Analysis of nonlocal equations:

One-sided weighted fractional Sobolev spaces and

Harnack inequality for fractional nondivergence form elliptic equations

by

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DEDICATION

To my parents, my first teachers.

To my husband, my favorite collaborator.

Sláinte

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ABSTRACT

This dissertation deals with the following two projects.

First, we characterize one-sided weighted Sobolev spaces $W^{1,p}(\mathbb{R},\omega)$, where ω is a one-sided Sawyer weight, in terms of a.e. and weighted L^p limits as $\alpha \to 1^-$ of Marchaud fractional derivatives of order $0 < \alpha < 1$. These are Bourgain–Brezis–Mironescu-type characterizations for one-sided weighted Sobolev spaces. Similar results for weighted Sobolev spaces $W^{2,p}(\mathbb{R}^n,\nu)$, where ν is an A_p-Muckenhoupt weight, are proved in terms of limits as $s \to 1^-$ of fractional Laplacians $(-\Delta)^s$. We also additionally study the a.e. and weighted L^p limits as $\alpha, s \to 0^+$.

Second, we define fractional powers of nondivergence form elliptic operators $(-a^{ij}(x)\partial_{ij})^s$ for $0 < s < 1$ with Hölder coefficients and characterize a Poisson problem driven by $(-a^{ij}(x)\partial_{ij})^s$ with a local degenerate extension problem. An interior Harnack inequality for nonnegative solutions to such an extension equation with bounded, measurable coefficients is proved. This in turn implies the interior Harnack inequality for the fractional problem.

CHAPTER 1. INTRODUCTION

This dissertation pertains to the mathematical analysis of functional spaces related to fractional derivatives and fractional Laplacians and regularity theory of solutions to equations driven by fractional powers of nondivergence form elliptic differential operators. More precisely, we solve the following problems.

1. We define appropriate one-sided weighted Sobolev spaces $W^{1,p}(\mathbb{R},\omega)$ which are conducive for working with one-sided fractional derivatives. The weights $\omega \in A_p^-(\mathbb{R})$ are one-sided Sawyer weights. We prove a Bourgain–Brezis–Mironescu-type characterization of the weighted spaces $W^{1,p}(\mathbb{R},\omega)$ and $L^p(\mathbb{R},\omega)$ by showing that the respective limits

$$\lim_{\alpha\to 1^-}(D_{\mathrm{left}})^\alpha u(t) = u'(t) \qquad \text{and} \qquad \lim_{\alpha\to 0^+}(D_{\mathrm{left}})^\alpha u(t) = u(t) \tag{1.0.1}$$

   hold in $L^p(\mathbb{R},\omega)$ and almost everywhere. Here, $(D_{\mathrm{left}})^\alpha$ denotes the Marchaud left-derivative of order $0<\alpha<1$. We prove a similar characterization of the two-sided weighted Sobolev spaces $W^{2,p}(\mathbb{R}^n,\nu)$ and $L^p(\mathbb{R}^n,\nu)$ using fractional Laplacians of order $0<s<1$. Here, $\nu\in A_p(\mathbb{R}^n)$ are classical Muckenhoupt weights.

2. We define fractional powers of nondivergence form elliptic operators

$$Lu = -a^{ij}(x)\partial_{ij}u, \quad \mathrm{Dom}(L) = \{u \in C_0(\Omega)\cap W^{2,n}_{\mathrm{loc}}(\Omega) : Lu \in C_0(\Omega)\},$$

   using the method of semigroups, where $\Omega\subset\mathbb{R}^n$ is a bounded, Lipschitz domain and $a^{ij}(x)$ are Hölder continuous and uniformly elliptic. We prove Harnack inequality for nonnegative solutions $u\in \mathrm{Dom}(L)$ to the problem $L^s u = 0$, $0<s<1$, in a ball $B\subset\subset\Omega$. We characterize the Poisson problem driven by $L^s u$ with a local, degenerate extension equation.

1.1 Introduction

Fractional derivatives were first introduced as a purely mathematical concept. When referring to the nth order derivative $\frac{d^n u(t)}{dt^n}$, G. L'Höpital asked G. Leibniz in his 1695 letter [45]

"*What if* n *is* 1/2?"

Leibniz replied,

"*It will lead to a paradox. From this apparent paradox,*

one day useful consequences will be drawn."

Many years later, in 1819, S. F. Lacroix devoted only two pages of his 700 page textbook on differential and integral calculus to providing the first definition of a fractional derivative, with no apparent physical interpretations:

$$\frac{d^{1/2}}{dt^{1/2}}(t^p) = \frac{\Gamma(p+1)}{\Gamma(p+1/2)}\, t^{p-1/2}, \quad p > 0$$

where Γ denotes the Gamma function [41]. The first known application of fractional derivatives appeared shortly later in 1823 when N. H. Abel used fractional derivatives to model the tautochrone problem [1]. Since then, many scientists, like Riemann, Liouville, Riesz, and Weyl, have attempted to define derivatives of fractional order as seen in the very complete monograph by Samko–Kilbas–Marichev [64]. Moreover, within the last 20 years, there has been an explosion of interest of fractional derivatives in applied sciences. For example, fractional derivatives are used to capture the avalanche-like behavior and trapping effects of particles due to eddies in plasmas [22].

In contrast to classical derivatives for which, to compute $u'(t)$, we only need to know the values of u in a small neighborhood of t, fractional derivatives are nonlocal in nature. For example, the Marchaud left-fractional derivative of a function $u = u(t) : \mathbb{R} \to \mathbb{R}$ is given by

$$(D_{\text{left}})^\alpha u(t) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{t} \frac{u(h) - u(t)}{(t-h)^{1+\alpha}}\, dh, \quad 0 < \alpha < 1, \tag{1.1.1}$$

see [52]. This operator satisfies the Fourier transform identity

$$\widehat{(D_{\text{left}})^\alpha u}(\xi) = (i\xi)^\alpha \widehat{u}(\xi), \quad \xi \in \mathbb{R}.$$

By taking $\alpha = 1$, we can see that the definition agrees with the usual derivative. Notice from (1.1.1) that to determine $(D_{\text{left}})^\alpha u(t)$, one needs to know the entire history of u, i.e. all the values of u in $(-\infty, t]$. Fractional derivatives can be used to refine existing models in order to capture memory effects such as with world population growth and blood-alcohol level systems in humans [5].

While fractional derivatives can be regarded as nonlocal in *time*, fractional powers of second-order differential operators can be seen as nonlocal in *space*. Perhaps the most important nonlocal operator in space is the fractional Laplacian on $\mathbb{R}^n$. For a function $u = u(x) : \mathbb{R}^n \to \mathbb{R}$, we define $(-\Delta)^s u$ by

$$(-\Delta)^s u(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy, \quad 0 < s < 1, \tag{1.1.2}$$

where $P.V.$ denotes that the integral is taken in the principle value sense and $c_{n,s} > 0$ is a normalizing constant. This operator satisfies the Fourier transform identity

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\, \widehat{u}(\xi) \qquad \xi \in \mathbb{R}^n.$$

By taking $s = 1$, we can see that the definition agrees with the classical Laplacian and hence gives the name “fractional Laplacian.” Like (1.1.1), fractional Laplacians are nonlocal, but, in contrast to $(D_{\text{left}})^\alpha u(t)$ for which we need to know the values of u in one direction, to compute $(-\Delta)^s u(x)$, we need to know the values of u everywhere.

Fractional powers of differential operators are classical objects that have been studied from several points of view, including harmonic analysis [70], potential analysis [42], probability [10], functional analysis [80], pseudo-differential operator theory [77], fractional calculus [64], among others. Nonlocal equations of fractional order have received a lot of attention in recent years in the field of partial differential equations, mainly due to the work of Luis Caffarelli and collaborators [9, 17, 18, 19, 20, 21]. Indeed, over 90% of the papers on Google Scholar with the phrase “fractional Laplacian” in the title appear after Luis Silvestre’s thesis in 2005 [68].

Despite all this activity, there are still many open questions in the theory of fractional operators that need to be addressed. Given a model, the theory of PDEs requires refinements of the model to prove desired estimates. Just to mention one instance, the theory of pseudo-differential operators has proven to be insufficient for obtaining some fine regularity estimates, like Harnack inequalities, on Lipschitz domains.

1.2 Applications of fractional operators

We present some models that are the main motivation for the work presented in this dissertation.

1.2.1 Fractional derivatives

I. *Anomalous diffusion.* Let $u = u(x,t)$ be the probability of finding an object in position x at time t. The normal diffusion equation is given by

$$\frac{\partial}{\partial t}u = D\frac{\partial^2}{\partial x^2}u$$

where D is the diffusion constant. Spider monkeys, however, have been observed to remain in motion for a long period of time without changing direction. In this case, displacement grows faster than normal diffusion and is modeled by

$$\frac{\partial^\alpha}{\partial t^\alpha}u = D\frac{\partial^2}{\partial x^2}u$$

where $\alpha > 1$ is non-integer. On the other hand, when displacement grows slower than normal diffusion, such as for proteins diffusing across a cell or the movement of contaminants in ground water, the model utilizes a fractional time derivative of order $0 < \alpha < 1$. These are so-called anomalous diffusions. See [39] for these and more details.

II. *Viscoelasticity.* In continuum mechanics, when modeling how a continuous materials behaves under deformations, we need a relationship between the internal forces (stress, $\sigma = \sigma(t)$) and the measure of the deformation of the material (strain, $\varepsilon = \varepsilon(t)$). For elastic materials, such as springs, this relationship follows Hooke's law in that stress is proportional to strain:

$$\sigma(t) = E\varepsilon(t)$$

where E is material constant, called the Young's modulus. Viscous fluids, such as water or honey, follow Newton's law in that stress is proportional to the velocity of strain:

$$\sigma(t) = -\eta \frac{d}{dt}\varepsilon(t)$$

where η is a material constant, called the viscosity.

Viscoelastic materials, such as rubber and concrete, exhibit both elastic and viscous characteristics when undergoing deformations. Classical models of viscoelastic materials, such as the Maxwell and Kelvin-Voigt models, use combinations of elastic and viscous relations. Nevertheless, for some viscoelastic materials, such as gels and biological tissues, these turn out to be rather inadequate models. It has been observed experimentally that the stress-strain relationship for such materials can be effectively modeled with a fractional derivative relation

$$\sigma = C\frac{d^\alpha}{dt^\alpha}\varepsilon,$$

which cannot be expressed as a finite combination of strain and derivatives of strain (see [51, 57]).

As the result of an internship at Oak Ridge National Laboratory in 2018, my mentor at the lab Pablo Seleson, my major professor Pablo Raúl Stinga, and myself have established a model using fractional derivatives to describe viscoelastic solids with growing cracks. The work presented in this document is on the theory of Sobolev spaces that takes into account the history of fractional derivatives and is foundational towards the analysis of our model.

1.2.2 Fractional powers of nonlocal nondivergence form operators

III. *Random walks.* Consider a particle moving randomly in a bounded domain $\Omega \subset \mathbb{R}^n$ that is terminated at the boundary. We can describe this behavior with a Wiener process W_t (also called Brownian motion) that is killed at the first exit time τ of W from Ω. This process is generated by the Dirichlet Laplacian $-\Delta$.

Suppose now that we want to describe a particle that is randomly jumping in Ω and is killed when it tries to cross the boundary. In particular, we subordinate the process W_t with an s-stable symmetric Lévy subordinator T_t. This subordinated process is generated by the fractional power of the Dirichlet Laplacian $(-\Delta)^s$.

By considering the particle jumping in heterogeneous media, the process is generated by $-a^{ij}(x)\partial_{ij}$, for some elliptic coefficients $a^{ij}(x)$. The corresponding subordinated process is generated by the fractional operator $(-a^{ij}(x)\partial_{ij})^s$. See [36, 37, 69] for more details.

IV. *The fractional Monge–Ampère equation.* The classical Monge–Ampère equation, given by

$$\det D^2 u = f \quad \text{for a convex solution } u : \Omega \subset \mathbb{R}^n \to \mathbb{R},$$

appears in the theories of prescribed Gauss curvature, optimal transport, and fluid dynamics. See [28, 34] for more on applications and theory. Consider the class $\mathcal{M}$ of $n \times n$ symmetric, positive definite matrices A with constant coefficients such that $\det(A) = 1$. For a convex, C^2 function u, one can check that

$$\begin{aligned} n \det(D^2 u(x))^{1/n} &= \inf\{\Delta(u \circ A)(A^{-1}x) : A \in \mathcal{M}\} \\ &= \inf\{\operatorname{trace}(A^2 D^2 u(x)) : A \in \mathcal{M}\}. \end{aligned}$$

The infimum is attained at $A^2 = \det(D^2 u)^{1/n}(D^2 u)^{-1}$.

Define the operators L_A for $A \in \mathcal{M}$ by

$$L_A u(x) = -\Delta(u \circ A)(A^{-1}x) = -\operatorname{trace}(A^2 D^2 u(x)).$$

Caffarelli–Charro [15] defined a fractional Monge–Ampère operator by

$$\mathcal{D}_s u(x) = \inf\{-(-\Delta)^s(u \circ A)(A^{-1}x) : A \in \mathcal{M}\}, \quad 0 < s < 1.$$

Stinga–Jhaveri [76] showed that this definition coincides with

$$\mathcal{D}_s u(x) = \inf\{-(L_A)^s u(x) : A \in \mathcal{M}\},$$

where $(L_A)^s$ is the fractional power of L_A.

We remark that both the local and fractional Monge–Ampère operators are degenerate elliptic. Indeed, in dimension 2, the matrices

$$A_\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix} \quad \text{for } \varepsilon > 0$$

are in the class $\mathcal{M}$ but degenerate as $\varepsilon \to 0^+$. Nevertheless, for the local Monge–Ampère equation, if u is convex, $D^2_{ee}u \leq M_0$, and $\det(D^2u) \geq \eta_0 > 0$, then one can take the matrices $A > \lambda I$ in the computation of the infimum and the equation becomes uniformly elliptic (see [28]). In the fractional setting, if u is Lipschitz, semiconcave, and $\mathcal{D}_s u \geq \eta_0 > 0$ then, again, one can take the matrices $A > \lambda I$ in the computation of the infimum and the equation becomes uniformly elliptic (see [15]). Under these assumptions, it follows that

$$\mathcal{D}_s u(x) = -(-a^{ij}(x)\partial_{ij})^s u(x)$$

for some bounded, measurable, uniformly elliptic coefficients $a^{ij}(x)$.

1.3 Description of results

The results of this dissertation are contained in Chapter 2 and Chapter 3. Here, we describe those results.

1.3.1 Chapter 2: One-sided weighted fractional Sobolev spaces

The work in this chapter has been published in *Nonlinear Analysis*, 2020 [75].

Since L'Höpital's letter to Leibniz, many "derivatives of fractional order" have been defined [64]. It is my opinion that any reasonable definition of derivative D^α of fractional order $0 < \alpha < 1$ should at least satisfy the relations $D^\alpha[D^\beta u](t) = D^{\alpha+\beta}u(t)$, and

$$\lim_{\alpha \to 1^-} D^\alpha u(t) = u'(t) \qquad \text{and} \qquad \lim_{\alpha \to 0^+} D^\alpha u(t) = u(t) \tag{1.3.1}$$

whenever u is a sufficiently smooth function. Recall that the Marchaud left fractional derivative, given by

$$(D_{\text{left}})^\alpha u(t) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^t \frac{u(\tau) - u(t)}{(t-\tau)^{1+\alpha}}\, d\tau, \tag{1.3.2}$$

takes into account the values of u to the left of t (the *past*). Similarly, the Marchaud right fractional derivative

$$(D_{\text{right}})^{\alpha}u(t) = \frac{1}{\Gamma(-\alpha)} \int_t^{\infty} \frac{u(\tau) - u(t)}{(\tau - t)^{1+\alpha}}\, d\tau \tag{1.3.3}$$

looks at u only to the right of t (the *future*). These were first introduced by André Marchaud in his 1927 dissertation [52] (see also, for example, [2, 3, 4, 8, 64] for theory and applications). One can check that if u is a Schwartz class function, then (1.3.1) holds.

On the other hand, in 2001, Bourgain–Brezis–Mironescu famously characterized the Sobolev space $W^{1,p}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, in terms of the limit as $s \to 1^-$ of fractional Gagliardo seminorms, namely, the seminorms of the fractional Sobolev spaces $W^{s,p}(\Omega)$ [11]:

$$\lim_{s \to 1^-} (1 - s) \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy = c_{n,p} \int_\Omega |Du|^p\, dx.$$

In 2002, Maz'ya–Shaposhnikavo complimented their work by using the limit as $s \to 0^+$ to characterize the space $L^p(\mathbb{R})$ [58]:

$$\lim_{s \to 0^+} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy = c_{n,p} \int_{\mathbb{R}^n} |u|^p\, dx.$$

These characterizations embody the same flavor as showing (1.3.1) in L^p.

We define, for the first time, weighted Sobolev spaces that take into account the one-sided behavior of fractional derivatives. We prove characterizations of such spaces by studying the limits of fractional derivatives in the almost everywhere and L^p senses.

In this regard, we remark that classical Sobolev spaces make no distinction between left and right classical derivatives. Indeed, we recall that hidden within the limit definition of the usual derivative, $u'(t)$, are the following one-sided limits

$$\frac{d^-}{dt}u(t) = \lim_{h \to 0^-} \frac{u(t + h) - h(t)}{h}, \qquad \frac{d^+}{dt}u(t) = \lim_{h \to 0^+} \frac{u(t + h) - h(t)}{h}.$$

The property that a function is differentiable from only one-side, say the left, is lost in the weak setting since, for a test function $\varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \frac{d^-}{dt}u(t)\varphi(t)\, dt = -\int_{\mathbb{R}} u(t)\frac{d^+}{dt}\varphi(t)\, dt = -\int_{\mathbb{R}} u(t)\varphi'(t)\, dt = u'(\varphi).$$

In other words, $\frac{d^-}{dt}u = u'$ in the sense of distributions.

We believe that a more natural, appropriate class of functions to consider is the weighted Sobolev space $W^{1,p}(\mathbb{R},\omega)$, $1 \le p < \infty$, but where ω is now a *one-sided* Sawyer weight in $A_p^-(\mathbb{R})$ (for left-sided fractional derivatives) or in $A_p^+(\mathbb{R})$ (for right-sided fractional derivatives). We define these one-sided fractional Sobolev spaces as

$$W^{1,p}(\mathbb{R},\omega) = \big\{u \in L^p(\mathbb{R},\omega) : u' \in L^p(\mathbb{R},\omega)\big\}$$

with the norm

$$\|u\|^p_{W^{1,p}(\mathbb{R},\omega)} = \|u\|^p_{L^p(\mathbb{R},\omega)} + \|u'\|^p_{L^p(\mathbb{R},\omega)}$$

for $1 \le p < \infty$. The Sawyer weights $\omega \in A_p^-(\mathbb{R})$ are the good weights for the original one-sided Hardy–Littlewood maximal function [35, p. 92]:

$$M^- u(t) = \sup_{h>0} \frac{1}{h} \int_{t-h}^{t} |u(\tau)|\, d\tau.$$

Indeed, M^- is bounded in $L^p(\mathbb{R},\omega)$ if and only if $\omega \in A_p^-(\mathbb{R})$, $1 < p < \infty$, see [66], and M^- is bounded from $L^1(\mathbb{R},\omega)$ into weak-$L^1(\mathbb{R},\omega)$ if and only if $\omega \in A_1^-(\mathbb{R})$, see [55]. For more details, see Section 2.2 and also [54].

We first develop one-sided distributional spaces in which fractional derivatives have sense. Then we show that in such a setting one can always define $(D_{\text{left}})^\alpha u$ as a distribution for any function $u \in L^p(\mathbb{R},\omega)$, $\omega \in A_p^-(\mathbb{R})$. We additionally prove that smooth functions with compact support are dense in $W^{1,p}(\mathbb{R},\omega)$.

As done by Silvestre for the fractional Laplacian [68], we show that if $u \in C^{\alpha+\varepsilon}(I)$ for some $\varepsilon > 0$ and open set $I \in \mathbb{R}$, then $(D_{\text{left}})^\alpha u$, which *a priori* is a distribution, is a continuous function in I and coincides with (1.3.2) in I.

One of the main results of this chapter is that our one-sided weighted Sobolev spaces $W^{1,p}(\mathbb{R},\omega)$ can be characterized by the almost everywhere and L^p limits of fractional derivatives. We show that $u \in W^{1,p}(\mathbb{R}^n,\omega)$, $\omega \in A_p^-(\mathbb{R})$, if and only if

$$\lim_{\alpha\to 1^-} (D_{\text{left}})^\alpha u = u' \quad \text{in } L^p(\mathbb{R},\omega) \text{ and a.e. in } \mathbb{R}.$$

If $u \in W^{1,p}(\mathbb{R}^n, \omega)$, we also show that

$$\lim_{\alpha \to 0^+} (D_{\text{left}})^{\alpha} u = u \quad \text{in } L^p(\mathbb{R}, \omega) \text{ and a.e. in } \mathbb{R}.$$

This part of my thesis is largely a study in harmonic analysis and function spaces. In general, statements involving a.e. convergence are proved by considering the underlying maximal operators, see, for example, [27, Chapter 2]. A key estimate that we are able to deduce is the pointwise maximal inequality

$$\sup_{0<\alpha<1} \left| \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{t} \frac{u(\tau) - u(t)}{(t-\tau)^{1+\alpha}} \, d\tau \right| \leq C\big(M^{-}(u')(t) + M^{-}u(t)\big) \quad \text{a.e. in } \mathbb{R} \tag{1.3.4}$$

for any $u \in W^{1,p}(\mathbb{R}, \omega)$, where the constant $C > 0$ is independent of u, p, t, and α.

As a consequence of (1.3.4), the pointwise formula (1.3.2) is well-defined for $u \in W^{1,p}(\mathbb{R}, \omega)$ when $1 < p < \infty$. By a distributional argument, we are able to show that $(D_{\text{left}})^{\alpha} u$ coincides with Marchaud's pointwise formula almost everywhere. Notice that the object on the left-hand side of (1.3.4) is a maximal operator taken with respect to the orders of the fractional derivatives.

The one-sided $L^p(\mathbb{R}, \omega)$ spaces, with $\omega \in A_p^{-}(\mathbb{R})$, are also natural for the Marchaud left fractional derivative in the sense of the Fundamental Theorem of Fractional Calculus. Indeed, let $u \in L^p(\mathbb{R}, \omega)$ and consider the left-sided Weyl fractional integral [64]

$$(D_{\text{left}})^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} \frac{u(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau.$$

It was proved in [8] that $(D_{\text{left}})^{\alpha} (D_{\text{left}})^{-\alpha} u(t) = u(t)$ in $L^p(\mathbb{R}, \omega)$ and for a.e. $t \in \mathbb{R}$, for any $0 < \alpha < 1$ and any $u \in L^p(\mathbb{R}, \omega)$. Our work complements their results and shows that the one-sided weighted Sobolev spaces $W^{1,p}(\mathbb{R}, \omega)$ are the correct spaces for one-sided fractional derivatives.

We also consider the weighted Sobolev spaces $W^{2,p}(\mathbb{R}^n, \nu)$ that we define by

$$W^{2,p}(\mathbb{R}^n, \nu) = \big\{ u \in L^p(\mathbb{R}^n, \nu) : Du, D^2 u \in L^p(\mathbb{R}^n, \nu) \big\}$$

with the norm

$$\|u\|_{W^{2,p}(\mathbb{R}^n,\nu)}^p = \|u\|_{L^p(\mathbb{R}^n,\nu)}^p + \|Du\|_{L^p(\mathbb{R}^n,\nu)}^p + \|D^2 u\|_{L^p(\mathbb{R}^n,\nu)}^p$$

where ν is a weight in the Muckenhoupt class $A_p(\mathbb{R}^n)$, $1 \leq p < \infty$. We recall that the $A_p(\mathbb{R}^n)$ Muckenhoupt weights are the good weights for the classical Hardy–Littlewood maximal function M on $\mathbb{R}^n$. We prove a similar characterization of our weighted Sobolev spaces $W^{2,p}(\mathbb{R}^n, \nu)$ with the limits

$$\lim_{s\to 1^-}(-\Delta)^s u = -\Delta u \quad \text{and} \quad \lim_{s\to 0^+}(-\Delta)^s u = u$$

in $L^p(\mathbb{R}^n, \nu)$ and almost everywhere. For this, we prove the following maximal estimate

$$\sup_{0<s<1}\sup_{\varepsilon>0}\left| c_{n,s}\int_{|x-y|>\varepsilon}\frac{u(y)-u(x)}{|x-y|^{n+2s}}\,dy\right| \leq C_n\big(M(D^2u)(x)+Mu(x)\big) \quad \text{a.e. in } \mathbb{R}^n, \tag{1.3.5}$$

for any $u \in W^{2,p}(\mathbb{R}^n, \nu)$, where the constant $C_n > 0$ depends only on dimension.

Other authors have considered similar questions for abstract settings, see for example [25, 79]. In particular, their results apply to Ahlfors-regular metric spaces. On the other hand, some weighted fractional spaces with power weights were defined in [26]. Nevertheless, neither are our weighted spaces Ahlfors-regular nor do our weighted spaces correspond to those in [26].

1.3.2 Chapter 3: Harnack inequality for fractional nondivergence form elliptic equations

The work in this chapter will soon be submitted for publication [74].

Harnack inequality is a very important regularity estimate in partial differential equations that was first stated by Axel von Harnack in the 1800's for nonnegative harmonic functions. The simplest statement of the theorem is as follows: there exists constant $C > 0$ depending only on dimension such that for any nonnegative harmonic function $u : B_1 \subset \mathbb{R}^n \to \mathbb{R}$, we have that

$$\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u.$$

It is essential to note that the estimate holds for *all* nonnegative harmonic functions in B_1 with the same constant C. The proof follows, for example, from the mean value property for harmonic functions.

Harnack inequality has since been studied for elliptic and parabolic PDEs in various settings. Moser proved it for divergence form operators [60, 61], Krylov–Safanov proved it for nondivergence form elliptic operators [40], and Caffarelli proved it for viscosity solutions to fully nonlinear elliptic equations [14, 12]. Other notable names include De Giorgi, Nash, Nirenberg, Serrin, Trudinger, among others, see [31].

An important consequence of Harnack inequality is that weak solutions to divergence form and viscosity solutions to nondivergence form elliptic equations with bounded, measurable coefficients are Hölder continuous. This has important consequences for nonlinear equations in proving higher regularity estimates [14, 31].

For our setting, let $L : \mathrm{Dom}(L) \to C_0(\Omega)$ be the operator defined by

$$Lu = -a^{ij}(x)\partial_{ij}u, \quad \mathrm{Dom}(L) = \{u \in C_0(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega) : Lu \in C_0(\Omega)\}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, Lipschitz domain and $a^{ij} : \Omega \to \mathbb{R}$ are Hölder continuous, uniformly elliptic coefficients. We prove interior Harnack inequality for nonnegative solutions $u \in \mathrm{Dom}(L)$ to

$$(-a^{ij}(x)\partial_{ij})^s u = 0 \quad \text{in } B \tag{1.3.6}$$

where $0 < s < 1$ and $B \subset\subset \Omega$ is a Euclidean ball. Indeed, we show that there exist positive constants $C_H = C_H(n, \lambda, \Lambda, s) > 1$, $\kappa = \kappa(n, \lambda, \Lambda, s) < 1$, and $\hat{K} = \hat{K}(n, \lambda, \Lambda, s) \geq 1$, such that if $B_R(x_0)$ is a ball such that $B_{\hat{K}R}(x_0) \subset\subset B \subset\subset \Omega$, then

$$\sup_{B_{\kappa R}(x_0)} u \leq C_H \inf_{B_{\kappa R}(x_0)} u.$$

Furthermore, solutions to (1.3.6) are locally Hölder continuous in B. We mention that Grubb [32, 33] and Seeley [67] studied fractional powers of nondivergence form operators, but their work utilizes the theory of pseudo-differentiable operators and does not contain our result.

The first difficulty is how to define the nonlocal operator $L^s u$ in an appropriate way. We use the method of semigroups to define $L^s u$, $0 < s < 1$, using the definition of Balakrishnan [7], by

$$L^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-tL} u(x) - u(x) \right) \frac{dt}{t^{1+s}}.$$

Here, $v = e^{-tL}u$ is the C_0-semigroup generated by L which solves the heat equation with initial data u:

$$\begin{cases} \partial_t v(x,t) = -Lv(x,t) & \text{in } \Omega \times (0,\infty) \\ v(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty) \\ v(x,0) = u(x) & \text{on } \Omega \times \{t=0\}. \end{cases}$$

For more details, see [80]. It can be seen from this definition that L^s is nonlocal in Ω.

To study regularity properties of equations with L^s, we extend the nonlocal equation one dimension to $\mathbb{R}^{n+1}$ with a local extension characterization by Galé–Miana–Stinga [30]. Indeed, for $u \in C_0(\Omega)$, the function U given by

$$U(x,z) = \frac{(2s)z}{4^s\Gamma(s)} \int_0^\infty e^{-\frac{s^2}{t} z^{\frac{1}{s}}} e^{-tL} u(x) \frac{dt}{t^{1+s}}$$

is a solution to

$$\begin{cases} a_{ij}(x)\partial_{ij}U(x,z) + z^{2-1/s}U_{zz}(x,z) = 0 & \text{in } \Omega \times [0,\infty) \\ U(x,z) = 0 & \text{on } \Omega \times (0,\infty) \\ U(x,0) = u(x) & \text{on } \Omega \times \{z=0\}. \end{cases} \tag{1.3.7}$$

It turns out that if $u \in \mathrm{Dom}(L)$, then

$$-\lim_{z\to 0+} \partial_z U(x,z) = c_s(-a_{ij}(x)\partial_{ij})^s u(x)$$

for some constant $c_s > 0$. Observe that the Neumann condition recovers the fractional operator. Hence, to prove Harnack inequality for the nonlocal equation (1.3.6), we study the local, degenerate equation (1.3.7) and take the trace at $z = 0$. We mention that the extension characterization in [30] is more general than the one of Caffarelli–Silvestre for $(-\Delta)^s$ [19]. See also the work of Stinga–Torrea for an extension characterization of fractional operators in Hilbert spaces [73].

Towards this end, we define the even reflection of U by $\tilde{U}(x,z) = U(x,|z|)$. For convenience, we continue to use the notation U. We prove Harnack inequality for nonnegative solutions $U \in C^2(\Omega \times [\mathbb{R} \setminus \{z=0\}]) \cap C(\overline{\Omega} \times \mathbb{R})$, such that $U_z \in C([0,\infty); C_0(\Omega))$, to

$$\begin{cases} a^{ij}(x)\partial_{ij}U(x,z) + |z|^{2-1/s}U_{zz}(x,z) = 0 & \text{in } B \times \{z \neq 0\} \\ -\partial_{z+}U(x,0) = 0 & \text{on } B \times \{z=0\} \end{cases} \tag{1.3.8}$$

where $a^{ij}(x)$ are bounded, measurable, elliptic coefficients and $B \subset\subset \Omega$ is a Euclidean ball. As a consequence of (1.3.7), we obtain Harnack inequality for (1.3.6).

Next, we recast (1.3.8) as an equation comparable to a linearized Monge–Ampère equation. For this, we recall some details about the Monge–Ampère equation. The Monge-Ampère equation, given by

$$\det D^2\psi = F, \tag{1.3.9}$$

is a fully nonlinear second-order partial differential equation. One approach to studying regularity estimates of (1.3.9) is to take the directional derivative ∂_e in the direction of e to obtain

$$\operatorname{trace}(\det(D^2\psi)(D^2\psi)^{-1}D^2(\partial_e\psi)) = \partial_e F.$$

Notice that $A_\psi = \det(D^2\psi)(D^2\psi)^{-1}$ is the matrix of cofactors of $D^2\psi$. If we define $u = \partial_e\psi$ and $f = \partial_e F$, then u solves the linearized Monge-Ampère equation

$$\operatorname{trace}(A_\psi(x)D^2u) = f. \tag{1.3.10}$$

This equation is a linear, nondivergence form equation and is elliptic as soon as $D^2\psi > 0$ and $f > 0$. However, (1.3.10) is not uniformly elliptic in general because the eigenvalues of A_ψ are not controlled.

For our degenerate equation (1.3.7), we define function $\Phi = \Phi(x,z) : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\Phi(x,z) = \frac{1}{2}|x|^2 + \frac{s^2}{1-s}|z|^{\frac{1}{s}}.$$

Notice that Φ is strictly convex and $C^1(\mathbb{R}^{n+1})$. This function was also considered by Maldonado–Stinga in [50] to study the extension problem for the fractional nonlocal linearized Monge–Ampère equation $L^s_\varphi u = f$ associated to a convex function $\varphi \in C^3(\mathbb{R}^n)$. Since the Hessian of Φ is

$$D^2\Phi(x,z) = \begin{pmatrix} I & 0 \\ 0 & |z|^{\frac{1}{s}-2} \end{pmatrix},$$

the linearized Monge–Ampère operator for $D^2\Phi$ is

$$\operatorname{trace}((D^2\Phi)^{-1}D^2U) = \Delta_x U + |z|^{2-\frac{1}{s}}\partial_{zz}U. \tag{1.3.11}$$

As the coefficients $a^{ij}(x)$ are uniformly elliptic, there exist constants $0 < \lambda < \Lambda < \infty$ such that

$$\lambda\,|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda\,|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{z=0\},\ x \in \mathbb{R}^n,$$

from which we can see that the coefficients in (1.3.7) are comparable to the coefficients in (1.3.11).

An important feature of the linearized Monge–Ampère equation is the intrinsic geometry that was first discovered by Caffarelli–Guitérrez [16]. They proved Harnack inequality for nonnegative solutions to (1.3.10) with $f \equiv 0$ where the Euclidean balls are replaced by Monge–Ampère sections. The Monge–Ampère sections associated to a convex, C^1 function ψ are the sublevel sets of $\psi - \ell$ where ℓ is a linear function. The Monge–Ampère measure associated to ψ of a Borel set E is given by $\mu_\psi(E) = |D\psi(E)|$.

It turns out that the geometry of our degenerate equation (1.3.7) is given by the Monge–Ampère sections S_R associated to Φ, that is, the sublevel sets of $\Phi - \ell$. In many instances, we find it more natural to consider Monge–Ampère cubes Q_R. We will first prove Harnack inequality in these cubes (see Section 3.5.5).

Harnack inequality has been studied for the linearized Monge–Ampère equation by Caffarelli–Gutierrez [16], Gutierrez [34], Le [43], Maldonado [48], Le–Savin [44], among others. In each case, they either assume that the matrix D^2u is bounded away from zero and infinity or that the convex function ψ is sufficiently regular. In [49], Maldonado proved Harnack inequality for certain degenerate elliptic equations, but his techniques are different than the ones presented in this dissertation and do not include the case in which $1/2 < s < 1$.

We develop a method of sliding paraboloids as Savin did in the Euclidean case for uniformly elliptic equations [65]. Similar approaches were used by Le when $\lambda I \le D^2\psi \le \Lambda I$, $\psi \in C^2$ (making the underlying Monge–Ampère measure comparable to the Lebesgue measure) [43] and recently by Mooney for the uniformly elliptic case [59]. For our setting, we use the Monge–Ampère geometry which brings additional challenges since Φ is only C^1 and $D^2\Phi$ is degenerate/singular. Moreover, we cannot use any divergence form structure. Harnack inequality for (1.3.8) was proved by Maldonado–Stinga for the fractional linearized Monge–Ampère equation when the matrix $a^{ij}(x)$

is the identity matrix or comes from the matrix of cofactors $D^2\psi$ for a C^3 function ψ [50]. Their proof however relies on the Monge–Ampère structure in divergence form.

For this method, we define paraboloids P of opening $a > 0$ with vertex $(x_v, z_v) \in \mathbb{R}^{n+1}$ by

$$P(x, z) = -a(\Phi(x, z) - \Phi(x_v, z_v) - \langle D\Phi(x_v, z_v), (x, z) - (x_v, z_v)\rangle) + c_v.$$

We lift these paraboloids from below until they touch the graph of U in a cube Q_R for the first time, and we measure the set contact points with the underlying Monge–Ampère measure. Then we show that by increasing the opening of these paraboloids, they almost cover Q_R in measure, which ultimately leads to Harnack inequality. More precisely, the proof relies on three key lemmas.

The first lemma is similar to the Alexandroff–Bakelman–Pucci estimate for fully nonlinear equations (see [14]). We prove that if we lift paraboloids of fixed opening $a > 0$ with vertices in a closed, bounded set from below until they touch the graph of U, then, by using the equation, the Monge–Ampère measure of the contact points is a universal proportion of the Monge–Ampère measure of the vertices.

The second lemma is a measure estimate. Suppose that U can be touched from below with paraboloid P of opening $a > 0$ in a cube Q_r. We show that the set in which U can be touched from below by paraboloids of increased opening $Ca > 0$ in a smaller cube $Q_{\eta r}$, $0 < \eta < 1$, make up a universal proportion of Q_r. The proof relies on a delicate barrier argument to localize the equation and control the growth of $U - P$ in $Q_{\eta r}$.

The third lemma is similar to the Calderón-Zygmund decomposition. We use a Vitali covering lemma in the Monge–Ampère geometry to show that, as we increase the openings of the paraboloids, the measure of the contact points almost cover the domain in measure. By also controlling the height of the paraboloids, we consequently show that the measure of the set where U is large is small.

These ingredients allow us to prove Harnack inequality for the extension (1.3.8). By restricting back to $z = 0$, we obtain Harnack inequality for the fractional Poisson problem (1.3.6).

Stinga–Maldonado also proved a critical-density measure estimate and local boundedness for the fractional linearized Monge–Ampère equation for the fractional linearized Monge–Ampère equation [50]. We show that those results hold in our case and the proof is identical.

CHAPTER 2. ONE SIDED WEIGHTED FRACTIONAL SOBOLEV SPACES

2.1 Main results

We formally state the main results of this chapter.

The one-sided Sawyer weights $A_p^-(\mathbb{R})$ are the good weights for the original one-sided Hardy-Littlewood maximal functions M^- and M^+:

$$M^-u(t) = \sup_{h>0} \frac{1}{h} \int_{t-h}^{t} |u(\tau)|\, d\tau \quad \text{and} \quad M^+u(t) = \sup_{h>0} \frac{1}{h} \int_{t}^{t+h} |u(\tau)|\, d\tau$$

Indeed, $\omega \in A_p^-$ for $1 < p < \infty$ if and only if M^- is bounded from $L^p(\mathbb{R},\omega)$ into $L^p(\mathbb{R},\omega)$. Also, $\omega \in A_1^-$ if and only if $L^1(\mathbb{R},\omega)$ into weak-$L^1(\mathbb{R},\omega)$. For more details, see Section 2.2.

We define the one-sided weighted Sobolev spaces $W^{1,p}(\mathbb{R},\omega)$, $\omega \in A_p^-(\mathbb{R})$, $1 \le p < \infty$ by

$$W^{1,p}(\mathbb{R},\omega) = \{u \in L^p(\mathbb{R},\omega) : u' \in L^p(\mathbb{R},\omega)$$

with the norm

$$\|u\|^p_{W^{1,p}(\mathbb{R},\omega)} = \|u\|^p_{L^p(\mathbb{R},\omega)} + \|u'\|^p_{L^p(\mathbb{R},\omega)}.$$

Theorem 2.1.1 ($W^{1,p}(\mathbb{R},\omega)$ and limits of left fractional derivatives)**.** *Let $u \in L^p(\mathbb{R},\omega)$, where $\omega \in A_p^-(\mathbb{R})$, for $1 \le p < \infty$.*

(a) If $u \in W^{1,p}(\mathbb{R},\omega)$, then the distribution $(D_{\text{left}})^\alpha u$ coincides with a function in $L^p(\mathbb{R},\omega)$ and

$$(D_{\text{left}})^\alpha u(t) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{t} \frac{u(\tau) - u(t)}{(t-\tau)^{1+\alpha}}\, d\tau \quad \text{for a.e. } t \in \mathbb{R} \tag{2.1.1}$$

with

$$\|(D_{\text{left}})^\alpha u\|_{L^p(\mathbb{R},\omega)} \le C_{p,\omega}\big(\|u\|_{L^p(\mathbb{R},\omega)} + \|u'\|_{L^p(\mathbb{R},\omega)}\big) \tag{2.1.2}$$

for some constant $C_{p,\omega} > 0$. Moreover,

$$\lim_{\alpha \to 1^-} (D_{\text{left}})^\alpha u = u' \quad \text{in } L^p(\mathbb{R},\omega) \text{ and a.e. in } \mathbb{R} \tag{2.1.3}$$

and

$$\lim_{\alpha\to 0^+} (D_{\text{left}})^\alpha u = u \quad a.e. \ in\ \mathbb{R}. \tag{2.1.4}$$

Furthermore, the limit in (2.1.4) holds also in $L^p(\mathbb{R},\omega)$ *when* $1 < p < \infty$*, and in* $L^1(\mathbb{R},\omega)$ *when* $p = 1$ *and* $M^- u, M^- u' \in L^1(\mathbb{R},\omega)$*.*

(*b*) *Conversely, suppose that* $(D_{\text{left}})^\alpha u \in L^p(\mathbb{R},\omega)$ *and that* $(D_{\text{left}})^\alpha u$ *converges in* $L^p(\mathbb{R},\omega)$ *as* $\alpha \to 1^-$*. Then* $u \in W^{1,p}(\mathbb{R},\omega)$ *and (2.1.3) holds.*

(*c*) *Alternatively, suppose that* $(D_{\text{left}})^\alpha u \in L^p(\mathbb{R},\omega)$ *and that* $(D_{\text{left}})^\alpha u$ *converges in* $L^p(\mathbb{R},\omega)$ *as* $\alpha \to 0^+$*. Then (2.1.4) holds and, as a consequence,* $(D_{\text{left}})^\alpha u \to u$ *in* $L^p(\mathbb{R},\omega)$ *as* $\alpha \to 0^+$*.*

Theorem 2.1.2. *There exists a universal constant* $C > 0$ *such that for any* $u \in W^{1,p}(\mathbb{R},\omega)$*,* $\omega \in A_p^-(\mathbb{R})$*,* $1 \le p < \infty$*, we have*

$$\sup_{0<\alpha<1} \left| \frac{1}{\Gamma(-\alpha)} \int_0^\infty (u(t-\tau) - u(t)) \frac{d\tau}{\tau^{1+\alpha}} \right| \le C \left(M^-(u')(t) + M^- u(t) \right)$$

for a.e. $t \in \mathbb{R}$*.*

Though we established Theorems 2.1.1 and 2.1.2 for the left fractional derivative, all the arguments carry on by replacing D_{left} by D_{right} and $A_p^-(\mathbb{R})$ by $A_p^+(\mathbb{R})$. Hence, for the rest of the chapter, we will only consider the case of D_{left} and left-sided Sawyer weights.

The class of Muckenhoupt weights $A_p(\mathbb{R})$ are the good weights for the two-sided Hardy-Littlewood maximal functions M:

$$Mu(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |u(y)|\, dy$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x. Indeed, $\omega \in A_p$ for $1 < p < \infty$ if and only if M is bounded from $L^p(\mathbb{R}^n,\nu)$ into $L^p(\mathbb{R}^n,\nu)$. Also, $\omega \in A_1$ if and only if $L^1(\mathbb{R}^n,\nu)$ into weak-$L^1(\mathbb{R}^n,\nu)$. For more details, see Section 2.4.

We define the weighted Sobolev spaces $W^{2,p}(\mathbb{R}^n,\nu)$, $\nu \in A_p(\mathbb{R}^n)$, $1 \le p < \infty$ by

$$W^{2,p}(\mathbb{R}^n,\nu) = \{u \in L^p(\mathbb{R}^n,\nu) : u' \in L^p(\mathbb{R},\nu)$$

with the norm

$$\|u\|^p_{W^{2,p}(\mathbb{R}^n,\nu)} = \|u\|^p_{L^p(\mathbb{R}^n,\nu)} + \|Du\|^p_{L^p(\mathbb{R}^n,\nu)} + \|D^2u\|^p_{L^p(\mathbb{R}^n,\nu)}$$

In the following statement, $\{e^{t\Delta}\}_{t\geq 0}$ denotes the heat semigroup generated by the Laplacian on $\mathbb{R}^n$.

Theorem 2.1.3 ($W^{2,p}(\mathbb{R}^n,\nu)$ and limits of fractional Laplacians)**.** *Let $u \in L^p(\mathbb{R}^n,\nu)$, where $\nu \in A_p(\mathbb{R}^n)$, for $1 \leq p < \infty$.*

(a) *If $u \in W^{2,p}(\mathbb{R}^n,\nu)$, then the distribution $(-\Delta)^s u$ coincides with a function in $L^p(\mathbb{R}^n,\nu)$ and*

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta}u(x) - u(x)\right) \frac{dt}{t^{1+s}} \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{2.1.5}$$

In addition,

$$(-\Delta)^s u(x) = c_{n,s} \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and in } L^p(\mathbb{R}^n,\nu) \tag{2.1.6}$$

with

$$\|(-\Delta)^s u\|_{L^p(\mathbb{R}^n,\nu)} \leq C_{n,p,\nu}\big(\|u\|_{L^p(\mathbb{R}^n,\nu)} + \|\Delta u\|_{L^p(\mathbb{R}^n,\nu)}\big) \tag{2.1.7}$$

for some constant $C_{n,p,\nu} > 0$. Moreover,

$$\lim_{s\to 1^-} (-\Delta)^s u = -\Delta u \quad \text{in } L^p(\mathbb{R}^n,\nu) \text{ and a.e. in } \mathbb{R}^n \tag{2.1.8}$$

and

$$\lim_{s\to 0^+} (-\Delta)^s u = u \quad \text{a.e. in } \mathbb{R}^n. \tag{2.1.9}$$

Furthermore, the limit in (2.1.9) holds also in $L^p(\mathbb{R}^n,\nu)$ when $1 < p < \infty$, and in $L^1(\mathbb{R}^n,\nu)$ when $p = 1$ and $Mu, M(D^2u) \in L^1(\mathbb{R}^n,\nu)$.

(b) *Conversely, suppose that $(-\Delta)^s u \in L^p(\mathbb{R}^n,\nu)$ and that $(-\Delta)^s u$ converges in $L^p(\mathbb{R}^n,\nu)$ as $s \to 1^-$. If $1 < p < \infty$ then $u \in W^{2,p}(\mathbb{R}^n,\nu)$ and (2.1.8) holds. If $p = 1$, then $D^2u \in$ weak-$L^1(\mathbb{R}^n,\nu)$.*

(c) *Alternatively, suppose that $(-\Delta)^s u \in L^p(\mathbb{R}^n,\nu)$ and that $(-\Delta)^s u$ converges in $L^p(\mathbb{R}^n,\nu)$ as $s \to 0^+$. Then (2.1.9) holds and, as a consequence, $(-\Delta)^s u \to u$ in $L^p(\mathbb{R}^n,\nu)$ as $s \to 0^+$.*

Theorem 2.1.4. *There exists a constant $C_n > 0$ such that for any $u \in W^{2,p}(\mathbb{R}^n, \nu)$, $\nu \in A_p(\mathbb{R}^n)$, $1 \le p < \infty$, we have*

$$\sup_{0<s<1} \sup_{\varepsilon>0} \left| c_{n,s} \int_{|y|>\varepsilon} \frac{u(x-y) - u(x)}{|y|^{n+2s}} \, dy \right| \le C_n \left(M(D^2 u)(x) + Mu(x) \right)$$

for almost every $x \in \mathbb{R}^n$.

This rest of the chapter is organized as follows. Section 2.2 contains preliminary results on one-sided Sawyer weights, the new distributional setting for one-sided fractional derivatives, and the proof of the Theorem 2.1.2. Theorem 2.1.1 is proved in Section 2.3. The fractional Laplacian in weighted Lebesgue spaces is studied in detail in Section 2.4, where we also show the proof of Theorem 2.1.4 Finally, Section 2.5 contains the proof of Theorem 2.1.3.

We denote by $\mathcal{S}(\mathbb{R}^n)$ the class of Schwartz functions on $\mathbb{R}^n$. We always take $0 < \alpha, s < 1$. We will use the following inequality: for any fixed $\rho > 0$ there exists $C_\rho > 0$ such that, for every $r > 0$,

$$e^{-r} r^{\rho} \le C_\rho e^{-r/2}. \tag{2.1.10}$$

For a measure space (X, μ), we define the space weak-$L^1(X, \mu)$ as the set of measurable functions $u : X \to \mathbb{R}$ such that the quasi-norm $\|\cdot\|_{\text{weak-}L^1(X,\mu)}$, defined by

$$\|u\|_{\text{weak-}L^1(X,\mu)} = \sup_{\lambda>0} \lambda \mu(\{x \in X : |u(x)| > \lambda\}),$$

is finite.

2.2 Fractional derivatives and one-sided spaces

Let $u = u(t) \in \mathcal{S}(\mathbb{R})$ and define

$$D_{\text{left}} u(t) = \lim_{\tau \to 0^+} \frac{u(t) - u(t-\tau)}{\tau} \quad \text{and} \quad D_{\text{right}} u(t) = \lim_{\tau \to 0^+} \frac{u(t) - u(t+\tau)}{\tau}.$$

Observe that $D_{\text{left}} u = -D_{\text{right}} u = u'$. From the Fourier transform identities

$$\widehat{D_{\text{left}} u}(\xi) = (i\xi)\widehat{u}(\xi) \quad \text{and} \quad \widehat{D_{\text{right}} u}(\xi) = (-i\xi)\widehat{u}(\xi),$$

one can define

$$\widehat{(D_{\text{left}})^\alpha u}(\xi) = (i\xi)^\alpha \widehat{u}(\xi) \quad \text{and} \quad \widehat{(D_{\text{right}})^\alpha u}(\xi) = (-i\xi)^\alpha \widehat{u}(\xi). \tag{2.2.1}$$

Using the semigroup of translations, it is shown in [8], see also [64], that $(D_{\text{left}})^\alpha u(t)$ and $(D_{\text{right}})^\alpha u(t)$ are given by the pointwise formulas in (1.3.2) and (1.3.3), respectively. For completeness, we provide the details for $(D_{\text{left}})^\alpha u(t)$.

Using the Cauchy Integral Theorem, we can write

$$\Gamma(-\alpha) = \int_{\text{ray}(\theta)} (e^{-z} - 1) \frac{dz}{z^{1+\alpha}}, \quad \text{for any } 0 < \alpha < 1, \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

see [8, Lemma 2.1]. Choose $\theta = \pi/2$, parameterize the integral, and rearrange to obtain

$$(i\xi)^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left(e^{-i\xi\tau} - 1\right) \frac{dt}{\tau^{1+s}}, \quad \xi \in \mathbb{R}.$$

Therefore, $\widehat{(D_{\text{left}})^\alpha u}(\xi)$ can be written as

$$\widehat{(D_{\text{left}})^\alpha u}(\xi) = (i\xi)^\alpha \widehat{u}(\xi) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left(e^{-i\xi\tau} \widehat{u}(\xi) - \widehat{u}(\xi)\right) \frac{d\tau}{\tau^{1+\alpha}}.$$

Since translations $T_\tau u(t) = u(t - \tau)$ correspond to multiplication by $e^{-i\tau\xi}$ when taking the Fourier transform $\widehat{T_\tau u}(\xi) = e^{-i\tau\xi}\widehat{u}(\xi)$, we take the inverse Fourier transform to obtain

$$\begin{aligned} (D_{\text{left}})^\alpha u(t) &= \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{u(t-\tau) - u(t)}{\tau^{1+\alpha}} \, d\tau \\ &= \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^t \frac{u(\tau) - u(t)}{(t-\tau)^{1+\alpha}} \, d\tau. \end{aligned}$$

We mention that the family of shift operators $\{T_\tau\}_{\tau \geq 0}$ forms a semigroup and that $v = T_\tau u$ solves the transport equation with initial data u on $\mathbb{R}$:

$$\begin{cases} \partial_\tau v = -D_{\text{left}} v & \text{for } t \in \mathbb{R}, \ \tau > 0 \\ v(t, 0) = u(t) & \text{for } t \in \mathbb{R}. \end{cases}$$

This is consistent with properties we will see for the fractional Laplacian in Section 2.4.

2.2.1 Distributional setting

If $u, \varphi \in \mathcal{S}(\mathbb{R})$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} (D_{\text{left}})^{\alpha} u(t)\, \varphi(t)\, dt &= \frac{1}{\Gamma(-\alpha)} \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} \int_{-\infty}^{\infty} \frac{u(t-\tau) - u(t)}{\tau^{1+\alpha}} \varphi(t)\, dt\, d\tau \\
&= \frac{1}{\Gamma(-\alpha)} \lim_{\varepsilon \to 0^+} \left(\int_{\varepsilon}^{\infty} \int_{-\infty}^{\infty} \frac{u(t-\tau)\varphi(t)}{\tau^{1+\alpha}}\, dt\, d\tau - \int_{\varepsilon}^{\infty} \int_{-\infty}^{\infty} \frac{u(t)\varphi(t)}{\tau^{1+\alpha}}\, dt\, d\tau \right) \\
&= \frac{1}{\Gamma(-\alpha)} \lim_{\varepsilon \to 0^+} \left(\int_{\varepsilon}^{\infty} \int_{-\infty}^{\infty} \frac{u(r)\varphi(t+r)}{\tau^{1+\alpha}}\, dr\, d\tau - \int_{\varepsilon}^{\infty} \int_{-\infty}^{\infty} \frac{u(r)\varphi(r)}{\tau^{1+\alpha}}\, dr\, d\tau \right) \\
&= \frac{1}{\Gamma(-\alpha)} \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} \int_{-\infty}^{\infty} u(r) \frac{\varphi(r+\tau) - \varphi(r)}{\tau^{1+\alpha}}\, dr\, d\tau \\
&= \int_{-\infty}^{\infty} u(r) (D_{\text{left}})^{\alpha} \varphi(r)\, dr.
\end{aligned}$$

We will use this identity to define $(D_{\text{left}})^{\alpha} u$ in the sense of distributions. Notice that if $u \in \mathcal{S}'(\mathbb{R})$, then a natural definition would be

$$((D_{\text{left}})^{\alpha} u)(\varphi) = u\left((D_{\text{right}})^{\alpha} \varphi\right).$$

Nevertheless, it is straightforward from (2.2.1) to see that, in general, $(D_{\text{right}})^{\alpha} \varphi \notin \mathcal{S}(\mathbb{R})$, so we need to consider a different space of test functions and distributions.

We define the class

$$\mathcal{S}_{-} = \{\varphi \in \mathcal{S}(\mathbb{R}) : \operatorname{supp} \varphi \subset (-\infty, A],\ \text{for some } A \in \mathbb{R}\}.$$

We denote by $\mathcal{S}_{-}^{\alpha}$ the set of functions

$$\varphi \in C^{\infty}(\mathbb{R}) \ \text{such that} \ \operatorname{supp} \varphi \subset (-\infty, A] \text{ and } \left| \frac{d^k}{dt^k} \varphi(t) \right| \leq \frac{C}{1 + |t|^{1+\alpha}}$$

for all $k \geq 0$, for some $A \in \mathbb{R}$ and $C > 0$.

Lemma 2.2.1. *If $\varphi \in \mathcal{S}_{-}$ then $(D_{\text{right}})^{\alpha} \varphi \in \mathcal{S}_{-}^{\alpha}$.*

Proof. Clearly, if $\varphi \in \mathcal{S}_{-}$ with $\operatorname{supp} \varphi \in (-\infty, A]$, then $(D_{\text{right}})^{\alpha} \varphi$ also has support in $(-\infty, A]$, see (1.3.3). Since $(D_{\text{right}})^{\alpha} \frac{d^k}{dt^k} \varphi = \frac{d^k}{dt^k} (D_{\text{right}})^{\alpha} \varphi$, we know $(D_{\text{right}})^{\alpha} \varphi \in C^{\infty}(\mathbb{R})$ and only need to estimate $(D_{\text{right}})^{\alpha} \varphi$.

If $t > A$, then the estimate holds trivially because $(D_{\text{right}})^\alpha \varphi(t) = 0$.

Suppose that $-1 < t < A$. The estimate holds because φ is smooth and bounded. Indeed,

$$\begin{aligned}
|(D_{\text{right}})^\alpha \varphi(t)| &\le \int_t^\infty \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|^{1+\alpha}}\, d\tau \\
&\le 2\,\|\varphi\|_{L^\infty(\mathbb{R})} \int_{t+1}^\infty \frac{1}{(\tau - t)^{1+\alpha}}\, d\tau + \|\varphi'\|_{L^\infty(\mathbb{R})} \int_t^{t+1} \frac{1}{|\tau - t|^\alpha}\, d\tau \\
&= 2\,\|\varphi\|_{L^\infty(\mathbb{R})} \int_1^\infty r^{-1-\alpha}\, dr + \|\varphi'\|_{L^\infty(\mathbb{R})} \int_0^1 r^{-\alpha}\, dr \\
&= \frac{2\,\|\varphi\|_{L^\infty(\mathbb{R})}}{\alpha} + \frac{\|\varphi'\|_{L^\infty(\mathbb{R})}}{1-\alpha} \\
&= C\frac{|t|^{1+\alpha} + 1}{|t|^{1+\alpha} + 1} < \frac{C}{|t|^{1+\alpha} + 1}.
\end{aligned}$$

Suppose $-\infty < t < -1$ and write

$$\int_t^\infty \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|^{1+\alpha}}\, d\tau = \int_t^{t/2} \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|^{1+\alpha}}\, d\tau + \int_{t/2}^\infty \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|^{1+\alpha}}\, d\tau = I + II.$$

For I, note that

$$\begin{aligned}
|\varphi(\tau) - \varphi(t)| &\le |\varphi'(\xi)|\,|\tau - t| \\
&= |\varphi'(\xi)|(1 + |\xi|)^3 \frac{(\tau - t)}{(1 + |\xi|)^3} \le C_\varphi \frac{(\tau - t)}{(1 + |\xi|)^3}
\end{aligned}$$

where ξ is some point in between t and τ. Hence,

$$I \le \frac{C}{|t|^3} \int_t^{t/2} \frac{1}{(\tau - t)^\alpha}\, d\tau = \frac{C}{|t|^{2+\alpha}} \le \frac{C}{1 + |t|^{1+\alpha}}.$$

On the other hand, if $\tau > t/2$, then $\tau - t > -t/2 > 0$ and

$$\begin{aligned}
II &\le \int_{t/2}^\infty \frac{|\varphi(\tau)|}{(\tau - t)^{1+\alpha}}\, d\tau + |\varphi(t)| \int_{t/2}^\infty \frac{1}{(\tau - t)^{1+\alpha}}\, d\tau \\
&\le \frac{C}{|t|^{1+\alpha}}\, \|\varphi\|_{L^1(\mathbb{R})} + \frac{C}{|t|^{1+\alpha}} |t\varphi(t)| \le \frac{C}{1 + |t|^{1+\alpha}}.
\end{aligned}$$

Collecting all the terms, we get

$$|(D_{\text{right}})^\alpha \varphi(t)| \le C \int_t^\infty \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|^{1+\alpha}}\, d\tau \le \frac{C}{1 + |t|^{1+\alpha}}$$

for all $t \in \mathbb{R}$. Thus, $(D_{\text{right}})^\alpha \varphi \in \mathcal{S}_-^\alpha$. □

We endow $\mathcal{S}_-$ and $\mathcal{S}_-^\alpha$ with the families of seminorms

$$\rho_-^{\ell,k}(\varphi) = \sup_{t\in\mathbb{R}} |t|^\ell \left| \frac{d^k}{dt^k}\varphi(t) \right| \quad \text{for } \ell, k \geq 0,$$

and

$$\rho_-^{\alpha,k}(\varphi) = \sup_{t\in\mathbb{R}} (1+|t|^{1+\alpha}) \left| \frac{d^k}{dt^k}\varphi(t) \right| \quad \text{for } k \geq 0,$$

respectively. Let us denote by $(\mathcal{S}_-)'$ and $(\mathcal{S}_-^\alpha)'$ the corresponding dual spaces of $\mathcal{S}_-$ and $\mathcal{S}_-^\alpha$. Notice that $\mathcal{S}_- \subset \mathcal{S}_-^\alpha$, so that $(\mathcal{S}_-^\alpha)' \subset (\mathcal{S}_-)'$. It turns out that $(\mathcal{S}_-^\alpha)'$ is the appropriate class of distributions to extend the definition of the left fractional derivative.

Definition 2.2.1. *For $u \in (\mathcal{S}_-^\alpha)'$, we define $(D_{\text{left}})^\alpha u$ as the distribution in $(\mathcal{S}_-)'$ given by*

$$((D_{\text{left}})^\alpha u)(\varphi) = u((D_{\text{right}})^\alpha \varphi) \quad \textit{for any } \varphi \in \mathcal{S}_-.$$

Consider next the class of functions given by

$$L_-^\alpha = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}) : \int_{-\infty}^A \frac{|u(\tau)|}{1+|\tau|^{1+\alpha}}\, d\tau < \infty, \text{ for any } A \in \mathbb{R} \right\}.$$

We use the notation

$$\|u\|_A = \int_{-\infty}^A \frac{|u(\tau)|}{1+|\tau|^{1+\alpha}}\, d\tau \quad \text{for } A \in \mathbb{R}.$$

Any function $u \in L_-^\alpha$ defines a distribution in $(\mathcal{S}_-^\alpha)'$ in the usual way, so that $(D_{\text{left}})^\alpha u$ is well defined as an object in $(\mathcal{S}_-)'$. The following result is proved similarly as in the case of the fractional Laplacian (see Proposition 2.4.1).

Proposition 2.2.1. *Let $u \in L_-^\alpha$. Assume that $u \in C^{\alpha+\varepsilon}(I)$ for some $\varepsilon > 0$ and some open set $I \subset \mathbb{R}$. Then $(D_{\text{left}})^\alpha u \in C(I)$ and*

$$(D_{\text{left}})^\alpha u(t) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^t \frac{u(\tau)-u(t)}{(t-\tau)^{1+\alpha}}\, d\tau \qquad \textit{for all } t \in I.$$

Proof. Let $I_0 \subset\subset I$ be arbitrary. There exists a sequence $u_k \in \mathcal{S}_-$ such that f_k is uniformly bounded in $C^{\alpha+\varepsilon}(I)$, $f_k \to f$ uniformly in I_0, and $f_k \to f$ in L_-^α.

We will show that

$$(D_{\text{left}})^\alpha u_k(t) \to \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^t \frac{u(\tau)-u(t)}{(t-\tau)^{1+\alpha}}\, d\tau \quad \text{as } k \to \infty$$

uniformly in I_0. Fix $\bar{\varepsilon} > 0$. Let $\rho > 0$ be such that

$$\int_0^\rho \frac{M}{|\tau|^{1-\varepsilon}}\,d\tau \le \frac{\bar{\varepsilon}}{3}, \quad \text{where } M = \sup_k [u_k]_{C^{\alpha+\varepsilon}(I_0)}. \tag{2.2.2}$$

Write

$$\begin{aligned} I_\rho^k + II_\rho^k &:= \frac{1}{\Gamma(-\alpha)} \int_{t-\rho}^t \frac{u_k(\tau)-u_k(t)}{(t-\tau)^{1+\alpha}}\,d\tau + \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{t-\rho} \frac{u_k(\tau)-u_k(t)}{(t-\tau)^{1+\alpha}}\,d\tau \\ I_\rho + II_\rho &:= \frac{1}{\Gamma(-\alpha)} \int_{t-\rho}^t \frac{u(\tau)-u(t)}{(t-\tau)^{1+\alpha}}\,d\tau + \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{t-\rho} \frac{u(\tau)-u(t)}{(t-\tau)^{1+\alpha}}\,d\tau. \end{aligned}$$

First, observe that, by (2.2.2),

$$\begin{aligned} \left|I_\rho^k\right| &\le \frac{1}{|\Gamma(-\alpha)|} \int_{t-\rho}^t \frac{|u_k(\tau)-u_k(t)|}{|t-\tau|^{1+\alpha}}\,d\tau \\ &\le \frac{1}{|\Gamma(-\alpha)|} \int_{t-\rho}^t \frac{M\,|t-\tau|^{\alpha+\varepsilon}}{|t-\tau|^{1+\alpha}}\,d\tau = \frac{1}{|\Gamma(-\alpha)|} \int_0^\rho \frac{M}{|\tau|^{1-\varepsilon}}\,d\tau \le \frac{\bar{\varepsilon}}{3}. \end{aligned}$$

It follows similarly that $|I_\rho| \le \frac{\bar{\varepsilon}}{3}$. To study $II_\rho^k - II_\rho$, notice that

$$\begin{aligned} \frac{1}{|\Gamma(-\alpha)|} \int_{-\infty}^{t-\rho} \frac{|u_k(t)-u(t)|}{|t-\tau|^{1+\alpha}}\,d\tau &= |u_k(t)-u(t)|\,\frac{1}{|\Gamma(-\alpha)|} \int_\rho^\infty \frac{1}{|\tau|^{1+\alpha}}\,d\tau \\ &= |u_k(t)-u(t)|\,\frac{\rho^{-\alpha}}{\alpha\,|\Gamma(-\alpha)|} < \frac{\bar{\varepsilon}}{6} \end{aligned}$$

for k large because $u_k \to u$ uniformly in I_0. Since $u_k \to u$ in L_-^α, it follows that, for k large,

$$\begin{aligned} \frac{1}{|\Gamma(-\alpha)|} \int_{-\infty}^{t-\rho} \frac{|u_k(\tau)-u(\tau)|}{|t-\tau|^{1+\alpha}}\,d\tau &= \frac{1}{|\Gamma(-\alpha)|} \int_{-\infty}^{t-\rho} \frac{|u_k(\tau)-u(\tau)|}{|t-\tau|^{1+\alpha}} \frac{1+|\tau|^{1+\alpha}}{1+|\tau|^{1+\alpha}}\,d\tau \\ &\le \frac{C}{|\Gamma(-\alpha)|} \int_{-\infty}^{t-\rho} \frac{|u_k(\tau)-u(\tau)|}{1+|\tau|^{1+\alpha}}\,d\tau \le \frac{\bar{\varepsilon}}{6}. \end{aligned}$$

Here, we used that

$$\frac{1+|\tau|}{|t-\tau|} \le \frac{1}{|t-\tau|} + \frac{|t-\tau|}{|t-\tau|} + \frac{|t|}{|t-\tau|} \le \frac{1}{\rho} + 1 + \frac{|t|}{\rho} \le C.$$

Therefore,

$$\begin{aligned} \left|II_\rho^k - II_\rho\right| &= \left|\frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{t-\rho} \frac{(u_k(\tau)-u(\tau))-(u_k(t)-u(t))}{(t-\tau)^{1+\alpha}}\,d\tau\right| \\ &\le \frac{1}{|\Gamma(-\alpha)|} \int_{-\infty}^{t-\rho} \frac{|u_k(\tau)-u(\tau)|}{|t-\tau|^{1+\alpha}}\,d\tau + \frac{1}{|\Gamma(-\alpha)|} \int_{-\infty}^{t-\rho} \frac{|u_k(t)-u(t)|}{|t-\tau|^{1+\alpha}}\,d\tau < \frac{\bar{\varepsilon}}{3}. \end{aligned}$$

We conclude that, for k sufficiently large,

$$\left|(D_{\text{left}})^{\alpha}u_k(t) - \frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{t}\frac{u(\tau)-u(t)}{(t-\tau)^{1+\alpha}}\,d\tau\right| \le |I_\rho^k| + |I_\rho| + |II_\rho^k - II_\rho| < \bar{\varepsilon}$$

as desired.

Let $\varphi \in C_c^\infty(I_0)$. Let $A \in \mathbb{R}$ be such that $\operatorname{supp}\varphi \subset (-\infty, A] \subset I_0$, so that $\varphi \in \mathcal{S}_-$ and $(D_{\text{left}})^{\alpha}\varphi \in \mathcal{S}_-^{\alpha}$. By the above uniform convergence, we have that

$$((D_{\text{left}})^{\alpha}u_k)(\varphi) = \int_{-\infty}^{\infty}(D_{\text{left}})^{\alpha}u_k(t)\varphi(t)\,dt \to \int_{-\infty}^{\infty}\left(\frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{t}\frac{u(\tau)-u(t)}{(t-\tau)^{1+\alpha}}\,d\tau\right)\varphi(t)\,dt$$

as $k \to \infty$. Furthermore, since $u_k \to u$ in L_-^{α}, we know that $(D_{\text{left}})^{\alpha}u_k \to (D_{\text{left}})^{\alpha}u$ in $(\mathcal{S}_-)'$. Hence,

$$((D_{\text{left}})^{\alpha}u_k)(\varphi) = u_k((D_{\text{right}})^{\alpha}\varphi) \to u((D_{\text{right}})^{\alpha}\varphi) = ((D_{\text{left}})^{\alpha}u)(\varphi), \quad \text{as } k \to \infty.$$

By uniqueness of limits,

$$((D_{\text{left}})^{\alpha}u)(\varphi) = \int_{-\infty}^{\infty}\left(\frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{t}\frac{u(\tau)-u(t)}{(t-\tau)^{1+\alpha}}\,d\tau\right)\varphi(t)\,dt.$$

Since $\varphi \in C_c^\infty(I_0)$ was arbitrary, $(D_{\text{left}})^{\alpha}u$ must coincide with the pointwise formula in I_0. Moreover, $(D_{\text{left}})^{\alpha}u$ is continuous in I_0 as it is the uniform limit of a sequence of continuous functions.

Since I_0 was arbitrary, the result follows. □

Remark 2.2.1. *We have found that the one-sided class L_-^{α} is the appropriate space of locally integrable functions to define the left fractional derivative. This is a refinement with respect to the distributional definition presented in [8, Remark 2.6], which was two-sided in nature.*

2.2.2 One-sided weighted spaces

A nonnegative, locally integrable function $\omega = \omega(\tau)$ defined on $\mathbb{R}$ is in the left-sided Sawyer class $A_p^-(\mathbb{R})$, for $1 < p < \infty$, if there exists $C > 0$ such that

$$\left(\frac{1}{h}\int_a^{a+h}\omega\,d\tau\right)^{1/p}\left(\frac{1}{h}\int_{a-h}^{a}\omega^{1-p'}\,d\tau\right)^{1/p'} \le C$$

for all $a \in \mathbb{R}$ and $h > 0$, where $1/p + 1/p' = 1$. We then write $\omega \in A_p^-(\mathbb{R})$. By re-orienting the real line, one may similarly define the right-sided $A_p^+(\mathbb{R})$-condition: a weight $\tilde{\omega}$ belongs to $A_p^+(\mathbb{R})$

if there is a constant $C > 0$ such that

$$\left(\frac{1}{h}\int_{a-h}^{a} \tilde{\omega}\, d\tau\right)^{1/p} \left(\frac{1}{h}\int_{a}^{a+h} \tilde{\omega}^{1-p'}\, d\tau\right)^{1/p'} \leq C$$

for all $a \in \mathbb{R}$ and $h > 0$. In this way, $\omega \in A_p^-(\mathbb{R})$ if and only if $\omega^{1-p'} \in A_{p'}^+(\mathbb{R})$.

From the definition, one should note that, for $\omega \in A_p^-(\mathbb{R})$, there exist $-\infty \leq a < b \leq \infty$ such that $\omega = \infty$ in $(-\infty, a)$, $0 < \omega < \infty$ in (a, b), $\omega = 0$ in (b, ∞), and $\omega \in L^1_{\text{loc}}((a, b))$. For simplicity and without loss of generality, we will assume $(a, b) = \mathbb{R}$, so that $0 < \omega < \infty$ in $\mathbb{R}$.

The one-sided Hardy–Littlewood maximal functions M^- and M^+ are defined by

$$M^-u(t) = \sup_{h>0} \frac{1}{h}\int_{t-h}^{t} |u(\tau)|\, d\tau \quad \text{and} \quad M^+u(t) = \sup_{h>0} \frac{1}{h}\int_{t}^{t+h} |u(\tau)|\, d\tau$$

respectively. If $1 < p < \infty$, then $M^{\pm}$ is bounded on $L^p(\mathbb{R}, \omega)$ if and only if $\omega \in A_p^{\pm}(\mathbb{R})$, see [66]. When $p = 1$, $M^{\pm}$ is bounded from $L^1(\mathbb{R}, \omega)$ into weak-$L^1(\mathbb{R}, \omega)$ if and only if $\omega \in A_1^{\pm}(\mathbb{R})$, namely, there exists $C > 0$ such that

$$M^{\mp}\omega(t) \leq C\omega(t) \quad \text{for a.e. } t \in \mathbb{R}$$

see [55]. We refer to [46, 53, 54, 55, 56, 66] for these and more properties of one-sided weights.

It is clear that $A_p^-(\mathbb{R})$ is a larger family than the classical class of Muckenhoupt weights $A_p(\mathbb{R})$. In particular, any decreasing function is in $A_p^-(\mathbb{R})$, but there are decreasing functions that are not in $A_p(\mathbb{R})$ (see Section 2.4). For instance, $\omega(t) = e^{-t}$ belongs to $A_p^-(\mathbb{R})$ but not to $A_p(\mathbb{R})$ because it is not a doubling weight.

For a measurable set $E \subset \mathbb{R}$, we denote

$$\omega(E) = \int_E \omega\, d\tau.$$

An important property that we will use is the following.

Lemma 2.2.2 (See [46, Theorem 3]). *Let $\eta = \eta(t) \geq 0$ be a integrable function with support in $[0, \infty)$ and nonincreasing in $[0, \infty)$. Then, for any measurable function $u : \mathbb{R} \to \mathbb{R}$ and for almost all $t \in \mathbb{R}$, we have*

$$|u * \eta(t)| \leq M^-u(t) \int_0^\infty \eta(\tau)\, d\tau.$$

By changing the orientation of the real line, the analogue conclusion holds for nondecreasing η *supported in* $(-\infty,0]$ *with* M^+ *in place of* M^-.

Lemma 2.2.3 (See [56, Theorem 1])**.** *If* $\omega \in A_p^-(\mathbb{R})$, $1 \le p < \infty$, *then there exist* $C,\delta > 0$ *such that*

$$\frac{\omega(E)}{\omega((a,c))} \le C\left(\frac{|E|}{b-a}\right)^{\delta}$$

for all $a<b<c$ *and all measurable subsets* $E \subset (b,c)$.

Lemma 2.2.4. *If* $\omega \in A_1^-(\mathbb{R})$, *then there is a constant* $C>0$ *such that, for any* $0<a<b$,

$$\frac{\omega((-a,-a+(b-a)))}{2(b-a)} \le C \inf_{-b<t<-a} \omega(t).$$

Proof. Let $t \in (-b,-a)$. Since $(-a,-a+(b-a)) \subset (t,t+2(b-a))$, then, by the $A_1^-(\mathbb{R})$-condition, we get

$$\begin{aligned} C\omega(t) \ge M^+\omega(t) &\ge \frac{1}{2(b-a)}\int_t^{t+2(b-a)} \omega(\tau)\,d\tau \\ &\ge \frac{1}{2(b-a)}\int_{-a}^{-a+(b-a)} \omega(\tau)\,d\tau \\ &= \frac{\omega((-a,-a+(b-a)))}{2(b-a)} \end{aligned}$$

for almost every $t \in \mathbb{R}$. □

The following result says that $(D_{\text{left}})^{\alpha}u$ is well defined as a distribution in $(\mathcal{S}_-)'$ whenever $u \in L^p(\mathbb{R},\omega)$, for $\omega \in A_p^-(\mathbb{R})$, $1 \le p < \infty$.

Proposition 2.2.2. *If* $\omega \in A_p^-(\mathbb{R})$, $1 \le p < \infty$, *then* $L^p(\mathbb{R},\omega) \subset L_-^{\alpha}$, $\alpha \ge 0$, *and, for any* $A \in \mathbb{R}$, *there is a constant* $C = C_{A,\omega,p} > 0$ *such that*

$$\|u\|_A \le C\|u\|_{L^p(\mathbb{R},\omega)}.$$

In particular, $L^p(\mathbb{R},\omega) \subset L^1_{\text{loc}}(\mathbb{R})$.

Proof. Let $u \in L^p(\mathbb{R},\omega)$ and fix any $A \in \mathbb{R}$.

We first let $1 < p < \infty$. By Hölder's inequality,

$$\begin{aligned}\|u\|_A &= \int_{-\infty}^{A} \frac{|u(\tau)|}{1+|\tau|^{1+\alpha}}\,d\tau \\ &= \int_{-\infty}^{A} |u(\tau)|\,\omega(\tau)^{1/p}\,\frac{\omega(\tau)^{-1/p}}{1+|\tau|^{1+\alpha}}\,d\tau \\ &\le \|u\|_{L^p(\mathbb{R},\omega)} \left(\int_{-\infty}^{A} \frac{\omega(\tau)^{-p'/p}}{(1+|\tau|)^{p'}}\,d\tau\right)^{1/p'} \\ &= \|u\|_{L^p(\mathbb{R},\omega)} \cdot (I_A)^{1/p'}.\end{aligned}$$

Observe that $\tilde{\omega}(\tau) = \omega(\tau)^{-p'/p} = \omega(\tau)^{1-p'} \in A^+_{p'}(\mathbb{R})$. To conclude, it is enough to recall that

$$I = \int_{-\infty}^{0} \frac{\tilde{\omega}(\tau)}{(1+|\tau|)^{p'}}\,d\tau < \infty,$$

see [53, Lemma 4].

Now let $p = 1$. For convenience with the notation, we let $A = 0$ (the general case follows the same lines). First observe that, by the $A^-_1(\mathbb{R})$-condition and Lemma 2.2.4,

$$\begin{aligned}\int_{-1}^{0} \frac{|u(\tau)|}{1+|\tau|^{1+\alpha}}\,d\tau &\le \int_{-1}^{0} |u(\tau)|\,\omega(\tau)\omega(\tau)^{-1}\,d\tau \\ &\le \|u\|_{L^1(\mathbb{R},\omega)} \sup_{t\in(-1,0)} \omega(t)^{-1} \\ &= \|u\|_{L^1(\omega)} \left(\inf_{t\in(-1,0)} \omega(t)\right)^{-1} \le \|u\|_{L^1(\omega)} \frac{C}{\omega((-1,0))} < \infty.\end{aligned}$$

On the other hand, by Lemma 2.2.4,

$$\begin{aligned}\int_{-\infty}^{-1} \frac{|u(\tau)|}{1+|\tau|^{1+\alpha}}\,d\tau &\le \sum_{k=0}^{\infty} \int_{-2^{k+1}}^{-2^k} \frac{|u(\tau)|}{|\tau|}\,d\tau \\ &\le \sum_{k=0}^{\infty} \frac{1}{2^k} \int_{-2^{k+1}}^{-2^k} |u(\tau)|\,\omega(\tau)\omega(\tau)^{-1}\,d\tau \\ &\le \|u\|_{L^1(\mathbb{R},\omega)} \sum_{k=0}^{\infty} \frac{1}{2^k} \sup_{-2^{k+1}<t<-2^k} \omega(\tau)^{-1} \\ &\le \|u\|_{L^1(\mathbb{R},\omega)} \sum_{k=0}^{\infty} \frac{1}{2^k} \left(\inf_{-2^{k+1}<t<-2^k} \omega(t)\right)^{-1} \\ &\le C\,\|u\|_{L^1(\mathbb{R},\omega)} \sum_{k=0}^{\infty} \frac{1}{\omega((-2^k,0))}.\end{aligned}$$

Lemma 2.2.3 implies that there exist $C, \delta > 0$ such that

$$\frac{\omega((-1,0))}{\omega((-2^k,0))} \leq C\left(\frac{1}{2^k}\right)^{\delta}.$$

Whence,

$$\int_{-\infty}^{-1} \frac{|u(\tau)|}{1+|\tau|^{1+\alpha}}\,d\tau \leq \frac{C}{\omega((-1,0))}\,\|u\|_{L^1(\mathbb{R},\omega)} \sum_{k=0}^{\infty}\left(\frac{1}{2^k}\right)^{\delta} < \infty.$$

Thus, $u \in L^{\alpha}_{-}$ with the corresponding estimate. □

2.2.3 Density of smooth functions in $W^{1,p}(\mathbb{R},\omega)$

The proof of the following statement is similar to that of Lorente [46, Theorem 3]. Indeed, the idea is to bound $\psi \in C_c^{\infty}([0,\infty))$ by a measurable function η supported in $[0,\infty)$ which is nonincreasing in $[0,\infty)$, and follow the steps of the proof in [46]. We provide the proof for completeness.

Proposition 2.2.3. *Let $\omega \in A_p^-(\mathbb{R})$ and $u \in L^p(\mathbb{R},\omega)$ for $1 \leq p < \infty$. Let $\psi \in C_c^{\infty}([0,\infty))$ be a nonnegative function such that $\int_0^{\infty} \psi\,dt = 1$. Define $\psi_{\varepsilon}(t) = \frac{1}{\varepsilon}\psi\left(\frac{t}{\varepsilon}\right)$. Then the following hold.*

(1) $|u * \psi_{\varepsilon}(t)| \leq CM^{-}u(t)$ *for almost every* $t \in \mathbb{R}$.

(2) $\|u * \psi_{\varepsilon}\|_{L^p(\mathbb{R},\omega)} \leq C\,\|u\|_{L^p(\mathbb{R},\omega)}$.

(3) $\lim_{\varepsilon\to 0^+} u * \psi_{\varepsilon}(t) = u(t)$ *for almost every* $t \in \mathbb{R}$.

(4) $\lim_{\varepsilon\to 0^+} \|u * \psi_{\varepsilon} - u\|_{L^p(\mathbb{R},\omega)} = 0$.

It follows that $C^{\infty}(\mathbb{R}) \cap L^p(\mathbb{R},\omega)$ and $C_c^{\infty}(\mathbb{R})$ are dense in $L^p(\mathbb{R},\omega)$ for $\omega \in A_p^-(\mathbb{R})$, $1 \leq p < \infty$. Additionally, notice that if ψ is as in Proposition 2.2.3 and $u \in W^{1,p}(\mathbb{R},\omega)$, then

$$(u * \psi_{\varepsilon})'(t) = \int_{-\infty}^{\infty} u'(\tau)\psi_{\varepsilon}(t-\tau)\,d\tau = (u' * \psi_{\varepsilon})(t).$$

Hence $u * \psi_{\varepsilon} \to u$ as $\varepsilon \to 0^+$ in $W^{1,p}(\mathbb{R},\omega)$, so that $C^{\infty}(\mathbb{R}) \cap W^{1,p}(\mathbb{R},\omega)$ and $C_c^{\infty}(\mathbb{R})$ are dense in $W^{1,p}(\mathbb{R},\omega)$ for $\omega \in A_p^-(\mathbb{R})$, $1 \leq p < \infty$.

Proof of Proposition 2.2.3. Let $\eta \geq 0$ be an integrable function with support in $[0,\infty)$, which is nonincreasing in $[0,\infty)$ be such that $\psi \leq \eta$ everywhere.

To prove (1), we first estimate

$$
\begin{aligned}
|(u*\psi_\varepsilon)(t)| &\le \int_{-\infty}^{\infty} |u(t-\tau)| \, \frac{1}{\varepsilon}\psi\left(\frac{\tau}{\varepsilon}\right)\, d\tau \\
&= \int_{-\infty}^{\infty} |u(t-\varepsilon\tau)| \, \psi(\tau)\, d\tau \\
&\le \int_{0}^{\infty} |u(t-\varepsilon\tau)| \int_0^{\eta(\tau)} ds\, d\tau.
\end{aligned}
$$

Define $h(s)$ and $\bar{s}$ by

$$
h(s) = \sup\{\tau \ge 0 : \eta(\tau) \ge s\}, \quad \bar{s} = \sup\{s : h(s) \ge 0\}.
$$

Therefore, we have that

$$
\begin{aligned}
\int_0^{\infty} |u(t-\varepsilon\tau)| \int_0^{\eta(\tau)} ds\, d\tau &= \int_0^{\bar{s}} \int_0^{h(s)} |u(t-\varepsilon\tau)| \; d\tau\, ds \\
&= \int_0^{\bar{s}} h(s) \left(\frac{1}{\varepsilon\, h(s)} \int_{t-\varepsilon h(s)}^{t} |u(r)| \; dr \right) ds \\
&\le \int_0^{\bar{s}} h(s) M^- u(t)\, ds \\
&= M^- u(t) \int_0^{\infty} \int_{\{\tau \ge 0 : \eta(\tau) \ge s\}} d\tau\, ds \\
&= M^- u(t) \int_0^{\infty} \int_0^{\eta(\tau)} ds\, d\tau \\
&= M^- u(t) \int_0^{\infty} \eta(\tau) d\tau \\
&= C M^- u(t).
\end{aligned}
$$

Combing the estimates, we obtain (1).

We now prove (2). For $1 < p < \infty$, it follows from part (1) and the boundedness of M^- that

$$
\|u*\psi_\varepsilon\|_{L^p(\mathbb{R},\omega)} \le C \left\|M^- u\right\|_{L^p(\mathbb{R},\omega)} \le C \|u\|_{L^p(\mathbb{R},\omega)} .
$$

Let $p = 1$, we have that

$$\begin{aligned}\|u * \psi_\varepsilon\|_{L^1(\mathbb{R},\omega)} &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(t-\tau)|\,\psi_\varepsilon(\tau)\,d\tau\,\omega(t)\,dt \\ &= \int_{-\infty}^{\infty} \psi_\varepsilon(\tau)\int_{-\infty}^{\infty} |u(t-\tau)|\,\omega(t)\,dt\,d\tau \\ &= \int_{-\infty}^{\infty} \psi_\varepsilon(\tau)\int_{-\infty}^{\infty} |u(t)|\,\omega(t+\tau)\,dt\,d\tau \\ &= \int_{-\infty}^{\infty} |u(t)|\int_{-\infty}^{\infty} \omega(t+\tau)\psi_\varepsilon(\tau)\,d\tau\,dt.\end{aligned}$$

By the computation in part (1) with u and $-\varepsilon\tau$ replaced by ω and $\varepsilon\tau$, it follows that

$$\int_{-\infty}^{\infty} \omega(t+\tau)\psi_\varepsilon(\tau)\,d\tau \leq CM^+\omega(t)$$

which proves (2).

To prove (3), we see that

$$\begin{aligned}|u * \psi_\varepsilon(t) - u(t)| &\leq \int_{-\infty}^{\infty} |u(t-\varepsilon\tau) - u(t)|\,\psi(\tau)\,d\tau \\ &\leq \int_{0}^{\infty} |u(t-\varepsilon\tau) - u(t)|\,\eta(\tau)\,d\tau \\ &= \int_0^{\overline{s}}\int_0^{h(s)} |u(t-\varepsilon\tau) - u(t)|\;d\tau\,ds \\ &= \int_0^{\overline{s}} h(s)\left(\frac{1}{\varepsilon h(s)}\int_0^{\varepsilon h(s)} |u(t-\tau) - u(t)|\;d\tau\right)\,ds \to 0\end{aligned}$$

as $\varepsilon \to 0$ for almost every $t \in \mathbb{R}$ by the Dominated Convergence Theorem and Lebesgue Differentiation Theorem.

Lastly, we prove (4). For $1 < p < \infty$, by part (1),

$$|u * \psi_\varepsilon(t) - u(t)|^p\,\omega(t) \leq C\left(|Mu(t)|^p + |u|^p\right)\omega(t) \in L^1(\mathbb{R}).$$

Hence, by the Dominated Convergence Theorem and part (3)

$$\lim_{\varepsilon\to 0}\|u * \psi_\varepsilon - u\|_{L^p(\mathbb{R},\omega)} = \int_{-\infty}^{\infty}\lim_{\varepsilon\to 0}|u * \psi_\varepsilon(t) - u(t)|^p\,\omega(t)\,dt = 0.$$

For $p = 1$, we define the function g by

$$g(\tau) = \int_{-\infty}^{\infty} |u(t-\tau) - u(t)|\,\omega(t)\,dt.$$

If $u \in C_c^\infty(\mathbb{R})$, then g is continuous and we estimate

$$\begin{aligned}
\|u * \psi_\varepsilon - u\|_{L^1(\mathbb{R},\omega)} &= \int_{-\infty}^{\infty} |u * \psi_\varepsilon(t) - u(t)|\,\omega(t)\,dt \\
&\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(t-\tau) - u(t)|\,\psi_\varepsilon(\tau)\,d\tau\omega(t)\,dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(t-\tau) - u(t)|\,\omega(t)\,dt\,\psi_\varepsilon(\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} g(\tau)\psi_\varepsilon(\tau)\,d\tau \to g(0) = 0
\end{aligned}$$

as $\varepsilon \to 0$ since $\{\psi_\varepsilon\}_{\varepsilon>0}$ is an approximation of the identity.

Since $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R},\omega)$ (see [63, Theorem 3.14]), the result follows in general. Indeed, for $u \in L^1(\mathbb{R},\omega)$, let $u_k \in C_c^\infty(\mathbb{R})$ such that $u_k \to u$ in $L^1(\mathbb{R},\omega)$. Let $\delta > 0$. By part (2), we get

$$\begin{aligned}
\|u * \psi_\varepsilon - u\|_{L^1(\mathbb{R},\omega)} &\leq \|u * \psi_\varepsilon - u_k * \psi_\varepsilon\|_{L^1(\mathbb{R},\omega)} + \|u_k * \psi_\varepsilon - u_k\|_{L^1(\mathbb{R},\omega)} + \|u_k - u\|_{L^1(\mathbb{R},\omega)} \\
&= \|(u - u_k) * \psi_\varepsilon\|_{L^1(\mathbb{R},\omega)} + \|u_k * \varphi_\varepsilon - u_k\|_{L^1(\mathbb{R},\omega)} + \|u_k - u\|_{L^1(\mathbb{R},\omega)} \\
&\leq C\,\|u - u_k\|_{L^1(\mathbb{R},\omega)} + \|u_k * \varphi_\varepsilon - u_k\|_{L^1(\mathbb{R},\omega)} + \|u_k - u\|_{L^1(\mathbb{R},\omega)} < \delta
\end{aligned}$$

for k large and ε small. □

2.2.4 The maximal estimate (1.3.4)

Proof of Theorem 2.1.2. We begin by writing

$$I_\alpha + II_\alpha := \frac{1}{\Gamma(-\alpha)}\int_0^1 (u(t-\tau) - u(t))\,\frac{d\tau}{\tau^{1+\alpha}} + \frac{1}{\Gamma(-\alpha)}\int_1^\infty (u(t-\tau) - u(t))\,\frac{d\tau}{\tau^{1+\alpha}}. \tag{2.2.3}$$

To study I_α, notice that

$$\begin{aligned}
\int_0^1 |u(t-\tau) - u(t)|\,\frac{d\tau}{\tau^{1+\alpha}} &\leq \int_0^1 \tau \int_0^1 \left|u'(t-r\tau)\right| dr\,\frac{d\tau}{\tau^{1+\alpha}} \\
&= \int_0^1\int_0^1 \left|u'(t-r\tau)\right| \frac{d\tau}{\tau^\alpha}\,dr \\
&= \int_0^1 \left(\int_0^r \left|u'(t-\tau)\right| \frac{d\tau}{\tau^\alpha}\right) r^\alpha \frac{dr}{r} \\
&\leq \int_0^1 r^{\alpha-1}\int_0^1 \left|u'(t-\tau)\right| \frac{d\tau}{\tau^\alpha}\,dr \\
&= \frac{1}{\alpha}\int_0^1 \left|u'(t-\tau)\right| \frac{d\tau}{\tau^\alpha}.
\end{aligned} \tag{2.2.4}$$

Then, if we let $\eta(t) = t^{-\alpha}\chi_{(0,1)}(t)$, by Lemma 2.2.2,

$$\begin{aligned}|I_\alpha| &\leq \frac{1}{|\Gamma(-\alpha)\alpha|}(|u'| * \eta)(t) \\ &\leq \frac{1}{|\Gamma(1-\alpha)|} M^- u(t) \int_0^1 \tau^{-\alpha}\, d\tau \\ &= \frac{1}{|\Gamma(1-\alpha)|\,(1-\alpha)} M^- u(t) = C_1 M^- u(t)\end{aligned}$$

where

$$C_1 = \frac{1}{\Gamma(2-\alpha)}.$$

Considering now the second integral in (2.2.3), we observe that

$$II_\alpha = \frac{1}{\Gamma(-\alpha)} \int_1^\infty u(t-\tau) \frac{d\tau}{\tau^{1+\alpha}} + \frac{1}{\Gamma(1-\alpha)}\, u(t).$$

For the first term, we estimate using Lemma 2.2.2 with $\eta(t) = \chi_{(0,1]}(t) + t^{-1-\alpha}\chi_{(1,\infty)}(t)$,

$$\begin{aligned}\left| \frac{1}{\Gamma(-\alpha)} \int_1^\infty u(t-\tau) \frac{d\tau}{\tau^{1+\alpha}} \right| &\leq \frac{1}{|\Gamma(-\alpha)|}(|u| * \eta)(t) \\ &\leq \frac{1}{|\Gamma(-\alpha)|} M^- u(t) \left(\int_0^1 d\tau + \int_1^\infty \tau^{-1-\alpha}\, d\tau \right) \\ &= \frac{1}{|\Gamma(-\alpha)|} \left(1 + \frac{1}{\alpha}\right) M^- u(t) = C_2 M^- u(t)\end{aligned}$$

where

$$C_2 = \frac{1+\alpha}{\Gamma(1-\alpha)}$$

which is bounded independently of α. Therefore,

$$|II_\alpha| \leq C_2 M^- u(t) + C_3\, |u(t)| \leq (C_2 + C_3) M^- u(t)$$

where

$$C_3 = \frac{1}{|\Gamma(1-\alpha)|}.$$

The result follows since

$$C_2 \leq 1 \quad \text{and} \quad C_2 + C_3 = \frac{2+\alpha}{\Gamma(1-\alpha)} = \frac{(2+\alpha)(1-\alpha)}{\Gamma(2-\alpha)} \leq 3$$

□

2.3 Proof of Theorem 2.1.1

2.3.1 Proof of Theorem 2.1.1 (a)

The proof of part (a) is organized as follows. We first show that the formula in the right hand side of (2.1.1) is well-defined as a function in $L^p(\mathbb{R},\omega)$. It is then shown that the distribution $(D_{\text{left}})^\alpha u$ is indeed given by such pointwise formula using the fact that $C_c^\infty(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R},\omega)$. The $L^p(\mathbb{R},\omega)$ estimate in (2.1.2) follows immediately from these steps of the proof. Next, we show that the limit in (2.1.3) holds in $L^p(\mathbb{R},\omega)$ for $u \in C_c^\infty(\mathbb{R})$ and then use a density argument to show the result for $u \in W^{1,p}(\mathbb{R},\omega)$. The a.e. convergence of (2.1.3) is proved by showing that the set of functions in $W^{1,p}(\mathbb{R},\omega)$ such that (2.1.3) holds a.e. is closed in $W^{1,p}(\mathbb{R},\omega)$. The a.e. convergence of (2.1.4) follows similarly. Finally, the maximal estimate allows us to prove that (2.1.4) holds in $L^p(\mathbb{R},\omega)$, $1 < p < \infty$.

Step 1. The integral expression in (2.1.1) defines a function in $L^p(\mathbb{R},\omega)$.

First let $1 < p < \infty$. By Theorem 2.1.2 and the boundedness of M^- in $L^p(\mathbb{R},\omega)$ for $\omega \in A_p^-(\mathbb{R})$, it is immediate that

$$\left\| \frac{1}{\Gamma(-\alpha)} \int_0^\infty (u(t-\tau) - u(t)) \frac{d\tau}{\tau^{1+\alpha}} \right\|_{L^p(\mathbb{R},\omega)} \le C_\omega \left(\|u\|_{L^p(\mathbb{R},\omega)} + \|u'\|_{L^p(\mathbb{R},\omega)} \right). \tag{2.3.1}$$

For $p = 1$, we consider the terms I_α and II_α as in (2.2.3). We use (2.2.4) to observe that

$$\begin{aligned}
\|I_\alpha\|_{L^1(\mathbb{R},\omega)} &\le \frac{1}{|\Gamma(1-\alpha)|} \int_{-\infty}^\infty \int_0^1 \left|u'(t-\tau)\right| \frac{d\tau}{\tau^\alpha}\, \omega(t)\, dt \\
&= \frac{1}{|\Gamma(1-\alpha)|} \int_{-\infty}^\infty \int_{t-1}^t \frac{|u'(\tau)|}{(t-\tau)^\alpha}\, d\tau\, w(t)\, dt \\
&= \frac{1}{|\Gamma(1-\alpha)|} \int_{-\infty}^\infty \left|u'(\tau)\right| \int_\tau^{\tau+1} \frac{\omega(t)}{(t-\tau)^\alpha}\, dt\, d\tau \\
&= \frac{1}{|\Gamma(1-\alpha)|} \int_{-\infty}^\infty \left|u'(\tau)\right| \int_0^1 \frac{\omega(t+\tau)}{t^\alpha}\, dt\, d\tau.
\end{aligned}$$

Since $\omega \in A_1^-(\mathbb{R})$, for a.e. $\tau \in \mathbb{R}$ we can use Lemma 2.2.2 with $\eta(\tau) = |\tau|^{-\alpha} \chi_{(-1,0)}(\tau)$ to get

$$\begin{aligned}\int_0^1 \frac{\omega(t+\tau)}{t^\alpha}\,dt = \int_{-1}^0 \frac{\omega(\tau-t)}{|t|^\alpha}\,dt &= (\omega * \eta)(\tau)\\ &\leq M^+\omega(\tau)\int_{-1}^0 |t|^{-\alpha}\,dt\\ &= \frac{1}{(1-\alpha)}M^+\omega(\tau)\\ &\leq \frac{C}{(1-\alpha)}\omega(\tau).\end{aligned}$$

Therefore,

$$\|I_\alpha\|_{L^1(\mathbb{R},\omega)} \leq C_\omega C_1 \left\|u'\right\|_{L^1(\mathbb{R},\omega)}$$

where C_1 is as in the proof of Theorem 2.1.2. Moving to the second term in (2.2.3), we write

$$II_\alpha = \frac{1}{\Gamma(-\alpha)}\int_1^\infty u(t-\tau)\frac{d\tau}{\tau^{1+\alpha}} + \frac{1}{\Gamma(1-\alpha)}u(t)$$

and estimate

$$\begin{aligned}\left\|\int_1^\infty u(t-\tau)\frac{d\tau}{\tau^{1+\alpha}}\right\|_{L^1(\mathbb{R},\omega)} &\leq \int_{-\infty}^\infty\int_1^\infty \frac{|u(t-\tau)|}{\tau^{1+\alpha}}d\tau\,\omega(t)\,dt\\ &= \int_{-\infty}^\infty\int_{-\infty}^{t-1} \frac{|u(\tau)|}{(t-\tau)^{1+\alpha}}d\tau\,\omega(t)\,dt\\ &= \int_{-\infty}^\infty |u(\tau)|\int_{\tau+1}^\infty \frac{\omega(t)}{(t-\tau)^{1+\alpha}}\,dt\,d\tau\\ &= \int_{-\infty}^\infty |u(\tau)|\int_1^\infty \frac{\omega(t+\tau)}{t^{1+\alpha}}\,dt\,d\tau.\end{aligned}$$

By using again the $A_1^-(\mathbb{R})$-condition and Lemma 2.2.2 with $\eta(\tau) = \chi_{[-1,0)}(\tau) + |\tau|^{-1-\alpha}\chi_{(-\infty,-1)}(\tau)$, for a.e. $\tau \in \mathbb{R}$,

$$\begin{aligned}\int_1^\infty \frac{\omega(t+\tau)}{t^{1+\alpha}}\,dt &= \int_{-\infty}^{-1} \frac{\omega(\tau-t)}{|t|^{1+\alpha}}\,dt\\ &\leq (\omega*\eta)(\tau)\\ &\leq M^+\omega(\tau)\left(\int_{-1}^0 dt + \int_{-\infty}^{-1}|t|^{-1-\alpha}\,dt\right)\\ &\leq \frac{1+\alpha}{\alpha}M^+\omega(\tau)\\ &\leq C\frac{1+\alpha}{\alpha}\omega(\tau).\end{aligned}$$

Therefore, by collecting terms,

$$\begin{aligned}\|II_\alpha\|_{L^1(\mathbb{R},\omega)} &\le \frac{1}{|\Gamma(-\alpha)|}\left\|\int_1^\infty u(t-\tau)\frac{d\tau}{\tau^{1+\alpha}}\right\|_{L^1(\mathbb{R},\omega)} + \frac{1}{|\Gamma(1-\alpha)|}\|u\|_{L^1(\mathbb{R},\omega)} \\ &\le C\frac{1+\alpha}{|\Gamma(-\alpha)|\,\alpha}\|u\|_{L^1(\mathbb{R},\omega)} + \frac{1}{|\Gamma(1-\alpha)|}\|u\|_{L^1(\mathbb{R},\omega)} \\ &\le C_\omega(C_2+C_3)\|u\|_{L^1(\mathbb{R},\omega)}\end{aligned} \tag{2.3.2}$$

where $C_2, C_3 > 0$ are as in the proof of Theorem 2.1.2. Thus,

$$\left\|\frac{1}{\Gamma(-\alpha)}\int_0^\infty (u(t-\tau)-u(t))\,\frac{d\tau}{\tau^{1+\alpha}}\right\|_{L^1(\mathbb{R},\omega)} \le C_\omega\left(\|u\|_{L^1(\mathbb{R},\omega)} + \|u'\|_{L^1(\mathbb{R},\omega)}\right). \tag{2.3.3}$$

Hence, the integral in (2.1.1) is in $L^p(\mathbb{R},\omega)$ for $1 \le p < \infty$.

<u>**Step 2.**</u> The distribution $(D_{\text{left}})^\alpha u$ coincides with the integral formula in (2.1.1). Therefore $(D_{\text{left}})^\alpha u$ is in $L^p(\mathbb{R},\omega)$ and, by (2.3.1) and (2.3.3), (2.1.2) holds.

To show (2.1.1), let $u_k \in C_c^\infty(\mathbb{R})$ such that $u_k \to u$ in $W^{1,p}(\mathbb{R},\omega)$ as $k \to \infty$. We may write

$$(D_{\text{left}})^\alpha u_k(t) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty (u_k(t-\tau)-u_k(t))\,\frac{d\tau}{\tau^{1+\alpha}}.$$

Using (2.3.1) and (2.3.3), we can show that the formulas converge in norm. Indeed,

$$\begin{aligned}&\left\|\frac{1}{\Gamma(-\alpha)}\int_0^\infty (u_k(t-\tau)-u_k(t))\,\frac{d\tau}{\tau^{1+\alpha}} - \frac{1}{\Gamma(-\alpha)}\int_0^\infty (u(t-\tau)-u(t))\,\frac{d\tau}{\tau^{1+\alpha}}\right\|_{L^p(\mathbb{R},\omega)} \\ &\qquad \le C\left(\|u_k-u\|_{L^p(\omega)} + \|u_k'-u'\|_{L^p(\mathbb{R},\omega)}\right) \to 0 \quad \text{as } k\to\infty.\end{aligned}$$

If $\varphi \in C_c^\infty(\mathbb{R})$ and A is such that $\operatorname{supp}\varphi \subset (-\infty, A]$, then $\varphi \in \mathcal{S}_-$ and $(D_{\text{right}})^\alpha\varphi \in \mathcal{S}_-^\alpha$ with $\operatorname{supp}((D_{\text{right}})^\alpha\varphi) \subset (-\infty, A]$. Now, by Definition 2.2.1,

$$\begin{aligned}((D_{\text{left}})^\alpha u)(\varphi) &= \int_{-\infty}^\infty u(t)\,(D_{\text{right}})^\alpha\varphi(t)\,dt \\ &= \lim_{k\to\infty}\int_{-\infty}^\infty u_k(t)\,(D_{\text{right}})^\alpha\varphi(t)\,dt \\ &= \lim_{k\to\infty}\int_{-\infty}^\infty (D_{\text{left}})^\alpha u_k(t)\,\varphi(t)\,dt \\ &= \lim_{k\to\infty}\int_{-\infty}^\infty \left(\frac{1}{\Gamma(-\alpha)}\int_0^\infty (u_k(t-\tau)-u_k(t))\,\frac{d\tau}{\tau^{1+\alpha}}\right)\varphi(t)\,dt \\ &= \int_{-\infty}^\infty \left(\frac{1}{\Gamma(-\alpha)}\int_0^\infty (u(t-\tau)-u(t))\,\frac{d\tau}{\tau^{1+\alpha}}\right)\varphi(t)\,dt.\end{aligned} \tag{2.3.4}$$

In the second identity above we used that, by Proposition 2.2.2,

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} u_k(t)\,(D_{\text{right}})^{\alpha}\varphi(t)\,dt - \int_{-\infty}^{\infty} u(t)\,(D_{\text{right}})^{\alpha}\varphi(t)\,dt\right| \\
&\le \int_{-\infty}^{A} |u_k(t)-u(t)|\;|(D_{\text{right}})^{\alpha}\varphi(t)|\;dt \\
&= \int_{-\infty}^{A} \frac{|u_k(t)-u(t)|}{1+|t|^{1+\alpha}}\;|(D_{\text{right}})^{\alpha}\varphi(t)|\,(1+|t|^{1+\alpha})\,dt \\
&\le C\int_{-\infty}^{A} \frac{|u_k(t)-u(t)|}{1+|t|^{1+\alpha}}\,dt \\
&\le C\,\|u_k-u\|_{L^p(\mathbb{R},\omega)} \to 0 \quad \text{as } k\to\infty
\end{aligned}
$$

and in the last equality we observed that

$$
\begin{aligned}
\Bigg|\int_{-\infty}^{\infty}(D_{\text{left}})^{\alpha}u_k(t)\varphi(t)\,dt &- \int_{-\infty}^{\infty}\left(\frac{1}{\Gamma(-\alpha)}\int_0^{\infty}(u(t-\tau)-u(t))\,\frac{d\tau}{\tau^{1+\alpha}}\right)\varphi(t)\,dt\Bigg| \\
&\le C\int_{-\infty}^{A}\left|\int_0^{\infty}(u_k(t-\tau)-u_k(t))\,\frac{d\tau}{\tau^{1+\alpha}} - \int_0^{\infty}(u(t-\tau)-u(t))\,\frac{d\tau}{\tau^{1+\alpha}}\right|\frac{1}{1+|t|^{1+\alpha}}\,dt \\
&\le C\left\|\int_0^{\infty}(u_k(\cdot-\tau)-u_k(\cdot))\,\frac{d\tau}{\tau^{1+\alpha}} - \int_0^{\infty}(u(\cdot-\tau)-u(\cdot))\,\frac{d\tau}{\tau^{1+\alpha}}\right\|_{L^p(\mathbb{R},\omega)} \to 0
\end{aligned}
$$

as $k\to\infty$. Therefore, since φ was arbitrary in (2.3.4),

$$
(D_{\text{left}})^{\alpha}u(t) = \frac{1}{\Gamma(-\alpha)}\int_0^{\infty}(u(t-\tau)-u(t))\,\frac{d\tau}{\tau^{1+\alpha}} \quad \text{a.e. in } \mathbb{R}.
$$

Step 3. The limit as $\alpha\to 1^-$ in (2.1.3) holds in $L^p(\mathbb{R},\omega)$ for $u\in C_c^{\infty}(\mathbb{R})$.

Suppose that $u\in C_c^{\infty}(\mathbb{R})$ and write $(D_{\text{left}})^{\alpha}u(t) = I_\alpha + II_\alpha$ as in (2.2.3). For $1<p<\infty$, we see from the proof of Theorem 2.1.2 that

$$
\begin{aligned}
\|II_\alpha\|_{L^p(\mathbb{R},\omega)} &\le (C_2+C_3)\,\|M^- u\|_{L^p(\mathbb{R},\omega)} \\
&\le \left(\frac{1+\alpha}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\right)C_\omega\,\|u\|_{L^p(\mathbb{R},\omega)} \to 0
\end{aligned}
$$

as $\alpha\to 1^-$. For $p=1$, by (2.3.2) in Step 1, we similarly obtain

$$
\|II_\alpha\|_{L^p(\omega)} \le C_w(C_2+C_3)\,\|u\|_{L^1(\omega)} \to 0 \quad \text{as } \alpha\to 1^-.
$$

Next, observe that

$$
\begin{aligned}
I_\alpha - u'(t) &= \frac{1}{\Gamma(-\alpha)} \int_0^1 \left(-\int_0^\tau u'(t-r)\,dr \right) \frac{d\tau}{\tau^{1+\alpha}} - u'(t) \\
&= \frac{1}{\Gamma(-\alpha)} \int_0^1 \int_0^\tau (u'(t) - u'(t-r))\,dr\,\frac{d\tau}{\tau^{1+\alpha}} + \left(\frac{\alpha}{\Gamma(2-\alpha)} - 1 \right) u'(t) \\
&= \frac{1}{\Gamma(-\alpha)} \int_0^1 \int_0^\tau \int_0^r u''(t-\mu)\,d\mu\,dr\,\frac{d\tau}{\tau^{1+\alpha}} + \left(\frac{\alpha}{\Gamma(2-\alpha)} - 1 \right) u'(t).
\end{aligned}
$$

Let K be such that $\operatorname{supp} u''(\cdot - \mu) \subset [-K, K]$ for all $\mu \in [0,1]$. Then, for $1 \le p < \infty$,

$$
\left\| u''(\cdot - \mu) \right\|_{L^p(\mathbb{R},\omega)} \le \left\| u'' \right\|_{L^\infty(\mathbb{R})} \omega([-K,K])^{1/p} = c
$$

where $c > 0$ is independent of α. Therefore,

$$
\begin{aligned}
&\|I_\alpha - u'\|_{L^p(\mathbb{R},\omega)} \\
&\quad\le \frac{1}{|\Gamma(-\alpha)|} \int_0^1 \int_0^\tau \int_0^r \left\| u''(t-\mu) \right\|_{L^p(\mathbb{R},\omega)} d\mu\,dr\,\frac{d\tau}{\tau^{1+\alpha}} + \left| \frac{\alpha}{\Gamma(2-\alpha)} - 1 \right| \left\| u' \right\|_{L^p(\mathbb{R},\omega)} \\
&\quad\le \frac{c}{|\Gamma(-\alpha)|} \int_0^1 \int_0^\tau \int_0^r d\mu\,dr\,\frac{d\tau}{\tau^{1+\alpha}} + \left| \frac{\alpha}{\Gamma(2-\alpha)} - 1 \right| \left\| u' \right\|_{L^p(\mathbb{R},\omega)} \\
&\quad\le \frac{c}{|\Gamma(-\alpha)|\,(2-\alpha)} + \left| \frac{\alpha}{\Gamma(2-\alpha)} - 1 \right| \left\| u' \right\|_{L^p(\mathbb{R},\omega)} \\
&\quad= c\frac{\alpha(1-\alpha)}{|\Gamma(3-\alpha)|} + \left| \frac{\alpha}{\Gamma(2-\alpha)} - 1 \right| \left\| u' \right\|_{L^p(\mathbb{R},\omega)} \to 0 \quad \text{as } \alpha \to 1^-.
\end{aligned}
$$

Hence,

$$
\left\| (D_{\text{left}})^\alpha u - u' \right\|_{L^p(\mathbb{R},\omega)} \le \|II_\alpha\|_{L^p(\mathbb{R},\omega)} + \left\| I_\alpha - u' \right\|_{L^p(\mathbb{R},\omega)} \to 0,
$$

as $\alpha \to 1^-$.

<u>**Step 4.**</u> The limit as $\alpha \to 1^-$ in (2.1.3) holds in $L^p(\mathbb{R},\omega)$ for $u \in W^{1,p}(\mathbb{R},\omega)$.

Let $u_k \in C_c^\infty(\mathbb{R})$ be such that $u_k \to u$ in $W^{1,p}(\mathbb{R},\omega)$ as $k \to \infty$. We just observe that, by the L^p estimate (2.1.2) (that was proved in Step 2), for $1 \le p < \infty$,

$$
\begin{aligned}
&\left\| (D_{\text{left}})^\alpha u - u' \right\|_{L^p(\mathbb{R},\omega)} \\
&\quad\le \|(D_{\text{left}})^\alpha (u - u_k)\|_{L^p(\mathbb{R},\omega)} + \left\| (D_{\text{left}})^\alpha u_k - u_k' \right\|_{L^p(\mathbb{R},\omega)} + \left\| u_k' - u' \right\|_{L^p(\mathbb{R},\omega)} \\
&\quad\le C \left(\|u - u_k\|_{L^p(\mathbb{R},\omega)} + \left\| (u - u_k)' \right\|_{L^p(\mathbb{R},\omega)} \right) + \left\| (D_{\text{left}})^\alpha u_k - u_k' \right\|_{L^p(\mathbb{R},\omega)}.
\end{aligned}
$$

Then take k large and choose α close to 1^- (see Step 3).

Step 5. The limit as $\alpha \to 1^-$ in (2.1.3) holds almost everywhere for $u \in W^{1,p}(\mathbb{R},\omega)$.

It follows from Theorem 2.1.2 and the properties of M^- that the operator T^* defined by

$$T^*u(t) = \sup_{0<\alpha<1} (D_{\text{left}})^\alpha u(t) \quad \text{for } u \in W^{1,p}(\mathbb{R},\omega)$$

satisfies the estimates

$$\|T^*u\|_{L^p(\mathbb{R},\omega)} \le C_{p,\omega}\, \|u\|_{W^{1,p}(\mathbb{R},\omega)} \quad \text{for any } u \in W^{1,p}(\mathbb{R},\omega),\ 1<p<\infty$$

and

$$\omega\big(\{t \in \mathbb{R} : |T^*u(t)| > \lambda\}\big) \le \frac{C_\omega}{\lambda} \|u\|_{W^{1,1}(\mathbb{R},\omega)} \quad \text{for any } u \in W^{1,1}(\mathbb{R},\omega).$$

In particular, T^* is bounded from $W^{1,p}(\mathbb{R},\omega)$ into weak-$L^p(\mathbb{R},\omega)$, for any $1 \le p < \infty$. To conclude, we need to see that the set

$$E = \{u \in W^{1,p}(\mathbb{R},\omega) : \lim_{\alpha\to 1^-} (D_{\text{left}})^\alpha u(t) = u'(t) \ a.e.\}$$

is closed in $W^{1,p}(\mathbb{R},\omega)$. Indeed, since $C_c^\infty(\mathbb{R}) \subset E$ and $C_c^\infty(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R},\omega)$, this claim gives that $E = W^{1,p}(\mathbb{R},\omega)$.

To check that E is a closed set, let $u_k \in E$ be a sequence such that $u_k \to u$ in $W^{1,p}(\mathbb{R},\omega)$, for some $u \in W^{1,p}(\mathbb{R},\omega)$. We will prove that $u \in E$. Let $\lambda > 0$ be arbitrary. By using the boundedness of T^* and Chebyshev's inequality, we find that

$$\begin{aligned}
&\omega(\{t \in \mathbb{R} : \limsup_{\alpha\to 1^-} \big|(D_{\text{left}})^\alpha u(t) - u'(t)\big| > \lambda\}) \\
&\qquad \le \omega(\{t \in \mathbb{R} : \limsup_{\alpha\to 1^-} |(D_{\text{left}})^\alpha u(t) - (D_{\text{left}})^\alpha u_k(t)| > \lambda/3\}) \\
&\qquad\quad + \omega(\{t \in \mathbb{R} : \limsup_{\alpha\to 1^-} \big|(D_{\text{left}})^\alpha u_k(t) - u_k'(t)\big| > \lambda/3\}) \\
&\qquad\quad + \omega(\{t \in \mathbb{R} : \big|u_k'(t) - u'(t)\big| > \lambda/3\}) \\
&\qquad \le \omega(\{t \in \mathbb{R} : |T^*(u-u_k)(t)| > \lambda/3\}) + \omega(\{t \in \mathbb{R} : \big|(u_k-u)'(t)\big| > \lambda/3\}) \\
&\qquad \le 2\left(\frac{3C}{\lambda}\, \|u-u_k\|_{W^{1,p}(\mathbb{R},\omega)}\right)^p \to 0
\end{aligned}$$

as $k \to \infty$. Therefore

$$\omega(\{t \in \mathbb{R} : \limsup_{\alpha\to 1^-} \left|(D_{\text{left}})^\alpha u(t) - u'(t)\right| > 0\})$$
$$\leq \sum_{n=1}^{\infty} \omega(\{t \in \mathbb{R} : \limsup_{\alpha\to 1^-} \left|(D_{\text{left}})^\alpha u(t) - u'(t)\right| > 1/n\}) = 0.$$

Since $\lambda > 0$ was arbitrary, we have $u \in E$.

<u>**Step 6.**</u> The limit as $\alpha \to 0^+$ in (2.1.4) holds almost everywhere for $u \in W^{1,p}(\mathbb{R}, \omega)$.

As in Step 5, one can check that the set

$$E' = \{u \in W^{1,p}(\mathbb{R}, \omega) : \lim_{\alpha\to 0^+} (D_{\text{left}})^\alpha u(t) = u(t) \text{ a.e.}\}$$

is closed in $W^{1,p}(\mathbb{R}, \omega)$. Since $C_c^\infty(\mathbb{R}) \subset E'$, by density, we get $E' = W^{1,p}(\mathbb{R}, \omega)$.

<u>**Step 7.**</u> The limit as $\alpha \to 0^+$ in (2.1.4) holds in $L^p(\mathbb{R}, \omega)$.

By Theorem 2.1.2, for any $0 < \alpha < 1$,

$$|(D_{\text{left}})^\alpha u(t) - u(t)|^p \, \omega(t) \leq \left(C(M^-(u')(t) + M^- u(t)) + |u(t)|\right)^p \omega(t)$$
$$\leq C_p \left((M^-(u')(t))^p + (M^- u(t))^p\right) \omega(t) \in L^p(\mathbb{R}).$$

Therefore, by Step 6 and the Dominated Convergence Theorem, (2.1.4) holds.

The proof of Theorem 2.1.1, part (a), is completed. □

2.3.2 Proof of Theorem 2.1.1 (b)

This is proved through a distributional argument.

Suppose that $(D_{\text{left}})^\alpha u \to v$ in $L^p(\mathbb{R}, \omega)$ as $\alpha \to 1^-$. Let $\varphi \in C_c^\infty(\mathbb{R})$. Let $A \in \mathbb{R}$ be such that $\operatorname{supp} \varphi \subset (-\infty, A]$, so that $\varphi \in \mathcal{S}_-$ and $(D_{\text{right}})^\alpha \varphi \in \mathcal{S}_-^\alpha$. By Proposition 2.2.2,

$$\left|\int_{-\infty}^{\infty} v(t)\, \varphi(t)\, dt - \int_{-\infty}^{\infty} (D_{\text{left}})^\alpha u(t)\, \varphi(t)\, dt\right| \leq \int_{-\infty}^{A} |v(t) - (D_{\text{left}})^\alpha u(t)| \, \frac{C}{1+|t|}\, dt$$
$$\leq C_{\varphi,A,\omega,p} \left\|v - (D_{\text{left}})^\alpha u\right\|_{L^p(\mathbb{R},\omega)} \to 0$$

as $\alpha \to 1^-$. With this and the definition of $(D_{\text{left}})^\alpha u$ we can write

$$\begin{aligned}\int_{-\infty}^{\infty} v\,\varphi\,dt &= \lim_{\alpha\to 1^-}\int_{-\infty}^{\infty}(D_{\text{left}})^\alpha u\,\varphi\,dt\\ &= \lim_{\alpha\to 1^-}\int_{-\infty}^{\infty} u\,(D_{\text{right}})^\alpha\varphi\,dt.\end{aligned}$$

Next, notice that, by Proposition 2.2.2,

$$\begin{aligned}|u(t)|\,|(D_{\text{right}})^\alpha\varphi+\varphi'| &\le |u(t)|\frac{C_\varphi}{1+|t|^{1+\alpha}}\chi_{(-\infty,A]}(t)\\ &\le C_\varphi\frac{|u(t)|}{1+|t|}\chi_{(-\infty,A]}(t)\in L^1(\mathbb{R}).\end{aligned}$$

Therefore, by the Dominated Convergence Theorem, as $(D_{\text{right}})^\alpha\varphi(t)\to-\varphi'(t)$ as $\alpha\to 0^+$,

$$\begin{aligned}&\lim_{\alpha\to 1^-}\left|\int_{-\infty}^{\infty}u(t)\,(D_{\text{right}})^\alpha\varphi(t)\,dt+\int_{-\infty}^{\infty}u(t)\,\varphi'(t)\,dt\right|\\ &\qquad\le\int_{-\infty}^{\infty}\lim_{\alpha\to 1^-}|u(t)|\left|(D_{\text{right}})^\alpha\varphi(t)+\varphi'(t)\right|\,dt=0\end{aligned}$$

Whence,

$$\begin{aligned}\int_{-\infty}^{\infty} v\,\varphi\,dt &= \lim_{\alpha\to 1^-}\int_{-\infty}^{\infty} u\,(D_{\text{right}})^\alpha\varphi\,dt\\ &= -\int_{-\infty}^{\infty} u\,\varphi'\,dt=\int_{-\infty}^{\infty}u'\,\varphi\,dt.\end{aligned}$$

Therefore $v=u'$ a.e. in $\mathbb{R}$. Since $u'=v\in L^p(\mathbb{R},\omega)$, we get $u\in W^{1,p}(\mathbb{R},\omega)$, and by Theorem 2.1.1$(a)$, the conclusion follows. □

2.3.3 Proof of Theorem 2.1.1 (c)

Using the exact same arguments as in part (b), we find that

$$\begin{aligned}\int_{-\infty}^{\infty} v\,\varphi\,dt &= \lim_{\alpha\to 0^+}\int_{-\infty}^{\infty}(D_{\text{left}})^\alpha u\,\varphi\,dt\\ &= \lim_{\alpha\to 0^+}\int_{-\infty}^{\infty} u\,(D_{\text{right}})^\alpha\varphi\,dt=\int_{-\infty}^{\infty}u\,\varphi\,dt.\end{aligned}$$

Therefore $v=u$ a.e. in $\mathbb{R}$ and the conclusion follows. □

2.4 Fractional Laplacians and Muckenhoupt weights

For $u \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform identity

$$\widehat{(-\Delta)u}(\xi) = |\xi|^2 \widehat{u}(\xi)$$

is used to define the fractional Laplacian as

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi) \quad \text{for } 0 < s < 1.$$

Using the heat diffusion semigroup $\{e^{t\Delta}\}_{t\geq 0}$ generated by $-\Delta$, it is shown in [72, 73] that the fractional Laplacian can be expressed using the semigroup formula (2.1.5) and that this is equivalent to the pointwise formula (2.1.6). For completeness, we provide some of the details.

Using the Gamma function, we can write

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\lambda t} - 1\right) \frac{dt}{t^{1+s}}, \quad \text{for any } 0 < s < 1,\ \lambda > 0.$$

Choosing $\lambda = |\xi|^2$ and multiplying by $\widehat{u}(\xi)$, we obtain

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-t|\xi|^2} \widehat{u}(\xi) - \widehat{u}(\xi)\right) \frac{dt}{t^{1+s}}.$$

Taking the inverse Fourier transform, the semigroup formula (2.1.5) holds. Here, $e^{t\Delta}$ is the operator that satisfies

$$\widehat{e^{t\Delta} u}(\xi) = e^{-t|\xi|^2} \widehat{u}(\xi).$$

It is well known that $e^{t\Delta}u(x) = (W_t * u)(x)$ where $W_t(x)$ is the Gauss–Weierstrass kernel

$$W_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)} \qquad \text{for } x \in \mathbb{R}^n,\ t > 0$$

and that $v = e^{t\Delta}u$ solves the heat equation in $\mathbb{R}^n$ with initial data u,

$$\begin{cases} \partial_t v = \Delta v & \text{for } x \in \mathbb{R}^n, t > 0 \\ v(x,0) = u(x) & \text{for } x \in \mathbb{R}^n. \end{cases}$$

By writing

$$\begin{aligned} (-\Delta)^s u(x) &= \frac{1}{\Gamma(-s)} \int_0^\infty ((W_t * u)(x) - u(x)) \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n} W_t(x-y)\,(u(y) - u(x))\, dy \frac{dt}{t^{1+s}} \end{aligned}$$

and applying Fubini's theorem, the pointwise formula (2.1.6) holds.

2.4.1 Distributional setting

The distributional setting for the fractional Laplacian was developed by Silvestre in [68]. Consider the function class

$$\mathcal{S}_s = \left\{ \varphi \in C^\infty(\mathbb{R}^n) : |D^\gamma \varphi(x)| \leq \frac{C}{1+|x|^{n+2s}},\ \text{for all } \gamma \in \mathbb{N}_0^n,\ x \in \mathbb{R}^n,\ \text{for some } C > 0 \right\}.$$

We endow $\mathcal{S}_s$ with the topology induced by the family of seminorms

$$\rho_\gamma^s(\varphi) = \sup_{x \in \mathbb{R}^n} (1 + |x|^{n+2s}) \left| D^\gamma \varphi(x) \right|, \qquad \text{for } \gamma \in \mathbb{N}_0^n.$$

Let $(\mathcal{S}_s)'$ be the dual space of $\mathcal{S}_s$. Notice that $\mathcal{S} \subset \mathcal{S}_s$, so that $(\mathcal{S}_s)' \subset \mathcal{S}'$. For $u \in (\mathcal{S}_s)'$, $(-\Delta)^s u$ is defined as a distribution on $\mathcal{S}'$ by

$$((-\Delta)^s u)(\varphi) = u\left((-\Delta)^s \varphi\right) \quad \text{for any } \varphi \in \mathcal{S}.$$

One can check that $L_s \subset (\mathcal{S}_s)'$, where

$$L_s = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\, dx < \infty \right\}.$$

Proposition 2.4.1 (Silvestre [68])**.** *Let Ω be an open set in $\mathbb{R}^n$ and $u \in L_s$. If $u \in C^{2s+\varepsilon}(\Omega)$ (or $C^{1,2s+\varepsilon-1}(\Omega)$ if $s \geq 1/2$) for some $\varepsilon > 0$, then $(-\Delta)^s u \in C(\Omega)$ and*

$$(-\Delta)^s u(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \qquad \textit{for every } x \in \Omega.$$

Here (see [72, 73])

$$c_{n,s} = \frac{4^s \Gamma(n/2+s)}{|\Gamma(-s)|\, \pi^{n/2}} \sim s(1-s) \quad \textit{as } s \to 0, 1. \tag{2.4.1}$$

2.4.2 Muckenhoupt weights

A function $\nu \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\nu > 0$ a.e., is called an $A_p(\mathbb{R}^n)$ Muckenhoupt weight, $1 < p < \infty$, if it satisfies the following condition: there exists $C > 0$ such that

$$\left(\frac{1}{|B|} \int_B \nu\, dx \right)^{1/p} \left(\frac{1}{|B|} \int_B \nu^{1-p'}\, dx \right)^{1/p'} \leq C \tag{2.4.2}$$

for any ball $B \subset \mathbb{R}^n$. If ν satisfies (2.4.2), we write $\nu \in A_p(\mathbb{R}^n)$. Observe that $\nu \in A_p(\mathbb{R}^n)$ if and only if $\nu^{1-p'} \in A_{p'}(\mathbb{R}^n)$. The Hardy–Littlewood maximal function is defined by

$$Mu(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |u(y)| \, dy$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x. For $1 < p < \infty$, the operator M is bounded on $L^p(\mathbb{R}^n, \nu)$ if and only if $\nu \in A_p(\mathbb{R}^n)$. When $p = 1$, M is bounded from $L^1(\mathbb{R}^n, \nu)$ into weak-$L^1(\mathbb{R}^n, \nu)$ if and only if $\nu \in A_1(\mathbb{R}^n)$, namely, there exists $C > 0$ such that

$$M\nu(x) \leq C\nu(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

For a measurable set $E \subset \mathbb{R}^n$ and a weight ν, we denote

$$\nu(E) = \int_E \nu \, dx.$$

See [27] for more details about Muckenhoupt weights.

Lemma 2.4.1 (See [27, Proposition 2.7]). *Let $\eta = \eta(x)$ be a function that is positive, radial, decreasing (as a function on $(0, \infty)$) and integrable. Then for any measurable function $u : \mathbb{R}^n \to \mathbb{R}$ and for almost every $x \in \mathbb{R}^n$, we have*

$$|u * \eta(x)| \leq \|\eta\|_{L^1(\mathbb{R}^n)} Mu(x).$$

Lemma 2.4.2 (See [27, Corollary 7.6]). *If $\nu \in A_p(\mathbb{R}^n)$, $1 \leq p < \infty$, then there exists $\delta > 0$ such that given a ball B and a measurable subset S of B,*

$$\frac{\nu(S)}{\nu(B)} \leq C \left(\frac{|S|}{|B|} \right)^{\delta}.$$

Our next result shows that for any function $u \in L^p(\mathbb{R}^n, \nu)$, $\nu \in A_p(\mathbb{R}^n)$, $1 \leq p < \infty$, the object $(-\Delta)^s u$ is well defined as a distribution in $\mathcal{S}'$.

Proposition 2.4.2. *If $u \in L^p(\mathbb{R}^n, \nu)$, $\nu \in A_p(\mathbb{R}^n)$, $1 \leq p < \infty$, then $u \in L_s$, $s \geq 0$, and there is a constant $C = C_{n,p,\nu} > 0$ such that*

$$\|u\|_{L_s} \leq C \|u\|_{L^p(\mathbb{R}^n, \nu)}.$$

In particular, $L^p(\mathbb{R}^n, \nu) \subset L^1_{\text{loc}}(\mathbb{R}^n)$.

Proof. Suppose first that $1 < p < \infty$. By Hölder's inequality,

$$\|u\|_{L_s} \leq \|u\|_{L^p(\mathbb{R}^n,\nu)} \left(C_n \int_{\mathbb{R}^n} \frac{\nu(x)^{1-p'}}{(1+|x|^n)^{p'}}\, dx \right)^{\frac{1}{p'}}.$$

Let $\tilde{\nu}(x) = \nu(x)^{1-p'} \in A_{p'}(\mathbb{R}^n)$. It is enough to show

$$\int_{\mathbb{R}^n} \frac{\tilde{\nu}(x)}{(1+|x|)^{np'}}\, dx < \infty.$$

Let $f(x) = \chi_{B_1}(x)$. If $|x| \leq 1$, then $Mf(x) = 1$. If $|x| \geq 1$, then $B_1 \subset B(x, 2|x|)$ and

$$Mf(x) \geq \frac{|B(0,1)|}{|B(x,2|x|)|} = \frac{1}{(2|x|)^n} \geq \frac{C_n}{(1+|x|)^n}.$$

Since M is bounded on $L^{p'}(\mathbb{R}^n, \tilde{\nu})$, for $\tilde{\nu} \in A_{p'}(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\tilde{\nu}(x)}{(1+|x|)^{np'}}\, dx &\leq C \int_{\mathbb{R}^n} (Mf(x))^{p'} \tilde{\nu}(x)\, dx \\ &\leq C \int_{\mathbb{R}^n} (f(x))^{p'} \tilde{\nu}(x)\, dx = C \int_{B_1} \tilde{\nu}(x)\, dx = C\nu^{1-p'}(B_1).\end{aligned}$$

Therefore,

$$\|u\|_{L_s} \leq C \|u\|_{L^p(\mathbb{R}^n,\nu)} \nu^{1-p'}(B_1) < \infty.$$

Now let $p = 1$. Observe that

$$\int_{|x|<1} \frac{|u(x)|}{1+|x|^{n+2s}}\, dx \leq \|u\|_{L^1(\mathbb{R}^n,\nu)} \sup_{x \in B_1} \nu(x)^{-1} \leq C_{n,\nu} \|u\|_{L^1(\mathbb{R}^n,\nu)}$$

where in the last inequality we used that, since $\nu \in A_1(\mathbb{R}^n)$,

$$\sup_{B_1} \nu^{-1} = \left(\inf_{B_1} \nu \right)^{-1} \leq C \left(\frac{\nu(B_1)}{|B_1|} \right)^{-1}.$$

On the other hand, let $B_j = B_{2^j}(0)$, $j \geq 0$. By using the $A_1(\mathbb{R}^n)$-condition and Lemma 2.4.2 with $S = B_1$ and $B = B_j$,

$$\begin{aligned}
\int_{|x|>1} \frac{|u(x)|}{1+|x|^{n+2s}}\,dx &\le \sum_{j=0}^{\infty} \int_{B_{j+1}\setminus B_j} \frac{|u(x)|}{|x|^n}\,dx \\
&\le c_n \sum_{j=1}^{\infty} \frac{1}{(2^j)^n} \int_{B_j} |u(x)|\,dx \\
&\le c_n \left\|u\right\|_{L^1(\mathbb{R}^n,\nu)} \sum_{j=1}^{\infty} \frac{1}{(2^j)^n} \sup_{x\in B_j} \nu^{-1}(x) \\
&\le C \left\|u\right\|_{L^1(\mathbb{R}^n,\nu)} \sum_{j=1}^{\infty} \frac{1}{(2^j)^n} \frac{|B_j|^\delta}{\nu(B_j)} |B_j|^{1-\delta} \\
&\le C \left\|u\right\|_{L^1(\mathbb{R}^n,\nu)} \frac{|B_1|^\delta}{\nu(B_1)} \sum_{j=1}^{\infty} \frac{(2^{jn})^{1-\delta}}{(2^j)^n} \\
&\le C_{n,\nu} \left\|u\right\|_{L^1(\mathbb{R}^n,\nu)}.
\end{aligned}$$

The result for $p = 1$ follows by combining the previous estimates. □

2.4.3 The heat semigroup on weighted spaces

Recall the definition of the classical heat semigroup $\{e^{t\Delta}\}_{t\ge 0}$ on $\mathbb{R}^n$:

$$e^{t\Delta}u(x) = \int_{\mathbb{R}^n} W_t(x-y)u(y)\,dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/(4t)} u(y)\,dy \tag{2.4.3}$$

for $x \in \mathbb{R}^n$, $t > 0$. We believe that the following result belongs to the folklore, but we provide a proof for the sake of completeness.

Theorem 2.4.1. *Let $\nu \in A_p(\mathbb{R}^n)$ and $u \in L^p(\mathbb{R}^n, \nu)$, $1 \le p < \infty$. The following hold.*

(1) *The integral defining $e^{t\Delta}u(x)$ in (2.4.3) is absolutely convergent for $x \in \mathbb{R}^n$, $t > 0$, and*

$$\sup_{t>0} \left|e^{t\Delta}u(x)\right| \le Mu(x)$$

for almost every $x \in \mathbb{R}^n$.

(2) *$e^{t\Delta}u(x) \in C^\infty((0,\infty)\times\mathbb{R}^n)$ and $\partial_t(e^{t\Delta}u) = \Delta(e^{t\Delta}u)$ in $\mathbb{R}^n \times (0,\infty)$.*

(3) *$\left\|e^{t\Delta}u\right\|_{L^p(\mathbb{R}^n,\nu)} \le C_{n,p,\nu} \left\|u\right\|_{L^p(\mathbb{R}^n,\nu)}$, where $C_{n,p,\nu} > 0$.*

(4) $\lim_{t\to 0^+} e^{t\Delta}u(x) = u(x)$ *for almost every* $x \in \mathbb{R}^n$.

(5) $\lim_{t\to 0^+} \left\|e^{t\Delta}u - u\right\|_{L^p(\mathbb{R}^n,\nu)} = 0$.

(6) *If* $u \in W^{2,p}(\mathbb{R}^n,\nu)$, *then* $e^{t\Delta}\Delta u = \Delta e^{t\Delta}u$.

(7) $\lim_{\varepsilon\to 0} \left\| \int_{|x-y|<\varepsilon} W_t(x-y)u(y)\,dy \right\|_{L^p(\mathbb{R}^n,\nu)} = 0$.

Proof. Let $u \in L^p(\mathbb{R}^n,\nu)$, $\nu \in A_p(\mathbb{R}^n)$, for $1 \le p < \infty$.

For (1), we apply Lemma 2.4.1 with $\eta(x) = W_t(x)$ and notice that $\|W_t\|_{L^1(\mathbb{R}^n)} = 1$, for each fixed $t > 0$ to estimate

$$\left|e^{t\Delta}u(x)\right| = |\eta * u(x)| \le \|\eta\|_{L^1(\mathbb{R}^n)} Mu(x) = Mu(x).$$

To prove (2), we recall that $W_t(x) \in C^\infty(\mathbb{R}^n \times (0,\infty))$, $\partial_t W_t = \Delta W_t$ in $\mathbb{R}^n \times (0,\infty)$ and that there exists $c > 0$ such that $|\partial_t W_t(x)| \le \frac{c}{t} W_{ct}(x)$ for each $t > 0$ and $x \in \mathbb{R}^n$. Thus, we can differentiate inside of the integral in (2.4.3) to find that $e^{t\Delta}u(x) \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and solves the heat equation.

If $1 < p < \infty$, then part (1) and the boundedness of the maximal function M show that $\left\|e^{t\Delta}u\right\|_{L^p(\mathbb{R}^n,\nu)} \le C\|u\|_{L^p(\mathbb{R}^n,\nu)}$. If $p = 1$, as in part (1) and by using the $A_1(\mathbb{R}^n)$-condition,

$$\begin{aligned}
\left\|e^{t\Delta}u\right\|_{L^1(\mathbb{R}^n,\nu)} &\le \int_{\mathbb{R}^n} |u(y)| \left(\int_{\mathbb{R}^n} W_t(x-y)\nu(x)\,dx\right) dy \\
&\le \int_{\mathbb{R}^n} |u(y)|\, M\nu(y)\,dy \\
&\le C \int_{\mathbb{R}^n} |u(x)|\,\nu(y)\,dy = C\|u\|_{L^1(\mathbb{R}^n,\nu)}.
\end{aligned}$$

Whence, (3) holds.

To verify the almost everywhere limit in (4), we only need to observe that $\lim_{t\to 0^+} e^{t\Delta}\varphi(x) = \varphi(x)$ for every $x \in \mathbb{R}^n$ whenever $\varphi \in C_c^\infty(\mathbb{R}^n)$, that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n,\nu)$ and that, by part (1), the maximal operator

$$T^*u(x) = \sup_{t>0} \left|e^{t\Delta}u(x)\right|$$

is bounded from $L^p(\mathbb{R}^n,\nu)$ into weak-$L^p(\mathbb{R}^n,\nu)$ (see, for instance, [27, Theorem 2.2]).

For (5), notice that if $\varphi \in C_c^\infty(\mathbb{R}^n)$, then, as in part (1),

$$\begin{aligned}\left|e^{t\Delta}\varphi(x) - \varphi(x)\right| &= \left|\int_0^t \partial_s e^{s\Delta}\varphi(x)\,ds\right| \\ &\leq \int_0^t \left|e^{s\Delta}\Delta\varphi(x)\right|\,ds \leq CM(\Delta\varphi)(x)\,t.\end{aligned}$$

For $1 < p < \infty$,

$$\left\|e^{t\Delta}\varphi - \varphi\right\|_{L^p(\mathbb{R}^n,\nu)} \leq C\left\|\Delta\varphi\right\|_{L^p(\mathbb{R}^n,\nu)}\,t \to 0$$

as $t \to 0^+$. If $p = 1$, then by part (3),

$$\begin{aligned}\left\|e^{t\Delta}\varphi - \varphi\right\|_{L^1(\mathbb{R}^n,\nu)} &= \int_{\mathbb{R}^n} \left|e^{t\Delta}\varphi(x) - \varphi(x)\right|\nu(x)\,dx \\ &\leq \int_{\mathbb{R}^n}\int_0^t \left|e^{s\Delta}\Delta\varphi(x)\right|\nu(x)\,ds\,dx \\ &= \int_0^t \left\|e^{s\Delta}\Delta\varphi\right\|_{L^1(\mathbb{R}^n,\nu)}\,ds \\ &\leq \int_0^t C\left\|\Delta\varphi\right\|_{L^1(\mathbb{R}^n,\nu)}\,ds = Ct\left\|\Delta\varphi\right\|_{L^1(\mathbb{R}^n,\nu)} \to 0\end{aligned}$$

as $t \to 0^+$. We then use the density of $C_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n,\nu)$.

For (6), let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and observe that

$$\begin{aligned}\int_{\mathbb{R}^n} \Delta e^{t\Delta}u(x)\varphi(x)\,dx &= \int_{\mathbb{R}^n} e^{t\Delta}u(x)\Delta\varphi(x)\,dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W_t(x-y)u(y)\Delta\varphi(x)\,dy\,dx \\ &= \int_{\mathbb{R}^n} W_t(z)\left[\int_{\mathbb{R}^n} u(y)\Delta_y\varphi(x+y)\,dy\right]dx \\ &= \int_{\mathbb{R}^n} W_t(z)\left[\int_{\mathbb{R}^n} \Delta u(y)\varphi(x+y)\,dy\right]dx \\ &= \int_{\mathbb{R}^n} e^{t\Delta}\Delta u(x)\,\varphi(x)\,dx.\end{aligned}$$

Then $\Delta e^{t\Delta}u(x) = e^{t\Delta}\Delta u(x)$, for almost every $x \in \mathbb{R}^n$.

Let us finally prove (7). Observe that, by part (1),

$$\int_{|x-y|<\varepsilon} W_t(x-y)\left|u(y)\right|\,dy \leq Mu(x).$$

For $1 < p < \infty$, $Mu \in L^p(\mathbb{R}^n, \nu)$ so, by the Dominated Convergence Theorem,

$$\lim_{\varepsilon \to 0} \left\| \int_{|x-y|<\varepsilon} W_t(x-y)\,|u(y)|\,dy \right\|_{L^p(\mathbb{R}^n,\nu)}^p = \int_{\mathbb{R}^n} \lim_{\varepsilon \to 0} \left(\int_{|x-y|<\varepsilon} W_t(x-y)\,|u(y)|\,dy \right)^p \nu(x)\,dx = 0.$$

For $p = 1$,

$$\left\| \int_{|x-y|<\varepsilon} W_t(x-y)u(y)\,dy \right\|_{L^1(\mathbb{R}^n,\nu)} \leq \int_{\mathbb{R}^n} \left[|u(y)| \int_{|x-y|<\varepsilon} W_t(x-y)\nu(x)\,dx \right] dy$$

and, by part (1),

$$|u(y)| \int_{|x-y|<\varepsilon} W_t(x-y)\nu(x)\,dx \leq |u(y)|\, M\nu(y) \leq C\,|u(y)|\,\nu(y) \in L^1(\mathbb{R}^n)$$

for a.e. $y \in \mathbb{R}^n$. Therefore, (7) holds for $p = 1$ by the Dominated Convergence Theorem. □

2.4.4 The maximal estimate (1.3.5)

Proof of Theorem 2.1.4. Define the operator $T_{s,\varepsilon}$ on $W^{2,p}(\mathbb{R}^n, \nu)$ by

$$T_{s,\varepsilon}u(x) = c_{n,s} \int_{|y|>\varepsilon} \frac{u(x-y) - u(x)}{|y|^{n+2s}}\,dy.$$

We will show that there is a constant $C = C_n > 0$ such that

$$|T_{s,\varepsilon}u(x)| \leq C \left(M(D^2u)(x) + Mu(x) \right) \quad \text{for a.e. } x \in \mathbb{R}^n$$

from which the statement follows. We write

$$T_{s,\varepsilon}u(x) = c_{n,s} \int_{\varepsilon<|y|<1} \frac{u(x-y) - u(x)}{|y|^{n+2s}}\,dy + c_{n,s} \int_{|y|>1} \frac{u(x-y) - u(x)}{|y|^{n+2s}}\,dy = I + II.$$

Let us first estimate the second term. Take $\eta(x) = \chi_{\{|x|\leq 1\}}(x) + |x|^{-n-2s} \chi_{\{|x|>1\}}(x)$ in Lemma 2.4.1 and use (2.4.1) to get

$$
\begin{aligned}
|II| &\le c_{n,s}\int_{|y|>1}\frac{|u(x-y)|}{|y|^{n+2s}}\,dy + c_{n,s}\,|u(x)|\int_{|y|>1}\frac{1}{|y|^{n+2s}}\,dy\\
&\le C_n s(1-s)\bigg((|u|*\eta)(x)+\frac{|u(x)|}{s}\bigg)\\
&\le C_n s(1-s)\bigg(\;\|\eta\|_{L^1(\mathbb{R}^n)}\,Mu(x)+\frac{|u(x)|}{s}\bigg)\\
&= C_n s(1-s)\bigg(\;\bigg(\frac{1+2s}{2s}\bigg)\,Mu(x)+\frac{|u(x)|}{s}\bigg)\\
&\le C_n Mu(x).
\end{aligned}
$$

Consider now the first term, that we rewrite as

$$
I = c_{n,s}\int_{\varepsilon<|y|<1}\frac{u(x-y)-u(x)+\nabla u(x)\cdot y}{|y|^{n+2s}}\,dy.
$$

Since $u \in W^{2,p}(\mathbb{R}^n,\nu)$ and (2.4.1) holds, for a.e. $x\in\mathbb{R}^n$ we can estimate

$$
\begin{aligned}
|I| &\le c_{n,s}\int_{\varepsilon<|y|<1}\frac{|u(x-y)-u(x)+\nabla u(x)\cdot y|}{|y|^{n+2s}}\,dy\\
&\le c_{n,s}\int_{\varepsilon<|y|<1}\frac{|y|^2}{|y|^{n+2s}}\int_0^1(1-t)\,\big|D^2u(x-ty)\big|\,dt\,dy\\
&= c_{n,s}\int_0^1(1-t)\int_{\varepsilon<|y|<1}\frac{\big|D^2u(x-ty)\big|}{|y|^{n-2(1-s)}}\,dy\,dt\\
&\le c_{n,s}\int_0^1(1-t)\,t^{-2(1-s)}\int_{|y|<t}\frac{\big|D^2u(x-y)\big|}{|y|^{n-2(1-s)}}\,dy\,dt\\
&\le c_{n,s}\int_0^1(1-t)\,t^{-2(1-s)}\sum_{k=0}^{\infty}\int_{2^{-(k+1)}t<|y|<2^{-k}t}\frac{\big|D^2u(x-y)\big|}{|y|^{n-2(1-s)}}\,dy\,dt\\
&\le c_{n,s}\int_0^1(1-t)\,t^{-2(1-s)}\sum_{k=0}^{\infty}\frac{1}{(2^{-(k+1)}t)^{n-2(1-s)}}\int_{|y|<2^{-k}t}\big|D^2u(x-y)\big|\,dy\,dt\\
&\le C_n s(1-s)2^{n-2(1-s)}M(D^2u)(x)\int_0^1(1-t)\bigg[\sum_{k=0}^{\infty}\frac{1}{\big(2^{2(1-s)}\big)^k}\bigg]\,dt\\
&\le C_n\frac{s(1-s)}{4^{1-s}-1}M(D^2u)(x)\le C_nM(D^2u)(x)
\end{aligned}
$$

where in the last line we applied the estimate $4^{1-s}-1\ge c(1-s)$, for any $0<s<1$. Therefore, $|T_{s,\varepsilon}u(x)|\le |I|+|II|\le C_n\left(M(D^2u)(x)+Mu(x)\right)$ for a.e $x\in\mathbb{R}^n$. $\square$

2.5 Proof of Theorem 2.1.3

2.5.1 Proof of Theorem 2.1.3 (a)

The steps in the proof of part (a) are similar to the steps in the proof of Theorem 2.1.1 (a).

Step 1. The semigroup formula in (2.1.5) defines a function in $L^p(\mathbb{R}^n, \nu)$.

Let us begin by writing

$$\begin{aligned}
&\frac{1}{\Gamma(-s)} \int_0^\infty |e^{t\Delta}u(x) - u(x)| \frac{dt}{t^{1+s}} \\
&\quad = \frac{1}{\Gamma(-s)} \int_0^1 |e^{t\Delta}u(x) - u(x)| \frac{dt}{t^{1+s}} + \frac{1}{\Gamma(-s)} \int_1^\infty |e^{t\Delta}u(x) - u(x)| \frac{dt}{t^{1+s}} \\
&\quad = I + II.
\end{aligned} \tag{2.5.1}$$

To study I, recall Theorem 2.4.1 and observe for $t \in [0, 1]$ that

$$\left\| e^{t\Delta}u - u \right\|_{L^p(\mathbb{R}^n,\nu)} \leq \int_0^t \left\| e^{r\Delta}(\Delta u) \right\|_{L^p(\mathbb{R}^n,\nu)} dr \leq C \left\| \Delta u \right\|_{L^p(\mathbb{R}^n,\nu)} t.$$

Therefore

$$\begin{aligned}
\|I\|_{L^p(\mathbb{R}^n,\nu)} &\leq \frac{1}{|\Gamma(-s)|} \int_0^1 \left\| e^{t\Delta}u - u \right\|_{L^p(\mathbb{R}^n,\nu)} \frac{dt}{t^{1+s}} \\
&= \frac{C}{|\Gamma(-s)|} \left\| \Delta u \right\|_{L^p(\mathbb{R}^n,\nu)} \int_0^1 t^{-s}\, dt \\
&= C \frac{s}{|\Gamma(2-s)|} \left\| \Delta u \right\|_{L^p(\mathbb{R}^n,\nu)}.
\end{aligned} \tag{2.5.2}$$

For II, in view of Theorem 2.4.1,

$$\begin{aligned}
\|II\|_{L^p(\mathbb{R}^n,\nu)} &\leq \frac{1}{|\Gamma(-s)|} \int_1^\infty \left(\left\| e^{t\Delta}u \right\|_{L^p(\mathbb{R}^n,\nu)} + \|u\|_{L^p(\mathbb{R}^n,\nu)} \right) \frac{dt}{t^{1+s}} \\
&\leq \frac{1}{|\Gamma(-s)|} \left(C \|u\|_{L^p(\mathbb{R}^n,\nu)} + \|u\|_{L^p(\mathbb{R}^n,\nu)} \right) \int_1^\infty \frac{dt}{t^{1+s}} \\
&= \frac{C(1-s)}{|\Gamma(2-s)|} \|u\|_{L^p(\mathbb{R}^n,\nu)}.
\end{aligned} \tag{2.5.3}$$

Therefore

$$\left\| \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta}u(x) - u(x) \right) \frac{dt}{t^{1+s}} \right\|_{L^p(\mathbb{R}^n,\nu)} \leq C \left(\|u\|_{L^p(\mathbb{R}^n,\nu)} + \|\Delta u\|_{L^p(\mathbb{R}^n,\nu)} \right) < \infty. \tag{2.5.4}$$

Step 2. The distribution $(-\Delta)^s u$ coincides with the semigroup formula in (2.1.5) for a.e. $x \in \mathbb{R}^n$. Therefore, $(-\Delta)^s u$ is in $L^p(\mathbb{R}, \nu)$ and, by (2.5.4), we see that (2.1.7) holds.

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{2,p}(\mathbb{R}^n,\nu)$ (see [78]), there exists a sequence $u_k \in C_c^\infty(\mathbb{R}^n)$ such that $u_k \to u$ in $W^{2,p}(\mathbb{R}^n,\nu)$. We consider the terms I and II as in (2.5.1) and, similarly,

$$\begin{aligned}(-\Delta)^s u_k(x) &= \frac{1}{\Gamma(-s)}\int_0^1 \left(e^{t\Delta}u_k(x) - u_k(x)\right)\frac{dt}{t^{1+s}} + \frac{1}{\Gamma(-s)}\int_1^\infty \left(e^{t\Delta}u_k(x) - u_k(x)\right)\frac{dt}{t^{1+s}} \\ &= I_k + II_k.\end{aligned}$$

By (2.5.2),

$$\|I_k - I\|_{L^p(\mathbb{R}^n,\nu)} \le C\frac{s}{|\Gamma(2-s)|}\|\Delta(u_k - u)\|_{L^p(\mathbb{R}^n,\nu)} \to 0 \quad \text{as } k \to \infty.$$

Similarly, by (2.5.3),

$$\|II_k - II\|_{L^p(\mathbb{R}^n,\nu)} = \frac{C(1-s)}{\Gamma(2-s)}\|u_k - u\|_{L^p(\mathbb{R}^n,\nu)} \to 0 \quad \text{as } k \to \infty.$$

Therefore,

$$(-\Delta)^s u_k(x) \to \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta}u(x) - u(x)\right)\frac{dt}{t^{1+s}} \tag{2.5.5}$$

in $L^p(\mathbb{R}^n,\nu)$ as $k\to\infty$.

Next, let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and note that $(-\Delta)^s\varphi \in \mathcal{S}_s$. By Proposition 2.4.2,

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} u_k(x)(-\Delta)^s\varphi(x)\,dx - \int_{\mathbb{R}^n} u(x)(-\Delta)^s\varphi(x)\,dx\right| \\ &\qquad \le C\int_{\mathbb{R}^n}\frac{|u_k(x)-u(x)|}{1+|x|^{n+2s}}\,dx \\ &\qquad \le C\,\|u_k - u\|_{L^p(\mathbb{R}^n,\nu)} \to 0 \quad \text{as } k\to\infty.\end{aligned}$$

In addition, by (2.5.5),

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}(-\Delta)^s u_k(x)\varphi(x)\,dx - \frac{1}{\Gamma(-s)}\int_{\mathbb{R}^n}\int_0^\infty \left(e^{t\Delta}u(x)-u(x)\right)\frac{dt}{t^{1+s}}\,\varphi(x)\,dx\right| \\ &\quad \le C\int_{\mathbb{R}^n}\left|(-\Delta)^s u_k(x) - \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta}u(x)-u(x)\right)\frac{dt}{t^{1+s}}\right|\frac{1}{1+|x|^{n+2s}}\,dx \\ &\quad \le C\left\|(-\Delta)^s u_k(x) - \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta}u(x)-u(x)\right)\frac{dt}{t^{1+s}}\right\|_{L^p(\mathbb{R}^n,\nu)} \to 0\end{aligned}$$

as $k \to \infty$. Therefore

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta)^s u(x)\,\varphi(x)\,dx &= \int_{\mathbb{R}^n} u(x)(-\Delta)^s\varphi(x)\,dx \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^n} u_k(x)(-\Delta)^s\varphi(x)\,dx \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^n} (-\Delta)^s u_k(x)\,\varphi(x)\,dx \\ &= \int_{\mathbb{R}^n}\left[\frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta}u(x)-u(x)\right)\frac{dt}{t^{1+s}}\right]\varphi(x)\,dx,\end{aligned}$$

and so we obtain

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta}u(x)-u(x)\right)\frac{dt}{t^{1+s}} \quad \text{for a.e. } x\in\mathbb{R}^n.$$

Step 3. The integral expression in (2.1.6) defines a function in $L^p(\mathbb{R}^n,\nu)$ for all $\varepsilon>0$.

For $\varepsilon>0$, define the operator T_ε on $L^p(\mathbb{R}^n,\nu)$ by

$$T_\varepsilon u(x) = c_{n,s}\int_{|x-y|>\varepsilon}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy. \tag{2.5.6}$$

We claim that $T_\varepsilon u(x)\in L^p(\mathbb{R}^n,\nu)$ for all $\varepsilon>0$. Indeed, for $1<p<\infty$ this is immediate by Theorem 2.1.4: there exists $C>0$ such that

$$\|T_\varepsilon u\|_{L^p(\mathbb{R}^n,\nu)} \le C\left(\left\|M(D^2u)\right\|_{L^p(\mathbb{R}^n,\nu)} + \|Mu\|_{L^p(\mathbb{R}^n,\nu)}\right) < \infty.$$

For $p=1$, we write

$$\begin{aligned}T_\varepsilon u(x) &= c_{n,s}u(x)\int_{|x-y|>\varepsilon}\frac{1}{|x-y|^{n+2s}}\,dy + c_{n,s}\int_{|x-y|>\varepsilon}\frac{u(y)}{|x-y|^{n+2s}}\,dy \\ &= c_{n,s}\frac{C_n\varepsilon^{-2s}}{2s}u(x) + c_{n,s}\int_{|x-y|>\varepsilon}\frac{u(y)}{|x-y|^{n+2s}}\,dy.\end{aligned}$$

We only need to study the second term above. By applying Lemma 2.4.1 with $\eta(y)=\chi_{\{|y|\le\varepsilon\}}(y)+|y|^{-n-2s}\,\chi_{\{|y|>\varepsilon\}}(y)$ and the $A_1(\mathbb{R}^n)$-condition on ν, we find

$$\begin{aligned}\left\|\int_{|x-y|>\varepsilon}\frac{u(y)}{|x-y|^{n+2s}}\,dy\right\|_{L^1(\mathbb{R}^n,\nu)} &\le \int_{\mathbb{R}^n}|u(y)|\int_{|x-y|>\varepsilon}\frac{\nu(x)}{|x-y|^{n+2s}}\,dx\,dy \\ &\le \int_{\mathbb{R}^n}|u(y)|(\nu*\eta)(y)\,dy \\ &\le C_{n,s,\varepsilon}\int_{\mathbb{R}^n}|u(y)|M\nu(y)\,dy \\ &\le C_{n,s,\varepsilon,\nu}\,\|u\|_{L^1(\mathbb{R}^n,\nu)} < \infty.\end{aligned}$$

Step 4. The principal value in (2.1.6) converges in $L^p(\mathbb{R}^n,\nu)$ to the function $(-\Delta)^s u$.

We write the semigroup formula (2.1.5) as

$$
\begin{aligned}
(-\Delta)^s u(x) &= \frac{1}{\Gamma(-s)}\int_0^1 \left(\int_{\mathbb{R}^n} W_t(x-y)\,(u(y)-u(x))\,dy\right)\frac{dt}{t^{1+s}}\\
&\quad+\frac{1}{\Gamma(-s)}\int_1^\infty \left(\int_{\mathbb{R}^n} W_t(x-y)\,(u(y)-u(x))\,dy\right)\frac{dt}{t^{1+s}}\\
&= I + II
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
&c_{n,s}\int_{|x-y|>\varepsilon}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\\
&\qquad=\frac{1}{\Gamma(-s)}\int_{|x-y|>\varepsilon}(u(y)-u(x))\left(\int_0^\infty W_t(x-y)\,\frac{dt}{t^{1+s}}\right)dy\\
&\qquad=\frac{1}{\Gamma(-s)}\int_0^1\int_{|x-y|>\varepsilon} W_t(x-y)\,(u(y)-u(x))\,dy\,\frac{dt}{t^{1+s}}\\
&\qquad\quad+\frac{1}{\Gamma(-s)}\int_1^\infty\int_{|x-y|>\varepsilon} W_t(x-y)\,(u(y)-u(x))\,dy\,\frac{dt}{t^{1+s}}\\
&\qquad= I_\varepsilon + II_\varepsilon.
\end{aligned}
$$

From Theorem 2.4.1 it follows that

$$
\begin{aligned}
&\|II-II_\varepsilon\|_{L^p(\mathbb{R}^n,\nu)}\\
&\quad=\left\|\frac{1}{\Gamma(-s)}\int_1^\infty\left[\left(\int_{|x-y|<\varepsilon}W_t(x-y)u(y)\,dy\right)+u(x)\int_{|z|<\varepsilon}W_t(z)\,dz\right]\frac{dt}{t^{1+s}}\right\|_{L^p(\mathbb{R}^n,\nu)}\\
&\quad\le C\int_1^\infty\left(\left\|\int_{|x-y|<\varepsilon}W_t(x-y)u(y)\,dy\right\|_{L^p(\mathbb{R}^n,\nu)}+\|u\|_{L^p(\mathbb{R}^n,\nu)}\int_{|z|<\varepsilon}W_t(z)\,dz\right)\frac{dt}{t^{1+s}}\to 0
\end{aligned}
$$

as $\varepsilon\to 0^+$. We next show $\|I-I_\varepsilon\|_{L^p(\mathbb{R}^n,\nu)}\to 0$ as $\varepsilon\to 0^+$ as well to conclude the proof. Indeed,

$$
\|I-I_\varepsilon\|_{L^p(\mathbb{R}^n,\nu)}=\left\|\frac{1}{\Gamma(-s)}\int_0^1\left(\int_{|y|<\varepsilon}W_t(y)\,(u(x-y)-u(x))\,dy\right)\frac{dt}{t^{1+s}}\right\|_{L^p(\mathbb{R}^n,\nu)}
$$

By Taylor's Remainder Theorem and (2.1.10),

$$
\begin{aligned}
&\left| \int_{|y|<\varepsilon} W_t(y)\,(u(x-y)-u(x))\;dy \right| \\
&\qquad \le \int_{|y|<\varepsilon} W_t(y)|y|^2 \left(\int_0^1 (1-r)\left|D^2u(x-ry)\right|\,dr \right) dy \\
&\qquad \le Ct \int_{|y|<\varepsilon} W_{2t}(y) \left(\int_0^1 (1-r)\left|D^2u(x-ry)\right|\,dr \right) dy \\
&\qquad = Ct \int_0^1 (1-r) \left(\int_{|y|<\varepsilon} W_{2t}(y)\left|D^2u(x-ry)\right|\,dy \right) dr \\
&\qquad = Ct \int_0^1 (1-r) \left(\int_{|y|<r\varepsilon} W_{2tr^2}(y)\left|D^2u(x-y)\right|\,dy \right) dr.
\end{aligned}
$$

In particular, since $D^2u \in L^p(\mathbb{R}^n,\nu)$, by Theorem 2.4.1,

$$
\left| \int_{|y|<\varepsilon} W_t(y)\,(u(x-y)-u(x))\;dy \right| \to 0 \quad \text{as } \varepsilon \to 0^+ \tag{2.5.7}
$$

a.e. in $\mathbb{R}^n$. We continue estimating by

$$
\begin{aligned}
&\left| \int_{|y|<\varepsilon} W_t(y)\,(u(x-y)-u(x))\;dy \right| \\
&\qquad \le Ct \int_0^1 (1-r) \left(\int_{\mathbb{R}^n} W_{2tr^2}(y)\left|D^2u(x-y)\right|\,dy \right) dr \\
&\qquad \le CtM(D^2u)(x) \int_0^1 (1-r)\,dr = CtM(D^2u)(x).
\end{aligned}
$$

Whence, for $1<p<\infty$, we have

$$
|I-I_\varepsilon| \le CM(D^2u)(x) \int_0^1 t\,\frac{dt}{t^{1+s}} \le CM(D^2u)(x) \in L^p(\mathbb{R}^n,\nu)
$$

where $C>0$ is independent of ε. Thus, by the Dominated Convergence Theorem and (2.5.7), $\lim_{\varepsilon\to 0^+} \|I-I_\varepsilon\|_{L^p(\mathbb{R}^n,\nu)} = 0$. When $p=1$, by following the computations above and by Theorem

2.4.1, we get

$$
\begin{aligned}
&\|I - I_\varepsilon\|_{L^1(\mathbb{R}^n,\nu)} \\
&\le C \int_{\mathbb{R}^n} \int_0^1 t \int_0^1 (1-r) \left(\int_{|y|<\varepsilon} W_{2tr^2}(x-y) \left|D^2 u(y)\right| \, dy \right) dr \frac{dt}{t^{1+s}} \nu(x)\, dx \\
&= C \int_0^1 \int_0^1 (1-r) \int_{|y|<\varepsilon} \left|D^2 u(y)\right| \left(\int_{\mathbb{R}^n} W_{2tr^2}(x-y) \nu(x)\, dx \right) dy\, dr \, \frac{dt}{t^s} \\
&\le C \int_0^1 \int_0^1 (1-r) \left(\int_{|y|<\varepsilon} \left|D^2 u(y)\right| M\nu(y) dy \right) dr \, \frac{dt}{t^s} \\
&\le C \int_0^1 \int_0^1 (1-r) \left(\int_{|y|<\varepsilon} \left|D^2 u(y)\right| \nu(y) dy \right) dr \, \frac{dt}{t^s} \\
&= C \int_{|y|<\varepsilon} \left|D^2 u(y)\right| \nu(y)\, dy \to 0 \quad \text{as } \varepsilon \to 0^+.
\end{aligned}
$$

Step 5. The principal value in (2.1.6) converges almost everywhere in $\mathbb{R}^n$ to $(-\Delta)^s u$.

It follows from Theorem 2.1.4 and the properties of M that the operator T^* defined by

$$T^* u(t) = \sup_{\varepsilon>0} |T_\varepsilon u(x)| \quad \text{for } u \in W^{2,p}(\mathbb{R}^n, \nu),$$

where T_ε is defined as in (2.5.6), satisfies the estimates

$$\|T^* u\|_{L^p(\mathbb{R}^n,\nu)} \le C \|u\|_{W^{2,p}(\mathbb{R}^n,\nu)} \quad \text{for any } u \in W^{2,p}(\mathbb{R}^n, \nu), \ 1 < p < \infty$$

and

$$\nu\big(\{x \in \mathbb{R}^n : |T^* u(x)| > \lambda\}\big) \le \frac{C}{\lambda} \|u\|_{W^{2,1}(\mathbb{R}^n,\nu)} \quad \text{for any } u \in W^{2,1}(\mathbb{R}^n, \nu), \ \lambda > 0$$

where $C > 0$ is independent of u. In particular, T^* is bounded from $W^{2,p}(\mathbb{R}^n, \nu)$ into weak-$L^p(\mathbb{R}^n, \nu)$, for any $1 \le p < \infty$. With these estimates, as in Step 5 of the proof of Theorem 2.1.1(a), we find that the set

$$E = \left\{ u \in W^{2,p}(\mathbb{R}^n, \nu) : \lim_{\varepsilon \to 0^+} T_\varepsilon u(x) = (-\Delta)^s u(x) \text{ a.e.} \right\}$$

is closed in $W^{2,p}(\mathbb{R}^n, \nu)$. Since $C_c^\infty(\mathbb{R}^n) \subset E$, by density, we obtain $E = W^{2,p}(\mathbb{R}^n, \nu)$.

Step 6. The limit as $s \to 1^-$ in (2.1.8) holds in $L^p(\mathbb{R}, \nu)$.

Fix $\varepsilon > 0$. By Theorem 2.4.1, there exists $\delta > 0$ such that

$$\left\|e^{t\Delta}\Delta u - \Delta u\right\|_{L_p(\mathbb{R}^n,\nu)} < \varepsilon \quad \text{when } \; |t| < \delta.$$

We write

$$\begin{aligned}(-\Delta)^s u(x) &= \frac{1}{\Gamma(-s)} \int_0^\delta \left(e^{t\Delta}u(x) - u(x)\right) \frac{dt}{t^{1+s}} + \frac{1}{\Gamma(-s)} \int_\delta^\infty \left(e^{t\Delta}u(x) - u(x)\right) \frac{dt}{t^{1+s}} \\ &= I_\delta + II_\delta.\end{aligned}$$

Looking at the second term, by Theorem 2.4.1,

$$\begin{aligned}\|II_\delta\|_{L^p(\mathbb{R}^n,\nu)} &\le \frac{1}{|\Gamma(-s)|} \int_\delta^\infty \left(\left\|e^{t\Delta}u\right\|_{L^p(\mathbb{R}^n,\nu)} + \|u\|_{L^p(\mathbb{R}^n,\nu)}\right) \frac{dt}{t^{1+s}} \\ &\le \frac{C\,\|u\|_{L^p(\mathbb{R}^n,\nu)}}{|\Gamma(-s)|} \int_\delta^\infty t^{-1-s}\,dt = C\,\|u\|_{L^p(\mathbb{R}^n,\nu)}\;\delta^{-s}\frac{(1-s)}{|\Gamma(2-s)|} \to 0\end{aligned}$$

as $s \to 1^-$. Next,

$$\begin{aligned}\|I_\delta - &(-\Delta)u\|_{L^p(\mathbb{R}^n,\nu)} \\ &= \left\|\frac{1}{\Gamma(-s)} \int_0^\delta \int_0^t \partial_r e^{r\Delta}u(x)dr\,\frac{dt}{t^{1+s}} + \Delta u(x)\right\|_{L^p(\mathbb{R}^n,\nu)} \\ &= \left\|\frac{1}{\Gamma(-s)} \int_0^\delta \int_0^t e^{r\Delta}\Delta u(x)dr\,\frac{dt}{t^{1+s}} + \Delta u(x)\right\|_{L^p(\mathbb{R}^n,\nu)} \\ &= \left\|\frac{1}{\Gamma(-s)} \int_0^\delta \int_0^t \left(e^{r\Delta}\Delta u(x) - \Delta u(x)\right) dr\,\frac{dt}{t^{1+s}} + \left(\frac{(-s)\,\delta^{1-s}}{\Gamma(2-s)} + 1\right)\Delta u(x)\right\|_{L^p(\mathbb{R}^n,\nu)} \\ &\le \frac{1}{|\Gamma(-s)|} \int_0^\delta \int_0^t \left\|e^{r\Delta}\Delta u - \Delta u\right\|_{L^p(\mathbb{R}^n,\nu)} dr\,\frac{dt}{t^{1+s}} + \left|\frac{(-s)\,\delta^{1-s}}{\Gamma(2-s)} + 1\right| \|\Delta u\|_{L^p(\mathbb{R}^n,\nu)} \\ &\le \varepsilon\,\delta^{1-s}\,\frac{s}{|\Gamma(2-s)|} + \left|\frac{(-s)\,\delta^{1-s}}{\Gamma(2-s)} + 1\right| \|\Delta u\|_{L^p(\mathbb{R}^n,\nu)} \to \varepsilon \quad \text{as } s \to 1^-.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, (2.1.8) follows in $L^p(\mathbb{R}^n, \nu)$.

Step 7. The limits as $s \to 1^-$ in (2.1.8) and as $s \to 0^+$ in (2.1.9) hold a.e. in $\mathbb{R}^n$.

This is proved as in Step 5. By noticing that $\sup_{0<s<1} |(-\Delta)^s u(x)|$ can be bounded by means of Theorem 2.1.4, one can check that the sets

$$E' = \{u \in W^{2,p}(\mathbb{R}^n, \nu) : \lim_{s\to 1^-} (-\Delta)^s u(x) = -\Delta u(x) \text{ a.e.}\}$$

and

$$E'' = \{u \in W^{2,p}(\mathbb{R}^n, \nu) : \lim_{s \to 0^+} (-\Delta)^s u(x) = u(x) \text{ a.e.}\}$$

are closed in $W^{2,p}(\mathbb{R}^n, \nu)$. Since $C_c^\infty(\mathbb{R}^n) \subset E'$ and $C_c^\infty(\mathbb{R}^n) \subset E''$, by density, we conclude that $E' = E'' = W^{2,p}(\mathbb{R}^n, \nu)$.

Step 8. The limit as $s \to 0^+$ in (2.1.9) holds in $L^p(\mathbb{R}^n, \nu)$.

By Theorem 2.1.4, for any $0 < s < 1$,

$$\begin{aligned} |(-\Delta)^s u(x) - u(x)|^p \nu(x) &\le \left(C_n(M(D^2u)(x) + Mu(x)) + |u(x)|\right)^p \nu(x) \\ &\le C_{n,p} \left((M(D^2u)(x))^p + (Mu(x))^p\right) \nu(x). \end{aligned}$$

Therefore, by Step 7 and the Dominated Convergence Theorem, (2.1.9) holds in $L^p(\mathbb{R}^n, \nu)$.

This completes the proof of Theorem 2.1.3, part (a). □

2.5.2 Proof of Theorem 2.1.3 (b)

Suppose $(-\Delta)^s u \to v$ in $L^p(\mathbb{R}^n, \nu)$ as $s \to 1^-$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and observe that

$$\begin{aligned} \int_{\mathbb{R}^n} v\varphi \, dx &= \lim_{s \to 1^-} \int_{\mathbb{R}^n} (-\Delta)^s u \varphi \, dx \\ &= \lim_{s \to 1^-} \int_{\mathbb{R}^n} u(-\Delta)^s \varphi \, dx \\ &= \int_{\mathbb{R}^n} u(-\Delta)\varphi \, dx = (-\Delta u)(\varphi). \end{aligned}$$

In the first line we used that, by Proposition 2.4.2 and the fact that $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} v(x)\varphi(x) \, dx - \int_{\mathbb{R}^n} (-\Delta)^s u(x)\varphi(x) \, dx \right| &\le \int_{\mathbb{R}^n} |v(x) - (-\Delta)^s u(x)| \frac{C_\varphi}{1 + |x|^n} \, dx \\ &\le C_{\varphi,n,p,\nu} \|v - (-\Delta)^s u\|_{L^p(\mathbb{R}^n,\nu)} \to 0 \end{aligned}$$

as $s \to 1^-$, while in the second to last identity we used the Dominated Convergence Theorem, the fact that $(-\Delta)^s \varphi \in \mathcal{S}_s$, and Proposition 2.4.2 in the case of L_0.

Therefore, $v = -\Delta u$ a.e. in $\mathbb{R}^n$. Since $v \in L^p(\mathbb{R}^n, \nu)$, we get that $\Delta u \in L^p(\mathbb{R}^n, \nu)$. Now we apply the weighted Calderón–Zygmund estimates (see [27]). Hence, if $1 < p < \infty$, then $u \in W^{2,p}(\mathbb{R}^n, \nu)$

and, as a consequence of part (a), (2.1.8) holds. On the other hand, if $p = 1$, then $D^2u \in$ weak-$L^1(\mathbb{R}^n, \nu)$. □

2.5.3 Proof of Theorem 2.1.3 (c)

Using the exact same arguments as in part (b), we find that

$$\begin{aligned}\int_{\mathbb{R}^n} v\varphi\, dx &= \lim_{s\to 0^+} \int_{\mathbb{R}^n} (-\Delta)^s u\varphi\, dx \\ &= \lim_{s\to 0^+} \int_{\mathbb{R}^n} u(-\Delta)^s \varphi\, dx = \int_{\mathbb{R}^n} u\varphi\, dx.\end{aligned}$$

Therefore, $u = v = \lim_{s\to 0^+}(-\Delta)^s u$ a.e. in $\mathbb{R}^n$ and the result follows. □

CHAPTER 3. HARNACK INEQUALITY FOR FRACTIONAL NONDIVERGENCE FORM ELLIPTIC EQUATIONS

3.1 Main results

We formally state the main results of this chapter.

Assume that $a^{ij} = a^{ij}(x)$ are bounded, measurable functions on $\mathbb{R}^n$ and are uniformly elliptic: there exist $0 < \lambda < \Lambda$ such that

$$\lambda \left|\xi\right|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda \left|\xi\right|^2 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\},\ x \in \mathbb{R}^n. \tag{3.1.1}$$

For a bounded, Lipschitz domain $\Omega \subset \mathbb{R}^n$, let L be the operator defined by

$$L = -a^{ij}(x)\partial_{ij}, \quad \mathrm{Dom}(L) = \{u \in C_0(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega) : Lu \in C_0(\Omega)\}.$$

If $a^{ij} \in C^\alpha(\Omega) \cap C(\overline{\Omega})$, $0 < \alpha < 1$, then the fractional operator $L^s = (-a^{ij}(x)\partial_{ij})^s$ given by

$$L^s u = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-tL}u - u\right) \frac{dt}{t^{1+s}}, \quad 0 < s < 1,$$

is well-defined on $\mathrm{Dom}(L)$. Here, $e^{-tL}u$ denotes the uniformly bounded C_0-semigroup generated by L. See Section 3.2 for more details.

Theorem 3.1.1. *Let $0 < s < 1$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded, Lipschitz domain and that $a^{ij} \in C^\alpha(\Omega) \cap C(\overline{\Omega})$ satisfy (3.1.1). There exist positive constants $C_H = C_H(n, \lambda, \Lambda, s) > 1$, $\kappa = \kappa(n, s) < 1$, and $\hat{K} = \hat{K}(n, s) > 1$ such that for every ball $B_R = B_R(x_0)$ satisfying $B_{\hat{K}R} \subset\subset \Omega$ and every nonnegative $u \in \mathrm{Dom}(L)$ satisfying*

$$(-a^{ij}(x)\partial_{ij})^s u = 0 \quad \text{in } B_{\hat{K}R}, \tag{3.1.2}$$

we have that

$$\sup_{B_{\kappa R}} u \leq C_H \inf_{B_{\kappa R}} u. \tag{3.1.3}$$

Furthermore, there exist positive constants $\alpha = \alpha(n, \lambda, \Lambda, s) < 1$, $\hat{C} = \hat{C}(n, \lambda, \Lambda, s)$, *and* $\hat{K}_0 = \hat{K}_0(n, s) < \hat{K}$ *such that for any* $u \in \mathrm{Dom}(L)$ *satisfying* (3.1.2), *we have that*

$$|u(x_0) - u(x)| \leq \hat{C}\, |x - x_0|^{\alpha} \, (\hat{K}_0 R)^{-\alpha} \sup_{\Omega} |u| \quad \text{for every } x \in B_{\hat{K}_0 R}. \tag{3.1.4}$$

To prove Theorem 3.1.1, we use a local, degenerate extension characterization (see Section 3.2). The extension equation can be recast as an equation comparable to a linearized Monge–Ampère equation, so we will use the following Monge–Ampère geometry.

Define the convex, C^1 function $\Phi = \Phi(x, z) : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\Phi(x, z) = \varphi(x) + h(z) \qquad \text{where} \quad \varphi(x) = \frac{1}{2}\, |x|^2, \quad h(z) = \frac{s^2}{1-s}\, |z|^{\frac{1}{s}}.$$

The Monge–Ampère quasi-distance associated to Φ is defined by

$$\delta_\Phi((x_0, z_0), (x, z)) = \Phi(x, z) - \Phi(x_0, z_0) - \langle D\Phi(x_0, z_0), (x, z) - (x_0, z_0) \rangle.$$

The Monge–Ampère sections associated to Φ are given by

$$S_R(x_0, z_0) = \{(x, z) : \delta_\Phi((x_0, z_0), (x, z)) < R\}.$$

The Monge–Ampère measure associated to Φ is

$$\mu_\Phi(E) = |D\Phi(E)| = \int_E h''(z)\, dx\, dz$$

for all Borel sets $E \subset \mathbb{R}^{n+1}$. For more details, see Section 3.3.

Theorem 3.1.2. *Let* $0 < s < 1$. *Assume that* $a^{ij} = a^{ij}(x)$ *are bounded, measurable functions on* $\mathbb{R}^n$ *and satisfy* (3.1.1). *There exist positive constants* $C_H = C_H(n, \lambda, \Lambda, s) > 1$, $\kappa_1 = \kappa_1(n, s) < 1$, *and* $\hat{K}_1 = \hat{K}_1(n, s) > 1$ *such that for every section* $S_R = S_R(x_0, z_0) \subset \mathbb{R}^{n+1}$ *and every nonnegative solution* $U \in C^2(S_{\hat{K}_1 R} \setminus \{z = 0\}) \cap C(\overline{S}_{\hat{K}_1 R})$ *such that* $U(x, z) = U(x, -z)$ *and* $U_z \in C(S_{\hat{K}_1 R} \cap \{z \geq 0\})$ *to*

$$\begin{cases} a^{ij}(x)\partial_{ij} U + |z|^{2-\frac{1}{s}}\, \partial_{zz} U = 0 & \text{in } S_{\hat{K}_1 R} \setminus \{z = 0\} \\ -\partial_{z+} U(x, 0) = 0 & \text{on } S_{\hat{K}_1 R} \cap \{z = 0\}, \end{cases} \tag{3.1.5}$$

we have that

$$\sup_{S_{\kappa_1 R}} U \leq C_H \inf_{S_{\kappa_1 R}} U. \tag{3.1.6}$$

Consequently, there exist positive constants $\alpha_1 = \alpha_1(n, \lambda, \Lambda, s) < 1/2$ *and* $\hat{C}_1 = \hat{C}_1(n, \lambda, \Lambda, s)$ *such that every solution* $U \in C^2(S_{\hat{K}_1 R} \setminus \{z = 0\}) \cap C(\overline{S}_{\hat{K}_1 R})$ *such that* $U(x,z) = U(x,-z)$ *and* $U_z \in C(S_{\hat{K}_1 R} \cap \{z \geq 0\})$ *to* (3.1.5)*, we have that*

$$|U(x_0, z_0) - U(x,z)| \leq \hat{C}_1 (\delta_\Phi((x_0,z_0),(x,z)))^{\alpha_1} (\hat{K}_1 R)^{-\alpha_1} \sup_{S_{\hat{K}_1 R}} |U| \tag{3.1.7}$$

for every $(x,z) \in S_{\hat{K}_1 R}$*.*

The rest of the chapter is organized as follows. In Section 3.2, fractional powers of nondivergence form operators are defined using the method of semigroups and the Poisson problem is characterized using a local, degenerate extension equation. Section 3.3 contains preliminaries on the underling Monge–Ampère structure of the problem. A sketch of the proof of a critical density estimate and of local boundness are in Section 3.4. Finally, Section 3.5 develops the paraboloids associated to Φ and contains the proofs three key lemmas that are used to prove Theorem 3.1.2 which, in turn, is used to prove Theorem 3.1.1.

3.2 Fractional powers L^s

3.2.1 Semigroups

It is not immediately obvious how to define $(-a^{ij}(x)\partial_{ij})^s$. For example, we saw in Chapter 2 that the fractional Laplacian $(-\Delta)^s$ can be defined using the Fourier transform. However, a nondivergence form operator $L = -a^{ij}(x)\partial_{ij}$ has no natural Hilbert space structure. We use the method of semigroups to define L^s.

The relation

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-\lambda t} - 1) \frac{dt}{t^{1+s}} \quad \text{for all } \lambda > 0,\ 0 < s < 1,$$

suggests that we define L^s for $L = -a^{ij}(x)\partial_{ij}$ by

$$L^s u = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-tL}u - u\right) \frac{dt}{t^{1+s}}, \quad 0 < s < 1$$

where $v = e^{-tL}u$ solves the heat equation generated by L with initial data u:

$$\begin{cases} \partial_t v(x,t) = a^{ij}(x)\partial_{ij} v(x,t) & \text{for } t > 0,\ x \in \Omega \\ v(x,t) = 0 & \text{for } t \geq 0,\ x \in \partial\Omega \\ v(x,0) = u(x) & \text{for } x \in \Omega. \end{cases}$$

In order to make the definition of L^s more precise we will state the following definitions and results as found in [62].

Definition 3.2.1. *Let X be a Banach space. A one parameter family $\{T_t\}_{0 \leq t < \infty}$ of bounded linear operators from X into X is a semigroup on X if*

$$T_0 = I \quad \textit{and} \quad T_{t_1+t_2} = T_{t_1} \circ T_{t_2} \quad \textit{for every } t_1, t_2 \geq 0.$$

The linear operator A defined by

$$\mathrm{Dom}(A) = \left\{ x \in X : \lim_{t \to 0^+} \frac{T_t x - x}{t} \textit{ exists in } X \right\}$$

and

$$Ax = \lim_{t \to 0^+} \frac{T_t x - x}{t} \quad \textit{for } x \in \mathrm{Dom}(A)$$

is the called the infinitesimal generator of T_t. We use the notation $T_t = e^{tA}$.

We say T_t is strongly continuous, also called a C_0-semigroup, if

$$\lim_{t \to 0^+} T_t x = x \quad \textit{for all } x \in X.$$

Theorem 3.2.1. *Let T_t be a C_0-semigroup and let A be its infinitesimal generator. If $x \in \mathrm{Dom}(A)$, then $T_t x \in \mathrm{Dom}(A)$,*

$$\frac{d}{dt} T_t x = A T_t x = T_t A x, \quad \textit{and} \quad T_t x - x = \int_0^t T_\tau A x \, d\tau.$$

Theorem 3.2.2. *If T_t is a C_0-semigroup, then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T_t\| \leq M e^{\omega t} \quad \textit{for } 0 \leq t < \infty.$$

If $\omega = 0$, we say that T_t is uniformly bounded and if moreover $M = 1$, we say that T_t is contractive.

Theorem 3.2.3 (Hille-Yosida)**.** *A linear operator A generates a C_0-semigroup if and only if*

1. *A is closed and* $\mathrm{Dom}(A)$ *is dense in X*
2. *$(\omega, \infty) \subset \rho(A)$, where ρ denotes the resolvent set of A, and*

$$\left\|(\lambda I - A)^{-n}\right\| \leq \frac{M}{(\lambda - \omega)^n} \quad \textit{for all } \lambda > \omega,\ n \in \mathbb{N}.$$

If A is the infinitesimal generator of a uniformly bounded C_0-semigroup on X, then, for $x \in \mathrm{Dom}(A)$, define the fractional operator A^s using the definition of Balakrishnan [7] by

$$A^s x = \frac{1}{\Gamma(-s)} \int_0^\infty (T_t x - x) \frac{dt}{t^{1+s}} \quad 0 < s < 1,$$

where the integral is taken in the Bochner sense. For $x \in \mathrm{Dom}(A)$,

$$\begin{aligned}
\|A^s x\|_X &\leq c_s \left(\int_0^1 \|T_t x - x\|_X \frac{dt}{t^{1+s}} + \int_1^\infty (\|T_t x\|_X + \|x\|_X) \frac{dt}{t^{1+s}} \right) \\
&= c_s \left(\int_0^1 \left\| \int_0^t T_\tau (Ax)\, d\tau \right\|_X \frac{dt}{t^{1+s}} + \int_1^\infty (\|T_t x\|_X + \|x\|_X) \frac{dt}{t^{1+s}} \right) \\
&\leq c_s \left(\int_0^1 \int_0^t M \|Ax\|_X \, d\tau \frac{dt}{t^{1+s}} + \int_1^\infty (M+1) \|x\|_X \frac{dt}{t^{1+s}} \right) \\
&\leq c_{s,M} (\|Ax\|_X + \|x\|_X) < \infty
\end{aligned}$$

which shows that $A^s : \mathrm{Dom}(A) \to X$ is well-defined.

Example 3.2.1. *For $L = -\Delta$, let $X = C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and* $\mathrm{Dom}(-\Delta) = C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *Then, $-\Delta$ is the generator of a C_0 semigroup. In particular, we know that $T_t u = W_t * u$ where W is the Gauss-Weistrass heat kernel (see Section 2.4).*

Define the Banach space $C_0(\Omega)$ by

$$C_0(\Omega) = \{u \in C(\overline{\Omega}) : u \equiv 0 \text{ on } \partial\Omega\}.$$

The following result by Arendt–Schätzle is, in part, the motivation for choosing $a^{ij} \in C^\alpha(\Omega)$ [6]. We also reference the reader to [47, Theorem 5.1.19] for similar results on semigroups generated by $-a^{ij}(x)\partial_{ij}$ with $a^{ij} \in C^\alpha(\Omega)$.

Theorem 3.2.4 (Proposition 4.7 in [6]). *For a bounded, Lipschitz domain $\Omega \subset \mathbb{R}^n$, assume that $a^{ij} \in C^\alpha(\Omega) \cap C(\overline{\Omega})$ satisfy (3.1.1). The operator*

$$L = -a^{ij}(x)\partial_{ij}, \quad \mathrm{dom}(L) = \{u \in C_0(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega) : Lu \in C_0(\Omega)\} \tag{3.2.1}$$

generates a contractive C_0-semigroup, denoted by $T_t = e^{-tL}$, on $C_0(\Omega)$ such that if $u \geq 0$, then $e^{-tL}u \geq 0$. Moreover,

$$\left\|e^{-tL}u\right\|_{C_0(\Omega)} \leq Me^{-\varepsilon t}\left\|u\right\|_{C_0(\Omega)}, \quad t \geq 0 \tag{3.2.2}$$

for some $M > 0$, $\varepsilon > 0$. The resolvent $(\lambda I - L)^{-1}$ is compact for all λ in $\rho(L)$.

Hence, the following definition of $(-a^{ij}(x)\partial_{ij})^s$ is well-defined.

Definition 3.2.2. *Let $0 < s < 1$. Assume that $\Omega \subset \mathbb{R}^n$ is bounded, Lipschitz domain and $a^{ij} \in C^\alpha(\Omega) \cap C(\overline{\Omega})$ satisfy (3.1.1). Suppose that $X = C_0(\Omega)$ and L is given by (3.2.1). We define the fractional operator $L^s : \mathrm{Dom}(L) \to C_0(\Omega)$ by*

$$L^s u = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{-tL}u - u\right)\frac{dt}{t^{1+s}}, \quad 0 < s < 1. \tag{3.2.3}$$

3.2.2 The extension problem

In view of (3.2.3) and the pointwise definition of fractional Laplacian (2.1.6), we see that fractional powers of differential operators are nonlocal which brings additional difficulties when proving regularity estimates. Caffarelli–Silvestre introduced an extension problem to characterize $(-\Delta)^s$ as the Dirichlet-to-Neumann map for a local PDE [19]. In particular, they showed that if $U = U(x,z) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is the solution to

$$\begin{cases} \Delta_x U + z^{2-\frac{1}{s}}U_{zz} = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ U(x,0) = u(x) & \text{on } \mathbb{R}^n \times \{z=0\} \\ \lim_{z\to\infty} U(x,z) = 0 & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{3.2.4}$$

then, for a multiplicative constant $c_s > 0$,

$$c_s(-\Delta)^s u(x) = -U_{z+}(x,0) \quad \text{for } x \in \mathbb{R}^n.$$

Note that $-\partial_{z+}U$ is the exterior normal derivative on $\partial(\mathbb{R}^n\times[0,\infty))$. Hence, to prove regularity estimates for solutions to the nonlocal equation $(-\Delta)^s u = f$ in $\mathbb{R}^n$, one may study the local, degenerate equation (3.2.4) and take the trace across $\{z=0\}$. The extension equation is indeed degenerate as the coefficient $z^{2-\frac{1}{s}}$ cannot be controlled by above or below, depending on the value of $0<s<1$:

$$\lim_{z\to 0^+} z^{2-\frac{1}{s}} = \begin{cases} \infty & \text{if } 0<s<1/2 \\ 1 & \text{if } s=1/2 \\ 0 & \text{if } 1/2<s<1 \end{cases} \qquad \lim_{z\to\infty} z^{2-\frac{1}{s}} = \begin{cases} 0 & \text{if } 0<s<1/2 \\ 1 & \text{if } s=1/2 \\ \infty & \text{if } 1/2<s<1. \end{cases}$$

The method of semigroups has been developed by Stinga–Torrea in [71, 72, 73] and Galé–Miana–Stinga in [30] to characterize fractional powers of more general differential operators with an extension equation in Hilbert and Banach spaces, respectively. The following is a particular case of [30, Theorem 1.1].

Theorem 3.2.5 (See [30])**.** *Let $0<s<1$ and let X be a Banach space. Suppose that A generates a uniformly bounded C_0-semigroup T_t on X. For $u\in X$, a solution $U\in C^\infty((0,\infty);\mathrm{Dom}(A))\cap C([0,\infty);X)$ to*

$$\begin{cases} AU(z) + z^{2-\frac{1}{s}}\partial_{zz}U(z) = 0 & \text{in } \{z>0\} \\ U(0) = u & \text{on } \{z=0\} \end{cases} \tag{3.2.5}$$

is given by

$$U(z) = \frac{(2s)z}{4^s\Gamma(s)}\int_0^\infty e^{-\frac{s^2}{t}z^{\frac{1}{s}}}T_t u\,\frac{dt}{t^{1+s}}$$

and satisfies

$$\|U(z)\|_X \le M\,\|u\|_X \quad \text{for some } M>0.$$

Furthermore, if $u\in\mathrm{Dom}(A)$, then $U_z\in C([0,\infty);X)$ and

$$-\partial_{z+}U(0) = c_s A^s u, \quad c_s = \frac{2s\Gamma(s)}{4^s\,|\Gamma(-s)|} > 0.$$

Corollary 3.2.1. *Let $0<s<1$. Assume that $\Omega\subset\mathbb{R}^n$ is bounded, Lipschitz domain and $a^{ij}\in C^\alpha(\Omega)\cap C(\overline{\Omega})$ satisfy (3.1.1). Suppose that $X=C_0(\Omega)$ and L is given by (3.2.1). If $u\in\mathrm{Dom}(L)$,* *then a solution* $U \in C^{\infty}((0,\infty); \mathrm{Dom}(L)) \cap C([0,\infty); C_0(\Omega))$ *to*

$$\begin{cases} a^{ij}(x)\partial_{ij}U + z^{2-\frac{1}{s}}U_{zz} = 0 & \text{in } \Omega \times \{z > 0\} \\ U(x,0) = u(x) & \text{on } \Omega \times \{z = 0\}. \end{cases} \tag{3.2.6}$$

is given by

$$U(x,z) = \frac{(2s)z}{4^s\Gamma(s)} \int_0^{\infty} e^{-\frac{s^2}{t}z^{\frac{1}{s}}} e^{-tL}u(x)\,\frac{dt}{t^{1+s}} \tag{3.2.7}$$

and satisfies

$$\|U(\cdot,z)\|_{C_0(\Omega)} \leq M\,\|u\|_{C_0(\Omega)} \quad \text{for some } M > 0.$$

Furthermore, $U_z \in C([0,\infty); C_0(\Omega))$ *and*

$$-\partial_{z+}U(x,0) = c_s L^s u(x) \in C_0(\Omega).$$

Moreover, by interior Schauder estimates (see [31]), the solution U *given by* (3.2.7) *is classical in the sense that* $U \in C^2(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$.

Let $u \in \mathrm{Dom}(L)$ be a solution to $L^s u = f$ in a ball $B \subset\subset \Omega$ where $f \in C_0(\Omega)$. By Corollary 3.2.1, the function U given by (3.2.7) is a solution to

$$\begin{cases} a^{ij}(x)\partial_{ij}U + z^{2-\frac{1}{s}}\partial_{zz}U = 0 & \text{in } \Omega \times (0,\infty) \\ -\partial_{z+}U = f & \text{on } \Omega \times \{z = 0\}. \end{cases} \tag{3.2.8}$$

and, up to a multiplicative constant, $U(x,0) = u(x)$. Therefore, to prove estimates for u, it is enough to study *a priori* estimate for solutions U to (3.2.8).

To study the trace across $\{z = 0\}$, we let $\tilde{U}$ be the even reflection of U:

$$\tilde{U}(x,z) = U(x,|z|), \quad \text{for all } (x,z) \in \Omega \times \mathbb{R}.$$

As long as the context is clear, we will continue to use U in place of $\tilde{U}$. Hence, for a Monge–Ampère section $S \subset\subset \Omega \times \mathbb{R}$ (see Section 3.3), we prove Harnack inequality for nonnegative, classical solutions $U \in C^2(\Omega \times \{z \neq 0\}) \cap C(\overline{\Omega} \times \mathbb{R})$, $U_z \in C([0,\infty); C_0(\Omega))$ to

$$\begin{cases} a^{ij}(x)\partial_{ij}U + |z|^{2-\frac{1}{s}}\partial_{zz}U = 0 & \text{in } S \cap \{z \neq 0\} \\ -\partial_{z+}U = f & \text{on } S \cap \{z = 0\} \end{cases}$$

where $-\partial_{z+}U = f$ is the exterior normal derivative on $\partial(\Omega \times (0,\infty))$. Note that the Neumann condition $-\partial_{z+}U = f$ does not apply if $S \cap \{z=0\} = \emptyset$.

If $f = 0$ in $B \subset \Omega$, then, by the symmetry of U across $\{z=0\}$, we have that

$$\begin{aligned} -\partial_{z-}U(x,0) &= -\lim_{h\to 0^-} \frac{U(x,h)-U(x,0)}{h} \\ &= -\lim_{h\to 0^+} \frac{U(x,-h)-U(x,0)}{-h} \\ &= \lim_{h\to 0^+} \frac{U(x,h)-U(x,0)}{h} = \partial_{z+}U(x,0) = 0. \end{aligned}$$

3.3 Monge–Ampère setting

Given $0 < s < 1$, we define the functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(z) = \frac{1}{2}|x|^2 \quad \text{and} \quad h(z) = \frac{s^2}{1-s}|z|^{\frac{1}{s}}$$

and notice that $\varphi \in C^\infty(\mathbb{R})$ and $h \in C^1(\mathbb{R}) \cap C^2(\mathbb{R}\setminus\{0\})$ are strictly convex. Set

$$\Phi(x,z) = \varphi(x) + h(z) \quad \text{for all } (x,z) \in \mathbb{R}^{n+1}.$$

We note that

$$h'(z) = \frac{s}{1-s}|z|^{\frac{1}{s}-2}z, \quad h''(z) = |z|^{\frac{1}{s}-2}, \quad D^2\Phi(x,z) = \begin{pmatrix} I & 0 \\ 0 & |z|^{\frac{1}{s}-2} \end{pmatrix}.$$

It is clear that

$$h'(-z) = -h'(z) \quad \text{and} \quad h'(0) = 0.$$

Definition 3.3.1. *The Monge-Ampère measure associated to a strictly convex function $\psi \in C^1(\mathbb{R}^n)$ is given by*

$$\mu_\psi(E) = |D\psi(E)| \quad \textit{for every Borel set } E \subset \mathbb{R}^n.$$

Since $D\varphi(x) = x$, it clear that $\mu_\varphi(E) = |E|$ is the Lebesgue measure of E.

Lemma 3.3.1. *For a Borel set $I \subset \mathbb{R}$,*

$$\mu_h(I) = \int_I h''(z)\,dz.$$

Consequently, for a Borel set $E \subset \mathbb{R}^{n+1}$,

$$\mu_\Phi(E) = \int_E h''(z)\, dz\, dx.$$

Proof. Consider an open (or closed) interval $(-a, a) \subset \mathbb{R}$. Note that h' is monotone increasing, injective, and $h'(z) = 0$ if and only if $z = 0$. Since h is C^2 and strictly convex in $\mathbb{R} \setminus \{z = 0\}$, we have that

$$\begin{aligned}
\mu_h((-a,a)) &= \left|h'((-a,a))\right| \\
&= \left|h'((-a,0)) \cup h'(0) \cup h'((0,a))\right| \\
&= \left|h'((-a,0))\right| + \left|h'(0)\right| + \left|h'((0,a))\right| \\
&= \int_{-a}^{0} h''(z)\, dz + 0 + \int_0^a h''(z)\, dz \\
&= \int_{-a}^{a} h''(z)\, dz.
\end{aligned}$$

The result follows for any interval and hence for any Borel set $I \subset \mathbb{R}$. □

Definition 3.3.2. *The Monge-Ampère (quasi)-distance associated to a strictly convex function $\psi \in C^1(\mathbb{R}^n)$ is given by*

$$\delta_\psi(x_0, x) = \psi(x) - \psi(x_0) - \langle D\psi(x_0), x - x_0\rangle.$$

We use the terminology quasi-distance when there exists a $K \geq 1$ such that

$$\delta_\psi(x_1, x_2) \leq K\left(\delta_\psi(x_1, x_3) + \delta_\psi(x_3, x_2)\right) \quad \text{for all } x_1, x_2, x_3 \in \mathbb{R}^n.$$

For our functions φ, h, and Φ, we have

$$\begin{aligned}
\delta_\varphi(x_0, x) &= \frac{1}{2}|x|^2 - \frac{1}{2}|x_0|^2 - \langle x_0, x - x_0\rangle = \frac{1}{2}|x - x_0|^2 \\
\delta_h(z_0, z) &= h(z) - h(z_0) - h'(z_0)(z - z_0) \\
\delta_\Phi((x_0, z_0), (x, z)) &= \delta_\varphi(x_0, x) + \delta_h(z_0, z).
\end{aligned}$$

We will show that δ_h, δ_φ, and δ_Φ are indeed quasi-distances (see Lemma 3.3.4).

Lemma 3.3.2. *If $\psi \in C^1(\mathbb{R}^n)$ is strictly convex, then δ_ψ is continuous.*

Proof. Suppose that $x_k \to x_0$ and $y_k \to y_0$ as $k \to \infty$. By the continuity of ψ and $D\psi$,

$$\begin{aligned}\delta_\psi(x_k, y_k) &= \psi(y_k) - \psi(x_k) - \langle D\psi(x_k), y_k - x_k\rangle \\ &\to \psi(y_0) - \psi(x_0) - \langle D\psi(x_0), y_0 - x_0\rangle = \delta_\psi(x_0, y_0) \quad \text{as } k \to \infty.\end{aligned}$$

□

Definition 3.3.3. *The Monge–Ampère section of radius r, centered at x_0 associated to a strictly convex function $\psi \in C^1(\mathbb{R}^n)$ is given by*

$$S_r(x_0) = \{x : \delta_\psi(x_0, x) < r\}.$$

When necessary, we use the notation $S_\psi(x_0, r)$.

The supporting hyperplane to ψ at x_0 is given by

$$\ell(x) = \psi(x_0) + \langle D\psi(x_0), x - x_0\rangle.$$

By writing

$$S_r(x_0) = \{x : \psi(x) < r + \psi(x_0) + \langle D\psi(x_0), x - x_0\rangle\} = \{x : \psi(x) - \ell(x) < r\},$$

we can see that the Monge–Ampère sections for ψ centered at x_0 are the sublevel sets of $\psi - \ell$. In the case of φ, the sections correspond to Euclidean balls with the same center

$$S_R(x_0) = \{x : \frac{1}{2}|x - x_0|^2 < R\} = \{x : |x - x_0| < \sqrt{2R}\} = B_{\sqrt{2R}}(x_0), \tag{3.3.1}$$

The sections for h with radius $r > 0$ are one-dimensional, so they correspond to intervals in $\mathbb{R}$. Moreover, they are are comparable to intervals of radius r^s (see Lemma 3.3.5).

Definition 3.3.4. *We say that μ_ψ is doubling with respect to the center of mass on the sections of ψ if there is a constant $C_d > 0$ such that*

$$\mu_\psi(S_R(x)) \le C_d \mu_\psi\left(\frac{1}{2}S_R(x)\right) \quad \textit{for all sections } S_R(x).$$

We write $\mu_\psi \in (DC)_\psi$*. Here, we use the notation*

$$\alpha S_R(x) = \{\alpha(y - x^*) + x^* : y \in S_R(x)\}, \quad \alpha > 0$$

where x^* *is the center of mass of* $S_R(x)$*.*

Definition 3.3.5. *We say that* ψ *satisfies the engulfing property if there is a constant* $\theta \geq 1$ *such that, for every section* $S_R(x_0)$*, if* $x_1 \in S_R(x_0)$*, then* $S_R(x_0) \subset S_{\theta R}(x_1)$*.*

Theorem 3.3.1 (Theorem 5 in [29])**.** *Let* $\psi : \mathbb{R}^n \to \mathbb{R}$ *be a differentiable, strictly convex function. Then the following are equivalent*

1. $\mu_\psi \in (DC)_\psi$*;*
2. ψ *satisfies the engulfing property;*
3. μ_ψ *satisfies*

$$cR^n \leq |S_R(x)|\, \mu_\psi(S_R(x)) \leq CR^n$$

for all sections $S_R(x)$ *and some constants* $c, C > 0$*.*

All statements are equivalent in the sense that the constants in each property only depend on each other.

Forzani–Maldonado in [29] further remark that $\mu_\psi \in (DC)_\psi$ is quantitatively equivalent to δ_ψ satisfying the quasi-triangle inequality.

Lemma 3.3.3 (Lemma 6 in [29])**.** *Fix* $m \in \mathbb{N}$*. For each* $j = 1, \ldots, m$*, let* $\psi_j : \mathbb{R}^{n_j} \to \mathbb{R}$ *be strictly convex, differentiable functions. Set* $n = \sum_{j=1}^m n_j$ *and define*

$$\psi(x) = \sum_{j=1}^{m} \psi_j(x_j), \quad x = (x_1, \ldots, x_m) \in \mathbb{R}^n, x_j \in \mathbb{R}^{n_j}.$$

Then

$$S_\psi(x, R) \subset \prod_{j=1}^{m} S_{\psi_j}(x_j, R) \subset S_\psi(x, mR)$$

for all $x = (x_1, \ldots, x_m) \in \mathbb{R}^n$ *and* $R > 0$*.*

In particular, if ψ_j *satisfy the engulfing property with corresponding constants* θ_j *for all* $j = 1, \dots m$*, then* ϕ *satisfies the engulfing property with* $\theta = m \max_j\{\theta_j\}$*. Conversely, if* ψ *satisfies the engulfing property for some* $\theta > 1$*, then* ψ_j *satisfies the engulfing property with constant* θ *for all* $j = 1, \dots, m$*.*

Consequently, we have that

$$S_\Phi((x,z),R) \subset S_\varphi(x,R) \times S_h(z,R) \subset S_\Phi((x,z),2R) \tag{3.3.2}$$

for all $(x,z) \in \mathbb{R}^{n+1}$ and $R > 0$.

We will show that $\mu_\varphi \in (DC)_\varphi$ and $\mu_h \in (DC)_h$ to obtain the following Lemma.

Lemma 3.3.4. *1.* $\mu_\Phi \in (DC)_\Phi$ *with corresponding doubling constant* $C_d = C_d(n,s)$*.*

2. Φ *satisfies the engulfing property with corresponding constant* $\theta = \theta(n,s)$*.*

3. μ_Φ *satisfies*

$$cR^{n+1} \le |S_R(x,z)|\, \mu_\Phi(S_R(x,z)) \le CR^{n+1}$$

for all sections $S_R(x,z)$ *and some positive constants* $C = C(n,s)$*,* $c = c(n,s)$*.*

4. there exists a constant $K = K(n,s)$ *such that*

$$\begin{aligned} \delta_\Phi((x_1,z_1),(x_2,z_2)) \le K\big(&\min\{\delta_\Phi((x_1,z_2),(x_3,z_3)), \delta_\Phi((x_3,z_3),(x_1,z_1))\} \\ &+ \min\{\delta_\Phi((x_2,z_2),(x_3,z_3)), \delta_\Phi((x_3,z_3),(x_2,z_2))\}\big). \end{aligned} \tag{3.3.3}$$

for all $(x_1,z_1),(x_2,z_2),(x_3,z_3) \in \mathbb{R}^{n+1}$*.*

Proof. By (3.3.1), we can write

$$\mu_\varphi(S_\varphi(x_0,R)) = \left|B\left(x_0,\sqrt{2R}\right)\right| = 2^{n/2}\left|B\left(x_0,\sqrt{R}\right)\right| = 2^{n/2}\mu_\varphi\left(S_\varphi\left(x_0,\frac{1}{2}R\right)\right).$$

Hence $\varphi \in (DC)_\varphi$ with doubling constant $C_d^\varphi = C_d^\varphi(n)$.

As discussed in [50, Section 7.1], $h''(z)$ is a Muckenhoupt $A_\infty(\mathbb{R})$ weight for all $0 < s < 1$. Consequently, h'' is doubling on the real line which is equivalent to $\mu_h \in (DC)_h$ with doubling constant $C_d^h = C_d^h(s)$.

It follows from (3.3.2) that $\mu_\Phi \in (DC)_\Phi$ with doubling constant $C_d = C_d(n,s)$. Items 2, 3, and 4 follow from Theorem 3.3.1 and the comment thereafter. □

As a consequence of the doubling property (see [50, Equation 7.8]), there is a constant $K_d = K_d(n,s)$ such that

$$\mu_\Phi(S((x,z),r_2) \leq K_d \left(\frac{r_2}{r_1}\right)^\nu \mu_\Phi(S((x,z),r_1) \quad \text{for all } 0 < r_1 < r_2, \tag{3.3.4}$$

where $\nu := \log_2 K_d$.

Lemma 3.3.5. *There exist constants $c_s, C_s > 0$, depending only on s, such that*

$$B_{c_s r^s}(z_0) \subset S_h(z_0,r) \subset B_{C_s r^s}(z_0)$$

for all $r > 0$ and all $z_0 \in \mathbb{R}$. Consequently, there exist constants $c_s', C_s' > 0$, depending only on s, such that

$$2c_s r^s \leq |S_h(z_0,r)| \leq 2C_s r^s \quad \text{and} \quad c_s' r^{1-s} \leq \mu_h(S_h(z_0,r)) \leq C_s' r^{1-s}$$

for all sections $S_r(z_0)$.

Proof. For the function

$$\varphi_s(z) = s\,|z|^{1/s}, \quad z \in \mathbb{R},$$

there exist constants $0 < c \leq 1 \leq C < \infty$, depending only on s, such that

$$B(z_0, cr^s) \subset S_{\varphi_s}(z_0,r) \subset B(z_0, Cr^s), \quad \text{for all } z_0 \in \mathbb{R},\ r > 0.$$

See [49, Section 11]. It is easy to check that

$$S_{\varphi_s}(z_0,t) = S_h\left(z_0, \frac{s}{1-s}t\right).$$

Substitute $t = \frac{1-s}{s}r$ and take $c_s = c\left(\frac{1-s}{s}\right)^s$, $C_s = C\left(\frac{1-s}{s}\right)^s$, so that

$$B\left(z_0, c_s r^s\right) = B\left(z_0, c\left(\frac{1-s}{s}r\right)^s\right) \subset S_h(z_0,r) \subset B\left(z_0, C\left(\frac{1-s}{s}r\right)^s\right) = B\left(z_0, C_s r^s\right)$$

for all $z_0 \in \mathbb{R}$ and all $r > 0$.

For a section $S_h(z_0, r)$,

$$2c_s r^s = |B_{c_s r^s}(z_0)| \leq |S_h(z_0, r)| \leq |B_{C_s r^s}(z_0)| = 2C_s r^s.$$

Since $\mu_h \in (DC)_h$, by Theorem 3.3.1, there exist constants $c, C > 0$, depending only on s, such that

$$cr \leq |S_h(z_0, r)| \, \mu_h(S_h(z_0, r)) \leq Cr.$$

Therefore,

$$\mu_h(S_h(z_0, r)) = \frac{\mu_h(S_h(z_0, r)) \, |S_h(z_0, r)|}{|S_h(z_0, r)|} \leq \frac{Cr}{2c_s r^s} = C_s' r^{1-s}$$
$$\mu_h(S_h(z_0, r)) = \frac{\mu_h(S_h(z_0, r)) \, |S_h(z_0, r)|}{|S_h(z_0, r)|} \geq \frac{cr}{2C_s r^s} = c_s' r^{1-s}.$$

□

Unless otherwise stated, we will always let $S_r(x_0, z_0)$ denote a section association to Φ. We will need the following lemmas written for the sections of Φ.

Lemma 3.3.6 (Theorem 3.3.10 in [34]). *There exist constants $C_0 > 0$, $p \geq 1$, depending only on n and s, such that for $0 < s_1 < s_2 \leq 1$, $t > 0$ and $(x_1, z_1) \in S_\Phi((x_v, z_v), s_1 t)$, we have that*

$$S_\Phi((x_1, z_1), C_0(s_2 - s_1)^p t) \subset S_\Phi((x_v, z_v), s_2 t).$$

Lemma 3.3.7 (Lemma 10.6 in [50]). *Given $S_R(x_0, z_0)$ with $\overline{S}_R(x_0, z_0) \cap \{z = 0\} \neq \emptyset$, there is $R_r \in (R, 2KR)$ such that*

$$S_R(x_0, z_0) \subset S_{2R_r}(x_0, 0) \subset S_{4R_r}(x_0, 0) \subset S_{\beta_s R/2}(x_0, z_0)$$

where $K \geq 1$ is the quasi-triangle constant and $\beta_s = 4K(1 + 8K)$ is a constant depending only on $0 < s < 1$.

Many of our proofs will rely on the fact that $\Phi(x, z) = \varphi(x) + h(z)$ has separated variables. It is therefore essential to some of our arguments to consider Monge-Ampère cubes associated to Φ.

Definition 3.3.6. *The Monge–Ampère cube of radius r, centered at $(x,z)\in\mathbb{R}^{n+1}$, associated to Φ is given by*

$$Q_r(x,z) = S_r(x_1)\times\cdots\times S_r(x_n)\times S_r(z)\subset\mathbb{R}^{n+1}$$

where $x=(x_1,\ldots,x_n)$ and $S_r(x_i)=S_\varphi(x_i,r)$ is a one-dimensional section in $\mathbb{R}$ associated to $\varphi:\mathbb{R}\to\mathbb{R}$ given by $\varphi(x_i)=\frac{1}{2}(x_i)^2$.

It follows from Lemma 3.3.3 that

$$S_r(x,z)\subset Q_r(x,z)\subset S_{(n+1)r}(x,z)\quad\text{for all } (x,z)\in\mathbb{R}^{n+1},\ r>0.$$

We also note that

$$Q_r(x,z) = Q_r(x)\times S_r(z).$$

3.4 Local boundedness and critical-density estimate

Theorem 3.4.1. *Let $0<s<1$. Assume that $a^{ij}=a^{ij}(x)$ are bounded measurable functions on $\mathbb{R}^n$ that satisfy (3.1.1). There exist positive constants $K'=K'(n,\lambda,\Lambda,s)>0$, $\kappa'=\kappa'(n,\lambda,\Lambda,s)<1$ such that for every $p>0$, there exists positive constants $C_{1,p},C_{2,p}>0$, depending on p,n,λ,Λ,s, such that for every section $S_R=S_R(x_0,z_0)\subset\mathbb{R}^{n+1}$ and every classical supersolution $U=U(x,z)=U(x,-z)$ to*

$$\begin{cases} a^{ij}(x)\partial_{ij}U+|z|^{2-\frac{1}{s}}U\le 0 & \text{in } S_{K'R}\setminus\{z=0\}\\ -\partial_{z+}U(x,0)\ge f(x) & \text{on } S_{K'R}\cap\{z=0\}\end{cases}$$

where $f\in L^\infty(S_{K'R}\cap\{z=0\})$, we have

$$\sup_{S_{\kappa'R}} U\le C_{1,p}\left(\frac{1}{\mu_\Phi(S_R)}\int_{S_R}U^p\,d\mu_\Phi\right)^{1/p}+C_{2,p}R^s\,\|f\|_{L^\infty(S_{K'R}\cap\{z=0\})}.$$

Theorem 3.4.2. *Let $0<s<1$. Assume that $a^{ij}=a^{ij}(x)$ are bounded measurable functions on $\mathbb{R}^n$ that satisfy (3.1.1). There exist positive constants $\theta_0=\theta_0(n,\lambda,\Lambda,s)<1$, $\varepsilon_0=\varepsilon(n,\lambda,\Lambda,s)<1$ and $M=M(n,s)>1$ such that for every section $S_R=S_R(x_0,z_0)\subset\mathbb{R}^{n+1}$ and every nonnegative, classical viscosity supersolution $U=U(x,z)=U(x,-z)$ to*

$$\begin{cases} a^{ij}(x)\partial_{ij}U+|z|^{2-1/s}\partial_{zz}U\le 0, & \text{in } S_{\beta_sR}\setminus\{z=0\}\\ -U_{z+}(x,0)\ge f(x) & \text{on } S_{\beta_sR}\cap\{z=0\}\end{cases}$$

where $f \in L^{\infty}(S_{\beta_s R} \cap \{z=0\})$, *the inequalities*

$$R^s \|f\|_{L^{\infty}(S_{\beta_s R} \cap \{z=0\})} \leq \theta_0 \quad \text{and} \quad \inf_{S_R} U \leq 1$$

imply

$$\mu_\Phi(\{(x,z) \in S_{\beta_s R} : U(x,z) < M\}) \geq \varepsilon_0 \mu_\Phi(S_{\beta_s R}).$$

Theorems 3.4.1 and 3.4.2 are proved exactly as in [50] for the fractional nonlocal linearized Monge–Ampère equation with slight modifications involving the coefficients a^{ij}. Before providing a sketch of the proof, we recall the definition of the convex envelope and its relation to the ABP estimate.

Definition 3.4.1. *We say that an affine function* ℓ *is a supporting hyperplane for* U *at* (x_0, z_0) *in a set* $\tilde{\Omega} \subset \mathbb{R}^{n+1}$ *if*

$$\ell(x_0, z_0) = U(x_0, z_0) \quad \text{and} \quad \ell(x,z) \leq U(x,z) \quad \text{for all } (x,z) \in \tilde{\Omega}.$$

The convex envelope of U *in* $\tilde{\Omega}$ *is*

$$\Gamma_U(x,z) = \sup_{\ell}\{\ell(x,z) : \ell \leq U \text{ in } \tilde{\Omega},\ \ell \text{ is affine}\}.$$

The corresponding contact set for U *in* $\tilde{\Omega}$ *is*

$$A_0(U) = \{(x,z) \in \tilde{\Omega} : U(x,z) = \Gamma_U(x,z)\}.$$

We remark that Γ_U is indeed a convex function and that, for each $(x_0, z_0) \in A_0(U)$, there exists a supporting hyperplane ℓ to U at (x_0, z_0).

The Aleksandrov-Bakelman-Pucci (ABP) estimate is a maximum principle which bounds a subsolution H in $\tilde{\Omega}$ by the $L^{n+1}(\tilde{\Omega})$ norm of the right-hand side of the equation. We will need the following variation which will be used to avoid the degeneracy at $z = 0$ in the extension equation.

Lemma 3.4.1 (Corollary 10.3 in [50])**.** *Let* $\tilde{\Omega} \subset \mathbb{R}^{n+1}$ *be open, convex, and bounded. Suppose that* $H \in C(\overline{\tilde{\Omega}})$ *satisfies the following conditions*

1. $H \geq 0$ *on* $\partial\tilde{\Omega}$

2. there is an open set $\tilde{\Omega}' \subset \tilde{\Omega}$ such that $H \in C^2(\tilde{\Omega}')$ and $A_0(-H^-) \subset \tilde{\Omega}'$ where $H^- = \max\{0, -H\}$.

Then

$$\max_{\tilde{\Omega}} H \leq c_n |\tilde{\Omega}|^{\frac{1}{n+1}} \left(\int_{A_0(-H^-)} \left| \det D^2 H(x,z) \right| \, dx \, dz \right)^{\frac{1}{n+1}}.$$

We sketch the proof of Theorems 3.4.1 and 3.4.2.

Proof. (Sketch.) Theorem 3.4.1 follows from Theorem 3.4.2 (see [50, Section 11]).

The proof of Theorem 3.4.2 follows exactly as in the proof of [50, Theorem 10.1]. If the section $S_{2R} = S_{2R}(x_0, z_0)$ is such that $S_R \cap \{z = 0\} = \emptyset$, then we define an auxiliary function

$$H(x,z) = U(x,z) + 4\left(\frac{\delta_\Phi((x_0,z_0),(x,z))}{2R} - 1 \right), \quad (x,z) \in S_{2R}.$$

Using the ellipticity of a^{ij}, we can show that H is a supersolution with right-hand side $\frac{2}{R}(n\Lambda + 1)$ and moreover that

$$0 \leq \det\left(D^2 H\right) \leq c_{n,\lambda,\Lambda} (2R)^{-(n+1)} \det D^2 \Phi.$$

By Lemma 3.4.1 with $\tilde{\Omega} = \tilde{\Omega}' = S_{2R}$ and with $\inf_{S_R} U \leq 1$,

$$\begin{aligned} 1 \leq \left(\max_{\Omega} H \right)^{n+1} &\leq c_n \left| S_{2R} \right| \int_{A_0(-H^-)} \left| \det D^2 H(x,z) \right| \, dx \, dz \\ &\leq c_{n,\lambda,\Lambda} \left| S_{2R} \right| (2R)^{-(n+1)} \mu_\Phi(A_0(-H^-)). \end{aligned}$$

By Lemma 3.3.4 and by estimating

$$A_0(-H^-) \subset \{(x,z) \in S_{2R} : U(x,z) < 4\},$$

we can conclude that

$$1 \leq C c_{n,\lambda,\Lambda} \mu_\Phi(S_{2R})^{-1} \mu_\Phi(\{(x,z) \in S_{2R} : U(x,z) < 4\})$$

which proves the estimate for $\varepsilon_1 = 1/(C c_{n,\lambda,\Lambda})$, $M_1 = 4$.

If the section $S_{2R} = S_{2R}(x_0, 0)$ is centered on $\{z = 0\}$, then we instead use the auxiliary function

$$\begin{aligned} H(x,z) = U(x,z) &+ Q_s \left(\frac{\delta_\Phi((x_0,0),(x,z))}{2R} - 1 \right) \\ &- \|f\|_{L^\infty(S_{2R})} |z| - \frac{1}{q_s R^s} |z| + \|f\|_{L^\infty(S_{2R})} q_s R^s + 1 \end{aligned}$$

for constants

$$q_s = \frac{2^s(1-s)^s}{s^{2s}} \quad \text{and} \quad Q_s = 4q_s \|f\|_{L^\infty(S_{2R})} R^s + 8.$$

Notice that $\delta_\Phi((x_0,0),(x,z)$ and thus H are symmetric with respect to z.

We claim that Γ_{-H^-} cannot touch $-H^-$ on Z_0 and that the contact set $A_0(-H^-)$ lies at a positive distance from the set $\{z=0\}$. Indeed, if $\Gamma(x_2,0) = -H^-(x_2,0)$ for $(x_2,0)\in S_{2R}$, then

$$\begin{aligned} 0 &\le \frac{\Gamma(x_2,\varepsilon)-\Gamma(x_2,0)}{\varepsilon} \\ &\le \frac{(-H^-)(x_2,\varepsilon)-(-H^-)(x_2,0)}{\varepsilon} \\ &\le \frac{U(x_2,\varepsilon)-U(x_2,0)}{\varepsilon} + \frac{Q_s}{2R}\frac{\delta_\Phi((x_0,0),(x_2,\varepsilon))-\delta_\Phi((x_0,0),(x_2,0))}{\varepsilon} - \|f\|_{L^\infty(S_R)} - \frac{1}{q_sR^s} \end{aligned}$$

As this holds for all $\varepsilon>0$ such that $(x_2,\varepsilon)\in S_{2R}$,

$$\begin{aligned} 0 &\le U_{z+}(x_2,0) + \frac{Q_s}{2R}\partial_z\delta_\Phi((x_0,0),(x_2,z))\big|_{z=0} - \|f\|_{L^\infty(S_R)} - \frac{1}{q_sR^s} \\ &= -f(x_2) + 0 - \|f\|_{L^\infty(S_R)} - \frac{1}{q_sR^s} < 0, \end{aligned}$$

a contradiction. Next, suppose, by way of contradiction that for each $n\in\mathbb{N}$, there exists $z_n \in \left(-\frac{1}{n},\frac{1}{n}\right)$ and x_n such that $(x_n,z_n)\in S_{2R}$ and

$$\Gamma(x_n,z_n) = (-H^-)(x_n,z_n).$$

Since $x_n \in S_\varphi(x_0,2R)$, there is a subsequence $x_{n_k} = x_k$ such that $x_k \to \bar{x}$. By the continuity of Γ and $-H^-$,

$$\Gamma(\bar{x},0) = \lim_{k\to\infty}\Gamma(x_k,z_k) = \lim_{k\to\infty}(-H^-)(x_k,z_k) = (-H^-)(\bar{x},0),$$

a contradiction. Therefore, the contact set must be a positive distance from $\{z=0\}$.

By Lemma 3.4.1 with $\tilde{\Omega}' = S_{2R}\setminus\{z=0\}$ and $\tilde{\Omega} = S_{2R}$, we obtain

$$\left(\max_{S_{2R}} H^-\right)^{n+1} \le c_n\,|S_{2R}| \int_{A_0(-H^-)} \left|\det D^2H(x,z)\right|\,dx\,dz.$$

Carefully following a similar argument as above, we estimate

$$\mu_\Phi\left(\{(x,z)\in S_{2R} : U(x,z) < M_2\}\right) \ge \varepsilon_2\mu_\Phi(S_{2R})$$

where $M_2 = 4q_s + 8$ and $\varepsilon_2 = 1/(2C16^{n+1}c_{n,\lambda,\Lambda})$.

Lastly, we suppose that $S_{\beta_s R} = S_{\beta_s R}(x_0, z_0)$ is such that $S_{\beta_s R} \cap \{s = 0\} \neq \emptyset$ but $z_0 \neq 0$. By Lemma 3.3.7, there is an $\hat{R} = (2R)_r \in (2R, 4KR)$ such that

$$S_{2R}(x_0, z_0) \subset S_{2\hat{R}}(x_0, 0) \subset S_{4\hat{R}}(x_0, 0) \subset S_{\beta_s R}(x_0, 0).$$

We can then apply the estimate from the case centered at $z_0 = 0$ and use the doubling condition to estimate

$$\mu_\Phi(\{(x, z) \in S_{\beta_s R}(x_0, z_0) : U(x, z) < M_2\}) \geq \varepsilon_2 \left(K_d \left(\beta_s\right)^\nu\right)^{-1} \mu_\Phi(S_{\beta_s R}(x_0, z_0))$$

Since $\beta_s > 2$, Theorem 3.4.2 follows for any section $S_R(x_0, z_0)$ with $\varepsilon_0 = \varepsilon_2 \left(K_d \left(\beta_s\right)^\nu\right)^{-1}$ and $M = M_2 > 4$. □

3.5 Harnack inequality

3.5.1 Paraboloids and preliminaries

3.5.1.1 Paraboloids associated to Φ

We define paraboloids P of opening $a > 0$ by

$$P(x, z) = -a\Phi(x, z) + \langle (y, w), (x, z) \rangle + b$$

for some $(y, w) \in \mathbb{R}^{n+1}$ and $b \in \mathbb{R}$. We say that (x_v, z_v) is the vertex of P if $DP(x_v, z_v) = 0$.

We say that P touches a function U from below at (x_0, z_0) in a convex set $\tilde{\Omega} \subset \mathbb{R}^{n+1}$ if

$$P(x_0, z_0) = V(x_0, z_0) \quad \text{and} \quad P(x, z) \leq V(x, z) \quad \text{for all } (x, z) \in \tilde{\Omega}.$$

Lemma 3.5.1. *A paraboloid P of opening a with vertex (x_v, z_v) can be written as*

$$P(x, z) = -a\delta_\Phi((x_v, z_v), (x, z)) + c$$

for some constant $c \in \mathbb{R}$. If $P(x_0, z_0) = U(x_0, z_0)$ for a function continuous function $U : \mathbb{R}^{n+1} \to \mathbb{R}$, then

$$P(x, z) = -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_0, z_0)) + U(x_0, z_0).$$

Proof. Since

$$0 = DP(x_v, z_v) = -aD\Phi(x_v, z_v) + (y, w),$$

we can write

$$P(x, z) = -a\Phi(x, z) + a\langle D\Phi(x_v, z_v), (x, z)\rangle + b.$$

Moreover, we have

$$\begin{aligned} P(x, z) &= -a\Phi(x, z) + a\langle D\Phi(x_v, z_v), (x, z)\rangle + b \\ &\quad + a\Phi(x_v, z_v) - a\Phi(x_v, z_v) - a\langle D\Phi(x_v, z_v), (x_v, z_v)\rangle + a\langle D\Phi(x_v, z_v), (x_v, z_v)\rangle \\ &= -a\left(\Phi(x, z) - \Phi(x_v, z_v) - \langle D\Phi(x_v, z_v), (x, z) - (x_v, z_v)\rangle\right) \\ &\quad - \Phi(x_v, z_v) + a\langle D\Phi(x_v, z_v), (x_v, z_v)\rangle + b \\ &= -a\delta_\Phi((x_v, z_v), (x, z)) + c. \end{aligned}$$

Suppose that $P(x_0, z_0) = U(x_0, z_0)$. Then

$$U(x_0, z_0) = -a\delta_\Phi((x_v, z_v), (x_0, z_0)) + c.$$

By solving for c, we conclude that

$$P(x, z) = -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_0, z_0)) + U(x_0, z_0).$$

□

Lemma 3.5.2. *Suppose that P is a paraboloid of opening $a > 0$ that touches a continuous function U from below at (x_0, z_0) in a convex set $\tilde{\Omega} \subset \mathbb{R}^{n+1}$. For any $\tilde{a} \geq a$, there exists a paraboloid $\tilde{P}$ of opening $\tilde{a} > 0$ that touches U from below at (x_0, z_0) in $\tilde{\Omega}$.*

Proof. Begin by writing

$$\begin{aligned}
P(x,z) &= -a\delta_\Phi((x_v,z_v),(x,z)) + a\delta_\Phi((x_v,z_v),(x_0,z_0)) + U(x_0,z_0) \\
&= -a\left(\Phi(x,z) - \Phi(x_v,z_v) - \langle D\Phi(x_v,z_v),(x,z)-(x_v,z_v)\rangle\right) \\
&\quad + a\left(\Phi(x_0,z_0) - \Phi(x_v,z_v) - \langle D\Phi(x_v,z_v),(x_0,z_0)-(x_v,z_v)\rangle\right) + U(x_0,z_0) \\
&= -a\Phi(x,z) + a\langle D\Phi(x_v,z_v),(x,z)\rangle + a\Phi(x_0,z_0) - a\langle D\Phi(x_v,z_v),(x_0,z_0)\rangle + U(x_0,z_0) \\
&= -a\left(\Phi(x,z) - \Phi(x_0,z_0) - \langle D\Phi(x_0,z_0),(x,z)-(x_0,z_0)\rangle\right) \\
&\quad + a\langle D\Phi(x_v,z_v),(x,z)-(x_0,z_0)\rangle + a\langle D\Phi(x_0,z_0),(x,z)-(x_0,z_0)\rangle + U(x_0,z_0) \\
&= -a\delta_\Phi((x_0,z_0),(x,z)) + a\langle D\Phi(x_v,z_v) - D\Phi(x_0,z_0),(x,z)-(x_0,z_0)\rangle + U(x_0,z_0).
\end{aligned}$$

Define the paraboloid $\tilde{P}$ of opening $\tilde{a}$ by

$$\tilde{P}(x,z) = -\tilde{a}\delta_\Phi((x_0,z_0),(x,z)) + a\langle D\Phi(x_v,z_v) - D\Phi(x_0,z_0),(x,z)-(x_0,z_0)\rangle + U(x_0,z_0).$$

Since $\tilde{P}(x_0,z_0) = U(x_0,z_0)$ and

$$\begin{aligned}
\tilde{P}(x,z) &\le -a\delta_\Phi((x_0,z_0),(x,z)) + a\langle D\Phi(x_v,z_v) - D\Phi(x_0,z_0),(x,z)-(x_0,z_0)\rangle + U(x_0,z_0) \\
&= P(x,z) \le U(x,z),
\end{aligned}$$

for every $(x,z)\in\tilde{\Omega}$, we conclude that $\tilde{P}$ touches U from below at (x_0,z_0) in $\tilde{\Omega}$. □

The next three lemmas provide some observations regarding how the symmetry of U effects the paraboloids that touch U from below.

Lemma 3.5.3. *Consider a continuous function $U = U(x,z) = U(x,-z)$. Let P be a paraboloid of opening a with vertex (x_v,z_v) that touches U from below at (x_0,z_0) in a convex set $\tilde{\Omega}\subset\mathbb{R}^{n+1}$. If $z_0>0$, then $z_v\ge 0$, and if $z_0<0$, then $z_v\le 0$.*

Proof. Assume that $z_0>0$. Write

$$P(x,z) = -a\delta_\Phi((x_v,z_v),(x,z)) + a\delta_\Phi((x_v,z_v),(x_0,z_0)) + U(x_0,z_0)$$

and note that

$$\begin{aligned}P(x_0,-z_0) &= -a\delta_\Phi((x_v,z_v),(x_0,-z_0)) + a\delta_\Phi((x_v,z_v),(x_0,z_0)) + U(x_0,z_0)\\ &= -a\delta_h(z_v,-z_0) + a\delta_h(z_v,z_0)) + U(x_0,-z_0).\end{aligned}$$

Then

$$\begin{aligned}0 &\le U(x_0,-z_0) - P(x_0,-z_0)\\ &= a\delta_h(z_v,-z_0) - a\delta_h(z_v,z_0))\\ &= a\left(h(-z_0) - h(z_v) - h'(z_v)(-z_0-z_v)\right) - a\left(h(z_0) - h(z_v) - h'(z_v)(z_0-z_v)\right)\\ &= a\left(h(-z_0) - h(z_0)\right) + 2ah'(z_v)z_0\\ &= 2ah'(z_v)z_0.\end{aligned}$$

Since $z_0 > 0$, it follows that

$$0 \le h'(z_v) = \frac{s}{1-s}\,|z_v|^{\frac{1}{s}-2}\,z_v.$$

Hence, $z_v \ge 0$, as desired. The case for $z_0 < 0$ follows similarly. □

Lemma 3.5.4. *Let $\eta \in \mathbb{R}$ and let $\tilde{\Omega} \subset \Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}$ be a convex set. Suppose that a continuous function $U = U(x,z) = U(x,-z)$ such that $U_z \in C([0,\infty); C(\Omega))$ is such that*

$$-\partial_{z+}U \ge \eta \quad \text{on } \tilde{\Omega} \cap \{z=0\}.$$

If $\eta > 0$, then U cannot be touched from below in $\tilde{\Omega}$ by the convex envelope or by any paraboloid on the set $\tilde{\Omega} \times \{z=0\}$. If $\eta \le 0$ and P is a paraboloid of opening $a > 0$ with vertex (x_v, z_v) that touches U from below in $\tilde{\Omega}$ at $(x_0, 0)$, then $|h'(z_v)| \le |\eta|/a$. Consequently, if $\eta = 0$, then $z_v = 0$.

Proof. Let $\eta > 0$. Suppose, by way of contradiction, that the convex envelope Γ touches U from below in $\tilde{\Omega}$ at a point $(x_0, 0) \in \tilde{\Omega} \cap \{z=0\}$. Let $\varepsilon > 0$. By convexity of Γ and symmetry of Γ across $\{z=0\}$, we have

$$\frac{\Gamma(x_0,\varepsilon)+\Gamma(x_0,\varepsilon)}{2} = \frac{\Gamma(x_0,-\varepsilon)+\Gamma(x_0,\varepsilon)}{2} \ge \Gamma(x_0,0)$$

which implies

$$0 \le \frac{\Gamma(x_0,\varepsilon)-\Gamma(x_0,0)}{\varepsilon} \le \frac{U(x_0,\varepsilon)-U(x_0,0)}{\varepsilon}.$$

As this holds for all $\varepsilon > 0$,

$$0 \le \partial_{z+}U(x_0,0) \le -\eta < 0$$

a contradiction. Thus, Γ cannot touch U from below in $\tilde{\Omega}$ at $(x_0,0)$.

Let $\eta \in \mathbb{R}$ and suppose that P is a paraboloid of opening $a > 0$ that touches U from below in $\tilde{\Omega}$ at $(x_0,0)$. Write

$$P(x,z) = -a\delta_\Phi((x_v,z_v),(x,z)) + a\delta_\Phi((x_v,z_v),(x_0,0)) + U(x_0,0).$$

Let $\varepsilon > 0$. Since $U - P$ attains a minimum of 0 at $(x_0,0)$, we know that

$$\frac{(U(x_0,\varepsilon)-P(x_0,\varepsilon))-(U(x_0,0)-P(x_0,0))}{\varepsilon} \ge 0.$$

Therefore, taking the limit as $\varepsilon \to 0^+$, we obtain

$$0 \le \partial_{z+}U(x_0,0) - \partial_z P(x_0,0) \le \eta - ah'(z_v). \tag{3.5.1}$$

Similarly, for $\varepsilon > 0$,

$$\frac{(U(x_0,-\varepsilon)-P(x_0,-\varepsilon))-(U(x_0,0)-P(x_0,0))}{-\varepsilon} = \frac{-U(x_0,-\varepsilon)+P(x_0,-\varepsilon)}{\varepsilon} \le 0.$$

Taking the limit as $\varepsilon \to 0^+$ and using the symmetry of the Neumann condition, we have that

$$0 \ge \partial_{z-}U(x_0,0) - \partial_z P(x_0,0) \ge -\eta - ah'(z_v). \tag{3.5.2}$$

Combining (3.5.1) and (3.5.2), we estimate

$$\eta \le -ah'(z_v) \le -\eta.$$

If $\eta > 0$, then this is a contradiction, so we cannot touch U from below in $\tilde{\Omega}$ by P at $(x_0,0)$. If $\eta \le 0$, then

$$-\frac{|\eta|}{a} \le h'(z_v) \le \frac{|\eta|}{a}$$

as desired. If $\eta = 0$, then $h'(z_v) = 0$ which implies that $z_v = 0$. □

Lemma 3.5.5. *Let $\tilde{\Omega} \subset \mathbb{R}^{n+1}$ be a convex set that is symmetric with respect to $\{z = 0\}$. Consider a continuous function $U = U(x, z) = U(x, -z)$. If U can be touched from below in $\tilde{\Omega}$ by a paraboloid P of opening $a > 0$ with vertex (x_v, z_v) at (x_0, z_0), then U can be touched from below in $\tilde{\Omega}$ by a paraboloid of opening a with vertex $(x_v, -z_v)$ at $(x_0, -z_0)$.*

Proof. Let $\tilde{P}(x, z) = P(x, -z)$. Then

$$\tilde{P}(x, z) = P(x, -z) \leq U(x, -z) = U(x, z) \quad \text{for all } (x, z) \in \tilde{\Omega}$$

and

$$\tilde{P}(x_0, -z_0) = P(x_0, z_0) = U(x_0, z_0) = U(x_0, -z_0).$$

Thus, $\tilde{P}$ is a paraboloid of opening $a > 0$ that touches U from below in $\tilde{\Omega}$ at $(x_0, -z_0)$.

Lastly, since

$$\partial_z \tilde{P}(x, z) = -\partial_z P(x, -z) = 0 \quad \text{when } z = -z_v,$$

the vertex of $\tilde{P}$ is $(x_v, -z_v)$. □

3.5.1.2 Lemmas for Harnack inequality

The proof of Theorem 3.1.2 relies on the following three key lemmas.

The first lemma is a measure estimate similar to the usual ABP estimate. We slide paraboloids of fixed opening $a > 0$ from below until they touch the graph of U for the first time. The set of contact points make up a universal proportion of the set of vertices.

Lemma 3.5.6. *Let $0 < s < 1$. Assume that $a^{ij} = a^{ij}(x)$ are bounded, measurable functions on $\mathbb{R}^n$ that satisfy (3.1.1). For a cube $Q_R = Q_R(\tilde{x}, \tilde{z}) \subset \mathbb{R}^{n+1}$, suppose that $U = U(x, z) = U(x, -z)$ is a classical supersolution to*

$$\begin{cases} a^{ij}(x)\partial_{ij}U + |z|^{2-\frac{1}{s}}\,\partial_{zz}U \leq 0 & \text{in } Q_R \cap \{z \neq 0\} \\ -\partial_{z+}U \geq 0 & \text{on } Q_R \cap \{z = 0\}. \end{cases}$$

Fix $a > 0$. For each $(x_v, z_v) \in B \subset \overline{Q}_R$, B a closed set, we slide paraboloids of opening a and vertex (x_v, z_v) from below until they touch the graph of U for the first time in a set $A \subset Q_R$. Then A is *compact and there is a positive constant $c = c(n, \lambda, \Lambda) < 1$ such that*

$$\mu_\Phi(A) \geq c\mu_\Phi(B).$$

Remark 3.5.1. *Lemma 3.5.6 also holds for subsolutions when we slide paraboloids of opening $-a > 0$ with vertices $(x_v, z_v) \in B \subset \overline{Q}_R$ from above until they touch the graph of U for the first time in a set $A \subset Q_R$.*

Before stating the second lemma, we need to define the following constants and sets.

First, we will define a constant $\hat{K}_2$ to be large enough so that if $Q_r(x_0, z_0) \subset Q_R(\tilde{x}, \tilde{z})$, then $Q_{2(n+1)r}(x_0, z_0) \subset Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z})$. For simplicity, consider when $Q_R(\tilde{x}, \tilde{z}) \subset \mathbb{R}^2$. The case in $\mathbb{R}^{n+1}$ follows similarly. Let $(x_1, z_1), (x_2, z_2) \in Q_r(x_0, z_0) \subset Q_R(\tilde{x}, \tilde{z})$. Then, by the quasi-triangle inequality,

$$\delta_\varphi(x_1, x_2) \leq K\left(\delta_\varphi(\tilde{x}, x_1) + \delta_\varphi(\tilde{x}, x_2)\right) < 2KR$$
$$\delta_h(z_1, z_2) \leq K\left(\delta_h(\tilde{z}, z_1) + \delta_h(\tilde{z}, z_2)\right) < 2KR.$$

Therefore, $r < 2KR$. If $(x, z) \in Q_{2r}(x_0, z_0)$, then, by the quasi-triangle inequality,

$$\delta_\varphi(\tilde{x}, x) \leq K\left(\delta_\varphi(\tilde{x}, x_0) + \delta_\varphi(x_0, x)\right) < K\left(R + 2(n+1)r\right) < K(1 + 4K(n+1))R$$
$$\delta_h(\tilde{z}, z) \leq K\left(\delta_h(\tilde{z}, z_0) + \delta_h(z_0, z)\right) < K(1 + 4K(n+1))R.$$

We will take $\hat{K}_2 = \hat{K}_2(n, s)$ to be

$$\hat{K}_2 = K(1 + 4K(n+1)). \tag{3.5.3}$$

Let $\hat{K}_3 = \hat{K}_3(n, s)$ be given by

$$\hat{K}_3 = 4K^2\hat{K}_2. \tag{3.5.4}$$

If $Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) \cap \{z = 0\} \neq \emptyset$, then by Lemma 3.3.7, we observe that

$$Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) = Q_{\hat{K}_2 R}(\tilde{x}) \times S_{\hat{K}_2 R}(\tilde{z}) \subset Q_{2K\hat{K}_2 R}(\tilde{x}) \times S_{2K\hat{K}_2 R}(0) = Q_{2K\hat{K}_2 R}(\tilde{x}, 0)$$

and similarly

$$Q_{2K\hat{K}_2 R}(\tilde{x}, 0) = Q_{2K\hat{K}_2 R}(\tilde{x}) \times S_{2K\hat{K}_2 R}(0) \subset Q_{4K^2\hat{K}_2 R}(\tilde{x}) \times S_{4K^2\hat{K}_2 R}(\tilde{z}) = Q_{\hat{K}_3 R}(\tilde{x}, \tilde{z}).$$

We define a vertex set $B_v \subset \overline{Q}_{\hat{K}_3 R}(\tilde{x}, \tilde{z})$ by

$$B_v = \begin{cases} \overline{Q}_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) & \text{if } \tilde{z} = 0 \text{ or if } \overline{Q}_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) \cap \{z = 0\} = \emptyset \\ \overline{Q}_{2K\hat{K}_2 R}(\tilde{x}, 0) & \text{if } \tilde{z} \neq 0 \text{ and } \overline{Q}_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) \cap \{z = 0\} \neq \emptyset. \end{cases}$$

so that B_v is centered with respect to the set $\{z = 0\}$ if $\overline{Q}_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) \cap \{z = 0\} \neq \emptyset$.

Define a contact set $A_{a,R}$ for U on $Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z})$ by

$$\begin{aligned} A_{a,R} := \Big\{ (x, z) \in Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) : U(x, z) \leq aR \text{ and there is } (x_v, z_v) \in B_v \text{ such that} \\ U \text{ can be touched from below at } (x, z) \text{ in } Q_{\hat{K}_3}(\tilde{x}, \tilde{z}) \\ \text{by a paraboloid of opening } a > 0 \text{ with vertex } (x_v, z_v) \Big\}. \end{aligned}$$

Define positive constants $K_0 > 1$ and $\eta < 1$ by

$$K_0 = 2K^2 + 2K \quad \text{and} \quad \eta = \frac{1}{K^2(2KK_0 + 1)} \tag{3.5.5}$$

where K is the quasi-distance constant in (3.3.3).

The second lemma uses a delicate barrier to localize the solution to a smaller section. We prove that if U can be touched from below in a cube Q_r by a paraboloid of opening $a > 0$, then in a smaller cube $Q_{\eta r}$, U can be touched from below by paraboloids of narrower opening $Ca > 0$.

Lemma 3.5.7. *Let $0 < s < 1$. Assume that $a^{ij} = a^{ij}(x)$ are bounded, measurable functions on $\mathbb{R}^n$ that satisfy (3.1.1). For a cube $Q_R = Q_R(\tilde{x}, \tilde{z}) \subset \mathbb{R}^{n+1}$, suppose that $U = U(x, z) = U(x, -z)$ is a nonnegative, classical supersolution to*

$$\begin{cases} a^{ij}(x)\partial_{ij} U + |z|^{2-\frac{1}{s}} \partial_{zz} U \leq 0 & \text{in } Q_{\hat{K}_3 R} \cap \{z \neq 0\} \\ -\partial_{z+} U \geq 0 & \text{on } Q_{\hat{K}_3 R} \cap \{z = 0\}. \end{cases}$$

where $\hat{K}_3$ is as in (3.5.4). Fix $a > 0$. Let $Q_r(x_0, z_0)$ be such that

$$\overline{Q}_r(x_0, z_0) \subset Q_R \quad \text{and} \quad \overline{Q}_r(x_0, z_0) \cap A_{a,R} \neq \emptyset.$$

There exists positive constants $C = C(n, \lambda, \Lambda, s) > 1$ and $c = c(n, \lambda, \Lambda, s) < 1$ such that

$$\mu_\Phi(A_{Ca,R} \cap Q_{\eta r}(x_0, z_0)) \geq c\mu_\Phi(Q_r(x_0, z_0)).$$

where $\eta = \eta(n, s) < 1$ is as in (3.5.5).

The third lemma is a Calderón-Zygmund-type covering lemma that allows us to use Lemma 3.5.7 across the whole domain.

Lemma 3.5.8. *Let $K_0 = K_0(n,s) > 1$ and $\eta = \eta(n,s) < 1$ be as in (3.5.5). Suppose the closed sets $D_k \subset \mathbb{R}^{n+1}$ satisfy the following properties:*

1) $D_0 \subset D_1 \subset \cdots \subset D_k \subset \overline{Q}_{R/K_0}(\tilde{x},\tilde{z})$, $D_0 \neq \emptyset$

2) for any (x,z), ρ such that

$$Q_\rho(x,z) \subset Q_R(\tilde{x},\tilde{z}), \quad Q_{\eta\rho}(x,z) \subset Q_{R/K_0}(\tilde{x},\tilde{z})$$
$$\overline{Q}_\rho(x,z) \cap D_k \neq \emptyset,$$

we have

$$\mu_\Phi(Q_{\eta\rho}(x,z) \cap D_{k+1}) \geq c\mu_\Phi(Q_\rho(x,z)).$$

Then

$$\mu_\Phi(Q_{R/K_0} \setminus D_k) \leq (1-c)^k \mu_\Phi(Q_{R/K_0}).$$

3.5.2 Proof of Lemma 3.5.6

Proof of Lemma 3.5.6. We first show that A is closed. Let $(x_k,z_k) \in A$ be such that $(x_k,z_k) \to (x_0,z_0)$ as $k \to \infty$. There exist corresponding polynomials P_k of opening a with vertices $(x_v^k, z_v^k) \in B$ such that P_k touches U from below in Q_R at (x_k,z_k). Since $B \subset \overline{Q}_R$ is closed, B is compact. Thus, there is a subsequence, which we will still notate by k, and a point $(x_v^0, z_v^0) \in B$ such that $(x_v^k, z_v^k) \to (x_v^0, z_v^0)$. By the continuity of δ_Φ and U, we have that

$$\begin{aligned} P_k(x,z) &= -a\delta_\Phi((x_v^k,z_v^k),(x,z)) + a\delta_\Phi((x_v^k,z_v^k),(x_k,z_k)) + U(x_k,z_k) \\ &\to -a\delta_\Phi((x_v^0,z_v^0),(x,z)) + a\delta_\Phi((x_v^0,z_v^0),(x_0,z_0)) + U(x_0,z_0) \end{aligned}$$

Define the polynomial P of opening a by

$$P(x,z) = -a\delta_\Phi((x_v^0,z_v^0),(x,z)) + a\delta_\Phi((x_v^0,z_v^0),(x_0,z_0)) + U(x_0,z_0).$$

Note that $P(x_0, z_0) = U(x_0, z_0)$. Since $P_k \leq U$ in Q_R and $P_k(x, z) \to P(x, z)$ for each $(x, z) \in Q_R$, it must be that $P \leq U$ in Q_R. Therefore, P is a paraboloid of opening a with vertex $(x_v^0, z_v^0) \in B$ that touches U from below in Q_R at (x_0, z_0). This shows that $(x_0, z_0) \in A$, so that A is closed and, moreover, compact.

Let $(x_0, z_0) \in A$. There exists a paraboloid P of opening a and vertex $(x_v, z_v) \in B$ that touches U from below in Q_R at (x_0, z_0) which may be written as

$$P(x, z) = -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_0, z_0)) + U(x_0, z_0).$$

Let $B_0 \subset B$ be the set of vertices such that the corresponding touching paraboloids touch on the set $\{z \neq 0\}$. Let $A_0 = A \setminus \{z = 0\}$ be the contact points of these paraboloids. Since the set $A \cap \{z = 0\}$ is a set of measure zero, we know that $\mu_\Phi(A_0) = \mu_\Phi(A)$. Suppose that $(x_0, 0) \in A \cap \{z = 0\}$. By Lemma 3.5.4, we know that $z_v = 0$. Since $B \cap \{z = 0\}$ has measure zero, we also have that $\mu_\Phi(B_0) = \mu_\Phi(B)$.

Let $(x_0, z_0) \in A_0$. Since $U - P \geq 0$ in Q_R and $U - P = 0$ at (x_0, z_0), we know that $U - P$ attains a minimum at (x_0, z_0). Hence

$$DU(x_0, z_0) = DP(x_0, z_0) = -a\left(D\Phi(x_0, z_0) - D\Phi(x_v, z_v)\right)$$

which implies

$$D\Phi(x_v, z_v) = D\Phi(x_0, z_0) + \frac{1}{a}DU(x_0, z_0) = D\left(\Phi + \frac{1}{a}U\right)(x_0, z_0).$$

This is how the vertices $(x_v, z_v) \in B_0$ are uniquely determined by the contact points $(x_0, z_0) \in A_0$. Define the map $T : A_0 \to T(A_0) = D\Phi(B_0)$ by

$$T(x_0, z_0) = D\left(\Phi + \frac{1}{a}U\right)(x_0, z_0).$$

Fix $\varepsilon > 0$ and define a compact set $A_\varepsilon \subset A_0$ by

$$A_\varepsilon = A_0 \setminus \{(x, z) : |z| < \varepsilon\}.$$

Since A_ε is a positive distance from the set $\{z = 0\}$, we have that $|D^2\Phi(x_0, z_0)|$ is uniformly bounded on A_ε. Since $|D^2V(x_0, z_0)|$ is also uniformly bounded on A_ε, we have that $T : A_\varepsilon \to T(A_\varepsilon)$ Lipschitz and injective on A_ε. By the area formula for Lipschitz maps,

$$\begin{aligned}\int_{T(A_\varepsilon)} dy\,dw &= \int_{A_\varepsilon} |\det(DT(x,z))|\,dx\,dz \\ &= \int_{A_\varepsilon} \left|\det\left(D^2\left(\Phi + \frac{1}{a}U\right)(x,z)\right)\right| dx\,dz.\end{aligned}$$

Let $(x_0, z_0) \in A_0$. We claim that there is a positive constant $C = C(n, \lambda, \Lambda)$ such that

$$-aD^2\Phi(x_0,z_0) \leq D^2U(x_0,z_0) \leq CaD^2\Phi(x_0,z_0), \quad D^2\Phi(x_0,z_0) = \begin{pmatrix} I & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix}. \tag{3.5.6}$$

The first inequality is straightforward. Since P touches U from below in Q_R at (x_0, z_0), we know that

$$D^2U(x_0,z_0) \geq D^2P(x_0,z_0) = -aD^2\Phi(x_0,z_0).$$

Suppose, by way of contradiction, that $D^2U(x_0,z_0) > CaD^2\Phi(x_0,z_0)$ for all $C > 0$. Consequently,

$$D^2U(x_0,z_0) > Ca\begin{pmatrix} e_k \otimes e_k & 0 \\ 0 & 0 \end{pmatrix} - a\begin{pmatrix} I & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix}$$

where $e_k,\ k = 1, \ldots, n$ are the standard basis vectors in $\mathbb{R}^n$ and $e_k \otimes e_k = e_k e_k^T$. Since

$$\tilde{A} = \begin{pmatrix} A(x_0) & 0 \\ 0 & 0 \end{pmatrix} \geq 0 \quad \text{and} \quad D^2U(x_0,z_0) - Ca\begin{pmatrix} e_k \otimes e_k & 0 \\ 0 & 0 \end{pmatrix} + a\begin{pmatrix} I & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix} \geq 0,$$

it must be that

$$\tilde{A}\left(D^2U(x_0,z_0) - Ca\begin{pmatrix} e_k \otimes e_k & 0 \\ 0 & 0 \end{pmatrix} + a\begin{pmatrix} I & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix}\right) \geq 0.$$

In particular,

$$\operatorname{trace}\left(\tilde{A}D^2U(x_0,z_0) - Ca\tilde{A}\begin{pmatrix} e_k \otimes e_k & 0 \\ 0 & 0 \end{pmatrix} + a\tilde{A}\begin{pmatrix} I & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix}\right) \geq 0.$$

Rewriting this expression and using ellipticity, we obtain

$$a^{ij}(x_0)\partial_{ij}U(x_0,z_0) \geq (Ca)a^{kk}(x_0) - (a)a^{ii}(x_0) \geq Ca\lambda - an\Lambda. \tag{3.5.7}$$

Next, note that

$$D^2U(x_0,z_0) \geq Ca \begin{pmatrix} 0 & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix} - a \begin{pmatrix} I & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix}.$$

From the definition of positive definite matrices,

$$\partial_{zz}U(x_0,z_0) \geq Ca\,|z_0|^{\frac{1}{s}-2} - a\,|z_0|^{\frac{1}{s}-2}.$$

Therefore,

$$|z_0|^{2-\frac{1}{s}}\,\partial_{zz}U(x_0,z_0) > Ca - a. \tag{3.5.8}$$

Combining (3.5.7) and (3.5.8), we have

$$\begin{aligned} 0 \geq a^{ij}(x_0)\partial_{ij}U(x_0,z_0) + |z_0|^{2-\frac{1}{s}}\,\partial_{zz}U(x_0,z_0) &> Ca\lambda - an\Lambda + Ca - a \\ &= a[C(\lambda+1) - (n\Lambda+1)] > 0 \end{aligned}$$

for $C = C(n,\lambda,\Lambda)$ large. This is a contradiction, so the claim holds.

From (3.5.6), we estimate

$$0 \leq D^2\left(\Phi + \frac{1}{a}U\right)(x,z) = D^2\Phi(x_0,z_0) + \frac{1}{a}D^2U(x_0,z_0) \leq (C+1)D^2\Phi(x_0,z_0)$$

for all $(x_0,z_0) \in A_0$. Therefore,

$$\begin{aligned} \int_{T(A_\varepsilon)} dy\,dw &\leq \int_{A_\varepsilon} \det\big((C+1)D^2\Phi(x,z)\big)\;dx\,dz \\ &= (C+1)^{n+1}\int_{A_\varepsilon} \det\big(D^2\Phi(x,z)\big)\;dx\,dz \\ &= (C+1)^{n+1}\mu_\Phi(A_\varepsilon) \\ &\leq (C+1)^{n+1}\mu_\Phi(A_0) = (C+1)^{n+1}\mu_\Phi(A). \end{aligned}$$

Taking the limit as $\varepsilon \to 0^+$,

$$\mu_\Phi(B) = \mu_\Phi(B_0) = \int_{T(A_0)} dy\,dw \leq (C+1)^{n+1}\mu_\Phi(A),$$

which completes the proof of the Lemma with $c = (C+1)^{-n-1}$. □

3.5.3 Proof of Lemma 3.5.7

Lemma 3.5.9. *$A_{a,R}$ is closed in $Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z})$.*

Proof. Let $(x_k, z_k) \in A_{a,R}$ be such that $(x_k, z_k) \to (x_0, z_0)$. Since $U(x_k, z_k) \leq aR$ and U is continuous, it follows that $U(x_0, z_0) \leq aR$. By the proof of Lemma 3.5.6 with $B = B_v$, we can touch U from below in $Q_{\hat{K}_3 R}$ at (x_0, z_0) by a paraboloid P of opening a with vertex $(x_v^0, z_v^0) \in B_v$. Therefore, $(x_0, z_0) \in A_{a,R}$ which shows that $A_{a,R}$ is closed in $Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z})$. □

We will need the following three lemmas to prove lemma 3.5.7.

3.5.3.1 The function $\mathcal{Q}$

Lemma 3.5.10. *Let $0 < s < 1$ and $z_0 \in \mathbb{R}$. Define the function $\mathcal{Q}(z, z_0) = \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by*

$$\mathcal{Q}(z, z_0) = \frac{(h'(z) - h'(z_0))^2}{\delta_h(z_0, z) h''(z)}.$$

If $z_0 = 0$, then $\mathcal{Q}(z, 0) = 1/(1 - s)$.

Suppose $z_0 > 0$ is fixed.

- *If $0 < s < 1/2$, then $\mathcal{Q}(z, z_0)$ is a decreasing in $\{z > 0\}$ and $\mathcal{Q} \geq 1$ for $z > 0$.*
- *If $s = 1/2$, then $\mathcal{Q}(z, z_0) = 2$.*
- *If $1/2 < s < 1$, then $\mathcal{Q}(0, z_0) = 0$ and $\mathcal{Q}(z, z_0)$ is increasing in $\{z > 0\}$.*

Suppose $z_0 < 0$ is fixed.

- *If $0 < s < 1/2$, then $\mathcal{Q}(z, z_0)$ is increasing in $\{z < 0\}$ and $\mathcal{Q} \geq 1$ for $z < 0$.*
- *If $s = 1/2$, then $\mathcal{Q}(z, z_0) = 2$.*
- *If $1/2 < s < 1$, then $\mathcal{Q}(0, z_0) = 0$ and $\mathcal{Q}(z, z_0)$ is decreasing in $\{z < 0\}$.*

Proof. If $z_0 = 0$ and $0 < s < 1$, then

$$\mathcal{Q}(z) = \frac{(h'(z))^2}{h(z)h''(z)} = \frac{\frac{s^2}{(1-s)^2} z^{\frac{2}{s}-2}}{\frac{s^2}{1-s} z^{\frac{1}{s}} z^{\frac{1}{s}-2}} = \frac{1}{1-s}.$$

If $s = 1/2$ and $z_0 \in \mathbb{R}$, then

$$\mathcal{Q}(z) = \frac{(z-z_0)^2}{\frac{1}{2}\left|z-z_0\right|^2 \cdot 1} = 2 \geq 1.$$

Let $z_0 \in \mathbb{R}$ and $0 < s < 1$. Since

$$h'(-z) = \frac{s}{1-s}\left|-z\right|^{\frac{1}{s}-2}(-z) = -\frac{s}{1-s}\left|z\right|^{\frac{1}{s}-2}z = -h'(z)$$

and

$$\begin{aligned}
\delta_h(-z_0,-z) &= h(-z) - h(-z_0) - h'(-z_0)(-z-(-z_0)) \\
&= h(z) - h(z_0) + h'(z_0)(z-z_0) \\
&= \delta_h(z_0,z),
\end{aligned}$$

we observe that

$$\begin{aligned}
\mathcal{Q}(-z,-z_0) &= \frac{(h'(-z)-h'(-z_0))^2}{\delta_h(-z_0,-z)h''(-z)} \\
&= \frac{(-h'(z)+h'(z_0))^2}{\delta_h(z_0,z)h''(z)} \\
&= \frac{(h'(z)-h'(z_0))^2}{\delta_h(z_0,z)h''(z)} = \mathcal{Q}(z,z_0).
\end{aligned}$$

Assume for any $\tilde{z}_0 > 0$ fixed that $\mathcal{Q}(z,\tilde{z}_0)$ is decreasing in $\{z > 0\}$. If $z_0 < 0$ is fixed and $z_2 < z_1 < 0$. Then, for $\tilde{z}_0 = -z_0$, we have that

$$\mathcal{Q}(z_1,z_0) = \mathcal{Q}(-z_1,-z_0) > \mathcal{Q}(-z_2,-z_0) = Q(z_2,z_0).$$

Therefore, $\mathcal{Q}(z,z_0)$ is increasing in $\{z < 0\}$. Similarly, if for any $\tilde{z}_0 > 0$ fixed, we have that $\mathcal{Q}(z,\tilde{z}_0)$ is increasing in $\{z > 0\}$, then, for $z_0 < 0$ fixed, we have that $\mathcal{Q}(z,z_0)$ is decreasing in $\{z < 0\}$. Therefore, it is enough to prove the lemma for $z_0, z > 0$.

Fix $z_0 > 0$. For $0 < s < 1$, observe that

$$\lim_{z\to 0^+} \mathcal{Q}(z,z_0) = \frac{(h(0)-h'(z_0))^2}{h(0)-h(z_0)-h'(z_0)(0-z_0)} \lim_{z\to 0^+} z^{2-\frac{1}{s}}$$
$$= \frac{s}{(1-s)^2} z_0^{\frac{1}{s}-2} \lim_{z\to 0^+} z^{2-\frac{1}{s}}$$
$$= \begin{cases} 2 & \text{if } s = 1/2 \\ \infty & \text{if } s < 1/2 \\ 0 & \text{if } s > 1/2. \end{cases}$$

and, by L'Höpital's rule,

$$\lim_{z\to z_0} \mathcal{Q}(z,z_0) = \frac{1}{h''(z_0)} \lim_{z\to z_0} \frac{(h'(z)-h'(z_0))^2}{h(z)-h(z_0)-h'(z_0)(z-z_0)}$$
$$= \frac{1}{h''(z_0)} \lim_{z\to z_0} \frac{2(h'(z)-h'(z_0))h''(z)}{h'(z)-h'(z_0)}$$
$$= \frac{1}{h''(z_0)} \lim_{z\to z_0} 2h''(z) = 2.$$

Let $0 < s < 1$ be such that $s \neq 1/2$. To study when $\mathcal{Q}(\cdot, z_0)$ is increasing/decreasing, we first compute

$$\partial_z \mathcal{Q}(z,z_0) = \frac{(h'(z)-h'(z_0))h''(z)}{(\delta_h(z_0,z)h''(z))^2}$$
$$\left(2\delta_h(z_0,z)h''(z) - (h'(z)-h'(z_0))^2 - \delta_h(z_0,z)(h'(z)-h'(z_0))\frac{h'''(z)}{h''(z)}\right).$$

Assume for now that $0 < z < z_0$, so $(h'(z)-h'(z_0)) < 0$. We will study the term

$$I + II + III := 2\delta_h(z_0,z)h''(z) - (h'(z)-h'(z_0))^2 - \delta_h(z_0,z)(h'(z)-h'(z_0))\frac{h'''(z)}{h''(z)}.$$

We will write this out explicitly in terms of z, z_0, and s as

$$I = 2\delta_h(z_0,z)h''(z)$$
$$= 2\left(\frac{s^2}{1-s} z^{\frac{2}{s}-2} - \frac{s^2}{1-s} z_0^{\frac{1}{s}} z^{\frac{1}{s}-2} - \frac{s}{1-s} z_0^{\frac{1}{s}-1} z^{\frac{1}{s}-1} + \frac{s}{1-s} z_0^{\frac{1}{s}} z^{\frac{1}{s}-2}\right)$$

$$II = -(h'(z)-h'(z_0))^2$$
$$= -\frac{s^2}{(1-s)^2} z^{\frac{2}{s}-2} + \frac{s^2}{(1-s)^2} z^{\frac{1}{s}-1} z_0^{\frac{1}{s}-1} - \frac{s^2}{(1-s)^2} z_0^{\frac{2}{s}-2}$$

$$
\begin{aligned}
III &= -\delta_h(z_0,z)(h'(z)-h'(z_0))\frac{h'''(z)}{h''(z)} \\
&= -\frac{s^3}{(1-s)^2}\left(\frac{1}{s}-2\right)z^{\frac{2}{s}-2} + \frac{s^3}{(1-s)^2}\left(\frac{1}{s}-2\right)z_0^{\frac{1}{s}-1}z^{\frac{1}{s}-1} + \frac{s^3}{(1-s)^2}\left(\frac{1}{s}-2\right)z_0^{\frac{1}{s}}z^{\frac{1}{s}-2} \\
&\quad - \frac{s^3}{(1-s)^2}\left(\frac{1}{s}-2\right)z_0^{\frac{2}{s}-1}z^{-1} + \frac{s^2}{(1-s)^2}\left(\frac{1}{s}-2\right)z_0^{\frac{1}{s}-1}z^{\frac{1}{s}-1} - \frac{s^2}{(1-s)^2}\left(\frac{1}{s}-2\right)z_0^{\frac{2}{s}-2}\frac{s}{1-s} \\
&\quad - \frac{s^2}{(1-s)^2}\left(\frac{1}{s}-2\right)z_0^{\frac{1}{s}}z^{\frac{1}{s}-2} + \frac{s^2}{(1-s)^2}\left(\frac{1}{s}-2\right)z_0^{\frac{2}{s}-1}z^{-1}.
\end{aligned}
$$

We add these together and combine like terms to obtain

$$
I+II+III = -\frac{s}{1-s}z_0^{\frac{2}{s}-2} + \frac{s}{1-s}z_0^{\frac{1}{s}}z^{\frac{1}{s}-2} - \frac{s(1-2s)}{1-s}z_0^{\frac{1}{s}-1}z^{\frac{1}{s}-1} + \frac{s(1-2s)}{1-s}z_0^{\frac{2}{s}-1}z^{-1}.
$$

Therefore, $I+II+III>0$ if and only if

$$
-\frac{s}{1-s}z_0^{\frac{2}{s}-2} + \frac{s}{1-s}z_0^{\frac{1}{s}}z^{\frac{1}{s}-2} - \frac{s(1-2s)}{1-s}z_0^{\frac{1}{s}-1}z^{\frac{1}{s}-1} + \frac{s(1-2s)}{1-s}z_0^{\frac{2}{s}-1}z^{-1} > 0.
$$

Multiplying both sides by $z_0 z(1-s)/s>0$, this is equivalent to

$$
\psi(z) := -z_0^{\frac{2}{s}-1}z + z_0^{\frac{1}{s}+1}z^{\frac{1}{s}-1} - (1-2s)z_0^{\frac{1}{s}}z^{\frac{1}{s}} + (1-2s)z_0^{\frac{2}{s}} > 0.
$$

Observe that $\psi(z_0)=0$ and

$$
\psi(0) = (1-2s)z_0^{\frac{2}{s}} \begin{cases} >0 & \text{if } 0<s<1/2 \\ <0 & \text{if } 1/2<s<1. \end{cases}
$$

Suppose that $0<s<1/2$. We will show that ψ is decreasing for all $z>0$. First, compute

$$
\psi'(z) = -z_0^{\frac{2}{s}-1} + \left(\frac{1}{s}-1\right)z_0^{\frac{1}{s}+1}z^{\frac{1}{s}-2} - \frac{(1-2s)}{s}z_0^{\frac{1}{s}}z^{\frac{1}{s}-1}.
$$

Therefore, $\psi'(z)<0$ if and only if

$$
-z_0^{\frac{2}{s}-1} + \left(\frac{1}{s}-1\right)z_0^{\frac{1}{s}+1}z^{\frac{1}{s}-2} - \frac{(1-2s)}{s}z_0^{\frac{1}{s}}z^{\frac{1}{s}-1} < 0.
$$

Multiplying both sides by $z_0^{-1/s}(1-s)/s>0$ and rearranging, this is equivalent to

$$
z_0^1 z^{\frac{1}{s}-2} < \left(\frac{s}{1-s}\right)z_0^{\frac{1}{s}-1} + \left(\frac{1-2s}{1-s}\right)z^{\frac{1}{s}-1}
$$

which is true by Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{where} \quad p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad a, b \geq 0.$$

Indeed, take

$$a = z_0, \quad b = z^{\frac{1}{s}-2}, \quad p = \frac{1}{s} - 1 = \frac{1-s}{s} > 1, \quad q = \frac{1-s}{1-2s} > 1.$$

Since ψ is decreasing for $z > 0$, $\psi(0) > 0$, and $\psi(z_0) = 0$, we conclude that

$$\psi > 0 \quad \text{for } 0 < z < z_0 \quad \text{and} \quad \psi < 0 \quad \text{for } z > z_0.$$

Equivalently,

$$I + II + III > 0 \quad \text{for } 0 < z < z_0 \quad \text{and} \quad I + II + III < 0 \quad \text{for } z > z_0.$$

Since

$$\partial_z \mathcal{Q}(z, z_0) = \frac{(h'(z) - h'(z_0))h''(z)}{(\delta_h(z_0, z)h''(z))^2} (I + II + III)$$

and

$$h'(z) - h'(z_0) < 0 \quad \text{for } 0 < z < z_0 \quad \text{and} \quad h'(z) - h'(z_0) > 0 \quad \text{for } z > z_0,$$

we conclude that $\partial_z \mathcal{Q}(z, z_0) < 0$ for all $z > 0$. Thus, when $0 < s < 1/2$, we have that $\mathcal{Q}(z, z_0)$ is decreasing for $z > 0$.

Next, assume that $1/2 < s < 1$. We will show that ψ is increasing for all $z > 0$. By similar considerations as above, $\psi'(z) > 0$ if and only if

$$z^{1-\frac{1}{s}} z_0^{\frac{1}{s}-2} \leq \left(\frac{1-s}{s}\right) z^{-1} + \left(\frac{2s-1}{s}\right) z_0^{-1}$$

which is true by Young's inequality with

$$a = z^{1-\frac{1}{s}}, \quad b = z_0^{\frac{1}{s}-2}, \quad p = \frac{s}{1-s} > 1, \quad q = \frac{s}{2s-1} > 1.$$

Since ψ is increasing for $z > 0$, $\psi(0) < 0$, and $\psi(z_0) = 0$, we conclude that

$$\psi < 0 \quad \text{for } 0 < z < z_0 \quad \text{and} \quad \psi > 0 \quad \text{for } z > z_0.$$

Equivalently,

$$I + II + III < 0 \quad \text{for } 0 < z < z_0 \quad \text{and} \quad I + II + III > 0 \quad \text{for } z > z_0.$$

Since

$$\partial_z \mathcal{Q}(z, z_0) = \frac{(h'(z) - h'(z_0))h''(z)}{(\delta_h(z_0, z)h''(z))^2} (I + II + III)$$

and

$$h'(z) - h'(z_0) < 0 \quad \text{for } 0 < z < z_0 \quad \text{and} \quad h'(z) - h'(z_0) > 0 \quad \text{for } z > z_0,$$

we conclude that $\partial_z \mathcal{Q}(z, z_0) > 0$ for all $z > 0$. Thus, when $1/2 < s < 1$, we conclude that $\mathcal{Q}(z, z_0)$ is increasing for $z > 0$.

Lastly, for any $0 < s < 1$, we observe that

$$\begin{aligned}
\lim_{z\to\infty} \mathcal{Q}(z, z_0) &= \lim_{z\to\infty} \frac{(h'(z) - h'(z_0))^2}{(h(z) - h(z_0) - h'(z_0)(z - z_0))h''(z)} \\
&= \lim_{z\to\infty} \frac{(h'(z) - h'(z_0))^2}{(h(z) - h(z_0) - h'(z_0)(z - z_0))h''(z)} \frac{z^{2-\frac{2}{s}}}{z^{2-\frac{2}{s}}} \\
&= \lim_{z\to\infty} \frac{\left(\frac{s}{1-s} - h'(z_0)z^{1-\frac{1}{s}}\right)^2}{\frac{s^2}{1-s} - h(z_0)z^{-\frac{1}{s}} - h'(z_0)\left(z^{1-\frac{1}{s}} - z_0 z^{-\frac{1}{s}}\right)} = \frac{1}{1-s} > 0.
\end{aligned}$$

Therefore, when $0 < s < 1/2$, since $\mathcal{Q}(z, z_0)$ is decreasing for $z > 0$, we conclude that

$$\mathcal{Q}(z, z_0) \geq \frac{1}{1-s} > 1.$$

□

3.5.3.2 Construction of a subsolution

The next lemma is the construction of a barrier ϕ which will be used to localize the solution U. For a set S, we introduce the notation

$$S^+ = S \cap \{z \geq 0\} \quad \text{and} \quad S^- = S \cap \{z \leq 0\}.$$

If $z_0 \geq 0$, then we will work in the partial ring $[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$. If $z_0 < 0$, then we will work in the partial ring $[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^-$. We will use the condensed notation

$$[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^{\pm} = \begin{cases} [S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ & \text{if } z_0 \geq 0 \\ [S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^- & \text{if } z_0 < 0. \end{cases}$$

Lemma 3.5.11. *Fix $0 < \gamma < 1$ and $0 < s < 1$. Let $S_r(x_0, z_0) \subset \mathbb{R}^{n+1}$.*

If $z_0 \geq 0$, then there there exist a classical subsolution $\phi = \phi(x, z)$ to

$$\begin{cases} a^{ij}(x)\partial_{ij}\phi + |z|^{2-\frac{1}{s}}\,\partial_{zz}\phi > a(n\Lambda + 1) & \text{in } [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z \neq 0\} \\ -\partial_{z+}\phi(x, 0) < 0 & \text{on } [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z = 0\}. \end{cases} \tag{3.5.9}$$

If $z_0 \leq 0$, then there exist a classical subsolution $\phi = \phi(x, z)$ to

$$\begin{cases} a^{ij}(x)\partial_{ij}\phi + |z|^{2-\frac{1}{s}}\,\partial_{zz}\phi > a(n\Lambda + 1) & \text{in } [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^- \cap \{z \neq 0\} \\ -\partial_{z-}\phi(x, 0) > 0 & \text{on } [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^- \cap \{z = 0\}. \end{cases} \tag{3.5.10}$$

In each case, $\phi > 0$ in $[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^{\pm}$, $\phi \leq 0$ on $[\partial S_{2r}(x_0, z_0)]^{\pm}$, and there is a constant $C = C(n, \lambda, \Lambda, \gamma) > 0$ such that $\phi \leq Car$ on $[\partial S_{\gamma r}(x_0, z_0)]^{\pm}$.

Proof. We begin by considering the function $(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha}$ for a large constant $\alpha = \alpha(n, \lambda, \Lambda, s) > 0$. Let $\mathcal{Q}(z, z_0)$ be the function defined in Lemma 3.5.10. For a point $(x, z) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^{\pm} \setminus \{z = 0\}$, we have that

$$\begin{aligned}
&a^{ij}(x)\partial_{ij}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} + |z|^{2-\frac{1}{s}}\,\partial_{zz}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} \\
&\quad = \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \\
&\qquad \left[(\alpha + 1)\big(a^{ij}(x)(x - x_0)_i(x - x_0)_j + |z|^{2-\frac{1}{s}}\,(h'(z) - h'(z_0))^2\big) - \big(a^{ii}(x) + 1\big)\,\delta_\Phi((x_0, z_0), (x, z))\right] \\
&\quad \geq \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \\
&\qquad \left[(\alpha + 1)\big(\lambda\,|x - x_0|^2 + |z|^{2-\frac{1}{s}}\,(h'(z) - h'(z_0))^2\big) - (n\Lambda + 1)\,\delta_\Phi((x_0, z_0), (x, z))\right] \\
&\quad = \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \\
&\qquad \left[(\alpha + 1)\Big(2\lambda\delta_\varphi(x_0, x) + \frac{(h'(z) - h'(z_0))^2}{h''(z)\delta_h(z_0, z)}\delta_h(z_0, z)\Big) - (n\Lambda + 1)\,(\delta_\varphi(x_0, x) + \delta_h(z_0, z))\right] \\
&\quad = \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \\
&\qquad \Big[(2\lambda(\alpha + 1) - (n\Lambda + 1))\,\delta_\varphi(x_0, x) + (\mathcal{Q}(z, z_0)(\alpha + 1) - (n\Lambda + 1))\,\delta_h(z_0, z)\Big].
\end{aligned}$$

If $z_0 = 0$ or if $0 < s \le 1/2$, then $\mathcal{Q}(z, z_0) \ge 1$ in $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^{\pm} \setminus \{z = 0\}$. Therefore,

$$\begin{aligned} &a^{ij}(x)\partial_{ij}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} + |z|^{2-\frac{1}{s}}\, \partial_{zz}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} \\ &\quad \ge \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \\ &\qquad \Big[(2\lambda(\alpha+1) - (n\Lambda+1))\, \delta_\varphi(x_0, x) + ((\alpha+1) - (n\Lambda+1))\, \delta_h(z_0, z)\Big]. \end{aligned} \tag{3.5.11}$$

However, if $1/2 < s < 1$ and $z_0 \neq 0$, then, as seen in Lemma 3.5.10, we cannot bound $\mathcal{Q}$ from below. Hence, we will build a subsolution for $0 < s \le 1/2$ and for $1/2 < s < 1$ separately.

<u>Case 1</u>: $z_0 > 0$ and $0 < s < 1/2$

Define

$$\phi(x, z) = \begin{cases} \alpha^{-1} a (2r)^{\alpha+1}[(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} - r^{-\alpha}] & \text{if } z \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \\ \alpha^{-1} a 2^{\alpha+1}(\gamma^{-\alpha} - 1) r & \text{if } z \in [S_{\gamma r}(x_0, z_0)]^+. \end{cases}$$

Note that $\phi \le 0$ when

$$\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} \le r^{-\alpha}$$

and $\phi < 0$ when

$$\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} > r^{-\alpha}.$$

That is, $\phi \le 0$ in $[S_{2r}(x_0, z_0) \setminus S_r(x_0, z_0)]^+$ and $\phi > 0$ in $[S_r(x_0, z_0)]^+$.

Let $(x, z) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \setminus \{z = 0\}$. From (3.5.11), we find that

$$\begin{aligned} &a^{ij}(x)\partial_{ij}\phi(x, z) + |z|^{2-\frac{1}{s}}\, \partial_{zz}\phi(x, z) \\ &\quad \ge a(2r)^{\alpha+1}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \\ &\qquad \Big[(2\lambda(\alpha+1) - (n\Lambda+1))\, \delta_\varphi(x_0, x) + ((\alpha+1) - (n\Lambda+1))\, \delta_h(z_0, z)\Big]. \end{aligned}$$

Choose $\alpha = \alpha(\gamma, n, \lambda, \Lambda)$ large so that

$$2\lambda(\alpha+1) - (n\Lambda+1) > 2\gamma^{-1}(n\Lambda+1) \quad \text{and} \quad (\alpha+1) - (n\Lambda+1) > 2\gamma^{-1}(n\Lambda+1).$$

If $\gamma r/2 < \delta_\varphi(x_0, x) < 2r$, then

$$
\begin{aligned}
&a^{ij}(x)\partial_{ij}\phi(x,z) + |z|^{2-\frac{1}{s}}\,\partial_{zz}\phi(x,z)\\
&\qquad \geq a(2r)^{\alpha+1}(\delta_\Phi((x_0,z_0),(x,z)))^{-\alpha-2}\Big[2\gamma^{-1}(n\Lambda+1)\delta_\varphi(x_0,x) + 0\Big]\\
&\qquad > a(2r)^{\alpha+1}(\delta_\Phi((x_0,z_0),(x,z)))^{-\alpha-2}\Big[2\gamma^{-1}(n\Lambda+1)\frac{\gamma r}{2}\Big]\\
&\qquad = a(n\Lambda+1)(2r)^{\alpha+2}(\delta_\Phi((x_0,z_0),(x,z)))^{-\alpha-2}\\
&\qquad \geq a(n\Lambda+1).
\end{aligned}
$$

If $\delta_\varphi(x_0, x) \geq \gamma r/2$, then $\gamma r/2 < \delta_h(z_0, z) < 2r$ and we estimate

$$
\begin{aligned}
&a^{ij}(x)\partial_{ij}\phi(x,z) + |z|^{2-\frac{1}{s}}\,\partial_{zz}\phi(x,z)\\
&\qquad \geq a(2r)^{\alpha+1}(\delta_\Phi((x_0,z_0),(x,z)))^{-\alpha-2}\Big[0 + 2\gamma^{-1}\,(n\Lambda+1)\,\delta_h(z_0,z)\Big]\\
&\qquad > a(2r)^{\alpha+1}(\delta_\Phi((x_0,z_0),(x,z)))^{-\alpha-2}\Big[2\gamma^{-1}\,(n\Lambda+1)\,2\gamma\Big]\\
&\qquad \geq a(n\Lambda+1).
\end{aligned}
$$

Therefore, for all $(x, z) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \setminus \{z = 0\}$,

$$
a^{ij}(x)\partial_{ij}\phi(x,z) + |z|^{2-\frac{1}{s}}\,\partial_{zz}\phi(x,z) > a(n\Lambda+1).
$$

We next check the Neumann condition for sections that $S_r(x_0, z_0)$ that intersect $\{z = 0\}$. Let $(x, 0) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z = 0\}$ and observe that

$$
\begin{aligned}
\partial_z\phi(x,0) &= -a(2r)^{\alpha+1}(\delta_\Phi((x_0,z_0),(x,z))^{-\alpha-1}(h'(z) - h'(z_0))\big|_{z=0}\\
&= a(2r)^{\alpha+1}(\delta_\Phi((x_0,z_0),(x,0))^{-\alpha-1}h'(z_0)\\
&\geq ah'(z_0) > 0
\end{aligned}
$$

since $z_0 > 0$. Therefore, ϕ defined in $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$ is a subsolution to (3.5.9). We remark that if $z_0 = 0$, then we do not obtain a strict inequality in the Neumann condition. The construction of a subsolution for $z_0 = 0$ is similar to case in which $1/2 < s < 1$. We will address this later (see Case 3).

Lastly, for $(x,z) \in [\partial S_{\gamma r}(x_0,z_0)]^{\pm}$, we have that

$$\phi(x,z) = \alpha^{-1} a 2^{\alpha+1}(\gamma^{-1}-1) r = Car.$$

where $C = C(\gamma, n, \lambda, \Lambda) > 0$.

<u>Case 2</u>: $z_0 \geq 0$ and $1/2 < s < 1$

We will use the ideas of Caffarelli–Gutiérrez [16, Theorem 2] (see also the work of Le [43, Theorem 4.2]) to bypass the points where $|z|^{2-\frac{1}{s}}$ is small with respect to the size of the section $S_{2r}(z_0) = S_h(z_0, 2r)$.

Let $\varepsilon > 0$ be small and define H_ε by

$$\begin{aligned} H_\varepsilon &= \left\{ z \in S_{2r}(z_0) : |z|^{2-\frac{1}{s}} \leq \varepsilon^{\frac{2s-1}{1-s}} r^{2s-1} \right\} \\ &= \left\{ z \in S_{2r}(z_0) : |z| \leq \varepsilon^{\frac{s}{1-s}} r^{s} \right\}. \end{aligned}$$

We first show that H_ε is small. Indeed, by Lemma 3.3.5,

$$\begin{aligned} \mu_h(H_\varepsilon) = \int_{H_\varepsilon} h''(z)\, dz &\leq \int_{-\varepsilon^{\frac{s}{1-s}} r^s}^{\varepsilon^{\frac{s}{1-s}} r^s} h''(z)\, dz \\ &= 2h'\left(\varepsilon^{\frac{s}{1-s}} r^s\right) = C\varepsilon r^{1-s} = C\varepsilon \mu_h(S_{2r}(z_0)) \end{aligned}$$

for a constant $C = C(s)$.

We will construct a function h_ε in $[S_{2r}(z_0)]^+$ that to bypass the points in H_ε. Let $\tilde{H}_\varepsilon$ be an open set such that

$$H_\varepsilon \subset \tilde{H}_\varepsilon \subset S_{2r}(z_0), \quad \mu_\Phi(\tilde{H}_\varepsilon \setminus H_\varepsilon) \leq \varepsilon\, \mu_h(S_{2r}(z_0)),$$

and let $\psi = \psi(z)$ be a smooth function satisfying

$$\psi = 1 \text{ in } H_\varepsilon, \quad \psi = \varepsilon \text{ in } S_{2r}(z_0) \setminus \tilde{H}_\varepsilon, \quad \varepsilon \leq \psi \leq 1 \text{ in } S_{2r}(z_0).$$

We use the notation

$$[S_{2r}(z_0)]^+ = (z_L, z_R), \quad \text{where } 0 \leq z_L < z_0 < z_R.$$

Note that $z_L = 0$ if $0 \in S_{2r}(z_0)$.

In $[S_{2r}(z_0)]^+$, let $h_\varepsilon = h_\varepsilon(z)$ be the convex solution to

$$\begin{cases} h_\varepsilon'' = 2(n\Lambda + 1)\psi h'' \quad \text{in } [S_{2r}(z_0)]^+ \\ h_\varepsilon(z_R) = 0 \\ h_\varepsilon'(z_L) = \varepsilon r^{1-s}. \end{cases}$$

We remark that $h_\varepsilon \in C^\infty([S_{2r}(z_0)]^+) \cap C^1([\overline{S}_{2r}(z_0)]^+)$. Since h_ε is strictly convex in $[S_{2r}(z_0)]^+$, it follows that $h_\varepsilon' > 0$ in $[S_{2r}(z_0)]^+$. Moreover, by the maximum principle h_ε achieves its maximum at $z = z_R$, so that $h_\varepsilon \leq 0$ in $[S_{2r}(z_0)]^+$.

We apply the ABP Lemma to obtain

$$\begin{aligned} |h_\varepsilon| &\leq C \operatorname{diam}([S_{2r}(z_0)]^+) \int_{[S_{2r}(z_0)]^+} h_\varepsilon''(z)\, dz \\ &= C \operatorname{diam}([S_{2r}(z_0)]^+) \int_{[S_{2r}(z_0)]^+} 2(n\Lambda + 1)\psi(z) h''(z)\, dz \\ &\leq C \operatorname{diam}(S_{2r}(z_0)) \int_{S_{2r}(z_0)} \psi\, d\mu_h. \end{aligned}$$

where $C = C(n, \Lambda) > 0$. We estimate

$$\operatorname{diam}(S_{2r}(z_0)) \leq \operatorname{diam}(B_{C_s(2r)^s}(z_0)) = 2C_s(2r)^s = C\left|B_{c_s(2r)^s}(z_0)\right| \leq C\left|S_{2r}(z_0)\right|$$

and

$$\begin{aligned} \int_{S_{2r}(z_0)} \psi\, d\mu_h &= \int_{H_\varepsilon} \psi\, d\mu_h + \int_{\tilde{H}_\varepsilon \setminus H_\varepsilon} \psi\, d\mu_h + \int_{S_{2r}(z_0)\setminus \tilde{H}_\varepsilon} \psi\, d\mu_h \\ &\leq \int_{H_\varepsilon} d\mu_h + \int_{\tilde{H}_\varepsilon \setminus H_\varepsilon} d\mu_h + \int_{S_{2r}(z_0)\setminus \tilde{H}_\varepsilon} \varepsilon\, d\mu_h \\ &= \mu_h(H_\varepsilon) + \mu_h(\tilde{H}_\varepsilon \setminus H_\varepsilon) + \varepsilon \mu_h(S_{2r}(z_0) \setminus \tilde{H}_\varepsilon) \\ &\leq C\varepsilon \mu_h(S_{2r}(z_0)) + \varepsilon \mu_h(S_{2r}(z_0)) + \varepsilon \mu_h(S_{2r}(z_0)) \\ &= C\varepsilon \mu_h(S_{2r}(z_0)). \end{aligned}$$

Hence, we obtain

$$|h_\varepsilon| \leq C\varepsilon\, |S_{2r}(z_0)|\, \mu_h(S_{2r}(z_0)) \leq C_1 \varepsilon r$$

for a constant $C_1 = C_1(n, \Lambda, s)$.

Next, let $\delta > 0$. For $z \in [S_{2r}(z_0)]^+$,

$$\begin{aligned}
\left|h_\varepsilon'(z)\right| = h_\varepsilon'(z) &= \int_{z_L+\delta}^{z} h_\varepsilon''(w)\,dw + h_\varepsilon'(z_L+\delta) \\
&= \int_{z_L+\delta}^{z} 2(n\Lambda+1)\psi h''(w)\,dw + h_\varepsilon'(z_L+\delta) \\
&\le 2(n\Lambda+1)\int_{S_{2r}(z_0)} \psi\,d\mu_h + h_\varepsilon'(z_L+\delta) \\
&\le 2(n\Lambda+1)C\varepsilon\mu_h(S_{2r}(z_0)) + h_\varepsilon'(z_L+\delta) \\
&\le C\varepsilon r^{1-s} + h_\varepsilon'(z_L+\delta).
\end{aligned}$$

Taking the limit as $\delta \to 0$,

$$h_\varepsilon'(z) \le C\varepsilon r^{1-s} + h_\varepsilon'(z_L) = C\varepsilon r^{1-s} + \varepsilon r^{1-s} = C_2\varepsilon r^{1-s}$$

for a constant $C_2 = C_2(n,\Lambda,s)$.

Suppose that $\gamma r/2 < \delta_h(z_0,z) < 2r$. By the convexity of $\delta_h(z_0,z)$, we obtain

$$0 = \delta_h(z_0,z_0) \ge \delta_h(z_0,z) + \partial_z\delta_h(z_0,z)\cdot(z_0-z).$$

Since $S_{2r}(z_0) \subset B_{C_s(2r)^s}(z_0)$, this implies

$$|\partial_z\delta_h(z_0,z)| \ge \frac{\delta_h(z_0,z)}{|z-z_0|} \ge \frac{\gamma r/2}{2C_s(2r)^s} = C_3 r^{1-s} \tag{3.5.12}$$

for a constant $C_3 = C_3(\gamma,s)$. Choose $\varepsilon = \varepsilon(\gamma,n,\Lambda,s) > 0$ small so that $C_2\varepsilon < C_3$. Then,

$$\left|\partial_z\delta_h(z_0,z) - h_\varepsilon'(z)\right| \ge |\partial_z\delta_h(z_0,z)| - \left|h_\varepsilon'(z)\right| \ge (C_3 - C_2\varepsilon)r^{1-s} > 0$$

and

$$(\partial_z\delta_h(z_0,z) - h_\varepsilon'(z))^2 \ge (C_3-C_2\varepsilon)^2 r^{2-2s} = C_4 r^{2-2s} \tag{3.5.13}$$

for a constant $C_4 = C_4(\gamma,n,\Lambda,s) > 0$.

For a large constant $\alpha = \alpha(\gamma,n,\lambda,\Lambda,s) > 0$, we define the function $\tilde\phi$ on $[S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^+$ by

$$\tilde\phi(x,z) = (\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha}.$$

Let $(x,z) \in [S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^+ \setminus \{z=0\}$. Since $h_\varepsilon \leq 0$, we first note that

$$\gamma r < \delta_\Phi((x_0,z_0),(x,z)) \leq \delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z) < 2r + C_1\varepsilon r = (2+C_1\varepsilon)r.$$

The equation for $\tilde{\phi}$ in $[S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^+ \setminus \{z=0\}$ is given by

$$\begin{aligned}
&a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\,\partial_{zz}\tilde{\phi} \\
&\quad = \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2} \\
&\quad\quad \Bigg((\alpha+1)\left[a^{ij}(x)(x-x_0)_i(x-x_0)_j + |z|^{2-\frac{1}{s}}\,(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h_\varepsilon'(z))^2\right] \\
&\quad\quad - (\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))\left[a^{ii}(x) + 1 - |z|^{2-\frac{1}{s}}\,h_\varepsilon''(z)\right]\Bigg) \\
&\quad = \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2} \\
&\quad\quad \Bigg((\alpha+1)\left[a^{ij}(x)(x-x_0)_i(x-x_0)_j + |z|^{2-\frac{1}{s}}\,(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h_\varepsilon'(z))^2\right] \\
&\quad\quad - (\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))\left[a^{ii}(x) + 1 - 2(n\Lambda+1)\psi\right]\Bigg).
\end{aligned}$$

Using ellipticity, we estimate

$$\begin{aligned}
&a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\,\partial_{zz}\tilde{\phi} \\
&\quad \geq \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2} \\
&\quad\quad \Bigg((\alpha+1)\left[\lambda\,|x-x_0|^2 + |z|^{2-\frac{1}{s}}\,(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h_\varepsilon'(z))^2\right] \\
&\quad\quad - (\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))\left[n\Lambda + 1 - 2(n\Lambda+1)\psi\right]\Bigg) \\
&\quad = \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2} \\
&\quad\quad \Bigg((\alpha+1)\left[2\lambda\delta_\varphi(x_0,x) + |z|^{2-\frac{1}{s}}\,(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h_\varepsilon'(z))^2\right] \\
&\quad\quad - (\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))(1-2\psi)(n\Lambda+1)\Bigg).
\end{aligned}$$

Suppose that $z \in H_\varepsilon$. Since $\psi(z) = 1$, we drop the nonnegative term with $(\alpha+1)$ to obtain

$$\begin{aligned}
&a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\,\partial_{zz}\tilde{\phi} \\
&\qquad \geq \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2}\Big((\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))(n\Lambda+1)\Big) \\
&\qquad = \alpha(n\Lambda+1)(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-1} \\
&\qquad \geq \alpha(n\Lambda+1)(2+C_1\varepsilon)^{-\alpha-1}r^{-\alpha-1}.
\end{aligned}$$

Next, suppose that $z \notin H_\varepsilon$. Since $\psi(z) > 0$ and $|z|^{2-\frac{1}{s}} > \varepsilon^{\frac{2s-1}{1-s}}r^{2s-1}$, we estimate

$$\begin{aligned}
&a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\,\partial_{zz}\tilde{\phi} \\
&\qquad \geq \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2} \\
&\qquad\quad \Big((\alpha+1)\Big[2\lambda\delta_\varphi(x_0,x) + \varepsilon^{\frac{2s-1}{1-s}}r^{2s-1}(\partial_z(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z)))^2\Big] \\
&\qquad\quad - (\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))(n\Lambda+1)\Big).
\end{aligned}$$

If $\gamma r/2 < \delta_\varphi(x_0,x) < 2r$, then

$$2\lambda\delta_\varphi(x_0,x) + \varepsilon^{\frac{2s-1}{1-s}}r^{2s-1}(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h_\varepsilon(z)))^2 \geq 2\lambda\delta_\varphi(x_0,x) > \lambda\gamma r.$$

Choose $\alpha = \alpha(\gamma,n,\lambda,\Lambda,s)$ large enough to guarantee that

$$(\alpha+1)\lambda\gamma - (n\Lambda+1)(2+C_1\varepsilon) > (n\Lambda+1)(2+C_1\varepsilon).$$

Then, we have that

$$\begin{aligned}
&a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\,\partial_{zz}\tilde{\phi} \\
&\qquad \geq \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2}\Big((\alpha+1)\lambda\gamma r - (n\Lambda+1)(2r+C_1\varepsilon)r\Big) \\
&\qquad > \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2}(n\Lambda+1)(2+C_1\varepsilon)r \\
&\qquad \geq \alpha(n\Lambda+1)(2+C_1\varepsilon)^{-\alpha-2}r^{-\alpha-2}(2+C_1\varepsilon)r \\
&\qquad = \alpha(n\Lambda+1)(2+C_1\varepsilon)^{-\alpha-1}r^{-\alpha-1}.
\end{aligned}$$

If $\delta_\varphi(x_0,x) \le \gamma r/2$, then $\gamma r/2 < \delta_h(z_0,z) < 2r$ and, by (3.5.13), we obtain

$$\begin{aligned}2\lambda\delta_\varphi(x_0,x) + \varepsilon^{\frac{2s-1}{1-s}} r^{2s-1}(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h_\varepsilon(z)))^2 \\ \ge \varepsilon^{\frac{2s-1}{1-s}} r^{2s-1}(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h'_\varepsilon(z))^2 \\ \ge \varepsilon^{\frac{2s-1}{1-s}} r^{2s-1} C_4 r^{2-2s} = C_4\varepsilon^{\frac{2s-1}{1-s}} r.\end{aligned}$$

Let $\alpha = \alpha(\gamma,n,\lambda,\Lambda,s) > 0$ be large so that

$$(\alpha+1)C_4\varepsilon^{\frac{2s-1}{1-s}} - (n\Lambda+1)(2+C_1\varepsilon) > (n\Lambda+1)(2+C_1\varepsilon).$$

Then, we have that

$$\begin{aligned}&a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\partial_{zz}\tilde{\phi} \\ &\quad\ge \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2}\left((\alpha+1)C_4\varepsilon^{\frac{2s-1}{1-s}} r - (n\Lambda+1)(2+C_1\varepsilon)r\right) \\ &\quad> \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2}(n\Lambda+1)(2+C_1\varepsilon)r \\ &\quad\ge \alpha(n\Lambda+1)(2+C_1\varepsilon)^{-\alpha-1}r^{-\alpha-1}.\end{aligned}$$

Hence, for all $(x,z) \in [S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^+ \setminus \{z=0\}$, there is an $\alpha = \alpha(\gamma,n,\lambda,\Lambda,s) > 0$ such that

$$a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\partial_{zz}\tilde{\phi} > \alpha(n\Lambda+1)(2+C_1\varepsilon)^{-\alpha-1}r^{-\alpha-1}.$$

We define the barrier ϕ on $[S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^+$ by

$$\phi(x,z) = \frac{a}{\alpha}(2+C_1\varepsilon)^{\alpha+1}r^{\alpha+1}\left((\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha} - (1+C_1\varepsilon)^{-\alpha}r^{-\alpha}\right).$$

For $(x,z) \in [S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^+ \setminus \{z=0\}$, it follows that

$$\begin{aligned}a^{ij}(x)\partial_{ij}\phi + |z|^{2-\frac{1}{s}}\partial_{zz}\phi &= \frac{a}{\alpha}(2+C_1\varepsilon)^{\alpha+1}r^{\alpha+1}\left(a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\partial_{zz}\tilde{\phi}\right) \\ &> \frac{a}{\alpha}(2+C_1\varepsilon)^{\alpha+1}r^{\alpha+1}\alpha(n+1)(2+C_1\varepsilon)^{-\alpha-1}r^{-\alpha-1} \\ &= a(n\Lambda+1),\end{aligned}$$

as desired.

We next check the Neumann condition for sections that $S_{2r}(x_0, z_0)$ that intersect $\{z = 0\}$. Let $(x, 0) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z = 0\}$ and observe that

$$\begin{aligned}
\partial_z \phi(x, 0) &= -a(2 + C_1\varepsilon)^{\alpha+1} r^{\alpha+1} (\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z))^{-\alpha-1} (h'(z) - h'(z_0) - h'_\varepsilon(z))\big|_{z=0} \\
&= a(2 + C_1\varepsilon)^{\alpha+1} r^{\alpha+1} (\delta_\Phi((x_0, z_0), (x, 0)) - h_\varepsilon(0))^{-\alpha-1} (h'(z_0) + \varepsilon r^{1-s}) \\
&\geq a(h'(z_0) + \varepsilon r^{1-s}) > 0
\end{aligned}$$

since $z_0 \geq 0$. Therefore, ϕ defined in $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$ is a subsolution to (3.5.9).

For $(x, z) \in [S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$, since

$$\gamma r < \delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z) < (1 + C_1\varepsilon) r,$$

we have that $\phi > 0$ in $[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$. Choose ε small so that $2 > 1 + C_1\varepsilon$. Then, $\phi \leq 0$ on $[\partial S_{2r}(x_0, z_0)]^+$. Indeed, for $(x, z) \in [\partial S_{2r}(x_0, z_0)]^+$, we have that

$$-h_\varepsilon(z) \geq 0 > (1 + C_1\varepsilon - 2) r$$

which implies

$$\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z) = 2r - h_\varepsilon(z) > (1 + C_1\varepsilon) r.$$

Thus, $\phi(x, z) \leq 0$.

Lastly, let $(x, z) \in [\partial S_{\gamma r}(x_0, z_0)]^+$ and observe that

$$\begin{aligned}
\phi(x, z) &= \frac{a}{\alpha}(2 + C_1\varepsilon)^{\alpha+1} r^{\alpha+1} \left((\gamma r - h_\varepsilon(z))^{-\alpha} - (1 + C_1\varepsilon)^{-\alpha} r^{-\alpha}\right) \\
&\leq \frac{a}{\alpha}(2 + C_1\varepsilon_2)^{\alpha+1} r^{\alpha+1} \left((\gamma r + 0)^{-\alpha} - 0\right) \\
&= \frac{\gamma^{-\alpha}}{\alpha}(2 + C_1\varepsilon)^{\alpha+1} a r \\
&= C a r
\end{aligned}$$

for $C = C(\gamma, n, \lambda, \Lambda, s) > 0$.

<u>Case 3</u>: $z_0 = 0$ and $0 < s \leq 1/2$

We saw in Case 1 that the inequality for the Neumann condition was not strict when $z_0 = 0$. Hence, we need to add a function h_ε to the distance function to adjust the barrier as we did in Case 2. Since $0 < s < 1/2$, we will take $H_\varepsilon = \emptyset$.

Let $(x,z) \in [S_{2r}(x_0,0) \setminus S_{\gamma r}(x_0,0)]^+$. Since $z \in S_{2r}(0)$, we know that

$$\frac{s^2}{1-s}|z|^{\frac{1}{s}} = \delta_h(0,z) < 2r.$$

Therefore,

$$|z| < C_1 r^s, \quad C_1 := \left(\frac{1-s}{s^2}2\right)^s.$$

Since $2 - \frac{1}{s} < 0$, we also know that

$$|z|^{2-\frac{1}{s}} > C_1^{\frac{2s-1}{s}} r^{2s-1}. \tag{3.5.14}$$

Define h_ε in $[S_{2r}(0)]^+$ by

$$h_\varepsilon(z) = \varepsilon r^{1-s} z - \varepsilon C_1 r.$$

For all $z \in [S_{2r}(0)]^+$, we have that $h_\varepsilon \leq 0$, that

$$|h_\varepsilon(z)| = \varepsilon C_1 r - \varepsilon r^{1-s} z \leq C_1 \varepsilon r,$$

and that

$$h_\varepsilon'(z) = \varepsilon r^{1-s} > 0, \quad h_\varepsilon''(z) = 0.$$

Choose $\varepsilon_2 = \varepsilon_2(\gamma, s)$ such that $\varepsilon_2 < C_3$ where C_3 is the constant from (3.5.12). For $z > 0$ such that $\gamma r/2 < \delta_h(0,z) < 2r$, we estimate, as in (3.5.13), to obtain

$$(\partial_z \delta_h(z_0,z) - h_\varepsilon'(z))^2 \geq (C_3 - \varepsilon_2)^2 r^{2-2s} = C_4 r^{2-2s} \tag{3.5.15}$$

for a constant $C_4 = C_4(\gamma, n, \Lambda, s) > 0$.

Let that $(x,z) \in [S_{2r}(x_0,0) \setminus S_{\gamma r}(x_0,0)]^+$. Since $-h_\varepsilon \geq 0$, we have that

$$\gamma r < \delta_\Phi((x_0,0),(x,z)) \leq \delta_\Phi((x_0,0),(x,z)) - h_\varepsilon(z) < 2r + \varepsilon C_1 r = (2 + \varepsilon C_1)r. \tag{3.5.16}$$

We define a function $\tilde{\phi}$ on $[S_{2r}(x_0,0) \setminus S_{\gamma r}(x_0,0)]^+$ by

$$\tilde{\phi}(x,z) = (\delta_\Phi((x_0,0),(x,z)) - h_\varepsilon(z))^{-\alpha}.$$

Let in $(x,z) \in [S_{2r}(x_0,0) \setminus S_{\gamma r}(x_0,0)]^+ \setminus \{z=0\}$ and estimate the equation for $\tilde{\phi}$ using (3.5.14)

$$\begin{aligned}
&a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\,\partial_{zz}\tilde{\phi}\\
&\quad= \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2}\\
&\quad\quad\Big((\alpha+1)\Big[a^{ij}(x)(x-x_0)_i(x-x_0)_j + |z|^{2-\frac{1}{s}}\,(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h'_\varepsilon(z))^2\Big]\\
&\quad\quad - (\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))\Big[a^{ii}(x) + 1\Big]\Big)\\
&\quad\geq \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2}\\
&\quad\quad\Big((\alpha+1)\Big[2\lambda\delta_\varphi(x_0,x) + C_1^{\frac{2s-1}{s}}r^{2s-1}(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h'_\varepsilon(z))^2\Big]\\
&\quad\quad - (\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))(n\Lambda+1)\Big).
\end{aligned}$$

If $\gamma r/2 < \delta_\varphi(x_0,x) < 2r$, then

$$2\lambda\delta_\varphi(x_0,x) + C_1^{\frac{2s-1}{s}}r^{2s-1}(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h'_\varepsilon(z))^2 \geq 2\lambda\delta_\varphi(x_0,x) > \lambda\gamma r.$$

Choose $\alpha = \alpha(\gamma,n,\lambda,\Lambda,s)$ large enough to guarantee that

$$(\alpha+1)\lambda\gamma - (n\Lambda+1)(2+C_1\varepsilon) > (n\Lambda+1)(2+C_1\varepsilon).$$

Then, we have that

$$\begin{aligned}
&a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\,\partial_{zz}\tilde{\phi}\\
&\quad\geq \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2}\Big((\alpha+1)\lambda\gamma r - (n\Lambda+1)(2+C_1\varepsilon)r\Big)\\
&\quad> \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2}(n\Lambda+1)(2+C_1\varepsilon)r\\
&\quad\geq \alpha(n\Lambda+1)(2+C_1\varepsilon_2)^{-\alpha-2}r^{-\alpha-2}(2+C_1\varepsilon)r\\
&\quad= \alpha(n\Lambda+1)(2+C_1\varepsilon)^{-\alpha-1}r^{-\alpha-1}.
\end{aligned}$$

If $\delta_\varphi(x_0,x) \leq \gamma r/2$, then $\gamma r/2 < \delta_h(z_0,z) < 2r$ and, by (3.5.15), we obtain

$$\begin{aligned}
2\lambda\delta_\varphi(x_0,x) &+ C_1^{\frac{2s-1}{s}}r^{2s-1}(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h_\varepsilon(z)))^2\\
&\geq C_1^{\frac{2s-1}{s}}r^{2s-1}(\partial_z(\delta_\Phi((x_0,z_0),(x,z))) - h'_\varepsilon(z))^2\\
&\geq C_1^{\frac{2s-1}{s}}r^{2s-1}C_4r^{2-2s} = C_4C_1^{\frac{2s-1}{s}}r.
\end{aligned}$$

Let $\alpha = \alpha(\gamma, n, \lambda, \Lambda, s) > 0$ be large so that

$$(\alpha+1)C_4C_1^{\frac{2s-1}{s}} - (n\Lambda+1)(2+C_1\varepsilon) > (n\Lambda+1)(2+C_1\varepsilon).$$

Then, we have that

$$\begin{aligned} &a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\,\partial_{zz}\tilde{\phi} \\ &\qquad \geq \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2}\left((\alpha+1)C_4C_1^{\frac{2s-1}{s}}r - (n\Lambda+1)(2r+C_1\varepsilon)r\right) \\ &\qquad > \alpha(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-2}(n\Lambda+1)(2+C_1\varepsilon)r \\ &\qquad \geq \alpha(n\Lambda+1)(2+C_1\varepsilon)^{-\alpha-1}r^{-\alpha-1}. \end{aligned}$$

Hence, for all $(x,z) \in [S_{2r}(x_0,0) \setminus S_{\gamma r}(x_0,0)]^+ \setminus \{z=0\}$, there is an $\alpha = \alpha(\gamma, n, \lambda, \Lambda, s) > 0$ such that

$$a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\,\partial_{zz}\tilde{\phi} > \alpha(n\Lambda+1)(2+C_1\varepsilon)^{-\alpha-1}r^{-\alpha-1}.$$

We define the barrier ϕ on $[S_{2r}(x_0,0) \setminus S_{\gamma r}(x_0,0)]^+$ by

$$\phi(x,z) = \frac{a}{\alpha}(2+C_1\varepsilon)^{\alpha+1}r^{\alpha+1}\left((\delta_\Phi((x_0,0),(x,z)) - h_\varepsilon(z))^{-\alpha} - (1+C_1\varepsilon)^{-\alpha}r^{-\alpha}\right).$$

For $(x,z) \in [S_{2r}(x_0,0) \setminus S_{\gamma r}(x_0,0)]^+ \setminus \{z=0\}$, it follows that

$$a^{ij}(x)\partial_{ij}\phi + |z|^{2-\frac{1}{s}}\,\partial_{zz}\phi > a(n\Lambda+1).$$

For $(x,0) \in [S_{2r}(x_0,0) \setminus S_{\gamma r}(x_0,0)]^+ \cap \{z=0\}$ and observe that

$$\begin{aligned} \partial_z\phi(x,0) &= -a(2+C_1\varepsilon)^{\alpha+1}r^{\alpha+1}(\delta_\Phi((x_0,z_0),(x,z)) - h_\varepsilon(z))^{-\alpha-1}(h'(z) - h'(0) - h_\varepsilon'(z))\big|_{z=0} \\ &= a(2+C_1\varepsilon)^{\alpha+1}r^{\alpha+1}(\delta_\Phi((x_0,0),(x,0)) - h_\varepsilon(0))^{-\alpha-1}\varepsilon r^{1-s} \\ &\geq a\varepsilon r^{1-s} > 0. \end{aligned}$$

Therefore, ϕ defined in $[S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^+$ is a subsolution to (3.5.9).

One can check that $\phi > 0$ in $[S_r(x_0,0) \setminus S_{\gamma r}(x_0,0)]^+$ and that $\phi \leq 0$ on $[\partial S_{2r}(x_0,0)]^+$ when $\varepsilon = \varepsilon(\gamma,s)$ is small enough to guarantee that $2 > 1 + C_2\varepsilon$. Moreover, there is a constant $C = C(\gamma, n, \lambda, \Lambda, s) > 0$ such that

$$\phi(x,z) \leq Car \quad \text{on } [\partial S_{\gamma r}(x_0,0)]^+.$$

Case 4: $z_0 \leq 0$ and $0 < s < 1$

Note that if $(x,z) \in [S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^-$, then $(x,-z) \in [S_{2r}(x_0,-z_0) \setminus S_{\gamma r}(x_0,-z_0)]^+$. Define ψ in $[S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^-$ to be the even reflection across $\{z=0\}$ of the solution ϕ to (3.5.9) in $[S_{2r}(x_0,-z_0) \setminus S_{\gamma r}(x_0,-z_0)]^+$:

$$\psi(x,z) = \phi(x,-z), \quad \text{for } (x,z) \in [S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^-.$$

Since $D^2\psi(x,z) = D^2\psi(x,-z)$, we know, for $(x,z) \in [S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^- \setminus \{z=0\}$, that

$$\begin{aligned} a^{ij}(x)\partial_{ij}\psi(x,z) + |z|^{2-\frac{1}{s}}\,\partial_{zz}\psi(x,z) &= a^{ij}(x)\partial_{ij}\phi(x,-z) + |z|^{2-\frac{1}{s}}\,\partial_{zz}\psi(x,-z) \\ &> a(n\Lambda+1). \end{aligned}$$

For $(x,0) \in [S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^- \cap \{z=0\}$, we estimate

$$\begin{aligned} -\partial_{z-}\psi(x,0) &= -\lim_{h\to 0^-} \frac{\psi(x,h)-\psi(x,0)}{h} \\ &= -\lim_{h\to 0^-} \frac{\phi(x,-h)-\phi(x,0)}{h} \\ &= \lim_{h\to 0^+} \frac{\phi(x,h)-\phi(x,0)}{h} = \partial_{z+}\psi(x,0) > 0. \end{aligned}$$

Therefore, ψ is a subsolution to (3.5.10). It is straightforward to check that $\psi > 0$ in $[S_r(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^-$ and that $\psi \leq 0$ on $[\partial S_{2r}(x_0,z_0)]^-$.

Lastly, if $(x,z) \in [\partial S_{\gamma r}(x_0,z_0)]^-$, then $(x,-z) \in [\partial S_{\gamma r}(x_0,-z_0)]^+$. This gives the desired estimate

$$\psi(x,z) = \phi(x,-z) \leq Car \quad \text{for } (x,z) \in [\partial S_{\gamma r}(x_0,z_0)]^-.$$

□

3.5.3.3 Localization

Lemma 3.5.12. *Let $0 < s < 1$ and fix $0 < \gamma < 1$. Assume that $a^{ij} = a^{ij}(x)$ are bounded, measurable functions on $\mathbb{R}^n$ and satisfy (3.1.1). Let $\hat{K}_3$ be as in (3.5.4). For a cube $Q_R = Q_R(\tilde{x},\tilde{z}) \subset \mathbb{R}^{n+1}$, suppose that $U = U(x,z) = U(x,-z)$ is a nonnegative, classical supersolution to*

$$\begin{cases} a^{ij}(x)\partial_{ij}U + |z|^{2-\frac{1}{s}}\,\partial_{zz}U \leq 0 & \text{in } Q_{\hat{K}_3 R} \cap \{z \neq 0\} \\ -\partial_{z+}U \geq 0 & \text{on } Q_{\hat{K}_3 R} \cap \{z = 0\}. \end{cases}$$

Assume that $Q_r(x_0,z_0) \subset Q_R$*. Suppose that* U *is touched from below in* $Q_{\hat{K}_3 R}$ *by paraboloid* P *of opening* $a>0$ *at* $(x_1,z_1) \in [S_r(x_0,z_0)]^{\pm} \cap A_{a,R}$*. Then, there exists a constant* $C = C(\gamma,n,\lambda,\Lambda,s) > 0$ *and a point* $(x_2,z_2) \in [\overline{S}_{\gamma r}(x_0,z_0)]^{\pm}$ *such that*

$$U(x_2,z_2) - P(x_2,z_2) \leq Car.$$

Proof. Assume that $z_0, z_1 \geq 0$. The case in which $z_0, z_1 \leq 0$ follows similarly, using symmetry.

We also assume that $(x_1,z_1) \in [S_r(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^+$. Indeed, if $(x_1,z_1) \in [\overline{S}_{\gamma r}(x_0,z_0)]^+$,

$$U(x_1,z_1) - P(x_1,z_1) = 0 \leq Car$$

for all $C>0$, so we take $(x_2,z_2) = (x_1,z_1)$.

Let $W = U - P$. For $(x,z) \in Q_{\hat{K}_3 R} \setminus \{z=0\}$, we have that

$$a^{ij}(x)\partial_{ij}P(x,z) + |z|^{2-\frac{1}{s}}\,\partial_{zz}P(x,z) = -a(a^{ii}(x)+1) \geq -a(n\Lambda+1)$$

which implies

$$a^{ij}(x)\partial_{ij}W(x,z) + |z|^{2-\frac{1}{s}}\,\partial_{zz}W(x,z) \leq a(n\Lambda+1).$$

Since $z_1 \geq 0$, by Lemmas 3.5.3 and 3.5.4, we know that $z_v \geq 0$. Therefore,

$$-\partial_{z+}W(x,0) \geq 0 + ah'(z_v) \geq 0.$$

Let ϕ be the subsolution found in Lemma 3.5.11. By choice of $\hat{K}_2$ (3.5.3), we have that $Q_r(x_0,z_0) \subset Q_R$ implies

$$S_{2r}(x_0,z_0) \subset Q_{2r}(x_0,z_0) \subset Q_{\hat{K}_2 R} \subset Q_{\hat{K}_3 R}$$

Therefore, $W - \phi$ is a supersolution to

$$\begin{cases} a^{ij}(x)\partial_{ij}(W-\phi) + |z|^{2-\frac{1}{s}}\,\partial_{zz}(W-\phi) < 0 & \text{in } [S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^+ \cap \{z \neq 0\} \\ -\partial_{z+}(W-\phi)(x,0) > 0 & \text{on } [S_{2r}(x_0,z_0) \setminus S_{\gamma r}(x_0,z_0)]^+ \cap \{z = 0\}. \end{cases} \tag{3.5.17}$$

Let $(x_2, z_2) \in [\overline{S}_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$ be such that

$$W(x_2, z_2) - \phi(x_2, z_2) = \min_{[\overline{S}_{2r}(x_0,z_0)\setminus S_{\gamma r}(x_0,z_0)]^+} (W - \phi).$$

By the Maximum Principle (see for instance [31, Theorem 3.1]), the minimum of $W - \phi$ occurs on the boundary $\partial[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$. That is,

$$(x_2, z_2) \in [\partial S_{2r}(x_0, z_0)]^+ \cup [\partial S_{\gamma r}(x_0, z_0)]^+ \cup [(S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)) \cap \{z = 0\}].$$

Since $(x_1, z_1) \in [S_r(x_0, z_0)]^+$, we know that $\phi(x_1, z_1) > 0$ which implies

$$W(x_1, z_1) - \phi(x_1, z_1) = 0 - \phi(x_1, z_1) < 0.$$

Moreover, since $\phi \leq 0$ on $[\partial S_{2r}(x_0, z_0)]^+$, we have that

$$W(x, z) - \phi(x, z) \geq 0 \quad \text{on } [\partial S_{2r}(x_0, z_0)]^+.$$

Therefore, the minimum is strictly negative and cannot occur on $[\partial S_{2r}(x_0, z_0)]^+$.

If $[S_{2r}(x_0, z_0)]^+ \cap \{z = 0\} \neq \emptyset$, then suppose, by way of contradiction, that the minimum occurs on $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z = 0\}$, i.e. $z_2 = 0$. Then

$$-\partial_{z+}(W - \phi)(x_2, 0) \leq 0,$$

which contradicts (3.5.17). Therefore, it must be that the minimum occurs at $(x_2, z_2) \in [\partial S_{\gamma r}(x_0, z_0)]^+ \subset [\overline{S}_{\gamma r}(x_0, z_0)]^+$. It follows from Lemma 3.5.11 that $\phi(x_2, z_2) \leq Car$. Since $W(x_2, z_2) - \phi(x_2, z_2) < 0$, this implies

$$U(x_2, z_2) - P(x_2, z_2) < \phi(x_2, z_2) \leq Car.$$

□

3.5.3.4 Proof of lemma

Proof of Lemma 3.5.7. Without loss of generality, let

$$(x_1, z_1) \in Q_r(x_0, z_0) \cap A_{a,R} \neq \emptyset.$$

Otherwise, replace r by $r + \varepsilon$ and take the limit as $\varepsilon \to 0$ at the end.

Since $(x_1, z_1) \in A_{a,R}$, there is a paraboloid P of opening $a > 0$ with vertex $(x_v, z_v) \in B_v$ such that P touches U from below in $Q_{\hat{K}_3 R}$ at (x_1, z_1). We write P as

$$P(x,z) = -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_1, z_1)) + U(x_1, z_1).$$

We may assume that $z_0, z_1 \geq 0$ or that $z_0, z_1 \leq 0$. Indeed, suppose that $z_0 \geq 0$ and $z_1 < 0$. If $\overline{Q}_{\hat{K}_2 R} \cap \{z = 0\} = \emptyset$, this is a contradiction. Suppose that $\overline{Q}_{\hat{K}_2 R} \cap \{z = 0\} \neq \emptyset$. By Lemma 3.5.5, $\bar{P}(x, z) = P(x, -z)$ touches U from below in $Q_{\hat{K}_3 R}$ at $(x_1, -z_1)$ with vertex $(x_v, -z_v) \in B_v$. Since

$$\begin{aligned}
\delta_h(z_0, -z_1) &= h(-z_1) - h(z_0) - h'(z_0)(-z_1 - z_0) \\
&= h(z_1) - h(z_0) + h'(z_0)z_1 + h'(z_0)z_0 \\
&< h(z_1) - h(z_0) - h'(z_0)z_1 + h'(z_0)z_0 \quad \text{since } z_0 \geq 0 \text{ and } z_1 < 0 < -z_1 \\
&= h(z_1) - h(z_0) - h'(z_0)(z_1 - z_0) \\
&= \delta_h(z_0, z_1) < r
\end{aligned}$$

It follows that $(x_1, -z_1) \in Q_r(x_0, z_0) \cap A_{a,R}$. We proceed with the proof of the lemma using $\bar{P}$ and $-z_1 > 0$ in place of P and $z_1 < 0$. The argument for $z_0 \leq 0$ and $z_1 > 0$ follows similarly.

Note that $(x_1, z_1) \in Q_r(x_0, z_0) \subset S_{(n+1)r}(x_0, z_0)$ and let $\gamma = \eta/(2\theta^2)$. By Lemma 3.5.12, there is a point $(x_2, z_2) \in \overline{S}_{\gamma r/(n+1)}(x_0, z_0) \subset S_{\gamma r}(x_0, z_0)$ and a constant $C = C(n, \lambda, \Lambda, s) > 0$ such that

$$U(x_2, z_2) - P(x_2, z_2) \leq Car.$$

Let $\alpha = \eta/(2\theta^3) < 1$ and let $C' = C'(n, \lambda, \Lambda, s) > 1$ be a large constant, to be determined. Slide, from below, the family of paraboloids

$$\tilde{P}(x, z) = P(x, z) - C'a\delta_\Phi((\bar{x}_v, \bar{z}_v), (x, z)) + d, \quad \text{for } (\bar{x}_v, \bar{z}_v) \subset S_{\alpha r}(x_2, z_2) \tag{3.5.18}$$

until they touch the graph of U in $Q_{\hat{K}_3 R}$ for the first time. Without tracking the constant term, we expand a paraboloid $\tilde{P}$ by

$$\begin{aligned}
\tilde{P}(x, z) &= P(x, z) - C'a\delta_\Phi((\bar{x}_v, \bar{z}_v), (x, z)) + d \\
&= -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_1, z_1)) + U(x_1, z_1) - C'a\delta_\Phi((\bar{x}_v, \bar{z}_v), (x, z)) + d \\
&= -a\delta_\varphi(x_v, x) - C'a\delta_\varphi(\bar{x}_v, x) - a\delta_h(z_v, z) - C'a\delta_h(\bar{z}_v, z) + d'.
\end{aligned}$$

Suppose that $\xi \in \mathbb{R}$ is such that

$$h'(\xi) = \frac{h'(z_v) + C'h'(\bar{z}_v)}{C'+1}.$$

in the variable z, we write

$$\begin{aligned}
&-a\delta_h(z_v, z) - C'a\delta_h(\bar{z}_v, z)\\
&\quad = -a\left(h(z) - h(z_v) - h'(z_v)(z - z_v)\right) - C'a\left(h(z) - h(\bar{z}_v) - h'(\bar{z}_v)(z - \bar{z}_v)\right)\\
&\quad = -(C'+1)ah(z) + ah(z_v) + ah'(z_v)z - ah'(z_v)z_v + C'ah(\bar{z}_v) + C'ah'(\bar{z}_v)z - C'ah'(\bar{z}_v)\bar{z}_v\\
&\quad = -(C'+1)ah(z) + \left(ah'(z_v) + C'ah'(\bar{z}_v)\right)z + d'\\
&\quad = -(C'+1)a\left(h(z) + \frac{h'(z_v) + C'h'(\bar{z}_v)}{C'+1}z\right) + d'\\
&\quad = -(C'+1)a\left(h(z) - h'(\xi)z\right) + d'\\
&\quad = -(C'+1)a\left(h(z) - h(\xi) - h'(\xi)(z - \xi)\right) + d''\\
&\quad = -(C'+1)a\delta_h(\xi, z) + d''.
\end{aligned}$$

Since

$$\frac{D\varphi(x_v) + C'D\varphi(\bar{x}_v)}{C'+1} = \frac{x_v + C'\bar{x}_v}{C'+1} = D\varphi\left(\frac{x_v + C'\bar{x}_v}{C'+1}\right),$$

in the variable x, we similarly write

$$-a\delta_\varphi(x_v, x) - C'a\delta_\varphi(\bar{x}_v, x) = -(C'+1)a\delta_\varphi(\frac{x_v + C'\bar{x}_v}{C'+1}, x) + d'''.$$

Therefore

$$\tilde{P}(x, z) = -(C'+1)a\delta_\Phi\left(\left(\frac{x_v + C'\bar{x}_v}{C'+1}, \xi\right), (x, z)\right) + d''''.$$

Hence, the opening of $\tilde{P}$ is $(C'+1)a$ and the vertex is of the form

$$\left(\frac{x_v + C'\bar{x_v}}{C'+1}, \xi\right) \quad \text{where } h'(\xi) = \frac{h'(z_v) + C'h'(\bar{z}_v)}{C'+1}.$$

Let B denote the set of vertices for the family of $\tilde{P}$'s and A denote the set of touching points.

Since $\tilde{P}(x_2, z_2) \le U(x_2, z_2)$, we have that

$$P(x_2, z_2) - C'a\delta_\Phi((\bar{x}_v, \bar{z}_v), (x_2, z_2)) + d \le U(x_2, z_2).$$

By the engulfing property, $S_{\alpha r}(x_2, z_2) \subset S_{\alpha\theta r}(\bar{x}_v, \bar{z}_v)$, so that

$$\delta((\bar{x}_v, \bar{z}_v), (x_2, z_2)) < \alpha\theta r.$$

Therefore,

$$d \le U(x_2, z_2) - P(x_2, z_2) + C'a\delta((\bar{x}_v, \bar{z}_v), (x_2, z_2)) \le Car + C'\alpha\theta ar.$$

Since $(x_2, z_2) \in S_{\alpha\theta r}(\bar{x}_v, \bar{z}_v) \subset S_{2\alpha\theta r}(\bar{x}_v, \bar{z}_v)$, we again use the engulfing property to see that

$$S_{2\alpha\theta r}(\bar{x}_v, \bar{z}_v) \subset S_{2\alpha\theta^2 r}(x_2, z_2).$$

Suppose that $(x, z) \in Q_{\hat{K}_3 R}$ is such that $\delta_\Phi((x_2, z_2), (x, z)) \ge 2\alpha\theta^2 r$. Then $\delta_\Phi((\bar{x}_v, \bar{z}_v), (x, z)) \ge 2\alpha\theta r$ and

$$\begin{aligned} \tilde{P}(x, z) &\le P(x, z) - C'a(2\alpha\theta r) + \big(Car + C'\alpha\theta ar\big) \\ &= P(x, z) + \big(C - C'\theta\alpha\big)\, ar < P(x, z) \le U(x, z) \end{aligned}$$

when C' is such that $C' > C/(\theta\alpha)$. Hence, the contact points for $\tilde{P}$ are inside $S_{2\alpha\theta^2 r}(x_2, z_2)$.

Recall that $(x_2, z_2) \in \overline{S}_{\gamma r}(x_0, z_0)$. Since $\gamma = \alpha\theta$, we use the engulfing property to obtain

$$\begin{aligned} \overline{S}_{\gamma r}(x_0, z_0) &= \overline{S}_{\alpha\theta r}(x_0, z_0) \subset \overline{S}_{\alpha\theta^2 r}(x_2, z_2) \\ &\subset S_{2\alpha\theta^2 r}(x_2, z_2) \subset S_{2\alpha\theta^3 r}(x_0, z_0) = S_{\eta r}(x_0, z_0). \end{aligned} \tag{3.5.19}$$

Therefore, the contact points for $\tilde{P}$ are inside $S_{\eta r}(x_0, z_0) \subset Q_{\eta r}(x_0, z_0)$. That is, $A \subset Q_{\eta r}(x_0, z_0)$.

For sufficiently large C', we estimate

$$\begin{aligned} \tilde{P}(x, z) &\le P(x, z) + d \\ &= -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_1, z_1)) + U(x_1, z_1) + d \\ &\le a\delta_\Phi((x_v, z_v), (x_1, z_1)) + U(x_1, z_1) + d \\ &\le a\delta_\Phi((x_v, z_v), (x_1, z_1)) + aR + (Car + C'\alpha\theta ar) \\ &\le aK\left(\delta_\Phi((\tilde{x}, \tilde{z}), (x_v, z_v)) + \delta_\Phi((\tilde{x}, \tilde{z}), (x_1, z_1))\right) + aR + (CaR + C'\alpha\theta aR) \\ &\le aK(\hat{K}_3 R + R) + aR + (CaR + C'\alpha\theta aR) \\ &= \Big((\hat{K}_3 + 1)K + 1 + C + C'\alpha\theta\Big)\, aR \le (C' + 1)aR. \end{aligned}$$

This shows that $A \subset A_{(C'+1)a,R}$.

Therefore, by Lemma 3.5.6,

$$\mu_\Phi(A_{(C'+1)a,R} \cap Q_{\eta r}(x_0, z_0)) \geq \mu_\Phi(A \cap Q_{\eta r}(x_0, z_0)) = \mu_\Phi(A) \geq c\mu_\Phi(B). \tag{3.5.20}$$

We claim that

$$c\mu_\Phi(B) \geq c'\mu_\Phi(Q_r(x_0, z_0))$$

for a small constant $c' = c'(n, \lambda, \Lambda, s) < 1$.

We can express the set B as

$$B = \left\{(x, z) : x = \frac{x_v + C'\bar{x}_v}{C'+1},\ h'(z) = \frac{h'(z_v) + C'h'(\bar{z}_v)}{C'+1},\ (\bar{x}_v, \bar{z}_v) \in S_{\alpha r}(x_2, z_2)\right\}.$$

We will first show that

$$\mu_\Phi(B) \geq \left(\frac{C'}{C'+1}\right)^{n+1} \mu_\Phi(S_{\frac{\alpha r}{2}}(x_2, z_2)).$$

First, note that

$$S_{\alpha r/2}(x_2, z_2) \subset S_\varphi\left(x_2, \frac{\alpha r}{2}\right) \times S_h\left(z_2, \frac{\alpha r}{2}\right) \subset S_{\alpha r}(x_2, z_2).$$

Define the sets B_x and B_z by

$$B_x = \left\{x = \frac{x_v + C'\bar{x}_v}{C'+1} : \bar{x}_v \in S_\varphi\left(x_2, \frac{\alpha r}{2}\right)\right\}$$
$$B_z = \left\{z = h^{-1}\left(\frac{h'(z_v) + C'h'(\bar{z}_v)}{C'+1}\right) : \bar{z}_v \in S_h\left(z_2, \frac{\alpha r}{2}\right)\right\}.$$

Therefore, $B_x \times B_z \subset B$ and

$$\mu_\Phi(B) = \int_B h''(z)\,dz \geq \int_{B_x}\int_{B_z} h''(z)\,dz\,dx = \int_{B_x} dx \int_{B_z} h''(z)\,dz.$$

By change of variables, we have that

$$\int_{B_x} dx = \left(\frac{C'}{C'+1}\right)^n \int_{S_\varphi\left(x_2, \frac{\alpha r}{2}\right)} d\bar{x}_v.$$

Notice that

$$\{\bar{z}_v : h'(\bar{z}_v) = -\frac{1}{C'}h'(z_v)\} = \{\bar{z}_v = 0\}.$$

Then, by change of variables, we have that

$$\begin{aligned}
&\int_{B_z} h''(z)\,dz \\
&= \int_{B_z\setminus\{z=0\}} h''(z)\,dz \\
&= \int_{S_h\left(z_2,\frac{\alpha r}{2}\right)\setminus\{\bar{z}_v=0\}} h''\left((h')^{-1}\left(\frac{h'(z_v)+C'h'(\bar{z}_v)}{C'+1}\right)\right)\partial_z(h')^{-1}\Big|_{\frac{h'(z_v)+C'h'(\bar{z}_v)}{C'+1}}\left(\frac{C'}{C'+1}\right)h''(\bar{z}_v)d\bar{z}_v \\
&= \frac{C'}{C'+1}\int_{S_h\left(z_2,\frac{\alpha r}{2}\right)\setminus\{\bar{z}_v=0\}} h''\left((h')^{-1}\left(\frac{h'(z_v)+C'h'(\bar{z}_v)}{C'+1}\right)\right)\frac{h''(\bar{z}_v)}{h''\left((h')^{-1}\left(\frac{h'(z_v)+C'h'(\bar{z}_v)}{C'+1}\right)\right)}d\bar{z}_v \\
&= \frac{C'}{C'+1}\int_{S_h\left(z_2,\frac{\alpha r}{2}\right)\setminus\{\bar{z}_v=0\}} h''(\bar{z}_v)d\bar{z}_v \\
&= \frac{C'}{C'+1}\int_{S_h\left(z_2,\frac{\alpha r}{2}\right)} h''(\bar{z}_v)d\bar{z}_v.
\end{aligned}$$

Combing these estimates, we obtain

$$\begin{aligned}
\mu_\Phi(B) &\geq \left(\frac{C'}{C'+1}\right)^{n+1}\int_{S_\varphi\left(x_2,\frac{\alpha r}{2}\right)\times S_h\left(z_2,\frac{\alpha r}{2}\right)} h''(z)\,dz\,dx \\
&\geq \left(\frac{C'}{C'+1}\right)^{n+1}\int_{S_{\frac{\alpha r}{2}}(x_2,z_2)} h''(z)\,dz\,dx = \left(\frac{C'}{C'+1}\right)^{n+1}\mu_\Phi(S_{\frac{\alpha r}{2}}(x_2,z_2))
\end{aligned}$$

as desired.

By the doubling estimate (3.3.4), we estimate

$$\mu_\Phi(S_{\gamma\theta r}(x_2,z_2)) \leq K_d\left(\frac{\gamma\theta r}{\alpha r/2}\right)^\nu \mu_\Phi(S_{\frac{\alpha r}{2}}(x_2,z_2)) = K_d\left(\frac{2\theta\gamma}{\alpha}\right)^\nu \mu_\Phi(S_{\frac{\alpha r}{2}}(x_2,z_2))$$

and

$$\mu_\Phi(S_{(n+1)r}(x_0,z_0)) \leq K_d\left(\frac{(n+1)r}{\gamma r}\right)^\nu \mu_\Phi(S_{\gamma r}(x_0,z_0)) = K_d\left(\frac{n+1}{\gamma}\right)^\nu \mu_\Phi(S_{\gamma r}(x_0,z_0)).$$

Since $(x_2, z_2) \in \overline{S}_{\gamma r}(x_0, z_0)$, the engulfing property gives $\overline{S}_{\gamma r}(x_0, z_0) \subset \overline{S}_{\gamma\theta r}(x_2, z_2)$. Hence, we have

$$\begin{aligned}
c\mu_\Phi(B) &\geq c\left(\frac{C'}{C'+1}\right)^{n+1} \mu_\Phi(S_{\frac{\alpha r}{2}}(x_2, z_2)) \\
&\geq c\left(\frac{C'}{C'+1}\right)^{n+1} \frac{1}{K_d}\left(\frac{\alpha}{2\theta\gamma}\right)^{\nu} \mu_\Phi(S_{\gamma\theta r}(x_2, z_2)) \\
&\geq c\left(\frac{C'}{C'+1}\right)^{n+1} \frac{1}{K_d}\left(\frac{\alpha}{2\theta\gamma}\right)^{\nu} \mu_\Phi(S_{\gamma r}(x_0, z_0)) \\
&\geq c\left(\frac{C'}{C'+1}\right)^{n+1} \frac{1}{K_d}\left(\frac{\alpha}{2\theta\gamma}\right)^{\nu} \frac{1}{K_d}\left(\frac{\gamma}{n+1}\right)^{\nu} \mu_\Phi(S_{(n+1)r}(x_0, z_0)) \\
&= c\left(\frac{C'}{C'+1}\right)^{n+1} \frac{1}{K_d^2}\left(\frac{\alpha}{2\theta(n+1)}\right)^{\nu} \mu_\Phi(S_{(n+1)r}(x_0, z_0)) \\
&\geq c'\mu_\Phi(Q_r(x_0, z_0)).
\end{aligned}$$

This completes the proof of the claim.

From (3.5.20), we conclude that

$$\mu_\Phi(A_{(C'+1)a,R} \cap Q_{\eta r}(x_0, z_0)) \geq c'\mu_\Phi(Q_r(x_0, z_0)).$$

□

3.5.4 Proof of Lemma 3.5.8

We will need the following variation of [23, Theorem 1.2] for cubes.

Theorem 3.5.1. *Let $E \subset \mathbb{R}^{n+1}$ be a bounded subset and $\{Q_{r_{x,z}}(x, z)\}$ be a covering of E. There exists a countable family $\{Q_{r_i}(x_i, z_i)\}$ of disjoint cubes such that the family $\{Q_{K_0 r_i}(x_i, z_i)\}$ covers E. The constant $K_0 > 3K$, where K is the quasi-distance constant.*

Corollary 3.5.1. *Let $E \subset \mathbb{R}^{n+1}$ be a bounded subset and assume that for each $x \in E$, we associate a cube $\{Q_{r_{x,z}}(x, z)\}$. Then, we can find a countable number of these sections $Q_{r_i}(x_i, z_i)$ such that*

$$E \subset \bigcup_{i=1}^{\infty} Q_{r_i}(x_i, z_0), \quad \textit{with } Q_{r_i/K_0}(x_i, z_i) \textit{ disjoint.}$$

Proof of Lemma 3.5.8. Let $(x_0, z_0) \in Q_{R/K_0}(\tilde{x}, \tilde{z}) \setminus D_k$ and let r be given by

$$r = \inf\{r_0 : Q_{r_0}(x_0, z_0) \cap D_k \neq \emptyset\}. \tag{3.5.21}$$

By Corollary 3.5.1, there is a countable collection of these cubes $\{Q_{r_i}(x_i, z_i)\}$ such that

$$Q_{R/K_0} \setminus D_k \subset \bigcup_i Q_{r_i}(x_i, z_i), \quad \text{with } Q_{r_i/K_0}(x_i, z_i) \text{ disjoint.}$$

Then,

$$\mu_\Phi(Q_{R/K_0} \setminus D_k) \le \mu_\Phi\left(\bigcup_i Q_{r_i}(x_i, z_i) \cap Q_{R/K_0}\right) \le \sum_i \mu_\Phi(Q_{r_i}(x_i, z_i) \cap Q_{R/K_0}).$$

We claim, for any $(x_0, z_0) \in Q_{R/K_0} \setminus D_k$, that and r given by (3.5.21),

$$\mu_\Phi(Q_r(x_0, z_0) \cap Q_{R/K_0}) \le \frac{1}{c}\mu_\Phi(Q_{r/K_0}(x_0, z_0) \cap D_{k+1}).$$

Suppose for now that the claim holds. Then

$$\begin{aligned}
\mu_\Phi(Q_{R/K_0} \setminus D_k) &\le \mu_\Phi\left(\bigcup_i Q_{r_i}(x_i, z_i) \cap Q_{R/K_0}\right) \\
&\le \sum_i \mu_\Phi(Q_{r_i}(x_i, z_i) \cap Q_{R/K_0}) \\
&\le \sum_i \frac{1}{c}\mu_\Phi(Q_{r_i/K_0}(x_i, z_i) \cap D_{k+1}) \\
&= \frac{1}{c}\mu_\Phi\left(\bigcup_i Q_{r_i/K_0}(x_i, z_i) \cap (D_{k+1} \setminus D_k)\right) \\
&\le \frac{1}{c}\mu_\Phi(D_{k+1} \setminus D_k).
\end{aligned}$$

Since

$$Q_{R/K_0} \setminus D_k = (Q_{R/K_0} \setminus D_{k+1}) \cup (D_{k+1} \setminus D_k),$$

we have that

$$\begin{aligned}
\mu_\Phi(Q_{R/K_0} \setminus D_{k+1}) &= \mu_\Phi(Q_{R/K_0} \setminus D_k) - \mu_\Phi(D_{k+1} \setminus D_k) \\
&\le \mu_\Phi(Q_{R/K_0} \setminus D_k) - c\mu_\Phi(Q_{R/K_0} \setminus D_k) \\
&= (1-c)\mu_\Phi(Q_{R/K_0} \setminus D_k).
\end{aligned}$$

We iterate to obtain

$$\begin{aligned}\mu_\Phi(Q_{R/K_0}\setminus D_k) &\le (1-c)\mu_\Phi(Q_{R/K_0}\setminus D_{k-1})\\ &\le (1-c)^2\mu_\Phi(Q_{R/K_0}\setminus D_{k-2})\\ &\vdots\\ &\le (1-c)^k\mu_\Phi(Q_{R/K_0}\setminus D_0)\\ &\le (1-c)^k\mu_\Phi(Q_{R/K_0})\end{aligned}$$

and complete the proof of the lemma.

It is left to prove the claim. We will present the proof for $n=1$ for which we have that

$$Q_t(x,z)=S_t(x)\times S_t(z)\subset\mathbb{R}^2.$$

The more general case following similarly.

First, let $(x_1,z_1),(x_2,z_2)\in Q_{R/K_0}(\tilde{x},\tilde{z})$. Then

$$\delta_\varphi(x_1,x_2)\le K\left(\delta_\varphi(\tilde{x},x_1)+\delta_\varphi(\tilde{x},x_2)\right)<K\left(\frac{R}{K_0}+\frac{R}{K_0}\right)=\frac{2KR}{K_0}.$$

Similarly, $\delta_h(z_1,z_2)<2KR/K_0$. Therefore, $r<2KR/K_0$.

Let $(x,z)\in Q_r(x_0,z_0)$. By the quasi-triangle inequality (3.3.3) and choice of K_0 (3.5.5),

$$\begin{aligned}\delta_\varphi(\tilde{x},x) &\le K\left(\delta_\varphi(\tilde{x},x_0)+\delta_\varphi(x_0,x)\right)\\ &< K\left(\frac{R}{K_0}+r\right)\\ &< K\left(\frac{R}{K_0}+\frac{2KR}{K_0}\right)=\frac{K+2K^2}{K_0}R\le R.\end{aligned}$$

Similarly, one can show that $\delta_h(\tilde{z},z)<R$. Therefore, we have that $Q_r(x_0,z_0)\subset Q_R(\tilde{x},\tilde{z})$.

Without loss of generality, assume that $x_0\le\tilde{x}$ and $z_0\le\tilde{z}$. We will break into cases based on how far $(\tilde{x},\tilde{z})$ is from (x_0,z_0).

Case 1. Suppose that $(\tilde{x},\tilde{z})\in Q_{r/K_0}(x_0,z_0)$.

We will show that $Q_r(x_0,z_0)$ satisfies the hypothesis of property 2):

$$Q_r(x_0,z_0)\subset Q_R(\tilde{x},\tilde{z}),\quad Q_{\eta r}(x_0,z_0)\subset Q_{R/K_0}(\tilde{x},\tilde{z}),\quad \overline{Q}_r(x_0,z_0)\cap D_k\ne\emptyset.$$

We have already shown that $Q_r(x_0, z_0) \subset Q_R(\tilde{x}, \tilde{z})$, and by the definition of r, we know that $\overline{Q}_r(x_0, z_0) \cap D_k \neq \emptyset$. Thus, it is left to show that $Q_{\eta r}(x_0, z_0) \subset Q_{R/K_0}(\tilde{x}, \tilde{z})$. Let $(x, z) \in Q_{\eta r}(x_0, z_0)$. By the quasi-triangle inequality (3.3.3) and by our choice of K_0 and η (3.5.5),

$$\begin{aligned}
\delta_\varphi(\tilde{x}, x) &\leq K\left(\delta_\varphi(x_0, \tilde{x}) + \delta_\varphi(x_0, x)\right) \\
&< K\left(\frac{r}{K_0} + \eta r\right) \\
&\leq K\left(\frac{1}{K_0} + \eta\right)\frac{2KR}{K_0} \leq \frac{R}{K_0}.
\end{aligned}$$

We can similarly show that $\delta_h(\tilde{z}, z) < R/K_0$. Hence, $Q_{\eta r}(x_0, z_0) \subset Q_{R/K_0}(\tilde{x}, \tilde{z})$.

Therefore, by property 2), we know that

$$\mu_\Phi(Q_{\eta r}(x_0, z_0) \cap D_{k+1}) \geq c\mu_\Phi(Q_r(x_0, z_0)).$$

Since $\eta \leq 1/K_0$, we obtain the desired estimate

$$\begin{aligned}
\mu_\Phi(Q_{r/K_0}(x_0, z_0) \cap D_{k+1}) &\geq \mu_\Phi(Q_{\eta r}(x_0, z_0) \cap D_{k+1}) \\
&\geq c\mu_\Phi(Q_r(x_0, z_0)) \\
&\geq c\mu_\Phi(Q_r(x_0, z_0) \cap Q_{R/K_0}(\tilde{x}, \tilde{z})).
\end{aligned}$$

Case 2. Suppose that $\tilde{x} \notin S_{r/K_0}(x_0)$, $\tilde{z} \in S_{r/K_0}(z_0)$.

From our previous work, we deduce that

$$S_r(z_0) \subset S_R(\tilde{z}), \quad S_{\eta r}(z_0) \subset S_{R/K_0}(\tilde{z}).$$

We will find an x_1 between x_0 and $\tilde{x}$ and a positive constant $\beta < 1$ such that

$$S_{\beta r}(x_1) \subset S_{r/K_0}(x_0) \cap S_{R/K_0}(\tilde{x}).$$

Let $x_1 > x_0$ be such that $\delta_\varphi(x_0, x_1) = r/(2KK_0)$. We first show that $S_{r/(2KK_0)}(x_1) \subset S_{r/K_0}(x_0)$. Indeed, for $x \in S_{r/(2KK_0)}(x_1)$, we have that

$$\delta_\varphi(x_0, x) \leq K\left(\delta_\varphi(x_0, x_1) + \delta_\varphi(x_1, x)\right) < K\left(\frac{r}{2KK_0} + \frac{r}{2KK_0}\right) = \frac{r}{K_0}.$$

Since

$$\frac{r}{2KK_0} = \delta_\varphi(x_0, x_1) \leq K\delta_\varphi(x_1, x_0) \leq K^2\delta_\varphi(x_0, x_1) = K^2\frac{r}{2KK^0},$$

we know that

$$\frac{r}{2K^2K_0} \leq \delta_\varphi(x_1, x_0) \leq \frac{r}{2K_0}.$$

Thus, $x_0 \notin S_{r/(2K^2K_0)}(x_1)$. Since the sections $S_{r/(2K^2K_0)}(x_1)$ and $S_{r/K_0}(x_0)$ are one-dimensional intervals, we can write them as

$$S_{r/(2K^2K_0)}(x_1) = (x_L, x_R) \quad \text{where } x_L < x_1 < x_R$$
$$S_{r/K_0}(x_0) = (x_L^0, x_R^0) \quad \text{where } x_L^0 < x_0 < x_R^0.$$

Since $\tilde{x} \notin S_{r/K_0}(x_0)$ and $x_0 < \tilde{x}$, we know that

$$x_L^0 < x_0 < x_R^0 < \tilde{x}.$$

Since $x_0 < x_1$ and $S_{r/(2K^2K_0)}(x_1) \subset S_{r/K_0}(x_0)$, we have that

$$x_0 < x_L < x_1 < x_R < x_R^0 < \tilde{x}.$$

Thus, for any $x \in S_{r/(2K^2K_0)}(x_1)$, we know that $x_0 < x < \tilde{x}$. By the convexity of φ,

$$\frac{\varphi(x) - \varphi(x_0)}{x - x_0} < \varphi'(\tilde{x})$$

which implies

$$\begin{aligned}
\delta_\varphi(\tilde{x}, x) &= \varphi(x) - \varphi(\tilde{x}) - \varphi'(\tilde{x})(x - \tilde{x}) \\
&< \varphi(x_0) - \varphi(\tilde{x}) - \varphi'(\tilde{x})(x_0 - \tilde{x}) \\
&= \delta_\varphi(\tilde{x}, x_0) \\
&< \frac{R}{K_0}
\end{aligned}$$

for any $x \in S_{r/(2K^2K_0)}(x_1)$. Hence, $S_{r/(2K^2K_0)}(x_1) \subset S_{R/K_0}(\tilde{x})$ and we proven the claim for $\beta = 1/(2KK_0)$.

Define

$$\rho = \left(K + \frac{1}{2K_0}\right) r.$$

We claim that $Q_r(x_0, z_0) \subset Q_\rho(x_1, z_0)$. Clearly $S_r(z_0) \subset S_\rho(z_0)$. Let $x \in S_r(x_0)$. Then,

$$\begin{aligned}\delta_\varphi(x_1, x) &\leq K\left(\delta_\varphi(x_0, x_1) + \delta_\varphi(x_1, x)\right)\\ &\leq K\left(\frac{r}{2KK_0} + r\right) = \rho.\end{aligned}$$

Hence, $S_r(x_0) \subset S_\rho(x_1)$. Therefore,

$$Q_r(x_0, z_0) = S_r(x_0) \times S_r(z_0) \subset S_\rho(x_1) \times S_\rho(z_0) = Q_\rho(x_1, z_0).$$

Since $\overline{Q}_r(x_0, z_0) \cap D_k \neq \emptyset$, we know that $\overline{Q}_\rho(x_1, z_0) \cap D_k \neq \emptyset$.

We will show that $Q_\rho(x_1, z_0)$ satisfies the assumptions on property 2):

$$Q_\rho(x_1, z_0) \subset Q_R(\tilde{x}, \tilde{z}), \quad Q_{\eta\rho}(x_1, z_0) \subset Q_{R/K_0}(\tilde{x}, \tilde{z}), \quad Q_{\eta\rho}(x_1, z_0) \subset Q_{r/K_0}(x_0, z_0).$$

First, let us check that $Q_\rho(x_1, z_0) \subset Q_R(\tilde{x}, \tilde{z})$. Take $(x, z) \in Q_\rho(x_1, z_0)$. Observe that

$$\begin{aligned}\delta_\varphi(\tilde{x}, x) &\leq K\left(\delta_\varphi(\tilde{x}, x_1) + \delta_\varphi(x_1, x)\right)\\ &< K\left(\frac{R}{K_0} + \rho\right)\\ &= K\left(\frac{R}{K_0} + \left(K + \frac{1}{2K_0}\right) r\right)\\ &< K\left(\frac{R}{K_0} + \left(K + \frac{1}{2K_0}\right)\frac{2KR}{K_0}\right)\\ &= \left(1 + 2K + \frac{1}{K_0}\right)\frac{KR}{K_0} \leq R\end{aligned}$$

by choice of K_0 (3.5.5). We can similarly show that $\delta_h(\tilde{z}, z) < R$. Hence, $Q_\rho(x_1, z_0) \subset Q_R(\tilde{x}, \tilde{z})$.

Next, we check that $Q_{\eta\rho}(x_1, z_0) \subset Q_{r/K_0}(x_0, z_0)$. By choice of η (3.5.5), we know that $S_{\eta\rho}(x_1) \subset S_{r/(2K^2K_0)}(x_1)$. Since $S_{r/(2K^2K_0)}(x_1) \subset S_{r/K_0}(x_0)$, we obtain

$$\begin{aligned}Q_{\eta\rho}(x_1, z_0) &= S_{\eta\rho}(x_1) \times S_{\eta\rho}(z_0)\\ &\subset S_{r/(2K^2K_0)}(x_1) \times S_{r/(2K^2K_0)}(z_0)\\ &\subset S_{r/K_0}(x_0) \times S_{r/K_0}(z_0) = Q_{r/K_0}(x_0, z_0).\end{aligned}$$

Lastly, we check that $Q_{\eta\rho}(x_1, z_0) \subset Q_{R/K_0}(\tilde{x}, \tilde{z})$. Indeed, for $z \in S_{\eta\rho}(z_0)$,

$$\begin{aligned}
\delta_h(\tilde{z}, z) &\leq K\left(\delta_h(z_0, \tilde{z}) + \delta_h(z_0, z)\right) \\
&< K\left(\frac{r}{K_0} + \eta\rho\right) \\
&= K\left(\frac{1}{K_0} + \eta\left(K + \frac{1}{2K_0}\right)\right) r \\
&< K\left(\frac{1}{K_0} + \eta\left(K + \frac{1}{2K_0}\right)\right) \frac{2KR}{K_0} \leq \frac{R}{K_0}.
\end{aligned}$$

by choice of K_0 and η (3.5.5). Therefore, $S_{\eta\rho}(z_0) \subset S_{R/K_0}(\tilde{z})$. Since $S_{\eta\rho}(x_1) \subset S_{r/(2K^2K_0)}(x_1) \subset S_{R/K_0}(\tilde{x})$, we obtain

$$\begin{aligned}
Q_{\eta\rho}(x_1, z_0) &= S_{\eta\rho}(x_1) \times S_{\eta\rho}(z_0) \\
&\subset S_{R/K_0}(\tilde{x}) \times S_{R/K_0}(\tilde{z}) = Q_{R/K_0}(\tilde{x}, \tilde{z}).
\end{aligned}$$

We have shown that $Q_\rho(x_1, z_0)$ satisfies the conditions of property 2). Therefore,

$$\mu_\Phi(Q_{\eta\rho}(x_1, z_0) \cap D_{k+1}) \geq c\mu_\Phi(Q_\rho(x_0, z_1)).$$

Since $\eta\rho \leq r/(2K^2K_0) \leq r/K_0$, we have that

$$\begin{aligned}
Q_{\eta\rho}(x_1, z_0) &= S_{\eta\rho}(x_1) \times S_{\eta\rho}(z_0) \\
&\subset S_{r/(2K^2K_0)}(x_1) \times S_{\eta\rho}(z_0) \\
&\subset S_{r/K_0}(x_0) \times S_{r/K_0}(z_0) = Q_{r/K_0}(x_0, z_0).
\end{aligned}$$

Therefore, we obtain the desired estimate

$$\begin{aligned}
\mu_\Phi(Q_{r/K_0}(x_0, z_0) \cap D_{k+1}) &\geq \mu_\Phi(Q_{\eta\rho}(x_1, z_0) \cap D_{k+1}) \\
&\geq c\mu_\Phi(Q_\rho(x_0, z_1)) \\
&\geq c\mu_\Phi(Q_r(x_0, z_0)).
\end{aligned}$$

Case 3. Suppose that $\tilde{x} \in S_{r/K_0}(x_0)$, $\tilde{z} \notin S_{r/K_0}(z_0)$.

This follows as in Case 2 by switching the $\tilde{x}$ and $\tilde{z}$ coordinates.

Case 4. Suppose that $\tilde{x} \notin S_{r/K_0}(x_0)$, $\tilde{z} \notin S_{r/K_0}(z_0)$.

This follows by combining Cases 2 and 3. □

3.5.5 Proof of Theorem 3.1.2

We will first prove the following variation of Harnack inequality then show that Theorem 3.1.2 follows.

Theorem 3.5.2. *Let $0 < s < 1$. Assume that $a^{ij} = a^{ij}(x)$ are bounded, measurable functions on $\mathbb{R}^n$ and satisfy (3.1.1). Let $\hat{K}_3$ be as in (3.5.4). There exist a positive constants $C_H = C_H(n, \lambda, \Lambda, s) > 1$ and $\kappa_2 = \kappa_2(n, \lambda, \Lambda, s) < 1$, such that for a cube $Q_R = Q_R(\tilde{x}, \tilde{z}) \subset \mathbb{R}^{n+1}$ and every nonnegative, classical solution $U = U(x, z) = U(x, -z)$ to*

$$\begin{cases} a^{ij}(x)\partial_{ij}U + |z|^{2-\frac{1}{s}}\,\partial_{zz}U = 0 & \text{in } Q_{\hat{K}_3 R} \setminus \{z = 0\} \\ -\partial_{z+}U(x, 0) = 0 & \text{on } Q_{\hat{K}_3 R} \cap \{z = 0\}, \end{cases}$$

we have that

$$\sup_{Q_{\kappa_2 R}} U \leq C_H U(\tilde{x}, \tilde{z}).$$

Proof. Let $a > 0$ be such that

$$\frac{aR}{2K_0} = U(\tilde{x}, \tilde{z}).$$

Slide the paraboloid

$$P(x, z) = -a\delta_\Phi((\tilde{x}, \tilde{z}), (x, z)) + C$$

from below in $Q_{\hat{K}_3 R}$ until it touches the graph of U, at say (x_0, z_0). We may write

$$P(x, z) = -a\delta_\Phi((\tilde{x}, \tilde{z}), (x, z)) + a\delta_\Phi((\tilde{x}, \tilde{z}), (x_0, z_0)) + U(x_0, z_0).$$

We claim that $(x_0, z_0) \in \overline{S}_{R/K_0}$. Indeed, if $\delta_\Phi((\tilde{x}, \tilde{z}), (x_0, z_0)) > R/K_0$, then

$$\frac{aR}{2K_0} > U(\tilde{x}, \tilde{z}) \geq P(\tilde{x}, \tilde{z}) = a\delta_\Phi((\tilde{x}, \tilde{z}), (x_0, z_0)) + U(x_0, z_0) > \frac{aR}{K_0},$$

which is a contradiction. Hence, $(x_0, z_0) \in \overline{S}_{R/K_0} \subset \overline{Q}_{R/K_0}$ and

$$A_{a,R} \cap \overline{Q}_{R/K_0} \neq \emptyset.$$

Define

$$D_k := A_{aC^k,R} \cap \overline{Q}_{R/K_0}, \quad k \geq 0$$

where C is the constant from Lemma 3.5.7. As a consequence of Lemma 3.5.2, we have

$$\emptyset \neq D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_k,$$

so that, by Lemma 3.5.7,

$$\mu_\Phi(D_k \cap Q_{\eta R/K_0}) = \mu_\Phi(A_{aC^k,R} \cap Q_{\eta R/K_0}) \geq c\mu_\Phi(Q_{R/K_0}).$$

By Lemma 3.5.8,

$$\mu_\Phi(Q_{R/K_0} \setminus D_k) \leq (1-c)^k \mu_\Phi(Q_{R/K_0}), \tag{3.5.22}$$

and from the definition of $A_{aC^k,R}$,

$$U(x,z) \leq aRC^k \quad \text{for } (x,z) \in D_k.$$

Define

$$\rho_k = c_0(1-c)^{k/\nu}, \quad c_0 = \frac{C^2\theta^2(n+1)}{K_0}\left(\frac{2K_d^2}{c}\right)^{1/\nu}.$$

where K_d is the doubling constant and $\nu = \log_2(K_d)$. For convenience in the notation, let

$$\beta = \frac{1}{3K_0}.$$

Suppose there exists a point $(x_k, z_k) \in Q_{\beta R/(n+1)} \subset S_{\beta R}$ such that

$$U(x_k, z_k) \geq aRC^{k+1}, \quad k \geq k_0$$

where $k_0 = k_0(n, \lambda, \Lambda, s)$ is a large constant, to be determined. We claim that there is a point $(x_{k+1}, z_{k+1}) \in \partial S_{\rho_k R}(x_k, z_k)$ such that

$$U(x_{k+1}, z_{k+1}) \geq aRC^{k+2}.$$

Suppose, by way of contradiction, that $U < aRC^{k+2}$ on $\partial S_{\rho_k R}(x_k, z_k)$. In the cylinder, $\{(x,z) : \delta_\Phi((x_k,z_k),(x,z)) \leq \rho_k R\}$, slide paraboloids of the form

$$P(x,z) = \frac{2aKC^{k+2}}{\rho_k}\delta_\Phi((x_v,z_v),(x,z)) + c_v, \quad (x_v,z_v) \in S_{\frac{\rho_k R}{\theta C^2}}(x_k,z_k)$$

from above until they touch the graph of U for the first time. Let A denote the set of contact points.

Slide the paraboloids further until they touch U at (x_k, z_k). Write these paraboloids as

$$\tilde{P}(x,z) = \frac{2aKC^{k+2}}{\rho_k}\delta_\Phi((x_v,z_v),(x,z)) - \frac{2aKC^{k+2}}{\rho_k}\delta_\Phi((x_v,z_v),(x_k,z_k)) + U(x_k,z_k).$$

Since $(x_v, z_v) \in S_{\frac{\rho_k R}{\theta C^2}}(x_k, z_k)$, we use the engulfing property to obtain

$$S_{\frac{\rho_k R}{\theta C^2}}(x_k,z_k) \subset S_{\frac{\rho_k R}{C^2}}(x_v,z_v).$$

In particular, we know $\delta_\Phi((x_v,z_v),(x_k,z_k)) \leq \frac{\rho_k R}{C^2}$. Therefore,

$$\begin{aligned}
\tilde{P}(x,z) &\geq \frac{2aKC^{k+2}}{\rho_k}\delta_\Phi((x_v,z_v),(x,z)) - \frac{2aKC^{k+2}}{\rho_k}\frac{\rho_k R}{C^2} + aRC^{k+1} \\
&= \frac{2aKC^{k+2}}{\rho_k}\delta_\Phi((x_v,z_v),(x,z)) + aRC^k\,(C-2K) \\
&\geq \frac{2aKC^{k+2}}{\rho_k}\delta_\Phi((x_v,z_v),(x,z)) + 2aRC^k.
\end{aligned}$$

Therefore, the height of U at the contact points is above $2aRC^k$ which shows

$$A \subset \{(x,z) \in S_{\rho_k R}(x_k,z_k) : U \geq 2aRC^k\}. \tag{3.5.23}$$

We will show that the contact points for the family of $\tilde{P}$'s are interior points of $S_{\rho_k R}(x_k, z_k)$. Let $(\bar{x}, \bar{z})$ be a contact point for $\tilde{P}$. Assume, by way of contradiction, that

$$\delta_\Phi((x_k,z_k),(\bar{x},\bar{z})) = \rho_k R.$$

By the quasi-triangle inequality (3.3.3),

$$\begin{aligned}
\rho_k R &\leq K\left(\delta_\Phi((x_k,z_k),(x_v,z_v)) + \delta_\Phi((x_v,z_v),(\bar{x},\bar{z}))\right) \\
&< K\left(\frac{\rho_k R}{\theta C^2} + \delta_\Phi((x_v,z_v),(\bar{x},\bar{z}))\right)
\end{aligned}$$

so that

$$\delta_\Phi((x_v,z_v),(\bar{x},\bar{z})) > \frac{\rho_k R}{K} - \frac{\rho_k R}{\theta C^2}.$$

Therefore,

$$\begin{aligned}\tilde{P}(\bar{x},\bar{z}) &\geq \frac{2aKC^{k+2}}{\rho_k}\delta_\Phi((x_v,z_v),(\bar{x},\bar{z})) + aRC^k\,(C-2K)\\ &> \frac{2aKC^{k+2}}{\rho_k}\left(\frac{\rho_k R}{K} - \frac{\rho_k R}{\theta C^2}\right) + aRC^k\,(C-2K)\\ &= 2aRC^{k+2} + 2aRC^k\left(C - 2K - \frac{K}{\theta}\right)\\ &> 2aRC^{k+2},\end{aligned}$$

which contradicts our assumption that $U < aRC^{k+2}$ on $\partial S_{\rho_k R}(x_k,z_k)$. Therefore, it must be that the contact points are interior points of $S_{\rho_k R}(x_k,z_k)$.

By Remark 3.5.1, it follows that

$$\mu_\Phi(A) \geq c\mu_\phi(S_{\frac{\rho_k R}{C^2\theta}}(x_k,z_k)).$$

Since $\beta < 1/K_0$, we have that

$$Q_{\beta R/(n+1)}(\tilde{x},\tilde{z}) \subset S_{\beta R}(\tilde{x},\tilde{z}) \subset S_{R/K_0}(\tilde{x},\tilde{z}) \subset Q_{R/K_0}(\tilde{x},\tilde{z}).$$

Since $(x_k,z_k) \in Q_{\beta R/(n+1)}(\tilde{x},\tilde{z}) \subset S_{R/K_0}(\tilde{x},\tilde{z})$, we use the engulfing property to obtain

$$S_{R/K_0}(\tilde{x},\tilde{z}) \subset S_{\theta R/K_0}(x_k,z_k).$$

As a consequence of the doubling property (3.3.4),

$$\begin{aligned}S_{\theta R/K_0}(x_k,z_k) &\leq K_d\left(\frac{\theta R/K_0}{\rho_k R/(C^2\theta)}\right)^\nu \mu_\phi(S_{\frac{\rho_k R}{C^2\theta}}(x_k,z_k))\\ &= K_d\left(\frac{C^2\theta^2}{\rho_k K_0}\right)^\nu \mu_\phi(S_{\frac{\rho_k R}{C^2\theta}}(x_k,z_k))\end{aligned}$$

and

$$\begin{aligned}\mu_\Phi(S_{R(n+1)/K_0}) &\leq K_d\left(\frac{R(n+1)/K_0}{R/K_0}\right)^\nu \mu_\Phi(S_{R/K_0})\\ &= K_d\,(n+1)^\nu\,\mu_\Phi(S_{R/K_0}).\end{aligned}$$

Thus, we use the above estimates to obtain

$$\begin{aligned}
\mu_\Phi(A) &\geq c\mu_\phi(S_{\frac{\rho_k R}{C^2\theta}}(x_k, z_k)) \\
&\geq cK_d^{-1}\left(\frac{\rho_k K_0}{C^2\theta^2}\right)^\nu \mu_\Phi(S_{\theta R/K_0}(x_k, z_k)) \\
&\geq cK_d^{-1}\left(\frac{\rho_k K_0}{C^2\theta^2}\right)^\nu \mu_\Phi(S_{R/K_0}(\tilde{x}, \tilde{z})) \\
&\geq cK_d^{-1}\left(\frac{\rho_k K_0}{C^2\theta^2}\right)^\nu K_d^{-1}(n+1)^{-\nu}\mu_\Phi(S_{R(n+1)/K_0}(\tilde{x}, \tilde{z})) \\
&= 2(1-c)^k\mu_\Phi(S_{R(n+1)/K_0}(\tilde{x}, \tilde{z})) \\
&\geq 2(1-c)^k\mu_\Phi(Q_{R/K_0}(\tilde{x}, \tilde{z})).
\end{aligned} \tag{3.5.24}$$

We next show that $S_{r_k R}(x_k, z_k) \subset\subset S_{R/K_0} \subset Q_{R/K_0}$ for $k \geq k_0$.

Since $(x_k, z_k) \in S_{\beta R}(\tilde{x}, \tilde{z})$, we know by Lemma 3.3.6 that there exist constants $C_0 > 0$ and $p > 1$ such that

$$\begin{aligned}
S((x_k, z_k), \rho_k R) &= S\left((x_k, z_k), C_0\left(\frac{\rho_k}{C_0}\right)R\right) \\
&= S\left((x_k, z_k), C_0\left(\beta + \left(\frac{\rho_k}{C_0}\right)^{1/p} - \beta\right)^p R\right) \\
&\subset S\left((\tilde{x}, \tilde{z}), \left(\beta + \left(\frac{\rho_k}{C_0}\right)^{1/p}\right)R\right).
\end{aligned}$$

Choose k_0 large so that

$$\sum_{j=k_0}^\infty \left(\frac{\rho_j}{C_0}\right)^{1/p} = \frac{1}{C_0^{1/p}}\sum_{j=k_0}^\infty \rho_j^{1/p} < \frac{1}{2K_0} - \frac{1}{3K_0} = \frac{1}{2K_0} - \beta.$$

Therefore,

$$S_{\rho_k R}(x_k, z_k) \subset S_{R/(2K_0)}(\tilde{x}, \tilde{z}) \subset\subset S_{R/K_0}(\tilde{x}, \tilde{z}) \subset Q_{R/K_0}(\tilde{x}, \tilde{z}). \tag{3.5.25}$$

Since $U(x, z) \leq aRC^k$ for all $(x, z) \in D_k$, we know that

$$D_k \subset \{(x, z) : U(x, z) \leq aRC^k\} \cap Q_{R/K_0}$$

so that

$$\{(x, z) : U(x, z) > aRC^k\} \cap Q_{R/K_0} \subset Q_{R/K_0} \setminus D_k.$$

Therefore, by (3.5.22), (3.5.23), (3.5.24), and (3.5.25),

$$\begin{aligned}
\mu_\Phi(\{U(x,z) > aRC^k\} \cap Q_{R/K_0}) &\le \mu_\Phi(Q_{R/K_0}(\tilde{x},\tilde{z}) \setminus D_k) \\
&\le (1-c)^k \mu_\Phi(Q_{R/K_0}) \\
&\le \frac{1}{2}\mu_\Phi(A) \\
&= \frac{1}{2}\mu_\Phi(A \cap Q_{R/K_0}) \\
&\le \frac{1}{2}\mu_\Phi(\{U \ge 2aRC^k\} \cap Q_{R/K_0}) \\
&\le \frac{1}{2}\mu_\Phi(\{U > aRC^k\} \cap Q_{R/K_0}),
\end{aligned}$$

a contradiction. This proves the claim.

We now use the claim to prove Harnack inequality. We want to show that

$$\sup_{Q_{\beta R/(n+1)}} U \le aRC^{k_0+1}.$$

Suppose, by way of contradiction, that there is a point $(x_{k_0}, z_{k_0}) \in Q_{\beta R/(n+1)}$ such that

$$\sup_{Q_{\beta R/(n+1)}} U \ge U(x_{k_0}, z_{k_0}) > aRC^{k_0+1}.$$

By the claim, there is a point $(x_{k_0+1}, z_{k_0+1}) \in \partial S_{\rho_{k_0}R}(x_{k_0}, z_{k_0})$ such that

$$U(x_{k_0+1}, z_{k_0+1}) > aRC^{k_0+2}.$$

Repeating this process, we can find a sequence $(x_{k+1}, z_{k+1}) \in \partial S_{\rho_k R}(x_k, z_k)$ such that

$$U(x_k, z_k) > aRC^{k+1} \quad \text{for } k \ge k_0.$$

Notice that

$$\begin{aligned}
S((x_k,z_k),\rho_k R) &= S\left((x_k,z_k), C_0\left(\beta + \sum_{j=k_0}^{k}\left(\frac{\rho_j}{C_0}\right)^{1/p} - \beta - \sum_{j=k_0}^{k-1}\left(\frac{\rho_j}{C_0}\right)^{1/p}\right)^p R\right) \\
&\subset S\left((\tilde{x},\tilde{z}), \left(\beta + \sum_{j=k_0}^{k}\left(\frac{\rho_j}{C_0}\right)^{1/p}\right) R\right) \\
&\subset S\left((\tilde{x},\tilde{z}), \frac{R}{2K_0}\right) \subset Q\left((\tilde{x},\tilde{z}), \frac{R}{2K_0}\right).
\end{aligned}$$

Therefore, $(x_k, z_k) \in Q_{R/(2K_0)}$ for all $k \ge k_0$.

We have shown that U is unbounded in $\overline{Q}_{R/(2K_0)}$, a contradiction. Letting $\kappa_2 = \beta/(n+1)$, we conclude that

$$\sup_{Q_{\kappa_2 R}} U \leq C^{k_0+1} aR$$
$$= C^{k_0+1}(2K_0)U(\tilde{x},\tilde{z}) = C_H U(\tilde{x},\tilde{z})$$

where $C_H = C_H(n,\lambda,\Lambda,s) > 1$. □

Proof of Theorem 3.1.2. Let $\hat{K}_2$ and $\hat{K}_3$ be as in (3.5.3) and (3.5.4), respectively.

Let $\hat{K}_1 = \hat{K}_1(n,s) > 1$ and $\kappa_1 = \kappa_1(n,s) < 1$ be such that

$$1 < (n+1)\theta\hat{K}_3 \leq \hat{K}_1 \quad \text{and} \quad \theta\kappa_1 < \kappa_2 < 1.$$

Let $(\tilde{x},\tilde{z}) \in S_{\kappa_1 R}(x_0,z_0)$. By the engulfing property,

$$S_{\kappa_1 R}(x_0,z_0) \subset S_{\theta\kappa_1 R}(\tilde{x},\tilde{z}) \subset Q_{\theta\kappa_1 R}(\tilde{x},\tilde{z}) \subset Q_{\kappa_2 R}(\tilde{x},\tilde{z}).$$

Again applying the engulfing property, we have

$$Q_{\hat{K}_3 R}(\tilde{x},\tilde{z}) \subset S_{(n+1)\hat{K}_3 R}(\tilde{x},\tilde{z}) \subset S_{(n+1)\theta\hat{K}_3 R}(x_0,z_0) \subset S_{\hat{K}_1 R}(x_0,z_0) \subset\subset \Omega \times \mathbb{R}.$$

By Theorem 3.5.2, we get

$$\sup_{S_{\kappa_1 R}(x_0,z_0)} U \leq \sup_{Q_{\kappa_2 R}(\tilde{x},\tilde{z})} U \leq C_H U(\tilde{x},\tilde{z})$$

Taking the infimum over all $(\tilde{x},\tilde{z}) \in S_{\kappa_1 R}(x_0,z_0)$, the desired Harnack inequality (3.1.6) holds.

It remains to prove the Hölder estimate (3.1.7). The proof follows by a standard argument (see, for example, [31, Section 8.9]). We provide the details for completeness. Let $0 < r \leq \hat{K}_1 R$ and define

$$M(r) = \sup_{S_r(x_0,z_0)} U \quad \text{and} \quad m(r) = \inf_{S_r(x_0,z_0)} U.$$

Apply (3.1.6) to $M(r) - U \geq 0$ in $S_r(x_0,z_0)$ to obtain

$$\sup_{S_{\kappa_1 r}(x_0,z_0)} (M(r) - U) \leq C_H \inf_{S_{\kappa_1 r}(x_0,z_0)} (M(r) - U).$$

Therefore,

$$M(r) - m(\kappa_1 r) \leq C_H \left(M(r) - M(\kappa_1 r)\right). \tag{3.5.26}$$

Similarly, applying (3.1.6) to $U - m(r) \geq 0$ in $S_r(x_0, z_0)$, we obtain

$$\sup_{S_{\kappa_1 r}(x_0,z_0)} (U - m(r)) \leq C_H \inf_{S_{\kappa_1 r}(x_0,z_0)} (U - m(r)),$$

so that

$$M(\kappa_1 r) - m(r) \leq C_H \left(m(\kappa_1 r) - m(r)\right). \tag{3.5.27}$$

Let $\omega(r) = M(r) - m(r)$. Adding (3.5.26) and (3.5.27) together, we get

$$\omega(r) + \omega(\kappa_1 r) \leq C_H \left(\omega(r) - \omega(\kappa_1 r)\right)$$

Rearranging, we obtain

$$\omega(\kappa_1 r) \leq \gamma \omega(r), \quad \gamma = \frac{C_H - 1}{C_H + 1} < 1.$$

Note that $\gamma = \gamma(n, \lambda, \Lambda, s)$.

By [31, Lemma 8.23], for any $\mu \in (0, 1)$, there are constants $\hat{C}_1 = \hat{C}_1(n, \lambda, \Lambda, s) > 0$ and $\alpha_1 = (1 - \mu) \log \gamma / \log \kappa$ such that

$$\omega(r) \leq C \left(\frac{r}{\hat{K}_1 R}\right)^{\alpha_1} \omega(\hat{K}_1 R) \leq 2C r^{\alpha_1} (\hat{K}_1 R)^{-\alpha_1} \sup_{S_{\hat{K}_1 R}(x_0,z_0)} |U|.$$

Choose μ so that $\alpha_1 < 1/2$.

By taking $r = \delta_\Phi((x_0, z_0), (x, z))$ and $\hat{C}_1 = 2C$, we estimate

$$|U(x_0, z_0) - U(x, z)| \leq \omega(r) \leq \hat{C}_1 (\delta_\Phi((x_0, z_0), (x, z)))^{\alpha_1} (\hat{K}_1 R)^{-\alpha_1} \sup_{S_{\hat{K}_1 R}(x_0,z_0)} |U|$$

for all $(x, z) \in S_{\hat{K}_1 R}(x_0, z_0)$. □

3.5.6 Proof of Theorem 3.1.1

Proof of Theorem 3.1.1. Let U be the solution to (3.2.6) given by (3.2.7) in $\Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}$. By Theorem 3.2.4, since $u \geq 0$, we know that $e^{-tL} u \geq 0$. It follows from (3.2.7) that $U \geq 0$.

Let $\kappa = \kappa(n, s) < 1$ and $\hat{K} = \hat{K}(n, s) > 1$ be such that

$$\kappa < \sqrt{\kappa_1} \quad \text{and} \quad \sqrt{2\hat{K}_1} < \hat{K}.$$

We recall from (3.3.1) that

$$B_r(x_0) = S_{r^2/2}(x_0) \quad \text{for any } r > 0.$$

Taking $r = \sqrt{\kappa_1} R$, we obtain

$$\begin{aligned} B_{\kappa R}(x_0) \times [S_{\kappa_1 R^2/2}(x_0, 0) \cap \{z = 0\}] &\subset B_{\sqrt{\kappa_1} R}(x_0) \times S_{\kappa_1 R^2/2}(x_0, 0) \\ &= S_{\kappa_1 R^2/2}(x_0, 0) \times S_{\kappa_1 R^2/2}(x_0, 0) \\ &\subset S_{\kappa_1 R^2}(x_0, 0). \end{aligned} \tag{3.5.28}$$

Taking $r = \sqrt{2\hat{K}_1} R$, we obtain

$$\begin{aligned} S_{\hat{K}_1 R^2}(x_0, 0) &\subset S_{\hat{K}_1 R^2}(x_0) \times S_{\hat{K}_1 R^2}(0) \\ &= B_{\sqrt{2\hat{K}_1} R}(x_0) \times S_{\hat{K}_1 R^2}(0) \\ &\subset B_{\hat{K} R}(x_0) \times S_{\hat{K}_1 R^2}(0) \subset\subset \Omega \times \mathbb{R}. \end{aligned}$$

Let $\tilde{U}$ be the even extension of U so that $\tilde{U}(x, z) = \tilde{U}(x, -z)$. We apply Theorem 3.1.2 to $\tilde{U}$ to obtain the following Harnack inequality

$$\sup_{S_{\kappa_1 R^2}(x_0,0)} \tilde{U} \leq C_H \inf_{S_{\kappa_1 R^2}(x_0,0)} \tilde{U}.$$

By (3.5.28), we have that

$$\begin{aligned} \sup_{B_{\kappa r}(x_0)} u &= \sup_{B_{\kappa R}(x_0) \times [S_{\kappa_1 R^2/2}(x_0,0) \cap \{z=0\}]} \tilde{U} \\ &\leq \sup_{S_{\kappa_1 R^2}(x_0,0)} \tilde{U} \\ &\leq C_H \inf_{S_{\kappa_1 R^2}(x_0,0)} \tilde{U} \\ &\leq C_H \inf_{B_{\kappa R}(x_0) \times [S_{\kappa_1 R^2/2}(x_0,0) \cap \{z=0\}]} \tilde{U} = C_H \inf_{B_{\kappa r}(x_0)} u \end{aligned}$$

which proves (3.1.3).

It remains to prove the Hölder estimate (3.1.4). By (3.1.7), we obtain the estimate

$$\left|\tilde{U}(x_0,0)-\tilde{U}(x,z)\right| \leq \hat{C}_1\delta_\Phi((x_0,0),(x,z))^{\alpha_1}\left(\hat{K}_1R^2\right)^{-\alpha_1}\sup_{S_{\hat{K}_1R^2}(x_0,0)}\tilde{U}$$

for every $(x,z)\in S_{\hat{K}_1R^2}(x_0,0)$. Since

$$B_{\sqrt{\hat{K}_1}R}(x_0)\times S_{\hat{K}_1R^2/2}(0)=S_{\hat{K}_1R^2/2}(x_0)\times S_{\hat{K}_1R^2/2}(0)\subset S_{\hat{K}_1R^2}(x_0,0),$$

we have, for any $x\in B_{\sqrt{K_1}R}(x_0)$, that

$$\begin{aligned}|u(x_0)-u(x)| &= \left|\tilde{U}(x_0,0)-\tilde{U}(x,0)\right| \\ &\leq \hat{C}_1\delta_\Phi((x_0,0),(x,0))^{\alpha_1}\left(\hat{K}_1R^2\right)^{-\alpha_1}\sup_{S_{\hat{K}_1R}(x_0,0)}\left|\tilde{U}\right| \\ &\leq \frac{\hat{C}_1}{2}|x-x_0|^{2\alpha_1}\left(\sqrt{\hat{K}_1}R\right)^{-2\alpha_1}\sup_{B_{\hat{K}R}(x_0)\times S_{\hat{K}_1R^2}(0)}\left|\tilde{U}\right|.\end{aligned}$$

For each fixed $z\geq 0$, by (3.2.2), we take the supremum in x to get

$$\begin{aligned}\|U(\cdot,z)\|_{L^\infty(B_{\hat{K}R}(x_0))} &= \left\|\frac{(2s)z}{4^s\Gamma(s)}\int_0^\infty e^{-\frac{s^2}{t}z^{\frac{1}{s}}}e^{-Lt}u(\cdot)\,\frac{dt}{t^{1+s}}\right\|_{L^\infty(B_{\hat{K}R}(x_0))} \\ &\leq \frac{(2s)z}{4^s\Gamma(s)}\int_0^\infty e^{-\frac{s^2}{t}z^{\frac{1}{s}}}\left\|e^{-Lt}u\right\|_{L^\infty(\Omega)}\frac{dt}{t^{1+s}} \\ &\leq \frac{(2s)z}{4^s\Gamma(s)}\int_0^\infty e^{-\frac{s^2}{t}z^{\frac{1}{s}}}M\|u\|_{L^\infty(\Omega)}\frac{dt}{t^{1+s}}=M\|u\|_{L^\infty(\Omega)}.\end{aligned}$$

Letting $\hat{K}_0=\sqrt{\hat{K}_1}$, $\hat{C}=M\hat{C}_1/2$, and $\alpha=2\alpha_1$, we conclude that

$$|u(x_0)-u(x)|\leq\hat{C}\,|x-x_0|^\alpha\,(\hat{K}_0R)^{-\alpha}\sup_\Omega|u|\quad\text{for all } x\in B_{\hat{K}_0R}(x_0).$$

□

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