}_\alpha]$  we have:*

$$\begin{aligned} \models f(s) \leftrightarrow f(t) &\Leftrightarrow \text{RCA}_\alpha \models s \doteq t, \\ \Vdash f(s) \leftrightarrow f(t) &\Leftrightarrow \text{CA}_\alpha \models s \doteq t. \end{aligned}$$

<sup>4</sup>Note: The classes  $\text{RCA}_\alpha$  and  $\text{RDf}_\alpha$  are called  $\text{IGs}_\alpha$  and  $\text{IGsdf}_\alpha$  respectively in [HMT71, HMT85]. Our definition is closer in character to the description on p. 172 of [HMT71]. Definition 5.1.33 is more precise (and consequently more complex) than necessary for our discussion.

<sup>5</sup>Especially important here, is Theorem 4.1.3 which establishes the nonfinite axiomatizability of  $\text{RCA}_\alpha$  for  $3 \leq \alpha \in \omega$ . This originally appeared in [Mon69] and the corresponding proof for  $\text{RDf}_\alpha$  (5.1.57) is analogous.

If  $s, t \in \text{Tm}[\text{Df}_\alpha] \subset \text{Tm}[\text{CA}_\alpha]$  then  $f(s), f(t) \in \Phi[L_\alpha^\neq]$  and:

$$\begin{aligned} \models f(s) \leftrightarrow f(t) & \Leftrightarrow \text{RDf}_\alpha \models s \doteq t, \\ \overline{|\!|}_{r,e} f(s) \leftrightarrow f(t) & \Leftrightarrow \text{Df}_\alpha \models s \doteq t. \quad \square \end{aligned}$$

[Theorem 28](#) provides a good indication<sup>6</sup> of the relationship between the algebras  $\text{CA}_\alpha$  and  $\text{Df}_\alpha$ , and the languages  $L_\alpha$  and  $L_\alpha^\neq$ . In light of the preceding paragraph, it also establishes the incompleteness of the latter because for  $\alpha \geq 3$  there are equations holding in the representable algebras which don't hold in their respective abstractions. From now on we shall concern ourselves primarily with  $\text{Df}_3$ . We conclude this section with the promised cylindric counterparts to [Theorems 22-24](#). The first of these is due to Don Pigozzi.

**Theorem 29** (2.4.53) *Every  $\mathfrak{A} \in \text{CA}_\alpha \cup \text{Df}_\alpha$  is semisimple.* □

**Theorem 30** (2.3.14, 16) *For every  $\mathfrak{A} \in \text{CA}_\alpha \cup \text{Df}_\alpha$ , the following are equivalent:*

- (i)  $\mathfrak{A}$  is simple,
- (ii) every subalgebra of  $\mathfrak{A}$  is simple,
- (iii) for every  $x \in A$ ,  $x \neq 0$  iff  $c_{(\alpha)}x = 1$ . □

As they appear in [HMT71], the above are actually a theorems about  $\text{CA}_\alpha$  only, but as noted in [HMT85, p. 187-8] the proofs carry through for  $\text{Df}_\alpha$  without changes. As a corollary of [Theorem 30](#), we obtain the following marvelous result.

**Theorem 31** *For every Boolean combination  $\varphi$  of (diagonal-free) cylindric algebraic equations, there is a correlated term  $\varphi^*$  such that  $\varphi$  and  $\varphi^* \doteq 0$  are equivalent in every simple  $\text{CA}_\alpha$  (or  $\text{Df}_\alpha$ ). The mapping  $*$  is defined thusly:*

- (i)  $(t_0 \doteq t_1)^* = t_0 \oplus t_1$ ,
- (ii)  $(\varphi \wedge \psi)^* = \varphi^* + \psi^*$ ,
- (iii)  $(\neg\varphi)^* = -c_{(\alpha)}\varphi^*$ . □

[Theorem 29](#) allows us to work exclusively with simple algebras (since they generate their parent varieties, and **HSP** preserves equational satisfaction). [Theorems 30-31](#) then allow us to code finitely many assumptions into a single equation. This is exemplified in [Proposition 55](#) below.

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<sup>6</sup>For a thorough and formal exposition of the connections between these formalisms and their corresponding algebras, we refer the reader to [HMT85, §4.3 and p. 203]



## CHAPTER 2. RELATION ALGEBRAIC REDUCTS

### 2.1 Definitions

Throughout this chapter we fix  $\mathfrak{A} = \langle A, +, -, c_\kappa \rangle_{\kappa \in 3} \in \text{Df}_3$  and  $e, q \in A$ . The parameter  $e$  simulates  $\{\mathbf{x} : x_0 = x_1 = x_2\}$  and  $q$  simulates  $\{\mathbf{x} : P_0(x_0) = x_1, P_1(x_0) = x_2\}$  where  $P_0$  and  $P_1$  are some conjugated quasi-projections. Occasionally, we shall informally refer to these “functions” in order to clarify certain concepts and definitions, but we emphasize that they *need not actually exist*. We frequently use finite sequences of 0’s and 1’s and let  $S_n$  be the set of such sequences having length less than  $n + 1$  and  $2^* = \cup_{n \in \omega} S_n$ . Concatenation of sequences shall be denoted by juxtaposition. All definitions pertinent to this chapter follow immediately. We recommend that the reader bookmark this page.

**Definition 32** For  $\{\kappa, \lambda, \mu\} = 3$  define the **diagonal elements**:

$$\begin{aligned} \mathbf{d}_{\kappa\lambda} &:= c_\mu e \\ \mathbf{d}_{\kappa\kappa} &:= c_\lambda c_\mu e. \end{aligned}$$

For  $\kappa, \lambda \in 3$  and  $x \in A$  define the  $\lambda$ -**for- $\kappa$  substitution** [HMT71, 1.5.1]:

$$s_\lambda^\kappa x := \begin{cases} x & \kappa = \lambda \\ c_\kappa (\mathbf{d}_{\kappa\lambda} \cdot x) & \kappa \neq \lambda. \end{cases}$$

Define the **quasi-projections** to be  $\mathbf{p}_0 = c_2 q$  and  $\mathbf{p}_1 = s_1^2 c_1 q$ , and for  $n \in 2$ , let

$$\mathbf{p}_n(\kappa, \lambda) := \begin{cases} s_\kappa^0 s_\lambda^1 \mathbf{p}_n & \kappa \leq \lambda \\ s_\kappa^2 s_\lambda^1 s_2^0 \mathbf{p}_n & \kappa > \lambda. \end{cases}$$

Let  $\{\kappa, \lambda, \mu\} = 3$ ;  $i, j \in 2^*$ ;  $n \in 2$ ; and  $\kappa' = \kappa +_3 1$  (where  $+_3$  denotes addition modulo 3). Then the **projectional diagonal elements** are defined as follows.

$$\mathbf{d}_{\kappa\langle \rangle \lambda \langle \rangle} := \mathbf{d}_{\kappa\lambda} \tag{2.1}$$

$$\mathbf{d}_{\kappa\langle n \rangle \lambda \langle \rangle} := \mathbf{d}_{\lambda \langle \rangle \kappa \langle n \rangle} := \mathbf{p}_n(\kappa, \lambda) \tag{2.2}$$

$$\mathbf{d}_{\kappa\langle in \rangle \lambda \langle \rangle} := \mathbf{d}_{\lambda \langle \rangle \kappa \langle in \rangle} := c_\mu (\mathbf{d}_{\kappa \langle i \rangle \mu \langle \rangle} \cdot \mathbf{d}_{\mu \langle n \rangle \lambda \langle \rangle}) \quad |i| \neq 0 \tag{2.3}$$

$$\mathbf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle} := \mathbf{c}_\mu(\mathbf{d}_{\mu\langle i\rangle\kappa\langle i\rangle} \cdot \mathbf{d}_{\mu\langle j\rangle\lambda\langle j\rangle}) \quad |i| \neq 0 \neq |j| \quad (2.4)$$

$$\mathbf{d}_{\kappa\langle i\rangle\kappa\langle j\rangle} := \mathbf{c}_{\kappa'}(\mathbf{d}_{\kappa'\langle i\rangle\kappa\langle i\rangle} \cdot \mathbf{d}_{\kappa'\langle j\rangle\kappa\langle j\rangle}). \quad (2.5)$$

Let  $\mathbf{Ax}$  to be the union of the following sets of equations.

$$\mathbf{Ax}_0 := \{\mathbf{d}_{\kappa\kappa} = 1 : \kappa \in 3\}$$

$$\mathbf{Ax}_1 := \{\mathbf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle} \cdot \mathbf{d}_{\lambda\langle j\rangle\mu\langle k\rangle} \leq \mathbf{d}_{\kappa\langle i\rangle\mu\langle k\rangle} : \kappa, \lambda, \mu \in 3; i, j, k \in S_3\}$$

$$\mathbf{Ax}_2 := \{\mathbf{d}_{\kappa\langle i\rangle\kappa\langle i\rangle} \cdot \mathbf{d}_{\lambda\langle j\rangle\lambda\langle j\rangle} \leq \mathbf{c}_\mu(\mathbf{d}_{\mu\langle 0\rangle\kappa\langle i\rangle} \cdot \mathbf{d}_{\mu\langle 1\rangle\lambda\langle j\rangle}) : \mu \neq \kappa, \lambda \in 3; i, j \in S_3\}$$

$$\mathbf{Ax}_3 := \{\mathbf{d}_{\kappa\langle i0\rangle\lambda\langle j0\rangle} \cdot \mathbf{d}_{\kappa\langle i1\rangle\lambda\langle j1\rangle} \leq \mathbf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle} : \kappa, \lambda \in 3; i, j \in S_2\}$$

For  $\kappa, \lambda \in 3$  and  $i \in 2^*$ , define the **projectional substitution**:

$$\mathbf{s}_{\lambda\langle i\rangle}^\kappa x := \begin{cases} x & \kappa = \lambda, i = \langle \rangle \\ \mathbf{c}_\kappa(\mathbf{d}_{\kappa\lambda\langle i\rangle} \cdot x) & \text{else.} \end{cases}$$

Finally, let  $\kappa, \lambda, \mu \in 3$  with  $\mu \neq \kappa$ .

$$\begin{aligned} \delta_{\kappa\langle i\rangle\lambda\langle j\rangle} &:= \mathbf{d}_{\kappa\langle i0\rangle\lambda\langle j00\rangle} \cdot \mathbf{d}_{\kappa\langle i1\rangle\lambda\langle j11\rangle} \cdot \mathbf{d}_{\lambda\langle j01\rangle\lambda\langle j10\rangle} & \epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle} &:= \mathbf{d}_{\kappa\langle i0\rangle\lambda\langle j1\rangle} \cdot \mathbf{d}_{\kappa\langle i1\rangle\lambda\langle j0\rangle} \\ \dot{\mathbf{i}} &:= \mathbf{d}_{0\langle 0\rangle 0\langle 0\rangle} & \dot{-}x &:= \dot{\mathbf{i}} \cdot \bar{x} \\ J_\kappa &:= \{x \in \text{Nr}_{\{\kappa\}} A : \mathbf{s}_\kappa^\mu \mathbf{s}_\mu^\kappa x = x\} & J &:= \{x \cdot \dot{\mathbf{i}} : x \in J_0\} \\ x \circ y &:= \mathbf{c}_1(\mathbf{s}_{1\langle 0\rangle}^0 x \cdot \mathbf{s}_{1\langle 1\rangle}^0 y \cdot \delta_{01}) & \check{x} &:= \mathbf{c}_1(\mathbf{s}_1^0 x \cdot \epsilon_{01}) \\ \dot{\mathbf{i}}' &:= \mathbf{d}_{0\langle 0\rangle 0\langle 1\rangle} \end{aligned}$$

We call  $\mathfrak{J}\mathfrak{A} := \langle J, +, \dot{-}, \circ, \check{\cdot}, \dot{\mathbf{i}}' \rangle$  the **Németi reduct** of  $\mathfrak{A}$ . □

The diagonal element  $\mathbf{d}_{\kappa\lambda}$  simulates  $\{\mathbf{x} : x_\kappa = x_\lambda\}$ . The  $\lambda$ -for- $\kappa$  substitution “moves” the  $\kappa$ -coordinate of  $\mathbf{x}$  to the  $\lambda$ -coordinate and cylindrifies the  $\kappa$ -coordinate. This mimics the process of substituting free occurrences of  $\mathbf{v}_\kappa$  with  $\mathbf{v}_\lambda$  in  $L_3$ .

The quasi-projection  $\mathbf{p}_n$  simulates  $\{\mathbf{x} : P_n(x_0) = x_1\}$ . We note that this manner of defining  $\mathbf{p}_0$  and  $\mathbf{p}_1$  requires that our quasi-projections have the same domain. We use  $\mathbf{p}_n(\kappa, \lambda)$  to simulate  $\{\mathbf{x} : P_n(x_\kappa) = x_\lambda\}$ . We require the rather complicated definition above because  $\text{CA}_\alpha \not\models \mathbf{s}_0^2 \mathbf{s}_1^0 \mathbf{s}_2^1 x = \mathbf{s}_1^2 \mathbf{s}_0^1 \mathbf{s}_2^0 x$ . Thus we must make some choice regarding the order of substitution (our definition is modified from Németi’s *substitution convention* [Ném85, p. 23]).

The projectional diagonal elements,  $\mathbf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}$  are the algebraic counterparts to Németi’s  $L_3$  formulas  $x_i \doteq x_j$  for  $\{x, y\} \subset \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$  [Ném85, p. 38]. They are intended to simulate the set of sequences  $\mathbf{x}$ , such that  $P_i(x_\kappa) = P_j(x_\lambda)$ —where  $P_{i_0 \dots i_n} = P_{i_n} \circ \dots \circ P_{i_0}$  and  $P_\langle \rangle$  represents the identity function. For

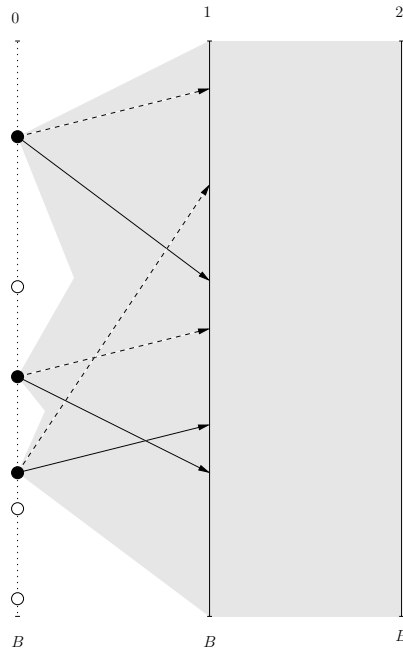
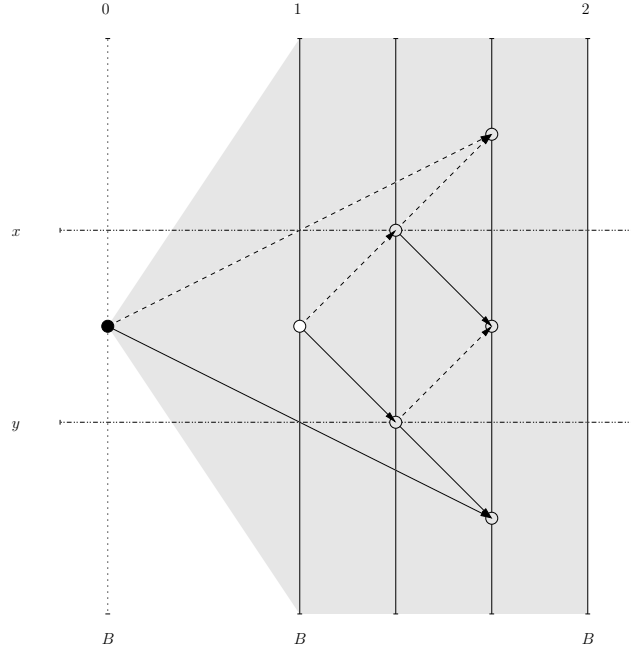


Figure 2.1 An illustration of  $\dot{1}$  (more precisely, a subset of  $\dot{1}$  in a full cylindric set algebra on  $B$ ). Vertical lines are copies of  $B$ , dotted arrows indicate that the endpoints are the first two coordinates of an element of  $\mathfrak{p}_0$ , and solid arrows indicate that the endpoints are the first two coordinates of an element of  $\mathfrak{p}_1$ . Our  $\dot{1}$  consists of all elements whose first coordinate is a closed circle, i.e. all ordered triples enclosed entirely in gray.

Figure 2.2 An illustration of  $x \circ y$ 

example, the element  $d_{0\langle 1 \rangle 1\langle 100 \rangle}$  could be thought of as the set  $\{\mathbf{x} : P_1(x_0) = P_0P_0P_1(x_1)\}$ . We typically write e.g.  $d_{\kappa\langle i \rangle \lambda}$  in place of  $d_{\kappa\langle i \rangle \lambda\langle \rangle}$ .

$Ax_1$  and  $Ax_2$  are modified slightly from [Ném86, p. 42].  $Ax_1$  says roughly that “equality” is a congruence with respect to the “projection functions”. When  $l$  is the empty sequence,  $Ax_1$  says that “ $P_i(x_\kappa) = P_j(x_\lambda)$  and  $P_j(x_\lambda) = P_k(x_\mu)$  imply that  $P_i(x_\kappa) = P_k(x_\mu)$ .” When  $l \in S_1$  is nonempty,  $Ax_1$  could be read as “ $P_i(x_\kappa) = P_j(x_\lambda)$  and  $P_lP_j(x_\lambda) = P_k(x_\mu)$  imply that  $P_lP_i(x_\kappa) = P_k(x_\mu)$ .”  $Ax_2$  expresses that any two generalized projections can be “coded” into a single element. In other words, “the existence of  $P_i(x_\kappa)$  and  $P_j(x_\lambda)$  imply there is some  $x_\mu$  such that  $P_0(x_\mu) = P_i(x_\kappa)$  and  $P_1(x_\mu) = P_j(x_\lambda)$ .” Finally,  $Ax_3$  expresses that “pairs” are unique—“ $P_0P_i(x_\kappa) = P_0P_j(x_\lambda)$  and  $P_1P_i(x_\kappa) = P_1P_j(x_\lambda)$  imply  $P_i(x_\kappa) = P_j(x_\lambda)$ .”

The axioms  $Ax$  are more powerful than we actually need. If we so desired, it would be a straightforward but time consuming task to read through all of the proofs and record which instances of the axioms we actually use. This would substantially reduce the number of assumptions, but drastically increase the length of their enumeration. Finiteness is required so that images under our translation function from  $\Sigma[\mathcal{L}]$  to  $Df_3$  will be finite in length. That  $Ax$  be finite (which it clearly is) and true for actual projection functions and equality relations are the only requirements we must satisfy.

**Remark 33** As demonstrated in [Ném86] and [AN11],  $Ax_3$  can be eliminated from our set of assumptions. Indeed the only places it will be used are in the proofs of (2.11), (2.17), (2.18), and in showing that our projection functions satisfy the uniqueness of pairs (so that  $\mathfrak{J}\mathfrak{A} \in Q^+RA \subset QRA$ ). If we eliminated  $Ax_3$  from  $Ax$  and chose  $J' := \{x : x \circ \dot{1} = x\}$  as the universe of our reduct, then we would not require  $Ax_3$  in the proof of Theorem 34.

In fact, the only necessary changes to the proof follow: condition I can be shown to be a consequence of Equation (2.8); condition II follows from the observation that  $J' \subset J$ ; condition III would be satisfied trivially; and the proof of condition IV from [Ném85, p. 51-4] could be easily replicated in  $\mathfrak{J}\mathfrak{A}$ . We could then show that  $\mathfrak{J}\mathfrak{A} \in QRA$  and that for each  $\mathfrak{R}\epsilon V \in QRA$  there is some  $\mathfrak{B} \in \text{Df}_3$  so that the Néméti reduct  $\mathfrak{J}\mathfrak{B}$  is isomorphic to  $\mathfrak{R}\epsilon V$ .  $\square$

**Theorem 34** *If  $e$  and  $q$  satisfy the equations in  $Ax$ , then  $\mathfrak{J}\mathfrak{A} \in RA$ . Moreover,  $\mathfrak{J}\mathfrak{A} \in Q^+RA$  with conjugated quasi-projections  $P_0 = d_{0(00)0(1)}$  and  $P_1 = d_{0(01)0(1)}$ .*  $\square$

We reserve our proof for §2.3 as it is quite long and requires several technical lemmas.

**Theorem 35** *Let  $\mathfrak{A} = \mathfrak{R}\epsilon V \in Q^+RA$  with quasi-projections functions  $p_0$  and  $p_1$ . Then there is some  $\mathfrak{B} \in \text{Df}_3$  with elements  $e$  and  $q$  such that the Néméti reduct  $\mathfrak{J}\mathfrak{B}$  is isomorphic to  $\mathfrak{A}$ .*  $\square$

PROOF Since  $p_0$  and  $p_1$  are functions (by the representability of  $\mathfrak{A}$ ), we use standard functional notation, writing  $p_k(x) = y$  in place of  $\langle x, y \rangle \in p_k$ . We may assume that  $p_0$  and  $p_1$  have the same domain. Let  $\mathfrak{B}'$  be the full cylindric set algebra of dimension 3 on  $V$  and  $\mathfrak{B}$  be its  $\text{Df}_3$  reduct. Choose parameters:

$$e = \{\mathbf{x} \in {}^3V : x_0 = x_1 = x_2\}$$

and

$$q = \{\mathbf{x} \in {}^3V : p_0(x_0) = x_1, p_1(x_0) = x_2\}.$$

For  $i = i_0 \cdots i_n \in 2^*$  and  $x \in V$  we define  $p_i(x) = p_{i_n} \cdots p_{i_0}(x)$ , so that according to Definition 32 we have:

$$d_{\kappa(i)\lambda(j)} = \{\mathbf{x} : p_i(x_\kappa) = p_j(x_\lambda)\},$$

$$\dot{1} = \{\mathbf{x} : x_0 \in \text{Dom } p_0\},$$

and  $J$  is the powerset of  $\dot{1}$  (since  $\mathfrak{B}' \in \text{RCA}_3 \models s_0^1 s_1^0 c_1 c_2 x = c_1 c_2 x$ ).

Let  $h : J \rightarrow A$  be defined by:

$$h(X) := \{\langle p_0(x_0), p_1(x_0) \rangle : \mathbf{x} \in X\}.$$

Note that  $h$  is well defined since for  $\mathbf{x} \in X \in J$ , we must also have  $\mathbf{x} \in \dot{1}$ , so  $x_0 \in \text{Dom } p_i$ . For any  $R \in A$  the element

$$h^{-1}(R) := \{\mathbf{x} \in \dot{1} : \langle p_0(x_0), p_1(x_0) \rangle \in R\}$$

maps to  $R$  under  $h$ , so  $h$  is onto. We note that  $h^{-1}(R)$  is guaranteed to exist since (1.2) and (1.4) guarantee that for each  $\langle u, v \rangle \in {}^2V$  there is some unique  $x \in V$  such that  $p_0(x) = u$  and  $p_1(x) = v$ . Now suppose that  $h(X) = h(Y)$  for some  $X, Y \in J$ . Then observe:

$$\begin{aligned} \mathbf{x} \in X &\Leftrightarrow \langle p_0(x_0), p_1(x_0) \rangle \in h(X) \\ &\Leftrightarrow \langle p_0(x_0), p_1(x_0) \rangle \in h(Y) \\ &\Leftrightarrow \mathbf{x} \in Y. \end{aligned}$$

Thus  $h$  is also injective and hence bijective, as required.

All that remains is to show that  $h$  respects the operations of  $\mathfrak{J}\mathfrak{B}$ . In what follows we sometimes write  $X_0 := \{x_0 : \mathbf{x} \in X\}$ , so that  $x_0 \in X_0$  iff  $\mathbf{x} \in X$  (since  $J \subseteq \text{Nr}_1 A$  by definition). For every  $X, Y \in J$  we have:

$$\begin{aligned} h(X + Y) &= \{\langle p_0(x_0), p_1(x_0) \rangle : \mathbf{x} \in X + Y\} \\ &= \{\langle p_0(x_0), p_1(x_0) \rangle : \mathbf{x} \in X\} \cup \{\langle p_0(x_0), p_1(x_0) \rangle : \mathbf{x} \in Y\} \\ &= h(X) \cup h(Y), \end{aligned}$$

and:

$$\begin{aligned} h(\dot{-}X) &= \{\langle p_0(x_0), p_1(x_0) \rangle : \mathbf{x} \in \dot{-}X\} \\ &= \{\langle p_0(x_0), p_1(x_0) \rangle : \mathbf{x} \in \dot{1} \cdot -X\} \\ &= \{\langle p_0(x_0), p_1(x_0) \rangle : x_0 \in \text{Dom } p_0, x_0 \notin X_0\} \\ &= \{\langle p_0(x_0), p_1(x_0) \rangle : x_0 \in \text{Dom } p_0\} \setminus \{\langle p_0(x_0), p_1(x_0) \rangle : x_0 \in X_0\} \\ &= {}^2V \setminus h(X) \\ &= \sim h(X). \end{aligned}$$

The second to last equality again follows from Equations (1.2) and (1.4). Then we have that  $h$  respects the Boolean substructure of  $\mathfrak{J}\mathfrak{B}$ . We now turn our attention to the relational operations. Observe that for every  $X \in J$  we have  $x \in X_0$  iff  $\langle p_0(x), p_1(x) \rangle \in h(X)$ , and that  $s_{\lambda(i)}^0 X = \{\mathbf{x} \in {}^3V : p_i(x_\lambda) \in X_0\}$ . For every  $X, Y \in J$  we have:

$$\begin{aligned} h(\check{X}) &= h(\mathbf{c}_1 (s_1^0 X \cdot \epsilon_{01})) \\ &= h(\mathbf{c}_1 \{\mathbf{x} : x_1 \in X_0, p_0(x_0) = p_1(x_1), p_1(x_0) = p_0(x_1)\}) \end{aligned}$$

$$\begin{aligned}
&= h(\mathbf{c}_1 \{ \mathbf{x} : \langle p_0(x_1), p_1(x_1) \rangle \in h(X), p_0(x_0) = p_1(x_1), p_1(x_0) = p_0(x_1) \}) \\
&= h(\{ \mathbf{x} : \langle p_1(x_0), p_0(x_0) \rangle \in h(X) \}) \\
&= h(\{ \mathbf{x} : \langle p_0(x_0), p_1(x_0) \rangle \in h(X)^{-1} \}) \\
&= h(X)^{-1},
\end{aligned}$$

and:

$$\begin{aligned}
&h(X \circ Y) \\
&= h\left(\mathbf{c}_1 \left( s_{1(0)}^0 X \cdot s_{1(1)}^0 Y \cdot \delta_{01} \right)\right) \\
&= h(\mathbf{c}_1 \{ \mathbf{x} : p_0(x_1) \in X_0, p_1(x_1) \in Y_0, p_0(x_0) = p_0 p_0(x_1), p_1(x_0) = p_1 p_1(x_1), p_0 p_1(x_1) = p_1 p_0(x_1) \}) \\
&= h(\mathbf{c}_1 \{ \mathbf{x} : \langle p_0 p_0(x_1), p_1 p_0(x_1) \rangle \in h(X), \langle p_0 p_1(x_1), p_1 p_1(x_1) \rangle \in h(Y), \\
&\quad p_0(x_0) = p_0 p_0(x_1), p_1(x_0) = p_1 p_1(x_1), p_0 p_1(x_1) = p_1 p_0(x_1) \}) \\
&= h(\mathbf{c}_1 \{ \mathbf{x} : \langle p_0(x_0), p_1 p_0(x_1) \rangle \in h(X), \langle p_0 p_1(x_1), p_1(x_0) \rangle \in h(Y), \\
&\quad p_0(x_0) = p_0 p_0(x_1), p_1(x_0) = p_1 p_1(x_1), p_0 p_1(x_1) = p_1 p_0(x_1) \}) \\
&= h(\{ \mathbf{x} : \exists y \in V[\langle p_0(x_0), y \rangle \in h(X), \langle y, p_1(x_0) \rangle \in h(Y)] \}) \\
&= h(\{ \mathbf{x} : \langle p_0(x_0), p_1(x_0) \rangle \in h(X) \mid h(Y) \}) \\
&= h(X) \mid h(Y)
\end{aligned}$$

Thus  $h$  respects the relational operations of  $\mathfrak{JB}$  as desired. Finally, we verify that  $h$  maps the identity of  $\mathfrak{JB}$  to the identity of  $\mathfrak{RcV}$ , i.e. that  $h(\dot{1}) = Id_V$ .

$$\begin{aligned}
h(\dot{1}) &= h(\{ \mathbf{x} : p_0(x_0) = p_1(x_0) \}) \\
&= h(\{ \mathbf{x} : \langle p_0(x_0), p_1(x_0) \rangle \in Id_V \}) \\
&= Id_V
\end{aligned}$$

Thus  $h$  is an isomorphism from  $\mathfrak{JA}$  to  $\mathfrak{A}$ , as was to be shown. ■

Next we turn to proving several lemmas which will make the proof of [Theorem 34](#) much shorter and much more readable.

## 2.2 Preliminary lemmas

For convenience we maintain the numbering system of [HMT71], but emphasize that the structure  $\langle A, +, -, \mathbf{c}_\kappa, \mathbf{d}_{\kappa\lambda} \rangle_{\kappa, \lambda \in \mathbb{3}}$  is *not necessarily* a  $\mathbf{CA}_3$ . Although  $\text{Ax}$  and the definitions of  $\mathbf{d}_{\kappa\lambda}$  do a good enough

job at simulating diagonal elements to establish that  $\mathfrak{J}\mathfrak{A} \in \text{RA}$ , we have not fully recovered all of the axioms of  $\text{CA}_3$ . In particular, the axiom  $C_7$  would require an infinite number of assumptions on our parameters because each element of the algebra would require a separate assumption.

Observe that by [Definition 32](#), when  $\kappa, \lambda \in 3$  and  $i, j \in 2^*$  we have  $\mathbf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle} = \mathbf{d}_{\lambda\langle j\rangle\kappa\langle i\rangle}$  (similar to 1.3.1). We shall use this fact throughout the remainder of this work without annotation. We begin by establishing some well-known  $\text{CA}_3$  identities.

**Lemma 36** *All of the identities from [Table 2.1](#) hold in  $\mathfrak{A}$ .* □

Table 2.1 Important  $\text{CA}_\alpha$  identities which hold in  $\mathfrak{A}$ .

$\mathbf{c}_\kappa x = 0 \Leftrightarrow x = 0$		1.2.1
$\mathbf{c}_\kappa \mathbf{c}_\kappa x = \mathbf{c}_\kappa x$		1.2.3
$\mathbf{c}_\kappa(x + y) = \mathbf{c}_\kappa x + \mathbf{c}_\kappa y$		1.2.6(ii)
$x \leq y \Rightarrow \mathbf{c}_\kappa x \leq \mathbf{c}_\kappa y$		1.2.7
$\mathbf{c}_\kappa x = x \Leftrightarrow \mathbf{c}_\kappa \bar{x} = \bar{x}$		1.2.12(i)
$\mathbf{d}_{\kappa\lambda} = \mathbf{d}_{\lambda\kappa}$		1.3.1
$\mathbf{c}_\kappa \mathbf{d}_{\kappa\lambda} = 1$		1.3.2
$\mathbf{s}_\lambda^\kappa(x + y) = \mathbf{s}_\lambda^\kappa x + \mathbf{s}_\lambda^\kappa y$		1.5.3(i)
$\mathbf{s}_\lambda^\kappa \mathbf{c}_\kappa x = \mathbf{c}_\kappa x$		1.5.8(i)
$\mathbf{s}_\lambda^\mu \mathbf{s}_\mu^\kappa \mathbf{c}_\mu x = \mathbf{s}_\lambda^\kappa \mathbf{c}_\mu x$	$\kappa \neq \lambda$	1.5.11(i)
$\mathbf{s}_\kappa^\lambda \mathbf{s}_\lambda^\kappa \mathbf{c}_\lambda \mathbf{c}_\mu x = \mathbf{s}_\kappa^\mu \mathbf{s}_\mu^\kappa \mathbf{c}_\lambda \mathbf{c}_\mu x$	$\kappa \neq \lambda, \mu$	1.5.11(ii)

PROOF Axioms  $C_0$  through  $C_4$  must hold since  $\mathfrak{A} \in \text{Df}_3$ . Thus the proofs of 1.2.1 through 1.2.12(i) found in [HMT71] remain valid for  $\mathfrak{A}$  (indeed, for every algebra in  $\text{Df}_3$ ). Theorem 1.3.1 holds by definition of  $\mathbf{d}_{\kappa\lambda}$ . Axiom  $C_5$  is guaranteed by  $\text{Ax}_0$  and  $C_6$  holds by the following argument.

$$\begin{aligned}
\mathbf{d}_{\lambda\mu} &= \mathbf{c}_\kappa e && \text{D32} \\
&\leq \mathbf{c}_\kappa (\mathbf{c}_\mu e \cdot \mathbf{c}_\lambda e) && C_2, 1.2.7 \\
&\leq \mathbf{c}_\kappa (\mathbf{d}_{\kappa\lambda} \cdot \mathbf{d}_{\kappa\mu}) && \text{D32} \\
\mathbf{c}_\kappa (\mathbf{d}_{\kappa\lambda} \cdot \mathbf{d}_{\kappa\mu}) &\leq \mathbf{c}_\kappa \mathbf{d}_{\lambda\mu} && \text{Ax}_1, 1.2.7 \\
&= \mathbf{c}_\kappa \mathbf{c}_\kappa e = \mathbf{c}_\kappa e = \mathbf{d}_{\lambda\mu} && 1.2.3, \text{D32}
\end{aligned}$$

The proof of 1.5.3(i) found in [HMT71] depends only on the axioms  $C_0$  through  $C_6$ , so it remains valid in  $\mathfrak{A}$  as well. We offer alternative proofs for those identities which require axiom  $C_7$ : namely 1.3.2, 1.5.8(i), and 1.5.11. As to 1.3.2, we have that  $\mathbf{c}_\kappa \mathbf{d}_{\kappa\lambda} = \mathbf{d}_{\lambda\lambda} = 1$  by definition (32) and  $\text{Ax}_0$ . The proof of 1.5.8(i) in [HMT71] does not depend directly on  $C_7$ , but rather on 1.3.2, so it remains valid in  $\mathfrak{A}$  as well.



To see 1.5.11(i), suppose that  $\kappa \neq \lambda$ . If  $\mu = \kappa$  or  $\mu = \lambda$  then the result is obvious by definition (32), so assume  $\kappa, \lambda, \mu$  are distinct.

$$\begin{aligned}
s_\lambda^\mu s_\mu^\kappa c_\mu x &= c_\mu (d_{\mu\lambda} \cdot c_\kappa (d_{\kappa\mu} \cdot c_\mu x)) && \text{D32} \\
&= c_\mu c_\kappa (d_{\mu\lambda} \cdot d_{\kappa\mu} \cdot c_\mu x) && \text{C}_3 \\
&= c_\kappa (c_\mu (d_{\mu\lambda} \cdot d_{\kappa\mu}) \cdot c_\mu x) && \text{C}_4, \text{C}_3 \\
&= c_\kappa (d_{\kappa\lambda} \cdot c_\mu x) && \text{C}_6 \\
&= s_\lambda^\kappa c_\mu x && \text{D32}
\end{aligned}$$

As to 1.5.11(ii), suppose  $\kappa \neq \lambda, \mu$ . If  $\lambda = \mu$  then the result is trivial, so again we assume that  $\kappa, \lambda, \mu$  are distinct.

$$\begin{aligned}
s_\kappa^\lambda s_\lambda^\kappa c_\lambda c_\mu x &= s_\kappa^\lambda s_\lambda^\mu s_\mu^\kappa c_\lambda c_\mu x && \text{C}_4, \text{1.5.11(i)} \\
&= s_\kappa^\lambda s_\lambda^\mu c_\lambda s_\mu^\kappa c_\mu x && \text{C}_3, \text{C}_4, \text{D32} \\
&= s_\kappa^\mu c_\lambda s_\mu^\kappa c_\mu x && \text{1.5.11(i)} \\
&= s_\kappa^\mu s_\mu^\kappa c_\lambda c_\mu x && \text{C}_3, \text{C}_4, \text{D32} \quad \blacksquare
\end{aligned}$$

Next we investigate some further properties of the substitution operations. As suggested by the notation, the operations  $s_\kappa^\lambda$  and  $s_\lambda^\kappa$  are natural inverses of one another. Indeed using axiom C<sub>7</sub> one can prove that  $s_\kappa^\lambda s_\lambda^\kappa x = s_\kappa^\lambda x$  [HMT71, 1.5.10(v)] and further, that if  $\lambda \notin \Delta(x)$  then  $s_\kappa^\lambda s_\lambda^\kappa x = x$  by 1.5.8(i). Remarkably, this is the only use of axiom C<sub>7</sub> which is required in the proof of [Ném85, Proposition 2.10] (on p. 49). The definition of  $J_\kappa$  is the key to recovering this identity and avoiding the use of axiom C<sub>7</sub>.

If  $\{\kappa, \lambda, \mu\} = 3$  and  $\Delta(x) \subseteq \{\kappa\}$ , then by 1.5.11(ii) we have that  $s_\kappa^\lambda s_\lambda^\kappa x = s_\kappa^\mu s_\mu^\kappa x$ , thus the definition of  $J_\kappa$  is unambiguous. We can generally make use of the inequality  $x \leq s_\kappa^\lambda s_\lambda^\kappa x$  for  $x \in \text{Nr}_{\{\kappa\}}A$ . Remarkably, when  $x = s_\kappa^\lambda u$  for some  $u \in \text{Nr}_{\{\lambda\}}A$ , we have  $x = s_\kappa^\lambda s_\lambda^\kappa x$  (thus  $x \in J_\kappa$ ). In other words, once we perform a single substitution on a 1-dimensional element “the damage is done” and subsequent substitutions cease to create more difficulties. This is particularly advantageous due to the definitions of  $x \circ y$  and  $\check{x}$ , which we shall see to be the images of a substitution in §2.3.

**Lemma 37** *Let  $\kappa \neq \lambda$ . If  $\Delta(x) \subseteq \{\kappa\}$  then  $x \leq s_\kappa^\lambda s_\lambda^\kappa x$ . Moreover, if  $x = s_\kappa^\lambda u$  for some  $u \in \text{Nr}_{\{\lambda\}}A$ , we have  $s_\kappa^\lambda s_\lambda^\kappa x = x \in J_\kappa$ .  $\square$*

PROOF Suppose first that  $\Delta(x) \subseteq \{\kappa\}$ .

$$\begin{aligned}
x &= s_\lambda^\kappa x && \text{1.5.8(i)} \\
&= s_\lambda^\kappa (d_{\kappa\lambda} \cdot x) && \text{BA}
\end{aligned}$$

$$\begin{aligned}
&\leq \mathfrak{s}_\lambda^\kappa \mathfrak{c}_\lambda (\mathfrak{d}_{\kappa\lambda} \cdot x) && \text{C}_2, 1.2.7 \\
&= \mathfrak{s}_\lambda^\kappa \mathfrak{s}_\kappa^\lambda x && \text{D32}
\end{aligned}$$

Now suppose further that  $x = \mathfrak{s}_\kappa^\lambda u$  for some  $u$  with  $\Delta(u) \subseteq \{\lambda\}$  and let  $\{\kappa, \lambda, \mu\} = 3$ .

$$\begin{aligned}
\mathfrak{s}_\kappa^\lambda \mathfrak{s}_\lambda^\kappa x &= \mathfrak{s}_\kappa^\mu \mathfrak{s}_\mu^\kappa x && 1.5.11(\text{ii}) \\
&= \mathfrak{s}_\kappa^\mu \mathfrak{s}_\mu^\kappa \mathfrak{s}_\kappa^\lambda u && \text{Def} \\
&= \mathfrak{s}_\kappa^\mu \mathfrak{s}_\mu^\lambda u && 1.5.11(\text{i}) \\
&= \mathfrak{s}_\kappa^\lambda u = x && 1.5.11(\text{i}), \text{Def} \quad \blacksquare
\end{aligned}$$

As an immediate consequence we can observe that if  $x \in J_\kappa$  then  $\mathfrak{s}_\lambda^\kappa x \in J_\lambda$ .

Our recovery of identity 1.5.10(v) via the definition of  $J_\kappa$  comes at a price. In [Ném85], the universe of the Néméti reduct is  $\{\dot{1} \cdot x : \mathfrak{c}_1 \mathfrak{c}_2 x = x\}$  which is quite easily seen to be closed under Boolean operations. That this is also true of  $J_\kappa$  is not a trivial fact. In fact, it was the realization and proof of Equation (2.8) below which first convinced me beyond a doubt that [Theorem 34](#) was an attainable result. Once again, using axiom C<sub>7</sub> it is straightforward to (equivalently) show that  $\mathfrak{s}_\lambda^\kappa \bar{x} = (\mathfrak{s}_\lambda^\kappa x)^-$  [HMT71, 1.5.3(ii)]. An alternative proof follows.

**Lemma 38** *The structure  $\langle J_\kappa, +, - \rangle$  is a Boolean algebra.* □

PROOF We only need establish closure under the Boolean operations. Closure of  $J_\kappa$  under  $+$  follows immediately from 1.5.3(i). For closure under complementation, we prove the following identities. Let  $\{\kappa, \lambda, \mu\} = 3$  and  $x \in J_\kappa$ .

$$(\mathfrak{d}_{\kappa\mu})^- + \mathfrak{d}_{\lambda\kappa} = (\mathfrak{d}_{\kappa\mu})^- + \mathfrak{d}_{\lambda\mu} \quad (2.6)$$

$$(\mathfrak{s}_\lambda^\kappa x)^- = \mathfrak{s}_\lambda^\mu (\mathfrak{s}_\mu^\kappa x)^- \quad (2.7)$$

$$(\mathfrak{s}_\lambda^\kappa x)^- = \mathfrak{s}_\lambda^\kappa \bar{x} \quad (2.8)$$

Equation (2.6) follows quickly from C<sub>6</sub> and the Boolean structure of  $\mathfrak{Nt}_{\{\kappa\}}\mathfrak{A}$ . We prove inequality in one direction, however, as  $\kappa, \lambda, \mu$  are arbitrary this is sufficient to establish equality.

$$\begin{aligned}
(\mathfrak{d}_{\kappa\mu})^- + \mathfrak{d}_{\lambda\kappa} &\geq (\mathfrak{d}_{\kappa\mu})^- + \mathfrak{d}_{\kappa\mu} \cdot \mathfrak{d}_{\lambda\mu} && \text{C}_6 \\
&= \left( (\mathfrak{d}_{\kappa\mu})^- + \mathfrak{d}_{\kappa\mu} \right) \cdot \left( (\mathfrak{d}_{\kappa\mu})^- + \mathfrak{d}_{\lambda\mu} \right) && \text{BA} \\
&= (\mathfrak{d}_{\kappa\mu})^- + \mathfrak{d}_{\lambda\mu} && \text{BA}
\end{aligned}$$

Equation (2.7) is a technical consequence of (2.6) which is quite useful in the proof of (2.8). It bears some resemblance to 1.5.11(i), but owing to the negations involved the proof is much more complex.

$$\begin{aligned}
(s_\lambda^\kappa x)^- &= (c_\kappa(d_{\kappa\lambda} \cdot x))^- && \text{D32} \\
&= c_\mu(d_{\lambda\mu} \cdot (c_\kappa(d_{\kappa\lambda} \cdot x))^-) && \text{1.5.8(i), 1.2.12(i)} \\
&= c_\mu((d_{\lambda\mu})^- + c_\kappa(d_{\kappa\lambda} \cdot x))^- && \text{BA} \\
&= c_\mu(c_\kappa((d_{\lambda\mu})^- + d_{\kappa\lambda} \cdot x))^- && \text{1.2.12(i), 1.2.6(ii)} \\
&= c_\mu(c_\kappa(((d_{\lambda\mu})^- + d_{\kappa\lambda}) \cdot ((d_{\lambda\mu})^- + x)))^- && \text{BA} \\
&= c_\mu(c_\kappa(((d_{\lambda\mu})^- + d_{\kappa\mu}) \cdot ((d_{\lambda\mu})^- + x)))^- && \text{(2.6)} \\
&= c_\mu(c_\kappa((d_{\lambda\mu})^- + d_{\kappa\mu} \cdot x))^- && \text{BA} \\
&= c_\mu((d_{\lambda\mu})^- + c_\kappa(d_{\kappa\mu} \cdot x))^- && \text{1.2.12(i), 1.2.6(ii)} \\
&= c_\mu(d_{\lambda\mu} \cdot (c_\kappa(d_{\kappa\mu} \cdot x))^-) && \text{BA} \\
&= s_\lambda^\mu(s_\mu^\kappa x)^- && \text{D32}
\end{aligned}$$

Finally, we are prepared to prove (2.8). Let  $u = s_\lambda^\kappa x$ , so that  $s_\kappa^\lambda u = x$  (by definition of  $J_\kappa$ ).

$$\begin{aligned}
(s_\lambda^\kappa x)^- &= s_\lambda^\mu(s_\mu^\kappa x)^- && \text{(2.7)} \\
&= s_\lambda^\mu(s_\mu^\kappa s_\kappa^\lambda u)^- && \text{Def} \\
&= s_\lambda^\mu(s_\mu^\lambda u)^- && \text{1.5.11(i)} \\
&= s_\lambda^\mu s_\mu^\kappa (s_\kappa^\lambda u)^- && \text{(2.7)} \\
&= s_\lambda^\mu s_\mu^\kappa \bar{x} && \text{Def} \\
&= s_\lambda^\kappa \bar{x} && \text{1.5.11(i)}
\end{aligned}$$

Applying Equation (2.8) twice (along with the definition of  $J_\kappa$ ) is sufficient to show closure under complementation.

$$\begin{aligned}
\bar{x} &= (s_\kappa^\lambda s_\lambda^\kappa x)^- && \text{D32} \\
&= s_\kappa^\lambda (s_\lambda^\kappa x)^- && \text{(2.8)} \\
&= s_\kappa^\lambda s_\lambda^\kappa \bar{x} && \text{(2.8)}
\end{aligned}$$

Thus  $\bar{x} \in J_\kappa$  by definition. ■

**Remark 39** Equation (2.6) is (a dual version of) an algebraization of the logically valid “ $x = y$  and  $x \neq z$  iff  $x = y$  and  $y \neq z$ ”. Equation (2.8) is a weaker formulation of axiom C<sub>7</sub>, holding only for

$x \in J_\kappa$ . Rewriting axiom  $C_7$  using substitutions yields  $s_\lambda^\kappa x \cdot s_\lambda^\kappa \bar{x} = 0$ , which follows immediately from Equation (2.8) when  $x \in J_\kappa$ . Conversely, (2.8) can be easily derived (for all  $x \in A$  as opposed to  $x \in J_\kappa$ ) using  $C_7$  by [HMT71, 1.5.3(ii)].  $\square$

Next, we establish that the dimension sets of  $d_{\kappa\langle i\rangle\lambda\langle j\rangle}$ ,  $\delta_{\kappa\langle i\rangle\lambda\langle j\rangle}$ ,  $\epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle}$ , and  $s_{\lambda\langle i\rangle}^\kappa x$  are as suggested by the notation. When we invoke the following lemma, we simply use  $\Delta$  as our annotation.

**Lemma 40 ( $\Delta$ )** *If  $x \in \{d_{\kappa\langle i\rangle\lambda\langle j\rangle}, \epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle}, \delta_{\kappa\langle i\rangle\lambda\langle j\rangle}\}$  then  $\Delta(x) \subseteq \{\kappa, \lambda\}$ . For  $x \in \text{Nr}_{\{\kappa\}}A$ ,  $\Delta(s_{\lambda\langle i\rangle}^\kappa x) \subseteq \{\lambda\}$ .*  $\square$

PROOF For the first part, it suffices to show that  $\Delta(d_{\kappa\langle i\rangle\lambda\langle j\rangle}) \subseteq \{\kappa, \lambda\}$  since  $\Delta(x \cdot y) \subseteq \Delta(x) \cup \Delta(y)$  (by  $C_3$ ). We proceed by case analysis on the definition of  $d_{\kappa\langle i\rangle\lambda\langle j\rangle}$ , examining each of (2.1) through (2.5) from Definition 32. In case (2.5), we note that  $d_{\kappa\langle i\rangle\kappa\langle j\rangle} = c_{\kappa+3}1(d_{\kappa+3}1\langle\kappa\langle j\rangle\rangle \cdot d_{\kappa+3}1\langle\kappa\langle j\rangle\rangle)$ , so  $\kappa+3 \notin \Delta(d_{\kappa\langle i\rangle\kappa\langle j\rangle})$  by 1.2.3. This reduces the proof to the case where  $\kappa \neq \lambda$  (by  $C_3$  and  $C_4$ ). In this case  $d_{\kappa\langle i\rangle\lambda\langle j\rangle}$  is defined by one of (2.1)-(2.4). We have already established for  $\{\kappa, \lambda, \mu\} = 3$  that  $c_\mu d_{\kappa\lambda} = d_{\kappa\lambda}$  (by Definition 32) so the result holds for (2.1). In the case of (2.3) or (2.4), we have  $\mu \notin \Delta(d_{\kappa\langle i\rangle\lambda\langle j\rangle})$  by 1.2.3. Finally in the case of (2.2). Note that  $\Delta(p_\kappa) \subseteq \{0, 1\}$  by Definition 32. Now we must use a case analysis for each possible choice of  $\kappa$  and  $\lambda$ —this can be found in Table 2.2. Finally, we note that when  $\kappa \neq \lambda$ ,  $\kappa \notin \Delta(s_\lambda^\kappa x)$  by 1.2.3 and Definition 32. Applying this principle to each of the possibilities for  $\kappa$  and  $\lambda$  concludes the proof that  $\Delta(d_{\kappa\langle i\rangle\lambda\langle j\rangle}) \subseteq \{\kappa, \lambda\}$ .

Now suppose  $\Delta(u) \subseteq \{\kappa\}$ . Then  $s_{\lambda\langle i\rangle}^\kappa u = c_\kappa(d_{\kappa\lambda\langle i\rangle} \cdot u)$  by Definition 32, so  $\kappa \notin \Delta(s_{\lambda\langle i\rangle}^\kappa u)$  by 1.2.3. Further, if  $\mu \neq \kappa, \lambda$

$$\begin{aligned} c_\mu s_{\lambda\langle i\rangle}^\kappa u &= c_\mu c_\kappa (d_{\kappa\lambda\langle i\rangle} \cdot u) && \text{D32} \\ &= c_\kappa c_\mu (d_{\kappa\lambda\langle i\rangle} \cdot u) && \text{C}_4 \\ &= c_\kappa (d_{\kappa\lambda\langle i\rangle} \cdot u) && \text{C}_3, \Delta. \end{aligned}$$

Thus  $\mu \notin \Delta(s_{\lambda\langle i\rangle}^\kappa u)$  either, concluding the proof.  $\blacksquare$

Now we show that the auxiliary dimension in the definition of  $d_{\kappa\langle i\rangle\kappa\langle j\rangle}$  need not be  $\kappa+3$ . This is hardly surprising, indeed we chose it quite arbitrarily. We require this fact in order to fully exploit the symmetry between dimensions. It won't be used directly during the course of our main proof, but rather the consequences which follow.

**Proposition 41** *For  $\mu \neq \kappa \in 3$  and  $i, j \in S_3$ ,  $d_{\kappa\langle i\rangle\kappa\langle j\rangle} = c_\mu (d_{\mu\kappa\langle i\rangle} \cdot d_{\mu\kappa\langle j\rangle})$ .*  $\square$

$\kappa$	$\lambda$	$\mathbf{p}_k(\kappa, \lambda)$	
0	1	$s_0^0 s_1^1 \mathbf{p}_k$	$= \mathbf{p}_k$
0	2	$s_0^0 s_2^1 \mathbf{p}_k$	$= s_2^1 \mathbf{p}_k$
1	0	$s_1^2 s_0^1 s_2^0 \mathbf{p}_k$	
1	2	$s_1^0 s_2^1 \mathbf{p}_k$	
2	0	$s_2^2 s_0^1 s_2^0 \mathbf{p}_k$	$= s_0^1 s_2^0 \mathbf{p}_k$
2	1	$s_2^2 s_1^1 s_2^0 \mathbf{p}_k$	$= s_2^0 \mathbf{p}_k$

Table 2.2 Case analysis for the substitution convention

PROOF Let  $\{\kappa, \lambda, \mu\} = 3$ . Then depending on the choice of  $\lambda$  and  $\mu$ ,  $\mathbf{d}_{\kappa\langle i \rangle \kappa\langle j \rangle}$  must be the left or right hand side of the following inequality. That is,  $\kappa +_3 1 \in \{\lambda, \mu\}$ .

$$\begin{aligned}
c_\lambda(\mathbf{d}_{\lambda\kappa\langle i \rangle} \cdot \mathbf{d}_{\lambda\kappa\langle j \rangle}) &= c_\lambda c_\mu(\mathbf{d}_{\mu\lambda} \cdot \mathbf{d}_{\lambda\kappa\langle i \rangle} \cdot \mathbf{d}_{\lambda\kappa\langle j \rangle}) && 1.5.8(i) \\
&\leq c_\lambda c_\mu(\mathbf{d}_{\mu\kappa\langle i \rangle} \cdot \mathbf{d}_{\mu\kappa\langle j \rangle}) && \text{Ax}_1, 1.2.7 \\
&= c_\mu(\mathbf{d}_{\mu\kappa\langle i \rangle} \cdot \mathbf{d}_{\mu\kappa\langle j \rangle}) && \Delta
\end{aligned}$$

The inequality in the opposite direction follows from a symmetric argument, so equality holds.  $\blacksquare$

The following is a generalization of 1.5.8(i) to our projectional substitutions. Specifically, we give conditions under which the operation  $s_{\lambda\langle i \rangle}^\kappa$  fixes an element of  $A$ .

**Proposition 42** *If  $\kappa \neq \lambda$ ,  $\kappa \notin \Delta(x)$ , and  $x \leq \mathbf{d}_{\lambda\langle i \rangle \lambda\langle i \rangle}$ , then  $s_{\lambda\langle i \rangle}^\kappa x = x$ .*  $\square$

PROOF Suppose  $x$  satisfies the above hypotheses.

$$\begin{aligned}
x &= x \cdot \mathbf{d}_{\lambda\langle i \rangle \lambda\langle i \rangle} && \text{BA} \\
&= x \cdot c_\kappa(\mathbf{d}_{\kappa\lambda\langle i \rangle}) && 41 \\
&= c_\kappa(\mathbf{d}_{\kappa\lambda\langle i \rangle} \cdot x) && C_3, \kappa \notin \Delta(x) \\
&= s_{\lambda\langle i \rangle}^\kappa x && \text{D32} \quad \blacksquare
\end{aligned}$$

**Remark 43** The hypothesis  $x \leq \mathbf{d}_{\lambda\langle i \rangle \lambda\langle i \rangle}$  from Proposition 42 may seem overly restrictive at first glance, but recall that  $\mathbf{d}_{\lambda\langle i \rangle \lambda\langle i \rangle}$  is meant to simulate  $\{\mathbf{x} : P_i(x_\lambda) = P_i(x_\lambda)\}$ . In other words, the hypothesis only guarantees that projection  $P_i$  is defined on the  $\lambda$ -coordinates of  $x$ .  $\square$

The following Proposition 44 will mostly be used in conjunction with  $\text{Ax}_2$ . The reason it is required is that the left hand side of equations in  $\text{Ax}_2$  will not show up frequently in our calculations. As we shall see however, they are upper bounds for the projectional diagonal elements which we **do** use frequently.

**Proposition 44** For  $\kappa, \lambda \in 3$  and  $i, j \in S_3$ ,  $\mathbf{d}_{\kappa\langle i \rangle \lambda\langle j \rangle} \leq \mathbf{d}_{\kappa\langle i \rangle \kappa\langle i \rangle}$ . □

PROOF Choose any  $\mu \neq \kappa, \lambda$ .

$$\begin{aligned} \mathbf{d}_{\kappa\langle i \rangle \lambda\langle j \rangle} &= \mathbf{c}_\mu (\mathbf{d}_{\mu\kappa\langle i \rangle} \cdot \mathbf{d}_{\mu\lambda\langle j \rangle}) && \text{D32, 41} \\ &\leq \mathbf{c}_\mu (\mathbf{d}_{\mu\kappa\langle i \rangle}) && \text{BA, 1.2.7} \\ &= \mathbf{d}_{\kappa\langle i \rangle \kappa\langle i \rangle} && \text{41} \end{aligned}$$

Now, let  $k \in 2$  and  $ik \in S_3$ .

$$\begin{aligned} \mathbf{d}_{\kappa\langle ik \rangle \kappa\langle ik \rangle} &= \mathbf{c}_\mu (\mathbf{d}_{\mu\kappa\langle ik \rangle}) && \text{41} \\ &= \mathbf{c}_\mu \mathbf{c}_\lambda (\mathbf{d}_{\kappa\langle i \rangle \lambda} \cdot \mathbf{d}_{\lambda\langle k \rangle \mu}) && \text{D32} \\ &\leq \mathbf{c}_\mu \mathbf{c}_\lambda (\mathbf{d}_{\kappa\langle i \rangle \lambda}) && \text{BA, 1.2.7} \\ &= \mathbf{c}_\lambda (\mathbf{d}_{\kappa\langle i \rangle \lambda}) && \Delta \\ &= \mathbf{d}_{\kappa\langle i \rangle \kappa\langle i \rangle} && \text{41} \end{aligned}$$

Applying these recursively yields the desired inequality. ■

**Remark 45** As regarding the previous proof: informally the first argument says “if  $P_i(x_\kappa) = P_j(x_\lambda)$  then  $P_i(x_\kappa) = P_i(x_\kappa)$ .” Intuitively “ $P_i(x_\kappa) = P_i(x_\kappa)$ ” simply expresses that  $P_i$  is defined on  $x_\kappa$ . This is not necessarily true of all elements as the functions  $P_i$  are not necessarily (or desirably) entire. The second argument could be read as “if  $P_{i_n} P_{i_{n-1}} \dots P_{i_0}(x_\kappa)$  exists then  $P_{i_{n-1}} \dots P_{i_0}(x_\kappa)$  exists as well.” □

Next, Proposition 46 establishes an identity similar to those in Ax<sub>1</sub> for the terms  $\epsilon_{\kappa\langle i \rangle \lambda\langle j \rangle}$  and  $\delta_{\kappa\langle i \rangle \lambda\langle j \rangle}$ . This is not surprising, as these terms are simply Boolean combinations of the projectional diagonal elements. Proposition 47 is a similar result relating to projectional substitutions.

**Proposition 46** When  $\kappa, \lambda, \mu \in 3$  and  $i \in S_2, j \in S_1$ ; we have the following:

$$\begin{aligned} \mathbf{d}_{\mu\kappa\langle i \rangle} \cdot \delta_{\mu\lambda\langle j \rangle} &\leq \delta_{\kappa\langle i \rangle \lambda\langle j \rangle} \\ \mathbf{d}_{\mu\lambda\langle j \rangle} \cdot \delta_{\kappa\langle i \rangle \mu} &\leq \delta_{\kappa\langle i \rangle \lambda\langle j \rangle} \\ \mathbf{d}_{\mu\kappa\langle i \rangle} \cdot \epsilon_{\mu\lambda\langle j \rangle} &\leq \epsilon_{\kappa\langle i \rangle \lambda\langle j \rangle} \end{aligned} \quad \square$$

PROOF These follow quickly by applying Ax<sub>1</sub> to the definitions of the terms.

$$\begin{aligned} \mathbf{d}_{\mu\kappa\langle i \rangle} \cdot \delta_{\mu\lambda\langle j \rangle} &= \mathbf{d}_{\mu\kappa\langle i \rangle} \cdot \mathbf{d}_{\mu\langle 0 \rangle \lambda\langle j00 \rangle} \cdot \mathbf{d}_{\mu\langle 1 \rangle \lambda\langle j11 \rangle} \cdot \mathbf{d}_{\lambda\langle j01 \rangle \lambda\langle j10 \rangle} && \text{D32} \\ &\leq \mathbf{d}_{\kappa\langle i0 \rangle \lambda\langle j00 \rangle} \cdot \mathbf{d}_{\kappa\langle i1 \rangle \lambda\langle j11 \rangle} \cdot \mathbf{d}_{\lambda\langle j01 \rangle \lambda\langle j10 \rangle} && \text{Ax}_1 \\ &= \delta_{\kappa\langle i \rangle \lambda\langle j \rangle} && \text{D32} \end{aligned}$$

$$\begin{aligned}
\mathbf{d}_{\mu\lambda\langle j\rangle}\delta_{\kappa\langle i\rangle\mu} &= \mathbf{d}_{\mu\lambda\langle j\rangle} \cdot \mathbf{d}_{\kappa\langle i0\rangle\mu\langle 00\rangle} \cdot \mathbf{d}_{\kappa\langle i1\rangle\mu\langle 11\rangle} \cdot \mathbf{d}_{\mu\langle 01\rangle\mu\langle 10\rangle} && \text{D32} \\
&\leq \mathbf{d}_{\kappa\langle i0\rangle\lambda\langle j00\rangle} \cdot \mathbf{d}_{\kappa\langle i1\rangle\lambda\langle j11\rangle} \cdot \mathbf{d}_{\lambda\langle j10\rangle\lambda\langle j01\rangle} && \text{Ax}_1 \\
&= \delta_{\kappa\langle i\rangle\lambda\langle j\rangle} && \text{D32} \\
\mathbf{d}_{\mu\kappa\langle i\rangle} \cdot \epsilon_{\mu\lambda\langle j\rangle} &= \mathbf{d}_{\mu\kappa\langle i\rangle} \cdot \mathbf{d}_{\mu\langle 0\rangle\lambda\langle j1\rangle} \cdot \mathbf{d}_{\mu\langle 1\rangle\lambda\langle j0\rangle} && \text{D32} \\
&\leq \mathbf{d}_{\kappa\langle i0\rangle\lambda\langle j1\rangle} \cdot \mathbf{d}_{\kappa\langle i1\rangle\lambda\langle j0\rangle} && \text{Ax}_1 \\
&= \epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle} && \text{D32} \quad \blacksquare
\end{aligned}$$

**Proposition 47** *Let  $\{\kappa, \lambda, \mu\} = 3$ ;  $x \in J_\kappa$ ; and  $i, j \in S_3$ . Then  $s_{\lambda\langle i\rangle}^\kappa x \cdot \mathbf{d}_{\lambda\langle i\rangle\mu\langle j\rangle} \leq s_{\mu\langle i\rangle}^\kappa x$ . Further, if  $ji \in S_3$  then  $s_{\lambda\langle i\rangle}^\kappa x \cdot \mathbf{d}_{\lambda\mu\langle j\rangle} \leq s_{\mu\langle ji\rangle}^\kappa x$ .  $\square$*

PROOF Assume  $i, j \in S_3$  for the first inequality, and further that  $ji \in S_3$  for the second inequality.

$$\begin{aligned}
s_{\lambda\langle i\rangle}^\kappa x \cdot \mathbf{d}_{\lambda\langle i\rangle\mu\langle j\rangle} &= c_\kappa(\mathbf{d}_{\kappa\lambda\langle i\rangle} \cdot x) \cdot \mathbf{d}_{\lambda\langle i\rangle\mu\langle j\rangle} && \text{D32} \\
&= c_\kappa(\mathbf{d}_{\kappa\lambda\langle i\rangle} \cdot x \cdot \mathbf{d}_{\lambda\langle i\rangle\mu\langle j\rangle}) && \Delta, C_3 \\
&\leq c_\kappa(\mathbf{d}_{\kappa\mu\langle j\rangle} \cdot x) && \text{Ax}_1, 1.2.7 \\
&= s_{\mu\langle j\rangle}^\kappa x && \text{D32} \\
s_{\lambda\langle i\rangle}^\kappa x \cdot \mathbf{d}_{\lambda\mu\langle j\rangle} &= c_\kappa(\mathbf{d}_{\kappa\lambda\langle i\rangle} \cdot x) \cdot \mathbf{d}_{\lambda\mu\langle j\rangle} && \text{Def} \\
&= c_\kappa(\mathbf{d}_{\kappa\lambda\langle i\rangle} \cdot x \cdot \mathbf{d}_{\lambda\mu\langle j\rangle}) && \Delta, C_3 \\
&\leq c_\kappa(\mathbf{d}_{\kappa\mu\langle ji\rangle} \cdot x) && \text{Ax}_1, 1.2.7 \\
&= s_{\mu\langle ji\rangle}^\kappa x && \text{D32} \quad \blacksquare
\end{aligned}$$

There is quite a bit of symmetry between cylindrified dimensions. In fact, we can cylindrify dimensions interchangeably provided we are careful not to change the dimension sets of the respective terms.

**Lemma 48** *Let  $\{\kappa, \lambda, \mu\} = 3$  and let  $X(\lambda)$  be a combination (via the  $\cdot$  operation) of some elements:  $\mathbf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}$ ,  $\delta_{\kappa\langle i\rangle\lambda\langle j\rangle}$ ,  $\epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle}$ , and  $s_{\lambda\langle i\rangle}^\kappa x$  (where  $x \in J_\kappa$ )—then  $c_\lambda(X(\lambda)) = c_\mu(X(\mu))$ .  $\square$*

PROOF A simple inductive argument using Ax<sub>1</sub> along with 46 and 47 shows that  $\Delta(X(\lambda)) \subseteq \{\lambda, \kappa\}$  and  $\mathbf{d}_{\lambda\mu} \cdot X(\lambda) \leq X(\mu)$ .

$$\begin{aligned}
c_\lambda X(\lambda) &= c_\lambda c_\mu(\mathbf{d}_{\lambda\mu} \cdot X(\lambda)) && 1.5.8(\text{i}) \\
&\leq c_\lambda c_\mu(X(\mu)) && \text{Induction} \\
&\leq c_\mu(X(\mu)) && \Delta
\end{aligned}$$

The two possible choices of  $\{\lambda, \mu\}$  yield opposite inequalities, so equality must hold.  $\blacksquare$

Finally, we establish that  $x$  can be “recovered” from  $s_{\lambda\langle i \rangle}^\kappa x$  when  $x \in J_\kappa$ . This is a generalized and algebraic version of (6) from [Ném85, Proposition 2.10]. The proof there invokes  $C_7$  to obtain  $s_0^2 s_2^0 x = x$ , which we have by the definition of  $J_0$ .

**Lemma 49** *Let  $\kappa \neq \lambda$ ,  $x \in J_\kappa$ , and  $i \in S_3$ . Then  $s_{\lambda\langle i \rangle}^\kappa x \cdot d_{\kappa\lambda\langle i \rangle} \leq x$ .* □

PROOF Let  $\{\kappa, \lambda, \mu\} = 3$ .

$$\begin{aligned}
s_{\lambda\langle i \rangle}^\kappa x \cdot d_{\kappa\lambda\langle i \rangle} &= c_\mu \left( d_{\kappa\mu} \cdot s_{\lambda\langle i \rangle}^\kappa x \cdot d_{\kappa\lambda\langle i \rangle} \right) && 1.5.8(i) \\
&\leq c_\mu \left( d_{\kappa\mu} \cdot d_{\mu\lambda\langle i \rangle} \cdot s_{\lambda\langle i \rangle}^\kappa x \right) && Ax_1, BA, 1.2.7 \\
&\leq c_\mu \left( d_{\kappa\mu} \cdot d_{\mu\lambda\langle i \rangle} \cdot c_\kappa \left( d_{\kappa\lambda\langle i \rangle} \cdot x \right) \right) && D32 \\
&\leq c_\mu \left( d_{\kappa\mu} \cdot c_\kappa \left( d_{\mu\lambda\langle i \rangle} \cdot d_{\kappa\lambda\langle i \rangle} \cdot x \right) \right) && C_3 \\
&\leq c_\mu \left( d_{\kappa\mu} \cdot c_\kappa \left( d_{\kappa\mu} \cdot x \right) \right) && Ax_1 \\
&\leq s_\kappa^\mu s_\mu^\kappa x \leq x && D32 \quad \blacksquare
\end{aligned}$$

We are now fully prepared for the proof of our main theorem.

### 2.3 $Q^+$ RA reducts in $Df_3$

We first prove that  $J$  is closed under all the operations of  $\mathfrak{J}\mathfrak{A}$ , and thusly that  $\mathfrak{J}\mathfrak{A}$  is an algebra. By  $Ax_1$  and [Lemma 37](#) we have that  $\dot{1} \in J$ , and since  $\langle J_0, +, - \rangle \in BA$ , we have that  $\langle J, +, \dot{-} \rangle \in BA$  as well (since it is a relativization to  $\dot{1}$ ). As to closure under the relational operations we note that by [Definition 32](#),  $x \in J$  if and only if  $s_0^2 s_2^0 x = x$  and  $x \leq \dot{1}$ . Let  $x, y \in J$  be arbitrary.

$$\begin{aligned}
x \circ y &\leq s_0^2 s_2^0 (x \circ y) && 37 \\
&\leq s_0^2 c_0 \left( d_{02} \cdot c_1 \left( s_{1\langle 0 \rangle}^0 x \cdot s_{1\langle 1 \rangle}^0 y \cdot \delta_{01} \right) \right) && D32 \\
&\leq s_0^2 c_0 c_1 \left( d_{02} \cdot s_{1\langle 0 \rangle}^0 x \cdot s_{1\langle 1 \rangle}^0 y \cdot \delta_{01} \right) && \Delta, C_3 \\
&\leq c_2 \left( d_{02} \cdot c_0 c_1 \left( s_{1\langle 0 \rangle}^0 x \cdot s_{1\langle 1 \rangle}^0 y \cdot \delta_{21} \right) \right) && 46, 1.2.7, D32 \\
&\leq c_2 c_1 \left( d_{02} \cdot s_{1\langle 0 \rangle}^0 x \cdot s_{1\langle 1 \rangle}^0 y \cdot \delta_{21} \right) && C_4, \Delta, C_3 \\
&\leq c_1 \left( s_{1\langle 0 \rangle}^0 x \cdot s_{1\langle 1 \rangle}^0 y \cdot \delta_{01} \right) && 46, C_2, C_4, \Delta \\
&\leq x \circ y && D32 \\
&\leq d_{0\langle 0 \rangle 0\langle 0 \rangle} \cdot d_{0\langle 1 \rangle 0\langle 1 \rangle} = \dot{1} && BA, 44, \Delta
\end{aligned}$$



$$\begin{aligned}
\check{x} &\leq s_0^2 s_2^0 \check{x} && 37 \\
&\leq s_0^2 c_0 (d_{02} \cdot c_1 (s_1^0 x \cdot \epsilon_{01})) && D32 \\
&\leq s_0^2 c_0 c_1 (d_{02} \cdot s_1^0 x \cdot \epsilon_{01}) && \Delta, C_3 \\
&\leq c_2 (d_{02} \cdot c_0 c_1 (s_1^0 x \cdot \epsilon_{21})) && 46, 1.2.7, D32 \\
&\leq c_2 c_1 (d_{02} \cdot s_1^0 x \cdot \epsilon_{21}) && C_4, \Delta, C_3 \\
&\leq c_1 (s_1^0 x \cdot \epsilon_{01}) && 46, C_4, \Delta \\
&\leq \check{x} && D32 \\
&\leq d_{0\langle 0 \rangle 0\langle 0 \rangle} \cdot d_{0\langle 1 \rangle 0\langle 1 \rangle} = \dot{1} && BA, 44, \Delta
\end{aligned}$$

We shall prove conditions I-IV from [JT51, Definition 4.1] to show that  $\mathfrak{J}\mathfrak{A} \in \text{RA}$ . These conditions are well-known to be equivalent to the RA axioms. For proof, we refer the reader to [CT51, Theorem 2.2] or [Mad06, 314].

$$(I) \langle J, +, \dot{\cdot} \rangle \in \text{BA}.$$

$$(II) (x \circ y) \circ z = x \circ (y \circ z) \text{ for any } x, y, z \in J.$$

$$(III) \dot{1} \circ x = x = x \circ \dot{1} \text{ for every } x \in J.$$

$$(IV) (x \circ y) \cdot z = 0 \Leftrightarrow (\check{x} \circ z) \cdot y = 0 \Leftrightarrow (z \circ \check{y}) \cdot x = 0 \text{ for any } x, y, z \in J.$$

Condition (I) has already been established (by Lemma 38), so we begin with condition (II).

PROOF (CONDITION II) To begin, we require some technical lemmas. These employ symmetry in the definitions in order to shorten the proof. Both are taken from [Ném85] and [Ném86], but are formulated there in logical terms. To begin, we work with terms of the form  $s_{1\langle i \rangle}^0(x \circ y)$  where  $i, j = \{0, 1\}$ . The following is proved as (4) in [Ném85, p. 45] and as (5) in [Ném86, p. 56].

$$\begin{aligned}
c_0 \left( d_{01\langle i \rangle} \cdot c_1 \left( s_{1\langle i \rangle}^0 x \cdot s_{1\langle j \rangle}^0 y \cdot \delta_{01} \right) \right) &\leq c_2 \left( d_{2\langle j \rangle 1\langle j \rangle} \cdot s_{2\langle i 0 \rangle}^0 x \cdot s_{2\langle i 1 \rangle}^0 y \cdot \delta_{1\langle i \rangle 2\langle i \rangle} \right) && (2.9) \\
c_0 \left( d_{01\langle i \rangle} \cdot c_1 \left( s_{1\langle i \rangle}^0 x \cdot s_{1\langle j \rangle}^0 y \cdot \delta_{01} \right) \right) & \\
\leq c_0 \left( d_{01\langle i \rangle} \cdot c_2 \left( s_{2\langle i \rangle}^0 x \cdot s_{2\langle j \rangle}^0 y \cdot \delta_{02} \right) \right) &&& 48 \\
\leq c_0 c_2 \left( d_{01\langle i \rangle} \cdot s_{2\langle i \rangle}^0 x \cdot s_{2\langle j \rangle}^0 y \cdot \delta_{02} \right) &&& C_3, \Delta \\
\leq c_2 \left( s_{2\langle i \rangle}^0 x \cdot s_{2\langle j \rangle}^0 y \cdot \delta_{1\langle i \rangle 2} \right) &&& 46, 1.2.7, \Delta
\end{aligned}$$

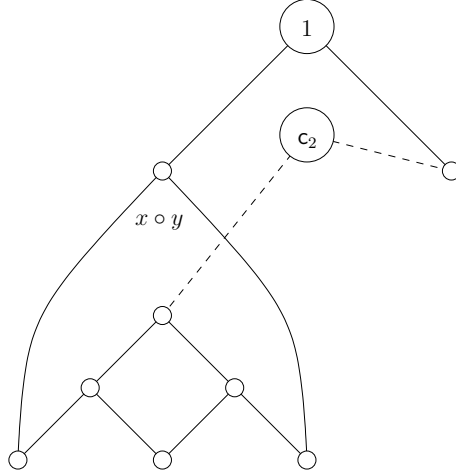


Figure 2.3 An illustration of (2.9) with  $i = 0$ . Circles containing a cylindrification operation represent cylindrified dimensions. Thus we can see the dimension set of the terms involved is  $\{1\}$ . Lines emanating from the left side of a point indicate the projection  $p_0$  and similarly  $p_1$  on the right. Dotted lines indicate the right hand side of the inequality.

$$\begin{aligned}
&\leq c_2 c_0 \left( d_{02} \cdot s_{2\langle i \rangle}^0 x \cdot s_{2\langle j \rangle}^0 y \cdot \delta_{1\langle i \rangle 2} \right) && 1.5.8(i) \\
&\leq c_2 c_0 \left( d_{02} \cdot s_{2\langle i \rangle}^0 x \cdot s_{2\langle j \rangle}^0 y \cdot \delta_{1\langle i \rangle 0} \right) && 46, 1.2.7 \\
&\leq c_0 \left( c_2 \left( d_{02} \cdot s_{2\langle i \rangle}^0 x \cdot s_{2\langle j \rangle}^0 y \right) \cdot \delta_{1\langle i \rangle 0} \right) && C_4, C_3, \Delta \\
&\leq c_0 \left( c_1 \left( d_{01} \cdot s_{1\langle i \rangle}^0 x \cdot s_{1\langle j \rangle}^0 y \right) \cdot \delta_{1\langle i \rangle 0} \right) && 48 \\
&\leq c_0 c_2 \left( d_{2\langle i \rangle 0} \cdot d_{2\langle j \rangle 1\langle j \rangle} \cdot c_1 \left( d_{01} \cdot s_{1\langle i \rangle}^0 x \cdot s_{1\langle j \rangle}^0 y \right) \cdot \delta_{1\langle i \rangle 0} \right) && Ax_2, C_3 \\
&\leq c_2 \left( d_{1\langle j \rangle 2\langle j \rangle} \cdot c_1 \left( d_{02\langle i \rangle} \cdot d_{01} \cdot s_{1\langle i \rangle}^0 x \cdot s_{1\langle j \rangle}^0 y \right) \cdot \delta_{1\langle i \rangle 2\langle i \rangle} \right) && 46, C_3 \\
&\leq c_2 \left( d_{1\langle j \rangle 2\langle j \rangle} \cdot c_0 c_1 \left( s_{2\langle ii \rangle}^0 x \cdot s_{2\langle ij \rangle}^0 y \right) \cdot \delta_{1\langle i \rangle 2\langle i \rangle} \right) && Ax_1, 47 \\
&\leq c_2 \left( d_{1\langle j \rangle 2\langle j \rangle} \cdot s_{2\langle ii \rangle}^0 x \cdot s_{2\langle ij \rangle}^0 y \cdot \delta_{1\langle i \rangle 2\langle i \rangle} \right) && \Delta
\end{aligned}$$

The next lemma enables us to recover the term  $\delta_{01}$  after invoking Equation (2.9). The following is assumed as (Ax<sub>8</sub>) in [Ném85, p. 48] and proved as (S13) in [Ném86, p. 53].

$$\begin{aligned}
&d_{0\langle i \rangle 2\langle iii \rangle} \cdot d_{0\langle j \rangle 2\langle jj \rangle} \cdot d_{2\langle ii \rangle 2\langle ij \rangle} \cdot d_{2\langle ij \rangle 2\langle ji \rangle} \leq \\
&\quad c_1 \left( d_{1\langle i \rangle 2\langle ii \rangle} \cdot c_0 \left( d_{01\langle j \rangle} \cdot c_1 \left( d_{1\langle i \rangle 2\langle ij \rangle} \cdot d_{1\langle j \rangle 2\langle j \rangle} \cdot \delta_{01} \right) \right) \cdot \delta_{01} \right) \quad (2.10)
\end{aligned}$$

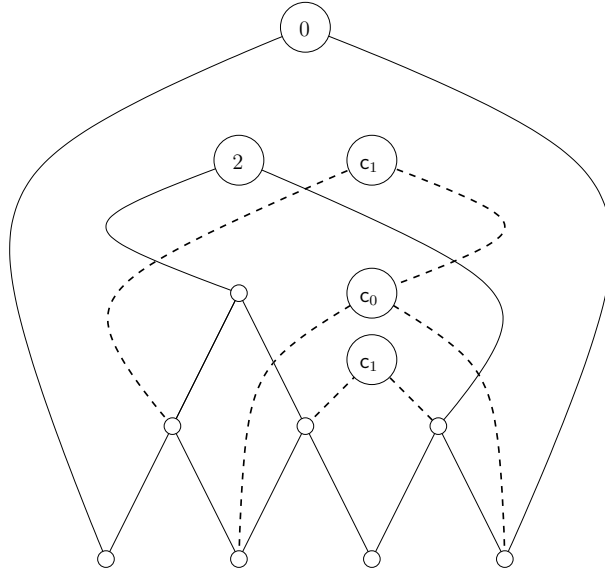


Figure 2.4 An illustration of (2.10) with  $i = 0$ . When we employ this lemma,  $x, y, z$  will be represented by the second row from the bottom. The corresponding figure for  $i = 1$  would be a mirrored reflection of this one.

$$\begin{aligned}
& \mathbf{d}_{0\langle i \rangle 2\langle iii \rangle} \cdot \mathbf{d}_{0\langle j \rangle 2\langle jj \rangle} \cdot \mathbf{d}_{2\langle ii j \rangle 2\langle i j i \rangle} \cdot \mathbf{d}_{2\langle i j j \rangle 2\langle j i \rangle} \\
& \leq \mathbf{d}_{0\langle i \rangle 2\langle iii \rangle} \cdot \mathbf{d}_{0\langle j \rangle 2\langle jj \rangle} \cdot \mathbf{d}_{2\langle ii j \rangle 2\langle i j i \rangle} \cdot \mathbf{d}_{2\langle i j j \rangle 2\langle j i \rangle} \\
& \quad \mathbf{c}_0 \left( \mathbf{d}_{0\langle i \rangle 2\langle i j i \rangle} \cdot \mathbf{d}_{0\langle j \rangle 2\langle j j \rangle} \cdot \mathbf{c}_1 \left( \mathbf{d}_{1\langle i \rangle 2\langle ii \rangle} \cdot \mathbf{d}_{1\langle j \rangle 0} \right) \cdot \mathbf{c}_1 \left( \mathbf{d}_{1\langle i \rangle 2\langle i j \rangle} \cdot \mathbf{d}_{1\langle j \rangle 2\langle j \rangle} \right) \right) \quad \text{Ax}_2 \\
& \leq \mathbf{d}_{0\langle i \rangle 2\langle iii \rangle} \cdot \mathbf{d}_{0\langle j \rangle 2\langle jj \rangle} \cdot \mathbf{d}_{2\langle ii j \rangle 2\langle i j i \rangle} \cdot \mathbf{d}_{2\langle i j j \rangle 2\langle j i \rangle} \cdot \\
& \quad \mathbf{c}_0 \left( \mathbf{d}_{0\langle i \rangle 2\langle i j i \rangle} \cdot \mathbf{d}_{0\langle j \rangle 2\langle j j \rangle} \cdot \mathbf{c}_1 \left( \mathbf{d}_{1\langle i \rangle 2\langle ii \rangle} \cdot \mathbf{d}_{1\langle j \rangle 0} \right) \cdot \mathbf{c}_1 \left( \mathbf{d}_{1\langle i \rangle 2\langle i j \rangle} \cdot \mathbf{d}_{1\langle j \rangle 2\langle j \rangle} \right) \right) \quad \text{C}_3 \\
& \leq \mathbf{d}_{0\langle i \rangle 2\langle iii \rangle} \cdot \mathbf{d}_{0\langle j \rangle 2\langle jj \rangle} \cdot \mathbf{d}_{2\langle ii j \rangle 2\langle i j i \rangle} \cdot \mathbf{d}_{2\langle i j j \rangle 2\langle j i \rangle} \cdot \\
& \quad \mathbf{c}_1 \mathbf{c}_0 \left( \mathbf{d}_{0\langle i \rangle 2\langle i j i \rangle} \cdot \mathbf{d}_{0\langle j \rangle 2\langle j j \rangle} \cdot \mathbf{d}_{1\langle i \rangle 2\langle ii \rangle} \cdot \mathbf{d}_{1\langle j \rangle 0} \cdot \mathbf{c}_1 \left( \mathbf{d}_{1\langle i \rangle 2\langle i j \rangle} \cdot \mathbf{d}_{1\langle j \rangle 2\langle j \rangle} \right) \right) \quad \text{C}_3, \text{C}_4 \\
& \leq \mathbf{c}_1 \left( \mathbf{d}_{0\langle i \rangle 2\langle iii \rangle} \cdot \mathbf{d}_{0\langle j \rangle 2\langle jj \rangle} \cdot \mathbf{d}_{2\langle ii j \rangle 2\langle i j i \rangle} \cdot \mathbf{d}_{2\langle i j j \rangle 2\langle j i \rangle} \cdot \right. \\
& \quad \left. \mathbf{c}_0 \left( \mathbf{d}_{0\langle i \rangle 2\langle i j i \rangle} \cdot \mathbf{d}_{0\langle j \rangle 2\langle j j \rangle} \cdot \mathbf{d}_{1\langle i \rangle 2\langle ii \rangle} \cdot \mathbf{d}_{1\langle j \rangle 0} \cdot \mathbf{c}_1 \left( \mathbf{d}_{1\langle i \rangle 2\langle i j \rangle} \cdot \mathbf{d}_{1\langle j \rangle 2\langle j \rangle} \right) \right) \right) \quad \Delta \\
& \leq \mathbf{c}_1 \left( \mathbf{d}_{0\langle i \rangle 2\langle iii \rangle} \cdot \mathbf{d}_{0\langle j \rangle 2\langle jj \rangle} \cdot \mathbf{d}_{2\langle ii j \rangle 2\langle i j i \rangle} \cdot \mathbf{d}_{1\langle i \rangle 2\langle ii \rangle} \cdot \right. \\
& \quad \left. \mathbf{c}_0 \left( \mathbf{d}_{0\langle i \rangle 2\langle i j i \rangle} \cdot \mathbf{d}_{0\langle j \rangle 2\langle j j \rangle} \cdot \mathbf{d}_{1\langle j \rangle 0} \cdot \mathbf{c}_1 \left( \mathbf{d}_{2\langle i j j \rangle 2\langle j i \rangle} \cdot \mathbf{d}_{1\langle i \rangle 2\langle i j \rangle} \cdot \mathbf{d}_{1\langle j \rangle 2\langle j \rangle} \right) \right) \right) \quad \text{C}_3
\end{aligned}$$

$$\begin{aligned}
&\leq c_1 \left( d_{0\langle i \rangle 2\langle iii \rangle} \cdot d_{0\langle j \rangle 2\langle jj \rangle} \cdot d_{2\langle iij \rangle 2\langle iji \rangle} \cdot d_{1\langle i \rangle 2\langle ii \rangle} \cdot \right. \\
&\quad c_0 \left( d_{1\langle j \rangle 0} \cdot d_{1\langle ji \rangle 2\langle iji \rangle} \cdot d_{1\langle jj \rangle 2\langle jj \rangle} \cdot \right. \\
&\quad \left. \left. c_1 \left( d_{0\langle i \rangle 2\langle iji \rangle} \cdot d_{0\langle j \rangle 2\langle jj \rangle} \cdot d_{2\langle ijj \rangle 2\langle jii \rangle} \cdot d_{1\langle i \rangle 2\langle ij \rangle} \cdot d_{1\langle j \rangle 2\langle j \rangle} \right) \right) \right) \quad \text{Ax}_1 \\
&\leq c_1 \left( d_{0\langle i \rangle 2\langle iii \rangle} \cdot d_{0\langle j \rangle 2\langle jj \rangle} \cdot d_{2\langle iij \rangle 2\langle iji \rangle} \cdot d_{1\langle i \rangle 2\langle ii \rangle} \cdot d_{1\langle ji \rangle 2\langle iji \rangle} \cdot d_{1\langle jj \rangle 2\langle jj \rangle} \cdot \right. \\
&\quad \left. c_0 \left( d_{1\langle j \rangle 0} \cdot c_1 \left( d_{0\langle i \rangle 2\langle iji \rangle} \cdot d_{0\langle j \rangle 2\langle jj \rangle} \cdot d_{2\langle ijj \rangle 2\langle jii \rangle} \cdot d_{1\langle i \rangle 2\langle ij \rangle} \cdot d_{1\langle j \rangle 2\langle j \rangle} \right) \right) \right) \quad \text{C}_3 \\
&\leq c_1 \left( d_{1\langle i \rangle 2\langle ii \rangle} \cdot c_0 \left( d_{01\langle j \rangle} \cdot c_1 \left( d_{1\langle i \rangle 2\langle ij \rangle} \cdot d_{1\langle j \rangle 2\langle j \rangle} \cdot \delta_{01} \right) \right) \cdot \delta_{01} \right) \quad \text{Ax}_1
\end{aligned}$$

We are now prepared to prove that  $\circ$  is associative in  $\mathfrak{J}\mathfrak{A}$ . Let  $x, y, z \in J$  be arbitrary.

$$\begin{aligned}
&c_1 \left( s_{1\langle i \rangle}^0 c_1 \left( s_{1\langle i \rangle}^0 x \cdot s_{1\langle j \rangle}^0 y \cdot \delta_{01} \right) \cdot s_{1\langle j \rangle}^0 z \cdot \delta_{01} \right) \\
&\leq c_1 \left( c_2 \left( d_{1\langle j \rangle 2\langle j \rangle} \cdot s_{2\langle ii \rangle}^0 x \cdot s_{2\langle ij \rangle}^0 y \cdot \delta_{1\langle i \rangle 2\langle i \rangle} \right) \cdot s_{1\langle j \rangle}^0 z \cdot \delta_{01} \right) \quad (2.9) \\
&\leq c_1 c_2 \left( s_{2\langle ii \rangle}^0 x \cdot s_{2\langle ij \rangle}^0 y \cdot s_{1\langle j \rangle}^0 z \cdot d_{1\langle j \rangle 2\langle j \rangle} \cdot \delta_{1\langle i \rangle 2\langle i \rangle} \cdot \delta_{01} \right) \quad \text{C}_3 \\
&\leq c_1 c_2 \left( s_{2\langle ii \rangle}^0 x \cdot s_{2\langle ij \rangle}^0 y \cdot s_{2\langle j \rangle}^0 z \cdot d_{1\langle j \rangle 2\langle j \rangle} \cdot \delta_{1\langle i \rangle 2\langle i \rangle} \cdot \delta_{01} \right) \quad 47 \\
&\leq c_1 c_2 \left( s_{2\langle ii \rangle}^0 x \cdot s_{2\langle ij \rangle}^0 y \cdot s_{2\langle j \rangle}^0 z \cdot d_{1\langle j \rangle 2\langle j \rangle} \cdot d_{1\langle ii \rangle 2\langle iii \rangle} \cdot d_{1\langle ij \rangle 2\langle ijj \rangle} \cdot d_{2\langle iij \rangle 2\langle iji \rangle} \cdot \right. \\
&\quad \left. d_{0\langle i \rangle 1\langle ii \rangle} \cdot d_{0\langle j \rangle 1\langle jj \rangle} \cdot d_{1\langle ij \rangle 1\langle jii \rangle} \right) \quad \text{Ax}_1, \text{D32} \\
&\leq c_2 \left( s_{2\langle ii \rangle}^0 x \cdot s_{2\langle ij \rangle}^0 y \cdot s_{2\langle j \rangle}^0 z \cdot d_{0\langle i \rangle 2\langle iii \rangle} \cdot d_{0\langle j \rangle 2\langle jj \rangle} \cdot d_{2\langle iij \rangle 2\langle iji \rangle} \cdot d_{2\langle ijj \rangle 2\langle jii \rangle} \right) \quad \text{Ax}_1, \Delta \\
&\leq c_2 \left( s_{2\langle ii \rangle}^0 x \cdot s_{2\langle ij \rangle}^0 y \cdot s_{2\langle j \rangle}^0 z \cdot \right. \\
&\quad \left. c_1 \left( d_{1\langle i \rangle 2\langle ii \rangle} \cdot c_0 \left( d_{01\langle j \rangle} \cdot c_1 \left( d_{1\langle i \rangle 2\langle ij \rangle} \cdot d_{1\langle j \rangle 2\langle j \rangle} \cdot \delta_{01} \right) \right) \cdot \delta_{01} \right) \right) \quad (2.10) \\
&\leq c_2 c_1 \left( s_{2\langle ii \rangle}^0 x \cdot d_{1\langle i \rangle 2\langle ii \rangle} \cdot \right. \\
&\quad \left. c_0 \left( d_{01\langle j \rangle} \cdot c_1 \left( s_{2\langle ij \rangle}^0 y \cdot d_{1\langle i \rangle 2\langle ij \rangle} \cdot s_{2\langle j \rangle}^0 z \cdot d_{1\langle j \rangle 2\langle j \rangle} \cdot \delta_{01} \right) \right) \cdot \delta_{01} \right) \quad \text{C}_3 \\
&\leq c_2 c_1 \left( s_{1\langle i \rangle}^0 x \cdot c_0 \left( d_{01\langle j \rangle} \cdot c_1 \left( s_{1\langle i \rangle}^0 y \cdot s_{1\langle j \rangle}^0 z \cdot \delta_{01} \right) \right) \cdot \delta_{01} \right) \quad 47 \\
&\leq c_1 \left( s_{1\langle i \rangle}^0 x \cdot s_{1\langle j \rangle}^0 c_1 \left( s_{1\langle i \rangle}^0 y \cdot s_{1\langle j \rangle}^0 z \cdot \delta_{01} \right) \cdot \delta_{01} \right) \quad \text{D32}, \Delta
\end{aligned}$$

Now, if  $i = 0$  and  $j = 1$  this inequality would read  $(x \circ y) \circ z \leq x \circ (y \circ z)$ . On the other hand, if  $i = 1$  and  $j = 0$  it would read  $z \circ (y \circ x) \leq (z \circ y) \circ x$ . As  $x, y, z$  are arbitrary, this is sufficient to establish that  $(x \circ y) \circ z = x \circ (y \circ z)$ .  $\blacklozenge$

**PROOF (CONDITION III)** We begin with two technical lemmas in order to exploit some symmetry in the argument. The following are assumed as  $(\text{Ax}_9)$  and  $(\text{Ax}_{10})$  respectively in [Ném85, p. 50]. Let  $\{i, j\} = \{0, 1\}$ .

$$s_{1\langle i \rangle}^0 \dot{1} \cdot \delta_{01} \leq d_{01\langle j \rangle} \quad (2.11)$$

$$\begin{aligned}
s_{1\langle i \rangle}^0 \dot{1}' \cdot \delta_{01} &\leq c_0 (d_{01\langle i \rangle} \cdot d_{0\langle i \rangle 0\langle j \rangle}) \cdot d_{0\langle i \rangle 1\langle ii \rangle} \cdot d_{0\langle j \rangle 1\langle jj \rangle} \cdot d_{1\langle ij \rangle 1\langle ji \rangle} && \text{Def} \\
&\leq d_{1\langle ii \rangle 1\langle ij \rangle} \cdot d_{0\langle i \rangle 1\langle ii \rangle} \cdot d_{0\langle j \rangle 1\langle jj \rangle} \cdot d_{1\langle ij \rangle 1\langle ji \rangle} && \text{Ax}_1, \Delta \\
&\leq d_{0\langle i \rangle 1\langle ij \rangle} \cdot d_{0\langle j \rangle 1\langle jj \rangle} \cdot d_{1\langle ij \rangle 1\langle ji \rangle} && \text{Ax}_1 \\
&\leq d_{0\langle i \rangle 1\langle ji \rangle} \cdot d_{0\langle j \rangle 1\langle jj \rangle} && \text{Ax}_1 \\
&\leq d_{01\langle j \rangle} && \text{Ax}_3
\end{aligned}$$

Next we provide a useful upper bound for  $\dot{1}$ . Since  $x \leq \dot{1}$  for all  $x \in J$ , we will have this as an upper bound for all elements of our universe as well. Let  $\{i, j\} = \{0, 1\}$ .

$$\dot{1} \leq c_1 (d_{01\langle j \rangle} \cdot \dot{1}'_{1\langle i \rangle} \cdot \delta_{01}) \quad (2.12)$$

$$\begin{aligned}
\dot{1} &= d_{0\langle i \rangle 0\langle i \rangle} \cdot d_{0\langle j \rangle 0\langle j \rangle} \\
&\leq c_2 (d_{2\langle i \rangle 0\langle i \rangle} \cdot d_{2\langle j \rangle 0\langle i \rangle}) && \text{Ax}_2 \\
&\leq c_2 c_1 (d_{1\langle i \rangle 2} \cdot d_{1\langle j \rangle 0} \cdot d_{2\langle i \rangle 0\langle i \rangle} \cdot d_{2\langle j \rangle 0\langle i \rangle}) && \text{Ax}_2, C_3 \\
&\leq c_2 c_1 (d_{01\langle j \rangle} \cdot d_{1\langle i \rangle 2} \cdot d_{1\langle ji \rangle 0\langle i \rangle} \cdot d_{1\langle jj \rangle 0\langle j \rangle} \cdot d_{2\langle i \rangle 0\langle i \rangle} \cdot d_{2\langle j \rangle 0\langle i \rangle}) && 44, \text{Ax}_1 \\
&\leq c_1 (d_{01\langle j \rangle} \cdot d_{1\langle ji \rangle 0\langle i \rangle} \cdot d_{1\langle jj \rangle 0\langle j \rangle} \cdot d_{1\langle ii \rangle 0\langle i \rangle} \cdot d_{1\langle ij \rangle 0\langle i \rangle}) && \text{Ax}_1, \Delta \\
&\leq c_1 (d_{01\langle j \rangle} \cdot d_{1\langle ii \rangle 1\langle ij \rangle} \cdot \delta_{01}) && \text{Ax}_1, D32 \\
&\leq c_1 (d_{01\langle j \rangle} \cdot c_0 (d_{01\langle i \rangle} \cdot d_{1\langle ii \rangle 1\langle ij \rangle}) \cdot \delta_{01}) && 44, \Delta, 42 \\
&\leq c_1 (d_{01\langle j \rangle} \cdot c_0 (d_{01\langle i \rangle} \cdot d_{0\langle i \rangle 0\langle j \rangle}) \cdot \delta_{01}) && \text{Ax}_1 \\
&\leq c_1 (d_{01\langle j \rangle} \cdot s_{1\langle i \rangle}^0 \dot{1}' \cdot \delta_{01}) && D32
\end{aligned}$$

The proof that  $\dot{1}'$  is an identity for  $\circ$  is now very short. Once again, let  $x \in J$  and  $\{i, j\} = \{0, 1\}$ .

$$\begin{aligned}
x &\leq x \cdot c_1 (d_{01\langle j \rangle} \cdot s_{1\langle i \rangle}^0 \dot{1}' \cdot \delta_{01}) && (2.12) \\
&\leq c_1 (d_{01\langle j \rangle} \cdot x \cdot s_{1\langle i \rangle}^0 \dot{1}' \cdot \delta_{01}) && C_3 \\
&\leq c_1 (c_0 (d_{01\langle j \rangle} \cdot x) \cdot s_{1\langle i \rangle}^0 \dot{1}' \cdot \delta_{01}) && C_2 \\
&\leq c_1 (s_{1\langle j \rangle}^0 x \cdot s_{1\langle i \rangle}^0 \dot{1}' \cdot \delta_{01}) && D32 \\
&\leq c_1 (s_{1\langle j \rangle}^0 x \cdot d_{01\langle j \rangle}) && (2.11) \\
&\leq c_1 x = x && 49, \Delta
\end{aligned}$$

The above simultaneously establishes that  $x = \dot{1}' \circ x$  (when  $i = 0$  and  $j = 1$ ) and that  $x = x \circ \dot{1}'$  (when  $i = 1$  and  $j = 0$ ). ◆

PROOF (CONDITION IV) We begin with a lemma from [Ném85], stated and proved therein as (10) on page 51. Roughly, we generalize the definition of  $\check{x}$  to elements of the form  $s_{\lambda\langle i \rangle}^0 x$ . Let  $\kappa, \lambda = \{1, 2\}$  and  $i, j \in S_2$ .

$$s_{\kappa\langle i \rangle}^0 x \cdot \epsilon_{\kappa\langle i \rangle\lambda\langle j \rangle} \leq s_{\lambda\langle j \rangle}^0 \check{x} \quad (2.13)$$

$$\begin{aligned} s_{\kappa\langle i \rangle}^0 x \cdot \epsilon_{\kappa\langle i \rangle\lambda\langle j \rangle} &= c_0 (d_{0\lambda\langle j \rangle} \cdot x) \cdot \epsilon_{\kappa\langle i \rangle\lambda\langle j \rangle} && \text{D32} \\ &\leq c_0 (d_{0\lambda\langle j \rangle} \cdot x \cdot \epsilon_{\kappa\langle i \rangle\lambda\langle j \rangle}) && \text{C}_3 \\ &\leq s_{\lambda\langle j \rangle}^0 (x \cdot \epsilon_{\kappa\langle i \rangle 0}) && \text{D32, 46} \\ &\leq s_{\lambda\langle j \rangle}^0 c_\lambda (d_{\lambda\kappa\langle i \rangle} \cdot x \cdot \epsilon_{\kappa\langle i \rangle 0}) && 42 \\ &\leq s_{\lambda\langle j \rangle}^0 c_\lambda (x \cdot \epsilon_{\lambda 0}) && 46 \\ &= s_{\lambda\langle j \rangle}^0 c_1 (x \cdot \epsilon_{10}) && 48 \\ &= s_{\lambda\langle j \rangle}^0 \check{x} && \text{D32} \end{aligned}$$

We also prove a lemma (not found in [Ném85] or [Ném86]) which allows us to recover the term  $\delta_{01}$  at one point in our proof.

$$d_{02\langle 1j \rangle} \cdot d_{1\langle j \rangle 2\langle 0 \rangle} \cdot \epsilon_{1\langle i \rangle 2\langle 1i \rangle} \cdot \delta_{2\langle 0 \rangle 2\langle 1 \rangle} \leq \delta_{01} \quad (2.14)$$

$$\begin{aligned} &d_{02\langle 1j \rangle} \cdot d_{1\langle j \rangle 2\langle 0 \rangle} \cdot \epsilon_{1\langle i \rangle 2\langle 1i \rangle} \cdot \delta_{2\langle 0 \rangle 2\langle 1 \rangle} \\ &= d_{02\langle 1j \rangle} \cdot d_{1\langle j \rangle 2\langle 0 \rangle} \cdot (d_{1\langle ii \rangle 2\langle 1ij \rangle} \cdot d_{1\langle ij \rangle 2\langle 1ii \rangle}) \cdot (d_{2\langle 0i \rangle 2\langle 1ii \rangle} \cdot d_{2\langle 0j \rangle 2\langle 1jj \rangle} \cdot d_{2\langle 1ij \rangle 2\langle 1ji \rangle}) && \text{D32} \\ &\leq (d_{0\langle i \rangle 2\langle 1ji \rangle} \cdot d_{0\langle j \rangle 2\langle 1jj \rangle}) \cdot (d_{1\langle j \rangle 2\langle 0i \rangle} \cdot d_{1\langle j \rangle 2\langle 0j \rangle}) \cdot \\ &\quad (d_{1\langle ii \rangle 2\langle 1ij \rangle} \cdot d_{1\langle ij \rangle 2\langle 1ii \rangle}) \cdot (d_{2\langle 0i \rangle 2\langle 1ii \rangle} \cdot d_{2\langle 0j \rangle 2\langle 1jj \rangle} \cdot d_{2\langle 1ij \rangle 2\langle 1ji \rangle}) && 44, \text{Ax}_1 \\ &= (d_{0\langle i \rangle 2\langle 1ji \rangle} \cdot d_{2\langle 1ij \rangle 2\langle 1ji \rangle} \cdot d_{1\langle ii \rangle 2\langle 1ij \rangle}) \cdot \\ &\quad (d_{0\langle j \rangle 2\langle 1jj \rangle} \cdot d_{2\langle 0j \rangle 2\langle 1jj \rangle} \cdot d_{1\langle j \rangle 2\langle 0j \rangle}) \cdot (d_{1\langle ij \rangle 2\langle 1ii \rangle} \cdot d_{2\langle 0i \rangle 2\langle 1ii \rangle} \cdot d_{1\langle ji \rangle 2\langle 0i \rangle}) && \text{BA} \\ &\leq d_{0\langle i \rangle 1\langle ii \rangle} \cdot d_{0\langle j \rangle 1\langle jj \rangle} \cdot d_{1\langle ij \rangle 1\langle ji \rangle} && \text{Ax}_1 \\ &= \delta_{01} && \text{D32} \end{aligned}$$

By 1.2.1, it is sufficient to show that  $c_0((x \circ y) \cdot z) = c_0((z \circ \check{y}) \cdot x) = c_0((\check{x} \circ z) \cdot y)$  (since  $c_0 t = 0$  iff  $t = 0$ ). To that end, we establish the following inequalities.

$$c_0((x \circ y) \cdot z) \leq c_0((\check{x} \circ z) \cdot y) \quad (2.15)$$

$$c_0((x \circ y) \cdot z) \leq c_0((z \circ \check{y}) \cdot x) \quad (2.16)$$

$$\check{\check{x}} \leq x \quad (2.17)$$

The proofs of (2.15) and (2.16) are quite similar in character. Let  $\{i, j\} = \{0, 1\}$ .

$$\begin{aligned}
& \mathbf{c}_0((x \circ y) \cdot z) \\
&= \mathbf{c}_0\left(\mathbf{c}_1\left(\mathbf{s}_{1\langle 0 \rangle}^0 x \cdot \mathbf{s}_{1\langle 1 \rangle}^0 y \cdot \delta_{01}\right) \cdot z\right) && \text{D32} \\
&\leq \mathbf{c}_0 \mathbf{c}_1\left(\mathbf{s}_{1\langle 0 \rangle}^0 x \cdot \mathbf{s}_{1\langle 1 \rangle}^0 y \cdot z \cdot \delta_{01}\right) && \text{C}_3 \\
&\leq \mathbf{c}_0 \mathbf{c}_1 \mathbf{c}_2\left(\mathbf{s}_{1\langle 0 \rangle}^0 x \cdot \mathbf{s}_{1\langle 1 \rangle}^0 y \cdot z \cdot \mathbf{d}_{2\langle 0 \rangle 0} \cdot \mathbf{d}_{2\langle 1 \rangle 1} \cdot \delta_{01}\right) && \text{Ax}_2, \text{C}_3 \\
&\leq \mathbf{c}_2\left(\mathbf{s}_{2\langle 10 \rangle}^0 x \cdot \mathbf{s}_{2\langle 11 \rangle}^0 y \cdot \mathbf{s}_{2\langle 0 \rangle}^0 z \cdot \delta_{2\langle 0 \rangle 2\langle 1 \rangle}\right) && \text{47, C}_2, \text{46, } \Delta \\
&\leq \mathbf{c}_2 \mathbf{c}_0\left(\mathbf{s}_{2\langle 10 \rangle}^0 x \cdot \mathbf{s}_{2\langle 11 \rangle}^0 y \cdot \mathbf{s}_{2\langle 0 \rangle}^0 z \cdot \epsilon_{02\langle 1i \rangle} \cdot \delta_{2\langle 0 \rangle 2\langle 1 \rangle}\right) && \text{Ax}_2, \text{C}_3 \\
&\leq \mathbf{c}_2 \mathbf{c}_0 \mathbf{c}_1\left(\mathbf{s}_{2\langle 10 \rangle}^0 x \cdot \mathbf{s}_{2\langle 11 \rangle}^0 y \cdot \mathbf{s}_{2\langle 0 \rangle}^0 z \cdot \mathbf{d}_{1\langle j \rangle 2\langle 0 \rangle} \cdot \mathbf{d}_{1\langle i \rangle 0} \cdot \epsilon_{02\langle 1i \rangle} \cdot \delta_{2\langle 0 \rangle 2\langle 1 \rangle}\right) && \text{Ax}_2, \text{C}_3 \\
&\leq \mathbf{c}_2 \mathbf{c}_1\left(\mathbf{s}_{2\langle 10 \rangle}^0 x \cdot \mathbf{s}_{2\langle 11 \rangle}^0 y \cdot \mathbf{s}_{2\langle 0 \rangle}^0 z \cdot \mathbf{d}_{1\langle j \rangle 2\langle 0 \rangle} \cdot \epsilon_{1\langle i \rangle 2\langle 1i \rangle} \cdot \delta_{2\langle 0 \rangle 2\langle 1 \rangle}\right) && \text{46, } \Delta \\
&\leq \mathbf{c}_2 \mathbf{c}_1 \mathbf{c}_0\left(\mathbf{s}_{2\langle 10 \rangle}^0 x \cdot \mathbf{s}_{2\langle 11 \rangle}^0 y \cdot \mathbf{s}_{2\langle 0 \rangle}^0 z \cdot \mathbf{d}_{02\langle 1j \rangle} \cdot \mathbf{d}_{1\langle j \rangle 2\langle 0 \rangle} \cdot \epsilon_{1\langle i \rangle 2\langle 1i \rangle} \cdot \delta_{2\langle 0 \rangle 2\langle 1 \rangle}\right) && \text{42} \\
&\leq \mathbf{c}_2 \mathbf{c}_1 \mathbf{c}_0\left(\mathbf{s}_{2\langle 10 \rangle}^0 x \cdot \mathbf{s}_{2\langle 11 \rangle}^0 y \cdot \mathbf{s}_{1\langle j \rangle}^0 z \cdot \mathbf{d}_{02\langle 1j \rangle} \cdot \epsilon_{1\langle i \rangle 2\langle 1i \rangle} \cdot \delta_{01}\right) && \text{47, (2.14)}
\end{aligned}$$

Now if  $i = 0$  and  $j = 1$  we have:

$$\begin{aligned}
\mathbf{c}_0((x \circ y) \cdot z) &\leq \mathbf{c}_2 \mathbf{c}_1 \mathbf{c}_0\left(\mathbf{s}_{2\langle 10 \rangle}^0 x \cdot \mathbf{s}_{2\langle 11 \rangle}^0 y \cdot \mathbf{s}_{1\langle 1 \rangle}^0 z \cdot \mathbf{d}_{02\langle 11 \rangle} \cdot \epsilon_{1\langle 0 \rangle 2\langle 10 \rangle} \cdot \delta_{01}\right) \\
&\leq \mathbf{c}_2 \mathbf{c}_1 \mathbf{c}_0\left(\mathbf{s}_{2\langle 10 \rangle}^0 x \cdot y \cdot \mathbf{s}_{1\langle 1 \rangle}^0 z \cdot \epsilon_{1\langle 0 \rangle 2\langle 10 \rangle} \cdot \delta_{01}\right) && \text{49} \\
&\leq \mathbf{c}_1 \mathbf{c}_0\left(\mathbf{s}_{1\langle 0 \rangle}^0 \check{x} \cdot y \cdot \mathbf{s}_{1\langle 1 \rangle}^0 z \cdot \delta_{01}\right) && \text{(2.13), } \Delta \\
&\leq \mathbf{c}_0\left(\mathbf{c}_1\left(\mathbf{s}_{1\langle 0 \rangle}^0 \check{x} \cdot \mathbf{s}_{1\langle 1 \rangle}^0 z \cdot \delta_{01}\right) \cdot y\right) && \text{C}_4, \text{C}_3 \\
&\leq \mathbf{c}_0((\check{x} \circ z) \cdot y) && \text{Def}
\end{aligned}$$

which concludes the proof of Equation (2.15). If instead  $i = 1$  and  $j = 0$  then we see:

$$\begin{aligned}
\mathbf{c}_0((x \circ y) \cdot z) &\leq \mathbf{c}_2 \mathbf{c}_1 \mathbf{c}_0\left(\mathbf{s}_{2\langle 10 \rangle}^0 x \cdot \mathbf{s}_{2\langle 11 \rangle}^0 y \cdot \mathbf{s}_{1\langle 0 \rangle}^0 z \cdot \mathbf{d}_{02\langle 10 \rangle} \cdot \epsilon_{1\langle 1 \rangle 2\langle 11 \rangle} \cdot \delta_{01}\right) \\
&\leq \mathbf{c}_2 \mathbf{c}_1 \mathbf{c}_0\left(x \cdot \mathbf{s}_{2\langle 11 \rangle}^0 y \cdot \mathbf{s}_{1\langle 0 \rangle}^0 z \cdot \epsilon_{1\langle 1 \rangle 2\langle 11 \rangle} \cdot \delta_{01}\right) && \text{49} \\
&\leq \mathbf{c}_1 \mathbf{c}_0\left(x \cdot \mathbf{s}_{1\langle 1 \rangle}^0 \check{y} \cdot \mathbf{s}_{1\langle 0 \rangle}^0 z \cdot \delta_{01}\right) && \text{(2.13), } \Delta \\
&\leq \mathbf{c}_0\left(\mathbf{c}_1\left(\mathbf{s}_{1\langle 0 \rangle}^0 z \cdot \mathbf{s}_{1\langle 1 \rangle}^0 \check{y} \cdot \delta_{01}\right) \cdot x\right) && \text{C}_4, \text{C}_3 \\
&\leq \mathbf{c}_0((z \circ \check{y}) \cdot x) && \text{D32}
\end{aligned}$$

concluding the proof of Equation (2.16). Next we turn our attention to Equation (2.17). From  $\text{Ax}_1$  and  $\text{Ax}_3$  we see that:

$$\epsilon_{01} \cdot \epsilon_{20} = \mathbf{d}_{0\langle 0 \rangle 1\langle 1 \rangle} \cdot \mathbf{d}_{0\langle 1 \rangle 1\langle 0 \rangle} \cdot \mathbf{d}_{2\langle 0 \rangle 0\langle 1 \rangle} \cdot \mathbf{d}_{2\langle 1 \rangle 0\langle 0 \rangle} \leq \mathbf{d}_{1\langle 0 \rangle 2\langle 0 \rangle} \mathbf{d}_{1\langle 1 \rangle 2\langle 1 \rangle} \leq \mathbf{d}_{12}. \quad (2.18)$$

Now let  $x \in J$ .

$$\begin{aligned}
\check{x} &\leq c_1 (s_1^0 c_1 (s_1^0 x \cdot \epsilon_{01}) \cdot \epsilon_{01}) && \text{D32} \\
&\leq c_2 (d_{02} \cdot c_1 (s_1^0 c_1 (s_1^0 x \cdot \epsilon_{01}) \cdot \epsilon_{01})) && 1.5.8(i) \\
&\leq c_2 c_1 (d_{02} \cdot s_1^0 c_1 (s_1^0 x \cdot \epsilon_{01}) \cdot \epsilon_{01}) && C_3 \\
&\leq c_2 c_1 (d_{02} \cdot c_0 (d_{01} \cdot c_1 (s_1^0 x \cdot \epsilon_{01})) \cdot \epsilon_{21}) && 46, \text{D32} \\
&\leq c_2 c_1 (d_{02} \cdot c_0 (d_{01} \cdot c_1 (s_1^0 x \cdot \epsilon_{01}) \cdot \epsilon_{21})) && C_3 \\
&\leq c_2 c_1 (d_{02} \cdot c_0 (d_{01} \cdot c_1 (s_1^0 x \cdot \epsilon_{01}) \cdot \epsilon_{20})) && 46 \\
&\leq c_2 c_1 (d_{02} \cdot c_0 (d_{01} \cdot c_1 (s_1^0 x \cdot \epsilon_{01} \cdot \epsilon_{20}))) && C_3 \\
&\leq c_2 c_1 (d_{02} \cdot c_0 (d_{01} \cdot c_1 (d_{12} \cdot s_1^0 x))) && (2.18) \\
&\leq c_1 s_0^2 s_1^0 s_2^1 s_1^0 x && C_4, \text{Def} \\
&\leq c_1 s_0^2 s_1^0 s_2^0 x && 1.5.11(i) \\
&\leq c_1 s_0^2 s_2^0 x && 1.5.8(i) \\
&= c_1 x = x && \text{D32}, \Delta
\end{aligned}$$

With the aid of these inequalities, we see that:

$$\begin{aligned}
c_0 ((x \circ y) \cdot z) &\leq c_0 ((\check{x} \circ z) \cdot y) && (2.15) \\
&\leq c_0 \left( \left( (\check{x} \circ y) \right) \cdot z \right) && (2.15) \\
&\leq c_0 ((x \circ y) \cdot z) && (2.17) \\
&\leq c_0 ((z \circ \check{y}) \cdot x) && (2.16) \\
&\leq c_0 \left( \left( (x \circ \check{y}) \right) \cdot z \right) && (2.16) \\
&\leq c_0 ((x \circ y) \cdot z) && (2.17)
\end{aligned}$$

concluding the proof of condition IV. ♦

Conditions I through IV have been proved, thus  $\mathfrak{J}\mathfrak{A} \in \text{RA}$  as desired. We begin our proof of Equations (1.1)–(1.3) by examining the elements  $\check{P}_0$  and  $\check{P}_1$ . Note that by definition we may write  $P_k = d_{0\langle 0k \rangle 0\langle 1 \rangle}$ .

$$\begin{aligned}
\check{P}_k &= c_1 (c_0 (d_{01} \cdot d_{0\langle 0k \rangle 0\langle 1 \rangle}) \cdot \epsilon_{01}) && \text{D32} \\
&\leq c_1 (d_{1\langle 0k \rangle 1\langle 1 \rangle} \cdot d_{0\langle 0 \rangle 1\langle 1 \rangle} \cdot d_{0\langle 1 \rangle 1\langle 0 \rangle}) && \text{Ax}_1, \text{D32} \\
&\leq d_{0\langle 1k \rangle 0\langle 0 \rangle} && \text{Ax}_1, \Delta \\
&\leq c_1 (d_{1\langle 0 \rangle 0\langle 1 \rangle} \cdot d_{1\langle 1 \rangle 0\langle 0 \rangle} \cdot d_{0\langle 1k \rangle 0\langle 0 \rangle}) && \text{Ax}_2
\end{aligned}$$



$$\begin{aligned}
&\leq c_1 (\epsilon_{01} \cdot \mathbf{d}_{1\langle 0k \rangle 1\langle 1 \rangle}) && \text{Ax}_1, \text{D32} \\
&\leq c_1 (c_0 (\mathbf{d}_{01} \cdot \mathbf{d}_{1\langle 0k \rangle 1\langle 1 \rangle}) \cdot \epsilon_{01}) && 1.5.8(i) \\
&\leq c_1 (\mathbf{s}_1^0 \mathbf{d}_{0\langle 0k \rangle 0\langle 1 \rangle} \cdot \epsilon_{01}) = \check{P}_k && \text{D32, Ax}_1, \text{D32}
\end{aligned}$$

Thus  $\check{P}_k = \mathbf{d}_{0\langle 1k \rangle 0\langle 0 \rangle}$ .

PROOF (EQUATION (1.1))

$$\begin{aligned}
\check{P}_k \circ P_k &= c_1 (\mathbf{s}_{1\langle 0 \rangle}^0 \mathbf{d}_{0\langle 1k \rangle 0\langle 0 \rangle} \cdot \mathbf{s}_{1\langle 1 \rangle}^0 \mathbf{d}_{0\langle 0k \rangle 0\langle 1 \rangle} \cdot \delta_{01}) && \text{D32} \\
&\leq c_1 (\mathbf{d}_{1\langle 01k \rangle 1\langle 00 \rangle} \cdot \mathbf{d}_{1\langle 10k \rangle 1\langle 11 \rangle} \cdot \mathbf{d}_{0\langle 0 \rangle 1\langle 00 \rangle} \cdot \mathbf{d}_{0\langle 1 \rangle 1\langle 11 \rangle} \cdot \mathbf{d}_{1\langle 01 \rangle 1\langle 10 \rangle}) && \text{Ax}_1, \Delta \\
&\leq c_1 (\mathbf{d}_{1\langle 01k \rangle 1\langle 00 \rangle} \cdot \mathbf{d}_{1\langle 10k \rangle 1\langle 11 \rangle} \cdot \mathbf{d}_{0\langle 0 \rangle 1\langle 00 \rangle} \cdot \mathbf{d}_{0\langle 1 \rangle 1\langle 11 \rangle} \cdot \mathbf{d}_{1\langle 01k \rangle 1\langle 10k \rangle}) && 44, \text{Ax}_1 \\
&\leq c_1 (\mathbf{d}_{1\langle 11 \rangle 1\langle 00 \rangle} \cdot \mathbf{d}_{0\langle 0 \rangle 1\langle 00 \rangle} \cdot \mathbf{d}_{0\langle 1 \rangle 1\langle 11 \rangle}) && \text{Ax}_1 \\
&\leq \mathbf{d}_{0\langle 0 \rangle 0\langle 1 \rangle} && \text{Ax}_1, \Delta
\end{aligned}$$

Thus  $(P_k)^\check{\circ} \circ P_k \leq \dot{1}$  as required, ( $\dot{1} = \mathbf{d}_{0\langle 0 \rangle 0\langle 1 \rangle}$  by Definition 32).  $\blacklozenge$

PROOF (EQUATION (1.2))

$$\begin{aligned}
\dot{1} &\leq c_1 c_2 (\mathbf{d}_{1\langle 0 \rangle 0\langle 0 \rangle} \cdot \mathbf{d}_{1\langle 1 \rangle 0} \cdot \mathbf{d}_{2\langle 0 \rangle 0} \cdot \mathbf{d}_{2\langle 1 \rangle 0\langle 1 \rangle}) && \text{Ax}_2, \text{C}_3 \\
&\leq c_1 c_2 (\mathbf{d}_{02\langle 0 \rangle} \cdot \mathbf{d}_{1\langle 1 \rangle 2\langle 0 \rangle} \cdot \mathbf{d}_{1\langle 0 \rangle 2\langle 00 \rangle} \cdot \mathbf{d}_{2\langle 1 \rangle 1\langle 11 \rangle}) && \text{Ax}_1 \\
&\leq c_1 c_2 (\mathbf{d}_{02\langle 0 \rangle} \cdot c_0 (\mathbf{d}_{0\langle 0 \rangle 1} \cdot \mathbf{d}_{0\langle 1 \rangle 2} \cdot \mathbf{d}_{1\langle 1 \rangle 2\langle 0 \rangle} \cdot \mathbf{d}_{1\langle 0 \rangle 2\langle 00 \rangle} \cdot \mathbf{d}_{2\langle 1 \rangle 1\langle 11 \rangle})) && \text{Ax}_2, \text{C}_3 \\
&\leq c_2 (\mathbf{d}_{02\langle 0 \rangle} \cdot c_0 (\mathbf{d}_{0\langle 1 \rangle 2} \cdot \mathbf{d}_{0\langle 01 \rangle 2\langle 0 \rangle} \cdot \mathbf{d}_{0\langle 00 \rangle 2\langle 00 \rangle} \cdot \mathbf{d}_{2\langle 1 \rangle 0\langle 011 \rangle})) && \text{Ax}_1, \Delta \\
&\leq c_2 (\mathbf{d}_{02\langle 0 \rangle} \cdot c_1 (\mathbf{d}_{1\langle 1 \rangle 2} \cdot \mathbf{d}_{1\langle 01 \rangle 2\langle 0 \rangle} \cdot \mathbf{d}_{1\langle 00 \rangle 2\langle 00 \rangle} \cdot \mathbf{d}_{2\langle 1 \rangle 1\langle 011 \rangle})) && 48 \\
&\leq c_1 c_2 (\mathbf{d}_{02\langle 0 \rangle} \cdot \mathbf{d}_{1\langle 1 \rangle 2} \cdot \mathbf{d}_{1\langle 01 \rangle 2\langle 0 \rangle} \cdot \mathbf{d}_{1\langle 00 \rangle 2\langle 00 \rangle} \cdot \mathbf{d}_{2\langle 1 \rangle 1\langle 011 \rangle}) && \text{C}_3 \\
&\leq c_1 (\mathbf{d}_{01\langle 10 \rangle} \cdot \mathbf{d}_{1\langle 01 \rangle 1\langle 10 \rangle} \cdot \mathbf{d}_{1\langle 00 \rangle 1\langle 100 \rangle} \cdot \mathbf{d}_{1\langle 11 \rangle 1\langle 011 \rangle}) && \text{Ax}_1, \Delta \\
&\leq c_1 (\mathbf{d}_{01\langle 10 \rangle} \cdot \delta_{01}) && \text{Ax}_1, \text{Def} \\
&\leq c_1 (c_0 (\mathbf{d}_{01\langle 0 \rangle}) \cdot c_0 (\mathbf{d}_{01\langle 1 \rangle}) \cdot \mathbf{d}_{01\langle 10 \rangle} \cdot \delta_{01}) && 44, 41 \\
&\leq c_1 (c_0 (\mathbf{d}_{01\langle 0 \rangle} \cdot \mathbf{d}_{1\langle 00 \rangle 1\langle 010 \rangle}) \cdot c_0 (\mathbf{d}_{01\langle 1 \rangle} \cdot \mathbf{d}_{1\langle 11 \rangle 1\langle 101 \rangle}) \cdot \delta_{01}) && \text{Ax}_1, \text{C}_3 \\
&\leq c_1 (c_0 (\mathbf{d}_{01\langle 0 \rangle} \cdot \mathbf{d}_{0\langle 0 \rangle 0\langle 10 \rangle}) \cdot c_0 (\mathbf{d}_{01\langle 1 \rangle} \cdot \mathbf{d}_{0\langle 1 \rangle 0\langle 01 \rangle}) \cdot \delta_{01}) && \text{Ax}_1 \\
&= \check{P}_0 \circ P_1 && \text{Def}
\end{aligned}$$

The opposite inequality follows from the fact that  $\text{RA} \models \check{P}_0 \circ P_1 \leq \dot{1}$ , therefore we conclude  $\check{P}_0 \circ P_1 = \dot{1}$  as desired.  $\blacklozenge$

PROOF (EQUATION (1.3))

$$\begin{aligned}
P_k \circ \check{P}_k &\leq c_1 \left( \check{s}_{1\langle 0 \rangle}^0 \mathbf{d}_{0\langle 0k \rangle 0\langle 1 \rangle} \cdot \check{s}_{1\langle 1 \rangle}^0 \mathbf{d}_{0\langle 1k \rangle 0\langle 0 \rangle} \cdot \delta_{01} \right) && \text{Def} \\
&\leq c_1 \left( \mathbf{d}_{1\langle 00k \rangle 1\langle 01 \rangle} \cdot \mathbf{d}_{1\langle 11k \rangle 1\langle 10 \rangle} \cdot \delta_{01} \right) && \text{Ax}_1, \Delta \\
&\leq \mathbf{d}_{0\langle 0k \rangle 0\langle 1k \rangle} && \text{Ax}_1, \Delta
\end{aligned}$$

Thus  $(P_0 \circ \check{P}_0) \cdot (P_1 \circ \check{P}_1) \leq \mathbf{d}_{0\langle 00 \rangle 0\langle 10 \rangle} \cdot \mathbf{d}_{0\langle 01 \rangle 0\langle 11 \rangle}$ , and by Ax<sub>3</sub> this is bounded by  $\mathbf{d}_{0\langle 0 \rangle 0\langle 1 \rangle} = \dot{1}$ .  $\blacklozenge$

As  $\mathfrak{J}\mathfrak{A} \in \text{RA}$  and Equations (1.1)–(1.3) have been verified, we conclude that  $\mathfrak{J}\mathfrak{A} \in \text{Q}^+\text{RA}$ , as desired.

## CHAPTER 3. FIRST-ORDER LOGIC

### 3.1 First-order logic in $Q^+RA$

**Definition 50** For  $\langle \mathfrak{A}, P_0, P_1 \rangle \in P^+RA$  and  $\kappa \in \omega$ , define:

$$P_0 := P_0, \quad P_{\kappa+1} := P_1; P_\kappa, \quad \mathbf{u} := 1' \cdot (P_0; 1 \cdot P_1; 1)^-.$$

Let  $X, Y \in \Phi[\mathcal{L}]$ ,  $\mu < \alpha$ , and  $\kappa, \lambda, \kappa_0, \dots, \kappa_{\rho(\mu)-1} < \omega$ .

$$\text{Dr}(X) := \begin{cases} 1 & \text{In}(X) = \emptyset \\ P_{\kappa_0}; \mathbf{u}; 1 \cdot \dots \cdot P_{\kappa_\lambda}; \mathbf{u}; 1 & \text{In}(X) = \{\kappa_0, \dots, \kappa_\lambda\} \end{cases}$$

$$\text{Tr}(v_\kappa = v_\lambda) := (P_\kappa; \mathbf{u} \cdot P_\lambda; \mathbf{u}); 1$$

$$\text{Tr}(R_\mu v_{\kappa_0} \cdots v_{\kappa_{\rho(\mu)-1}}) := (P_{\kappa_0}; \mathbf{u}; \check{P}_0 \cdot \dots \cdot P_{\kappa_{\rho(\mu)-1}}; \mathbf{u}; \check{P}_{\rho(\mu)-1}); \mathbf{w}_{\mu+1}; 1$$

$$\text{Tr}(\neg X) := (\text{Tr}(X))^- \cdot \text{Dr}(X)$$

$$\text{Tr}(X \vee Y) := (\text{Tr}(X) + \text{Tr}(Y)) \cdot \text{Dr}(X \vee Y)$$

$$\text{Tr}(\exists_{v_\kappa} X) := (P_{\lambda_0}; \check{P}_{\lambda_0} \cdot \dots \cdot P_{\lambda_\mu}; \check{P}_{\lambda_\mu}); \text{Tr}(X)$$

$$\text{In}(\exists_{v_\kappa} X) = \{\lambda_0, \dots, \lambda_\mu\}$$

$$\text{Tr}(\exists_{v_\kappa} X) := 1; \text{Tr}(X)$$

$$\text{In}(\exists_{v_\kappa} X) = \emptyset \quad \square$$

**Theorem 51** ([Mad89a, Theorem 21]) *Let  $U$  be a set with  $\mathfrak{B}(U) \in TPA$ . Suppose  $\mathbf{R} = \langle R_\mu : \mu \in \beta \rangle$  is a countable sequence of relations on  $U$  with  $R_\mu$  having rank  $\rho_\mu$ ,  $\mathfrak{A} = \langle U, \mathbf{R} \rangle \in RE[\mathcal{L}]$ ,  $X \in \Phi[\mathcal{L}]$ , and  $\mathbf{a} \in {}^\omega U$ . Define  $\mathbf{S} = \langle S_\kappa : \kappa \in \omega \rangle$  thusly:*

1.  $S_0 = \left\{ \langle v, v' \rangle \in \check{U} \times \check{U} : P_\kappa(v) = a_\kappa \text{ for every } \kappa \in \text{In}(X) \right\}$ ,
2.  $S_{\mu+1} = \left\{ \langle v, v' \rangle \in \check{U} \times \check{U} : \langle P_0(v), \dots, P_{\rho(\mu)-1}(v) \rangle \in R_\mu \right\}$  for  $\mu < \alpha$ , and
3.  $S_{\mu+1} = \emptyset$  whenever  $\alpha \leq \mu < \omega$ .

Then  $\mathfrak{A} \models X[\mathbf{a}]$  if and only if  $\mathfrak{B}(U) \models \mathbf{w}_0 \leq \text{Tr}(X)[\mathbf{S}]$ . In particular, if  $X \in \Sigma[\mathcal{L}]$ , then  $S_0 = \check{U} \times \check{U}$  and  $\mathfrak{A} \models X$  if and only if  $\mathfrak{B}(U) \models 1 = \text{Tr}(X)[\mathbf{S}]$ . □

PROOF Fix any  $\mathbf{a} \in {}^\omega U$ . Define  $\mathbf{b} : \ddot{U} \rightarrow {}^\omega U$  by  $(\mathbf{b}(v))_\kappa = P_\kappa(v)$  if  $P_\kappa(v) \in U$  and  $(\mathbf{b}(v))_\kappa = a_0$  otherwise. For each  $X \in \Phi[\mathcal{L}]$  define  $S(X) = \{v \in \ddot{U} : P_\kappa(v) = a_\kappa \text{ for } \kappa \in \text{In}(X)\}$ . Then  $\mathfrak{U} \models X[\mathbf{a}]$  if and only if  $\mathfrak{U} \models X[\mathbf{b}(v)]$  for each  $v \in S(X)$ , since the assignment of variables only matters for those variables which are free in  $X$ . Finally, define  $D, T : \Phi[\mathcal{L}] \rightarrow \text{Sb } \ddot{U}$  by

$$\begin{aligned} D(X) &:= \{v : P_\kappa(v) \in U \text{ for } \kappa \in \text{In}(X)\} \\ T(X) &:= \{v \in D(X) : \mathfrak{U} \models X[\mathbf{b}(v)]\}. \end{aligned}$$

Then  $S(X) \subseteq D(X)$ , and  $\mathfrak{U} \models X[\mathbf{a}]$  if and only if  $S(X) \subseteq T(X)$ .

By definition, we have that  $S_0 = S(X) \times \ddot{U}$  and  $\text{Dr}(X) = D(X) \times \ddot{U}$ . We proceed by proving that  $\text{Tr}(X)[\mathbf{S}] = T(X) \times \ddot{U}$ . We use induction on the construction of the formula  $X$ , supposing first that  $X$  is atomic.

PROOF (CASE  $X = v_\kappa = v_\lambda$ ) In this case we have  $\mathfrak{U} \models X[\mathbf{b}(v)]$  if and only if  $P_\kappa(v) = P_\lambda(v)$ . Thus

$$T(X) = \{v : P_\kappa(v) = P_\lambda(v) \in U\}.$$

We also have in this case that  $\text{Tr}(X)$  does not depend on  $\mathbf{S}$ . Indeed:

$$\text{Tr}(X) = \{\langle v, v' \rangle : P_\kappa(v) = P_\lambda(v) \in U\}.$$

Then clearly  $\text{Tr}(X)[\mathbf{S}] = T(X) \times \ddot{U}$  as desired. ♦

PROOF (CASE  $X = R_\mu v_{\kappa_0} \cdots v_{\kappa_{\rho(\mu)-1}}$ ) In this case  $\mathfrak{U} \models X[\mathbf{b}(v)]$  iff

$$\langle P_{\kappa_0}(v), \dots, P_{\kappa_{\rho(\mu)-1}}(v) \rangle \in R_\mu.$$

Then

$$T(X) = \{v : \langle P_{\kappa_0}(v), \dots, P_{\kappa_{\rho(\mu)-1}}(v) \rangle \in R_\mu\}.$$

On the other hand,

$$\begin{aligned} \text{Tr}(X)[\mathbf{S}] &= \left( P_{\kappa_0}; \mathbf{u}; \check{P}_0 \cdots \cdots P_{\kappa_{\rho(\mu)-1}}; \mathbf{u}; \check{P}_{\rho(\mu)-1} \right); S_{\mu+1}; 1 \\ &= \left( P_{\kappa_0}; \mathbf{u}; \check{P}_0 \cdots \cdots P_{\kappa_{\rho(\mu)-1}}; \mathbf{u}; \check{P}_{\rho(\mu)-1} \right); S_{\mu+1} \\ &= \{\langle v, u \rangle : P_{\kappa_i}(v) = P_i(u) \in U \text{ for } i \in \rho_\mu\} \mid \{\langle u, v' \rangle : \langle P_0(u), \dots, P_{\rho(\mu)-1}(u) \rangle \in R_\mu\} \\ &= \{\langle v, v' \rangle : \langle P_{\kappa_0}(v), \dots, P_{\kappa_{\rho(\mu)-1}}(v) \rangle \in R_\mu\}. \end{aligned}$$

Again, this is the same as  $T(X) \times \ddot{U}$  as desired. ♦

With  $\text{Tr}(X)[\mathbf{S}] = T(X) \times \ddot{U}$  established for atomic formulas, we proceed assuming that for some formulas  $X$  and  $Y$  we have  $\text{Tr}(X)[\mathbf{S}] = T(X) \times \ddot{U}$  and  $\text{Tr}(Y)[\mathbf{S}] = T(Y) \times \ddot{U}$ . Then we must prove that:

$$\begin{aligned}\text{Tr}(\neg X)[\mathbf{S}] &= T(\neg X) \times \ddot{U} \\ \text{Tr}(X \vee Y)[\mathbf{S}] &= T(X \vee Y) \times \ddot{U} \\ \text{Tr}(\exists_{v_\kappa} X)[\mathbf{S}] &= T(\exists_{v_\kappa} X) \times \ddot{U}.\end{aligned}$$

PROOF (CASE  $\neg X$ ) Notice that for  $v \in D(X)$ ,  $\mathfrak{U} \models \neg X[\mathbf{b}(v)]$  if and only if  $\mathfrak{U} \not\models X[\mathbf{b}(v)]$  if and only if  $v \notin T(X)$ . We conclude:

$$\begin{aligned}T(\neg X) &= \{v \in D(X) : v \notin T(X)\} \\ &= (T(X))^- \cap D(X).\end{aligned}$$

Turning our attention to  $\text{Tr}(X)$  we see:

$$\begin{aligned}\text{Tr}(\neg X)[\mathbf{S}] &= (\text{Tr}(X))^- \cap \text{Dr}(X)[\mathbf{S}] \\ &= (\text{Tr}(X)[\mathbf{S}])^- \cap (\text{Dr}(X)[\mathbf{S}]) \\ &= \left(T(X) \times \ddot{U}\right)^- \cap \left(D(X) \times \ddot{U}\right) \\ &= \left((T(X))^- \cap D(X)\right) \times \ddot{U},\end{aligned}$$

as desired. We note that the operation  $-$  is used in two different ways. Since  $\text{Tr}(X)$  and  $T(X) \times \ddot{U}$  are understood to be subsets of  $\ddot{U} \times \ddot{U}$  we mean the same interpretation for their complements. Similarly, since  $T(X)$  is a subset of  $\ddot{U}$ , we mean  $(T(X))^-$  to denote  $\ddot{U} \setminus T(X)$ .  $\blacklozenge$

PROOF (CASE  $X \vee Y$ ) Now  $\mathfrak{U} \models X \vee Y[\mathbf{b}(v)]$  if and only if either  $\mathfrak{U} \models X[\mathbf{b}(v)]$  or  $\mathfrak{U} \models Y[\mathbf{b}(v)]$  if and only if  $v \in T(X) \cup T(Y)$ . Then:

$$\begin{aligned}T(X \vee Y) &= \{v \in D(X \vee Y) : v \in T(X) \cup T(Y)\} \\ &= (T(X) \cup T(Y)) \cap D(X \vee Y)\end{aligned}$$

By definition  $\text{Tr}(X \vee Y)[\mathbf{S}] = (\text{Tr}(X) + \text{Tr}(Y)) \cdot \text{Dr}(X \vee Y)[\mathbf{S}]$ . This simplifies to:

$$\begin{aligned}\text{Tr}(X \vee Y)[\mathbf{S}] &= (\text{Tr}(X)[\mathbf{S}] \cup \text{Tr}(Y)[\mathbf{S}]) \cap \text{Dr}(X \vee Y)[\mathbf{S}] \\ &= \left(\left(T(X) \times \ddot{U}\right) \cup \left(T(Y) \times \ddot{U}\right)\right) \cap D(X \vee Y) \times \ddot{U} \\ &= \left((T(X) \cup T(Y)) \cap D(X \vee Y)\right) \times \ddot{U}.\end{aligned}$$

Thus  $\text{Tr}(X \vee Y) = T(X \vee Y) \times \ddot{U}$  as well.  $\blacklozenge$

PROOF (CASE  $\exists_{v_\kappa} X$ ) For the final case we have  $\mathfrak{U} \models \exists_{v_\kappa} X[\mathbf{b}(v)]$  if and only if there is some  $\mathbf{c} \in {}^\omega U$  such that  $c_\lambda = P_\lambda(v)$  for each  $\lambda \in \text{In}(\exists_{v_\kappa} X)$  and  $\mathfrak{U} \models X[\mathbf{c}]$ . The latter condition holds if and only if there is some  $u \in \ddot{U}$  such that  $P_\lambda(u) = P_\lambda(v)$  for each  $\lambda \in \text{In}(\exists_{v_\kappa} X)$  such that  $\mathfrak{U} \models X[\mathbf{b}(u)]$ . This in turn holds if and only if there is some  $u \in \ddot{U}$  such that  $P_\lambda(u) = P_\lambda(v)$  for each  $\lambda \in \text{In}(\exists_{v_\kappa} X)$  and  $u \in T(X)$ .

In case  $\text{In}(\exists_{v_\kappa} X) = \emptyset$ , the condition  $P_\lambda(u) = P_\lambda(v)$  for each  $\lambda \in \text{In}(\exists_{v_\kappa} X)$  is vacuously satisfied so the set of all  $\langle v, u \rangle$  which satisfy the condition is  $1 = \ddot{U} \times \ddot{U}$ . Otherwise the set of all such  $\langle v, u \rangle$  is denoted by  $\prod_{\lambda \in \text{In}(\exists_{v_\kappa} X)} P_\lambda; \check{P}_\lambda$ . Thus  $\text{Tr}(\exists_{v_\kappa} X)$  is the set of all  $\langle v, v' \rangle$  such that for some  $u \in \ddot{U}$  we have:  $P_\lambda(u) = P_\lambda(v)$  for each  $\lambda \in \text{In}(\exists_{v_\kappa} X)$  and  $\langle u, v' \rangle \in \text{Tr}(X)$ . But by the inductive hypothesis  $\langle u, v' \rangle \in \text{Tr}(X)$  if and only if  $u \in T(X)$ . We therefore conclude that  $\text{Tr}(\exists_{v_\kappa} X) = T(\exists_{v_\kappa} X) \times \ddot{U}$  as required.  $\blacklozenge$

We have therefore established that for any  $X \in \Phi[\mathcal{L}]$ ,  $\text{Tr}(X)[\mathbf{S}] = T(X) \times \ddot{U}$ . As we have already observed,  $S_0 = S(X) \times \ddot{U}$  and  $\mathfrak{U} \models X[\mathbf{a}]$  if and only if  $S(X) \subseteq T(X)$ . The theorem therefore follows from the fact that  $S_0 \subseteq \text{Tr}(X)[\mathbf{S}]$  if and only if  $S(X) \subseteq T(X)$ . The theorem's second assertion clearly follows from the first.  $\blacksquare$

The next theorem is also taken from [Mad89a, Theorem 22]. We have modified it slightly (formulated here for  $\text{P}^+\text{RA}$  instead of  $\text{PRA}$ ) and corrected an error in the proof.

**Theorem 52** *For  $X \in \Sigma[\mathcal{L}]$  the following are equivalent:*

1.  $X$  is logically valid,
2.  $\text{TPA} \models 1 = \text{Tr}(X)$ , and
3.  $\text{P}^+\text{RA} \models 1; \mathbf{u}; 1 = \text{Tr}(X)$ .  $\square$

PROOF Recall that in  $\text{P}^+\text{RA}$ , the conjugated quasi-projections satisfy (1.3) (thus their domains are identical).

PROOF (1 $\Rightarrow$ 2) Suppose  $X$  is logically valid, and let  $U$  be a nonempty set such that  $\ddot{U} \times \ddot{U} = \ddot{U} \setminus U$  (that is to say, let  $\mathfrak{B}(U) \in \text{TPA}$ ). Let  $\mathbf{S}' \in {}^\omega \text{Re}\ddot{U}$ . For every  $\mu < \omega$  define the relation  $R_\mu$  as follows.

$$R_\mu := \{ \langle P_0(v), \dots, P_{\rho(\mu)-1}(v) \rangle : v \in \text{Dom}(S'_{\mu+1}), P_0(v), \dots, P_{\rho(\mu)-1}(v) \in U \}$$

Let  $\mathbf{S}$  be as in Theorem 51 (thus  $S_0 = 1$  since  $X \in \Sigma[\mathcal{L}]$ ). The variable  $\mathbf{w}_{\mu+1}$  only occurs in  $\text{Tr}(X)$  as part of a subterm of the form:

$$(P_{\kappa_0}; \mathbf{u}; \check{P}_0 \cdot \dots \cdot P_{\kappa_{\rho(\mu)-1}}; \mathbf{u}; \check{P}_{\rho(\mu)-1}); \mathbf{w}_{\mu+1}; 1.$$

This subterm denotes the same relation in  $\mathfrak{B}(U)$  whether  $w_{\mu+1}$  is assigned to  $S'_{\mu+1}$  or  $S_{\mu+1}$ . Recall from the proof of Theorem 51 that in the latter case this subterm denotes:

$$\{\langle v, v' \rangle : \langle P_{\kappa_0}(v), \dots, P_{\kappa_{\rho(\mu)-1}}(v) \rangle \in R_{\mu+1}\}.$$

In case  $w_{\mu+1}$  is mapped to  $S'_{\mu+1}$  the same subterm is mapped to:

$$\begin{aligned} & \{\langle v, u \rangle : P_{\kappa_i}(v) = P_i(u) \in U \text{ for } i < \rho(\mu)\} \mid \{\langle u, v' \rangle : u \in \text{Dom}(S'_{\mu+1})\} \\ &= \{\langle v, v' \rangle : \text{for some } u \in \ddot{U}, P_{\kappa_i}(v) = P_i(u) \in U \text{ for } i < \rho(\mu) \text{ and } u \in \text{Dom}(S'_{\mu+1})\} \\ &= \{\langle v, v' \rangle : \text{for some } u \in \ddot{U}, P_{\kappa_i}(v) = P_i(u) \text{ for } i < \rho(\mu) \text{ and } \langle P_0(u), \dots, P_{\rho(\mu)-1}(u) \rangle \in R_{\mu+1}\} \\ &= \{\langle v, v' \rangle : \langle P_{\kappa_0}(v), \dots, P_{\kappa_{\rho(\mu)-1}}(v) \rangle \in R_{\mu+1}\}. \end{aligned}$$

Therefore  $\text{Tr}(X)[\mathbf{S}] = \text{Tr}(X)[\mathbf{S}']$ , so we conclude  $\mathfrak{B}(U) \models 1 = \text{Tr}(X)[\mathbf{S}']$  if and only if  $\mathfrak{B}(U) \models 1 = \text{Tr}(X)[\mathbf{S}]$ . By Theorem 51, the latter condition is equivalent to  $\mathfrak{U} \models X$  (where  $\mathfrak{U} = \langle U, \mathbf{R} \rangle$ ), which holds since  $X$  is assumed to be logically valid.  $\blacklozenge$

PROOF (2 $\Rightarrow$ 1) Suppose that  $\text{TPA} \models 1 = \text{Tr}(X)$ . Let  $\mathfrak{U} = \langle U, \mathbf{R} \rangle \in \text{RE}[\mathcal{L}]$ . We may assume that  $U$  satisfies  $\ddot{U} \times \ddot{U} = \ddot{U} \setminus U$ , because every  $\mathfrak{U} \in \text{RE}[\mathcal{L}]$  is isomorphic to some  $\mathfrak{U}' \in \text{RE}[\mathcal{L}]$  where  $U'$  is such. Let  $\mathbf{S} = \langle S_\kappa : \kappa < \omega \rangle$  be as in Theorem 51, so in particular  $S_0 = \ddot{U} \times \ddot{U}$ . By hypothesis  $\mathfrak{B}(U) \models 1 = \text{Tr}(X)[\mathbf{S}]$ , which implies  $\mathfrak{U} \models X$  by Theorem 51.  $\blacklozenge$

PROOF (2 $\Rightarrow$ 3) We prove the contrapositive. Suppose  $1; \mathbf{u}; 1 = \text{Tr}(X)$  fails in some  $\mathfrak{B} \in \text{P}^+\text{RA}$  (with quasi-projections  $P_0, P_1$ ). We will construct  $\mathfrak{B}(U) \in \text{TPA}$  such that  $\mathfrak{B}(U) \not\models 1 = \text{Tr}(X)$ . We may assume that  $\mathfrak{B}$  is simple by Lemma 25. By Tarski's QRA theorem,  $\mathfrak{B}$  is representable. Then  $\mathfrak{B} \cong \mathfrak{C} \leq \mathfrak{R}\mathfrak{e}V$  for some set  $V$  by [JT52, Theorem 4.28], so we assume without loss of generality that  $\mathfrak{B} \leq \mathfrak{R}\mathfrak{e}V$ . We may further assume that  $\ddot{V} \times \ddot{V} = \ddot{V} \setminus V$  since the structure of  $\mathfrak{R}\mathfrak{e}V$  does not depend on the structure of  $V$ , but rather only on its cardinality. Let  $U = V \setminus \text{Dom } P_0$  so that  $\ddot{U} \times \ddot{U} = \ddot{U} \setminus U$  (this property is easily seen to be inherited by subsets). If  $U = \emptyset$  then  $\text{Tr}(X)[\mathbf{S}] = \emptyset = 1; \mathbf{u}; 1$ , a contradiction, so we additionally assume that  $U \neq \emptyset$  and conclude  $\mathfrak{B} \models 1; \mathbf{u}; 1 = 1$ .

Define a binary operation  $f$  on  $V$  by  $f(x, y) = z$  where  $P_0(z) = x$  and  $P_1(z) = y$ . Such a  $z$  exists uniquely, since  $\mathfrak{A} \in \text{P}^+\text{RA}$ . Let  $W$  be the closure of  $U$  under  $f$ . Define  $P'_0 = P_0 \cap (W \times W)$  and  $P'_1 = P_1 \cap (W \times W)$ . Then  $\langle W, P'_0, P'_1 \rangle$  is isomorphic to  $\langle \ddot{U}, \mathfrak{p}_0^U, \mathfrak{p}_1^U \rangle$  by a recursively defined isomorphism  $h$  such that  $h(u) = u$  for each  $u \in U$  and  $h(f(x, y)) = \langle h(x), h(y) \rangle$  for  $x, y \in W$ . Then we assume without loss of generality that  $W = \ddot{U}$  and  $P'_i = \mathfrak{p}_i^U$  for  $i \in 2$ , so that  $\mathfrak{B}(U) = \mathfrak{R}\mathfrak{I}_{\ddot{U} \times \ddot{U}} \mathfrak{R}\mathfrak{e}V$ .

Let  $\mathbf{S} \in {}^\omega B$  such that  $\mathfrak{B} \not\models 1; \mathbf{u}; 1 = \text{Tr}(X)[\mathbf{S}]$ . For  $\kappa < \omega$  let  $S'_\kappa = S_\kappa \cap (\ddot{U} \times \ddot{U})$  so that  $\mathbf{S}'$  may be viewed as an assignment to  $\mathfrak{B}$  or to  $\mathfrak{B}(U)$ . Let  $R$  be the interpretation of  $\text{Tr}(X)$  in  $\mathfrak{B}(U)$  under

assignment  $\mathbf{S}'$  and  $S$  be the interpretation of  $\text{Tr}(X)$  in  $\mathfrak{B}$  under assignment  $\mathbf{S}$ . By monotonicity of the operations of  $\mathfrak{B}$ , we have  $R \subseteq S$ .

A simple inductive argument shows that  $S;1 = S$  (in fact, that  $\text{RA} \models \text{Tr}(X);1 = \text{Tr}(X)$ ). Further, since  $\text{In}(X) = \emptyset$  it must be that  $1;S = S$ . If  $X = Y \vee Z$  for  $Y, Z \in \Sigma[\mathcal{L}]$  then this follows from the distributivity of relative product over union [Mad06, 261]. If  $X = \neg Y$  for  $Y \in \Sigma[\mathcal{L}]$  then it follows from the fact that  $\text{RA} \models 1;(1;w_0)^- = (1;w_0)^-$  [Mad06, 305]. Finally, if  $X = \exists_{v_\kappa} Y$  for some  $Y$  with  $\text{In}(Y) \subseteq \{\kappa\}$  then it follows from  $\text{RA} \models 1;1;w_0 = 1;w_0$ . We conclude then, that  $1;S;1 = S$ . But recall that  $S \neq 1$  by construction, so since  $\mathfrak{B}$  is simple it must be that  $R = S = \emptyset$  and  $\mathfrak{B}(U) \not\models 1 = \text{Tr}(X)[\mathbf{S}']$  as desired.  $\blacklozenge$

PROOF (3 $\Rightarrow$ 2) Suppose that  $\text{P}^+\text{RA} \models 1;u;1 = \text{Tr}(X)$ . Every algebra in TPA is a  $\text{P}^+\text{RA}$  in which  $1;u;1 = 1$  holds, so every such algebra satisfies  $1 = \text{Tr}(X)$ .  $\blacklozenge$

Thus we have  $1 \Leftrightarrow 2 \Leftrightarrow 3$ , as desired.  $\blacksquare$

**Remark 53** The error in the proof given in [Mad89a] occurs during the proof that 2 $\Rightarrow$ 3. In particular, the assignment  $\mathbf{S} \in {}^\omega B$  under which  $\text{P}^+\text{RA} \models 1;u;1 \doteq \text{Tr}(X)[\mathbf{S}]$  fails, is there identified to be  $\mathbf{S} \in {}^\omega \ddot{U}$ . The existence of such an assignment is equivalent to the desired conclusion and cannot be assumed. Simply changing  $\ddot{U}$  to  $B$  is insufficient to fix the proof, since then the statement  $\mathfrak{B}(U) \models 1 \doteq \text{Tr}(X)[\mathbf{S}]$  occurring later in the proof ceases to make sense.  $\square$

### 3.2 First-order logic in $\text{Df}_3$

**Definition 54** Define  $\mathbf{d}_{\kappa\langle i \rangle \lambda\langle j \rangle}$ ,  $\dot{\mathbf{i}}$ ,  $\dot{\mathbf{c}}$ ,  $\circ$ ,  $\smile$ , and  $\dot{\mathbf{i}}$  be as in Definition 32—replacing every instance of  $e$  with  $w_0$  and every instance of  $q$  with  $w_1$ . Note that these are  $\text{Df}_3$ -terms with variables contained in  $\{w_0, w_1\}$  rather than elements and operations on a particular algebra,  $\mathfrak{A} \in \text{Df}_3$ .

$$\begin{aligned} \Omega_0 &:= \sum \left\{ (\mathbf{d}_{\kappa\kappa})^- : \kappa \in 3 \right\} \\ \Omega_1 &:= \sum \left\{ \mathbf{d}_{\kappa\langle i \rangle \lambda\langle j \rangle} \cdot \mathbf{d}_{\lambda\langle j l \rangle \mu\langle k \rangle} \cdot (\mathbf{d}_{\kappa\langle i l \rangle \mu\langle k \rangle})^- : \kappa, \lambda, \mu \in 3; i, j, k, l \in 2^*; i, j l, k l \in S_3 \right\} \\ \Omega_2 &:= \sum \left\{ \mathbf{d}_{\kappa\langle i \rangle \kappa\langle i \rangle} \cdot \mathbf{d}_{\lambda\langle j \rangle \lambda\langle j \rangle} \cdot (\mathbf{c}_\mu (\mathbf{d}_{\mu\langle 0 \rangle \kappa\langle i \rangle} \cdot \mathbf{d}_{\mu\langle 1 \rangle \lambda\langle j \rangle}))^- : \mu \neq \kappa, \lambda \in 3; i, j \in S_3 \right\} \\ \Omega_3 &:= \sum \left\{ \mathbf{d}_{\kappa\langle i 0 \rangle \lambda\langle j 0 \rangle} \cdot \mathbf{d}_{\kappa\langle i 1 \rangle \lambda\langle j 1 \rangle} \cdot (\mathbf{d}_{\kappa\langle i \rangle \lambda\langle j \rangle})^- : \kappa, \lambda \in 3; i, j \in S_2 \right\} \end{aligned}$$

Let  $X \in \Phi[\mathcal{L}]$  and let  $\{\kappa_i : i < \lambda\} = \text{Rel } X$  (the indices of relation symbols appearing in  $X$ ). Then define  $\Psi(X) \in \text{Tm}[\text{Df}_3]$  as follows.

$$\Psi(X) := c_0 c_1 c_2 \left( \sum_{i=0}^3 \Omega_i + \sum_{i=0}^{\lambda-1} ((s_0^1 s_1^0 c_1 c_2 w_{\kappa_i+2}) \oplus w_{\kappa_i+2} + (\dot{\mathbf{i}} \cdot w_{\kappa_i+2}) \oplus w_{\kappa_i+2}) \right)$$



Let  $u = \dot{1} \cdot (p_0 \circ \dot{1} + p_1 \circ \dot{1})^-$ ,  $P_0 = p_0$ , and  $P_{\kappa+1} = p_1 \circ P_\kappa$ . Finally, let  $\text{Tr}'$  and  $\text{Dr}'$  be as  $\text{Tr}$  and  $\text{Dr}$  are in Definition 50—replacing the relational operations and constants with our  $\text{Df}_3$  term operations and constants.  $\square$

**Proposition 55** *For every simple  $\mathfrak{A} \in \text{Df}_3$  and  $\mathbf{s} \in {}^\omega A$ , the following are equivalent:*

(i)  $\mathfrak{A} \models \Psi(X) = 0[\mathbf{s}]$ .

(ii) *both of the following conditions hold:*

(a)  $s_{\kappa_i+2} \in J$  for each  $\kappa_i \in \text{Rel } X$ .

(b)  $e = s_0$  and  $q = s_1$  satisfy the equations in  $\text{Ax}$ .

(iii)  $\mathfrak{A} \not\models \Psi(X) = 1[\mathbf{s}]$ .  $\square$

PROOF The equivalence of (i) and (iii) is a direct consequence of Theorem 30 since  $\mathfrak{A}$  is assumed to be simple. By Theorem 30 we have (i) if and only if  $\Omega_i = 0$  for  $i \leq 3$ ; and  $(s_0^1 s_1^0 c_1 c_2 s_{\kappa_i+2}) \oplus s_{\kappa_i+2} = 0$ ; and  $\dot{1} \cdot s_{\kappa_i+1} \oplus s_{\kappa_i+1} = 0$  for each  $i \in \text{Rel } X$ . By Theorem 31:

$$\begin{aligned} \dot{1} \cdot s_{\kappa_i+2} \oplus s_{\kappa_i+2} = 0 &\Leftrightarrow \dot{1} \cdot s_{\kappa_i+2} = s_{\kappa_i+2} \\ &\Leftrightarrow s_{\kappa_i+2} \leq \dot{1} \\ (s_0^1 s_1^0 c_1 c_2 s_{\kappa_i+2}) \oplus s_{\kappa_i+2} = 0 &\Leftrightarrow s_0^1 s_1^0 c_1 c_2 s_{\kappa_i+2} = s_{\kappa_i+2} \\ &\Leftrightarrow s_{\kappa_i+2} \in J_0. \end{aligned}$$

The above are equivalent to item (a) of condition (ii). We similarly deduce that  $\Omega_i = 0$  if and only if  $e = s_0$  and  $q = s_1$  satisfy all of the equations in  $\text{Ax}_i$  which is item (b) of condition (ii). Thus (ii) is equivalent to (i) as desired.  $\blacksquare$

**Theorem 56**  *$X \in \Sigma[\mathcal{L}]$  is logically valid if and only if  $\text{Df}_3 \models L(X)$  where  $L(X)$  is the equation*

$$1 \circ u \circ 1 + \Psi(X) \doteq \text{Tr}'(X) + \Psi(X).$$

PROOF By Theorem 52 it suffices to show that  $\text{Df}_3 \models L(X)$  if and only if  $\text{P}^+\text{RA} \models 1; u; 1 = \text{Tr}(X)$ .

Suppose first that  $\text{Df}_3 \not\models L(X)$  for some  $X$ . Then since  $\text{Df}_3$  is generated by its simple algebras (Theorem 29) there is some simple  $\mathfrak{A} \in \text{Df}_3$  and  $\mathbf{s} \in {}^\omega A$  such that  $\mathfrak{A} \not\models L(X)[\mathbf{s}]$ . It must be that  $\mathfrak{A} \models \Psi(X) \doteq 0[\mathbf{s}]$  (otherwise  $\mathfrak{A} \models \Psi(X) \doteq 1[\mathbf{s}]$ , hence  $\mathfrak{A} \models L(X)[\mathbf{s}]$  trivially). So we conclude that  $s_0$  and  $s_1$  are diagonal and quasi-projection parameters (respectively) satisfying the equations in  $\text{Ax}$  and further that  $s_{\kappa+2} \in J$  for every  $R_\kappa$  appearing in  $X$  by Proposition 55. By Theorem 34 we have that

$\mathfrak{J}\mathfrak{A} = \langle A, +, \dot{+}, \circ, \smile, \dot{\imath} \rangle \in \mathbf{Q}^+\mathbf{RA}$  so that  $\langle \mathfrak{J}\mathfrak{A}, P_0, P_1 \rangle \in \mathbf{P}^+\mathbf{RA}$ . Since  $\mathfrak{A} \not\models L(X)[\mathbf{s}]$  and  $\mathfrak{A} \models \Psi(X) \doteq 0$ , it follows that  $\mathfrak{A} \not\models 1 \circ \mathbf{u} \circ 1 \doteq \text{Tr}'(X)[\mathbf{s}]$ . If we define  $\mathbf{s}' \in {}^\omega J$  by  $s'_\kappa = s_{\kappa+2}$  for  $\kappa \in \omega$ , then this is equivalent to  $\langle \mathfrak{J}\mathfrak{A}, P_0, P_1 \rangle \not\models 1 \circ \mathbf{u} \circ 1 \doteq \text{Tr}(X)[\mathbf{s}']$ . We conclude that  $\mathbf{P}^+\mathbf{RA} \not\models 1; \mathbf{u}; 1 \doteq \text{Tr}(X)$ .

On the other hand, suppose  $\text{Df}_3 \models L(X)$ . Choose any  $\mathfrak{A} \in \mathbf{P}^+\mathbf{RA}$ . We may assume  $\mathfrak{A}$  is simple because  $\mathbf{P}^+\mathbf{RA}$  is generated by its simple algebras by [Lemma 25](#). Then by [JT52, Theorem 4.28] we have that the  $\mathbf{Q}^+\mathbf{RA}$  reduct of  $\mathfrak{A}$  is isomorphic to a subalgebra of  $\mathfrak{R}\mathfrak{e}V$  which is in turn isomorphic to the Némethi reduct of some  $\mathfrak{B} \in \text{Df}_3$  with diagonal and quasi-projection parameters  $e$  and  $q$  by [Theorem 35](#). Let  $f : \mathfrak{A} \rightarrow \mathfrak{J}\mathfrak{B}$  be such an isomorphism. Choose any  $\mathbf{S} \in {}^\omega A$ . Define  $\mathbf{T} \in {}^\omega J$  by  $T_\kappa = f(S_\kappa)$ , and  $\mathbf{T}'$  by  $T'_{\kappa+2} = T_\kappa$ ,  $T'_0 = e$ , and  $T'_1 = q$ . Then  $\mathfrak{A} \models 1; \mathbf{u}; 1 = \text{Tr}(X)[\mathbf{S}]$  if and only if  $\mathfrak{J}\mathfrak{B} \models 1 \circ \mathbf{u} \circ 1 = \text{Tr}(X)[\mathbf{T}]$  if and only if  $\mathfrak{B} \models L(X)[\mathbf{T}']$ . The last condition is true by assumption, so we conclude that  $\mathfrak{A} \models 1; \mathbf{u}; 1 = \text{Tr}(X)[\mathbf{S}]$  as was to be shown. ■

## BIBLIOGRAPHY

- [AN11] Hajnal Andréka and István Németi. Formalizing set theory in diagonal-free cylindric algebras, searching for the weakest logic with Gödel’s incompleteness property. Draft version, June 2011.
- [Bir35] Garrett Birkhoff. On the structure of abstract algebras. *Proc. Cambridge Phil. Soc.*, 31:433–454, 1935.
- [Bir44] Garrett Birkhoff. Subdirect unions in universal algebra. *Bull. Amer. Math. Soc.*, 50:764–768, 1944.
- [CT51] Louise H. Chin and Alfred Tarski. Distributive and modular laws in the arithmetic of relation algebras. *Univ. California Publ. Math. (N.S.)*, 1:341–384, 1951.
- [Dah98] Bernd I. Dahn. Robbins algebras are Boolean: a revision of McCune’s computer-generated solution of Robbins problem. *J. Algebra*, 208(2):526–532, 1998.
- [Hen67] Leon Henkin. *Logical systems containing only a finite number of symbols*. Séminaire de Mathématiques Supérieures, No. 21. Les Presses de l’Université de Montréal, Montreal, Que., 1967.
- [HMT71] Leon Henkin, J. Donald Monk, and Alfred Tarski. *Cylindric algebras. Part I. With an introductory chapter: General theory of algebras*. North-Holland Publishing Co., Amsterdam, 1971. Studies in Logic and the Foundations of Mathematics, Vol. 64.
- [HMT85] Leon Henkin, J. Donald Monk, and Alfred Tarski. *Cylindric algebras. Part II*, volume 115 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1985.
- [HT61] Leon Henkin and Alfred Tarski. Cylindric algebras. In *Proc. Sympos. Pure Math., Vol. II*, pages 83–113. American Mathematical Society, Providence, R.I., 1961.
- [Hun33a] Edward V. Huntington. Boolean algebra. A correction to: “New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell’s *principia*

- mathematica*” [Trans. Amer. Math. Soc. **35** (1933), no. 1, 274–304; 1501684]. *Trans. Amer. Math. Soc.*, 35(2):557–558, 1933.
- [Hun33b] Edward V. Huntington. New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell’s *principia mathematica*. *Trans. Amer. Math. Soc.*, 35(1):274–304, 1933.
- [JT48] Bjarni Jónsson and Alfred Tarski. Representation problems for relation algebras. *Bulletin of the American Mathematical Society* 54, 1948.
- [JT51] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators. I. *Amer. J. Math.*, 73:891–939, 1951.
- [JT52] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators. II. *Amer. J. Math.*, 74:127–162, 1952.
- [Löw15] Leopold Löwenheim. Über Möglichkeiten im Relativkalkül. *Math. Ann.*, 76(4):447–470, 1915.
- [Lyn50] Roger C. Lyndon. The representation of relational algebras. *Ann. of Math. (2)*, 51:707–729, 1950.
- [Mad78a] Roger D. Maddux. Some sufficient conditions for the representability of relation algebras. *Algebra Universalis*, 8(2):162–172, 1978.
- [Mad78b] Roger D. Maddux. *Topics in relation algebras*. ProQuest LLC, Ann Arbor, MI, 1978. Thesis (Ph.D.)—University of California, Berkeley.
- [Mad89a] Roger D. Maddux. Finitary algebraic logic. *Z. Math. Logik Grundlag. Math.*, 35(4):321–332, 1989.
- [Mad89b] Roger D. Maddux. Nonfinite axiomatizability results for cylindric and relation algebras. *J. Symbolic Logic*, 54(3):951–974, 1989.
- [Mad91] Roger D. Maddux. Introductory course on relation algebras, finite-dimensional cylindric algebras, and their interconnections. In *Algebraic logic (Budapest, 1988)*, volume 54 of *Colloq. Math. Soc. János Bolyai*, pages 361–392. North-Holland, Amsterdam, 1991.
- [Mad93] Roger D. Maddux. Finitary algebraic logic. II. *Math. Logic Quart.*, 39(4):566–569, 1993.
- [Mad06] Roger D. Maddux. *Relation algebras*, volume 150 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2006.

- [McC97] William McCune. Solution of the Robbins problem. *J. Automat. Reason.*, 19(3):263–276, 1997.
- [Mon64] J. Donald Monk. On representable relation algebras. *Michigan Math. J.*, 11:207–210, 1964.
- [Mon69] J. Donald Monk. Nonfinitizability of classes of representable cylindric algebras. *J. Symbolic Logic*, 34:331–343, 1969.
- [MSS92] Szabolcs Mikulás, Ildikó Sain, and András Simon. Complexity of equational theory of relational algebras with projection elements. *Bull. Sect. Logic Univ. Łódź*, 21(3):103–111, 1992.
- [Ném85] István Németi. Logic with three variables has Gödel’s incompleteness property, 1985. PrePrint.
- [Ném86] István Németi. *Free Algebras and Decidability in Algebraic Logic*. PhD thesis, Hungarian Academy of Sciences, 1986.
- [SA04] Tarek Sayed Ahmed. Tarskian Algebraic Logic. *JoRMiCS*, 1:3–26, 2004.
- [SA05] Tarek Sayed Ahmed. Algebraic logic, where does it stand today? *Bull. Symbolic Logic*, 11(4):465–516, 2005.
- [Sai00] Ildikó Sain. On the search for a finitizable algebraization of first order logic. *Log. J. IGPL*, 8(4):497–591, 2000.
- [Sim07] András Simon. Connections between quasi-projective relation algebras and cylindric algebras. *Algebra Universalis*, 56(3-4):263–301, 2007.
- [Sto36] M. H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40(1):37–111, 1936.
- [Tar41] Alfred Tarski. On the calculus of relations. *J. Symbolic Logic*, 6:73–89, 1941.
- [Tar43] Alfred Tarski. untitled. Manuscript on the calculus of relations, 1943.
- [Tar53a] Alfred Tarski. A formalization of set theory without variables. *The Journal of Symbolic Logic*, 18(2):pp. 189, 1953.
- [Tar53b] Alfred Tarski. Some metalogical results concerning the calculus of relations. *The Journal of Symbolic Logic*, 18(2):pp. 188–189, 1953.

- [Tar55] Alfred Tarski. Contributions to the theory of models. III. *Nederl. Akad. Wetensch. Proc. Ser. A.*, 58:56–64 = *Indagationes Math.* 17, 56–64 (1955), 1955.
- [TG87] Alfred Tarski and Steven Givant. *A formalization of set theory without variables*, volume 41 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1987.
- [Wol02] Stephen Wolfram. *A new kind of science*. Wolfram Media, Inc., Champaign, IL, 2002.