AFFIRMATIVE ACTION IN INDIA: RESTRICTED STRATEGY SPACE, COMPLEX CONSTRAINTS, AND DIRECT MECHANISM DESIGN[1]

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ABSTRACT. Since the 1950s, India has instituted an intricate affirmative action program through a meticulously designed reservation system. This system incorporates vertical and horizontal reservations to address historically marginalized groups' socioeconomic imbalances. Vertical reservations designate specific quotas of available positions for Scheduled Castes, Scheduled Tribes, Other Backward Classes, and Economically Weaker Sections. Concurrently, horizontal reservations are employed within each vertical category to allocate positions for additional subgroups, such as women and individuals with disabilities. In educational admissions, the legal framework recommended that unfilled positions reserved for the OBC category revert to unreserved status. Moreover, we document that individuals from vertically reserved categories have more complicated preferences over institution-vertical category position pairs, even though authorities only elicit their preferences over institutions. To address these challenges, the present paper proposes a novel class of *Generalized Lexicographic* (GL) choice rules. This class is comprehensive, subsuming the most salient priority structures discussed in the extant matching literature. Utilizing the GL choice rules and the deferred acceptance mechanism, we present a robust framework that generates equitable and effective solutions for resource allocation problems in the Indian context.

JEL CODES: $C78$, $D02$, $D47$, $I38$.

KEYWORDS: Market design, affirmative action, vertical reservations, horizontal reservations, de-reservations, India, lexicographic choice.

[1]This paper subsumes two earlier papers titled *"Designing Direct Matching Mechanisms for India with Comprehensive Affirmative Action"* and *"Matching with Generalized Lexicographic Choice Rules."* Some key concepts of this paper introduced in the second chapter of Aygün (2014), Ph.D. dissertation submitted to Boston College. We thank Itai Ashlagi, Eduardo Azevedo, David Delacrétaz, Battal Doğan, Federico Echenique, Aytek Erdil, Ajay Gudavarthi, Isa Hafalır, Onur Kesten, Aditya Kuvalekar, Kriti Manocha, Debasis Mishra, Josue Ortega, Assaf Romm, Al Roth, Arunava Sen, Rajesh Singh, Rakesh Vohra, Bumin Yenmez, Kemal Yildiz, as well as the participants at the ISU Market Design Conference, 2022 AMES in East and Southeast Asia, and 2022 Conference on Mechanism and Institution Design for their helpful comments and suggestions. We acknowledge that we used ChatGPT to improve readability and language of the intoduction section.

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1 Introduction

India has enforced the most sophisticated affirmative action program in allocating government jobs and admissions to public universities since the 1950s. Indian affirmative action program has been implemented via a highly regulated reserve system that consists of two different provisions: *vertical* and *horizontal*.[4] These terms were first used in the Supreme Court of India (SCI)'s judgment in Indra Shawney vs. Union of India (1992).[5] After this jurisprudential milestone, the operational modalities and interplay between vertical and horizontal reservations have been subject to ongoing regulatory refinement through successive SCI verdicts, thereby shaping the legal landscape of affirmative action in India since 1992.

By established legal frameworks, reservations for Scheduled Castes (SC), Scheduled Tribes (ST), Other Backward Classes (OBC), and Economically Weaker Sections (EWS) are categorized as vertical reservations. Under this schema, 15%, 7.5%, 27%, and 10% of available institutional positions are statutorily earmarked for these respective social strata.[6] Individuals not encompassed by these vertical categories are classified as members of the General Category (GC). The residual 40.5% of available positions fall under the open category. The governing statutes stipulate that any open-category positions secured by individuals belonging to SC, ST, OBC, or EWS are not to be deducted from their respective vertical reservations. Hence, vertical reservations operate under an "over-and-above" legal principle. Failure to declare membership in SC, ST, OBC, or EWS categories relegates candidates to the GC by default. Importantly, the legal framework provides that the disclosure of vertical category affiliation is discretionary.

Horizontal reservations are designated for additional marginalized groups, such as women and individuals with disabilities. The SCI's judgments provides partial elucidation on the intricate interaction between vertical and horizontal reservations.[7] Judicial mandates stipulate that horizontal reservations should be implemented in a compartmentalized fashion within each vertical category, extending as well to the open category. Furthermore, the Court has dictated that slots reserved horizontally should take precedence and be allocated prior to any unreserved positions, serving as "minimum guarantees." Nonetheless, the Court's rulings have left certain aspects ambiguously undefined (Sönmez and Yenmez (2022)). For instance, it remains an open question whether an individual eligible for multiple types of horizontal reservations is counted against a single category of her horizontal eligibility or multiple categories concurrently.

[4]Vertical reservations have been studied in the market design literature in Echenique and Yenmez (2015), Aygün and Turhan (2017), Aygün and Turhan (2020), and Aygün and Turhan (2022). Baswana et al. (2019) is the first paper that considered both vertical and horizontal reservations in Indian context from a market design perspective and designed a new joint seat allocation procedure for engineering colleges. Horizontal reservations and their interplay with vertical reservations are also studied in Sönmez and Yenmez (2022) where the authors relate Indian judiciary to matching theory.

[5]The case is available at https://indiankanoon.org/doc/1363234/.

[6]In 2019, the Indian legislative body instituted a 10% vertical reservation for a subset of the General Category (GC), specifically the Economically Weaker Section (EWS), defined by an annual income threshold below Rs. 8 lakhs.

[7]For an in-depth analysis, see Sönmez and Yenmez (2022).

In the realm of governmental employment and admissions to publicly-funded academic institutions, reservations for SC and ST are categorized as "hard" reserves, meaning that unfilled positions earmarked for these groups are non-transferable to other categories. While the reservations for OBC are treated as"hard" reserves in the context of governmental job allocations, they were suggested and have been implemented as "soft" reserves in academic admissions, pursuant to the landmark SCI decision in Ashoka Kumar Thakur vs. Union of India (2008).[8]

It should be noted that the SCI ruling did not proffer an explicit procedural framework for de-reserving unfilled OBC slots. This absence of judicial guidance has caused considerable ambiguity concerning the eligible beneficiaries of these reverted positions—whether they should be exclusively limited to GC candidates or be open to all, including those in reserved categories. The lack of clear rules has led to various inconsistent practices in school admissions across India.[9]

The complex interaction between vertical and horizontal reservations, along with OBC dereservation policies, significantly influences the selection processes of institutions when confronted with a pool of applicants. These processes are primarily guided by the SCI's jurisprudence, which seeks to rectify and optimize the criteria employed by institutions for selection. However, it is imperative to note that institutions represent only one side of the matching markets. The complementary component consists of individuals applying for positions in publicly-funded universities or governmental roles. A critical but often overlooked aspect concerns the language provided to them for expressing their preferences over institutions when they also report their vertical category membership.

In India authorities solicit applicants' preferences exclusively over institutions, operating under the presumption that candidates from reserved categories are indifferent between open and reserved-category positions within the same institution. In Appendix 5.1, we provide comprehensive evidence to challenge this assumption, demonstrating that many reserved category candidates exhibit distinct preferences for the specific position-category under which they are admitted.[10]

Disclosing vertical category membership is discretionary and can be strategically leveraged by applicants. Should reserved category candidates abstain from declaring their membership, they are exclusively considered for open category positions. Conversely, upon revealing their membership, these candidates become eligible for assignment to institutions via either an open or a reserved category position. To implement vertical reservations as over-and-above, institutions first fill open category positions considering all applicants including the members of reserved categories.

While applicants' preferences only over institutions are collected, the announced outcomes

[8]The judgment is available at https://indiankanoon.org/doc/1219385/.

[9]See Aygün and Turhan (2022) for details.

[10]Our evidence is drawn from a multiple sources including legal precedents, extant scholarly literature, discourse in online forums, and direct communications with experts all of which pertain to the preferences of reserved category members in various resource allocation scenarios within India.

encompass both the institution and the position category under which an individual is assigned.[11]

The strategic calculus involving the optional disclosure of reserved category membership gains prominence, particularly for individuals who have specific preferences over the type of positions. To elucidate, consider an OBC candidate with a rank-ordered list a-b-c over institutions $\{a, b, c\}$. Should she choose to disclose her OBC membership, her candidacy would be evaluated in accordance with the following sequence:

$$(a, Open) - (a, OBC) - (b, Open) - (b, OBC) - (c, Open) - (c, OBC)\,.$$

Conversely, if she opts to withhold her OBC membership while submitting identical preferences, her evaluation would proceed in the following manner:

$$(a, Open) - (b, Open) - (c, Open)\,.$$

If the applicant prefers $(b, Open)$ to (a, OBC), for example, she then may prefer to hide her category membership. This demonstrates the salient role of strategic disclosure in influencing the eventual matching outcomes within the constraints of the current preference language. Moreover, some preferences, such as the one below, cannot be expressed according to the current language.

$$(a, Open) - (b, Open) - (c, Open) - (a, OBC) - (b, OBC) - (c, OBC)\,.$$

A parallel issue emerges in the context of cadet-branch matching at the United States Military Academy (USMA), as elaborated by Greenberg et al. (2023). Previously, cadets are required to submit (1) a hierarchical ranking of military branches, and (2) optionally, sign a ”Branch-of-Choice” contract for a select number of branches, thereby committing to additional service years in exchange for elevated priority within those branches.[12] Given these inputs, a particular preference relation over branch-service time pairs is constructed for the implementation of the Deferred Acceptance (DA) algorithm. However, this mechanism is categorized as indirect, owing to the limited strategy space that precludes cadets from adequately expressing nuanced preferences over branch-service time pairs. Greenberg et al. (2023) reports the new design in which the preference language of cadets are expanded to branch-service time pairs.

[11]As elaborated in Appendix 5.1, the public nature of these outcomes significantly influences individual preferences over position categories. The extant stigma associated with reserved category positions compels individuals to strategically opt for open category positions. Such intricate preferences cannot be captured through the current preference language available to applicants. Concealing reserve category membership to secure an open-category position, although risky due to high cutoff scores, becomes a strategic action. It is important to clarify that this paper does not contend that extending the preference domain to institution-category pairs will mitigate the stigma issue, as that is beyond the scope of this study.

[12]The Branch-of-Choice incentive program allows cadets to gain increased priority for their preferred branches in exchange for a commitment to serve three additional years.

To rectify the implementation issues due to restricted preference language, one may consider expanding the strategy space to include institution-vertical category pairs in obeject allocation problems in India. Such an expansion would enable reserved category applicants, who may be indifferent between open and reserved slots but particular about institutional affiliations, to more explicitly state their preferences.

An objection that could be raised against the expansion of the strategy space, which arose in the context of cadet-branch matching, relates to the augmented complexity and elongation of elicited preferences. This apprehension, however, is largely inconsequential for multiple reasons. Firstly, in various allocation frameworks, such as assigning civil service roles, the array of institutions or divisions candidates must rank is generally constrained. Secondly, candidates are typically well-equipped with pertinent information regarding their prospective assignments as they submit their preferences after the public release of merit-based scores and rankings. Additionally, historical cutoff scores for each institution and corresponding category are made publicly available, thus enabling candidates in reserved categories to make well-informed decisions about utilizing reservation benefits.

1.1 Overview of Model and Results

In our analytical framework, we extend the preference domain to encompass both institutions and position categories, thereby providing individuals the possibility for more nuanced preferences. Absent such an expansive preference domain, designing a direct matching mechanism in the Indian context would be impossible. Moreover, this enlarged preference domain permits applicants to articulate their preferences solely in terms of institutions while simultaneously incorporating the current operational principle that preference is accorded to the former in instances of indifference between open and reserved category positions.

The contributions of this paper are both theoretical and practical. Theoretically, it introduces a class of *Generalized Lexicographic* (GL) priorities for institutions, thereby serving as a unifying framework that subsumes key choice rules delineated in the market design literature. From a practical standpoint, the paper rigorously models the complexities of resource allocation issues in India, incorporating the multifaceted affirmative action stipulations that characterize these environments. In doing so, it captures the full generality of these allocation problems, thereby bridging the gap between theoretical constructs and real-world policy applications. The manuscript presents a many-to-one matching model with contracts, featuring a complex priority structure for institutions, and offers a unifying framework for analysis. While the primary application of our model concentrates on object allocation issues in India, the general framework is versatile enough to accommodate a diverse array of real-world matching applications. These include cadet-branch matching, airline seat upgrades, and multidimensional reserves in Brazil, among others.

The institutional choice rules are structured as follows: Each institution consists of various (vertical) categories and a predetermined number of open positions. Categories are populated in accordance with a pre-established linear order. Within each category, a *sub-choice rule* is defined, which takes two variables as inputs: *the pool of available contracts* and *its dynamic capacity*. Both variables are contingent on the decisions made by preceding categories. The available contracts represent the residual options subsequent to the selections made by earlier categories. The dynamic capacity of a given category is determined by a *transfer function* that takes the number of unfilled positions from preceding categories as inputs. The *overall choice rule* for an institution is subsequently the aggregation of its categories' individual sub-choice rules.

A choice rule is in the *Generalized Lexicographic* (GL) family if (1) all sub-choice rules are substitutable, size monotonic, and quota monotonic, and (2) the capacity transfer function is monotonic. Substitutability (Kelso and Crawford (1982)) and size-monotonicity (Alkan and Gale (2003) and Hatfield and Milgrom (2005)) are widely used properties in matching literature. *Quota monotonicty* is new to this paper and says that

- when capacity increases, the choice rule selects all alternatives it was choosing under the initial capacity, and

- the difference between the number of chosen alternatives under the increased and initial capacities cannot exceed the increase in capacity.

The choice functions encompassed within the GL framework offer considerable latitude to integrate intricate constraints pertaining to diversity. Noteworthy instances of choice functions within this family, as delineated in extant market design literature, encompass a diverse array of approaches. These include the reserve-based choice rules articulated in Echenique and Yenmez (2015), the slot-specific priority-based choice functions of Kominers and Sönmez (2016), the dynamic reserves choice rules of Aygün and Turhan (2020), the meritorious horizontal choice rule of Sönmez and Yenmez (2022), and the dual-price choice rule as of Greenberg et al. (2023), among a plethora of other rules. The GL family provides a unified framework for market designers as it nests the rules listed above.

Our main theoretical result, Theorem 1, states that the Cumulative Offer Mechanism (COM) coupled with the GL priorities is stable and strategy-proof.

In addressing resource allocation challenges in India, our approach employs a tripartite design framework to formulate effective direct mechanisms. Initially, we introduce *Sub-Choice Rules* that amalgamate individuals' merit scores with horizontal reservations pertinent to each vertical category. Subsequently, *Overall Choice Rules* are formulated by synthesizing these sub-choice rules within the context of prevailing de-reservation policies. As the final component, we adopt the COM coupled with institutions' overall choice rules to govern the allocation process. This hierarchical design structure offers a comprehensive and integrative framework, which allows for the nuanced incorporation of multiple considerations such as merit, horizontal reservations, vertical reservations, and de-reservation policies into the resource allocation mechanisms.

Indian legal statutes remain ambiguous on how an individual with multiple horizontal types should be accounted for in allocation procedures, a point highlighted in Sönmez and Yenmez (2022). The literature presents two distinct methodologies to address this lacuna: the *One-to-One* approach (Sönmez and Yenmez (2022)) and the *One-to-All* approach (Aygün (2014) and Aygün (2017)). In the"One-to-One" approach, each candidate is attributed to a singular horizontal type for the purpose of allocation. Conversely, under the"One-to-All" framework, each candidate is counted against all their respective horizontal types. These methodologies offer divergent paradigms for interpreting and operationalizing horizontal types within the allocation process, each with its own implications for policy and practice.

In order to accommodate diverse policy objectives, our framework incorporates sub-choice rules that are consistent with both the One-to-One and One-to-All approaches. The linkage between these sub-choice rules for various vertical categories is governed by the de-reservation (or transfer) policy. For instance, in the context of government job allocations, unfilled positions designated for the Other Backward Classes (OBC) are not reallocated. Conversely, in the domain of college admissions, any unoccupied OBC slots are converted to open category positions. Given these differing policies, our design of the overarching choice rules is nuanced. Consequently, we introduce four distinct overall choice rules: (1) one-to-one horizontal matching with no transfers, (2) one-to-one horizontal matching with transfers, (3) all-to-one horizontal matching with no transfers, and (4) all-to-one horizontal matching with transfers. This taxonomy provides a comprehensive framework for the formulation of overall choice rules, effectively catering to the multifaceted nature of resource allocation problems within the Indian context.

We show that these overall choice rules are in the GL family and fair (Theorems 3 and 5). The fairness of a choice rule—first defined in Aygün and Bó (2021)—says that if an individual is rejected, then all chosen individuals must either have a higher merit score or claim a vertical or horizontal category membership that the rejected individual does not have.

1.2 Related Work

This paper contributes to the burgeoning body of literature that addresses the complexities of resource allocation issues in India, particularly within the framework of intricate affirmative action (AA) policies. Echenique and Yenmez (2015) stand as pioneers in applying market design principles to the Indian AA landscape, illustrating their insights through the lens of controlled school choice in college admissions. Subsequent work by Aygün and Turhan (2017) delves into the specific challenges encountered in admissions to the Indian Institutes of Technology (IITs). Notably,Aygün and Turhan (2020) offered a comprehensive treatment of both vertical reservations and de-reservations in the context of admissions to technical colleges in India. They introduced a novel family of *dynamic reserves* choice rules, albeit without addressing horizontal reservations. The family of GL choice rules presented in the current paper encapsulates these dynamic reserves choice rules, thereby extending their applicability. This generalization is of significance, as the dynamic reserves choice rules proposed by Aygün and Turhan (2020) were formulated on the basis of q-responsive sub-choice rules, which are inherently limited to incorporate horizontal reservations.

Sönmez and Yenmez (2022) also examine Indian AA through the lens of market design, situating court decisions regarding resource allocation under AA within the matching theory framework. The primary focus of their work is the conjoint implementation of vertical and horizontal reservations. The present paper diverges from Sönmez and Yenmez (2022) in three crucial dimensions. Firstly, the current paper extends the scope of analysis by taking into account individuals' preferences over both institutions and position categories. This contrasts with the approach of Sönmez and Yenmez (2022), who limit their consideration to preferences solely over institutions, operating under the assumption that all applicants in reserved categories are indifferent between open and reserved slots. Secondly, our paper introduces both priority design at the level of individual institutions and mechanism design for a centralized marketplace. In contrast, Sönmez and Yenmez (2022) constrain their focus solely to the priority design of a single institution. Lastly, Sönmez and Yenmez (2022) do not incorporate de-reservation policies into their model, thereby confining their analysis to the allocation of government jobs where OBC de-reservations are not applicable. Conversely, the present study offers a comprehensive framework that simultaneously considers (i) vertical reservations, (ii) horizontal reservations, and (iii) de-reservations, all within a setting that allows applicants to articulate their preferences across institution-category pairs.

Aygün and Turhan (2022) examine the concurrent implementation of vertical reservations and OBC de-reservations, specifically focusing on the recently revamped admissions procedures for technical universities in India, as originally designed and implemented by Baswana et al. (2019). Aygün and Turhan (2022) introduce a modification of the Deferred Acceptance (DA) algorithm, paired with a novel choice rule, to more effectively realize de-reservation policies. Their proposed mechanism exhibits favorable incentive properties and results in a Pareto improvement for the applicant pool. However, for the sake of simplicity, their model omits horizontal reservations and restricts applicants' preferences solely to institutions. The present paper extends Aygün and Turhan (2022) in two significant ways. First, it incorporates both horizontal and vertical reservations alongside de-reservations, thereby providing a more holistic approach to resource allocation in India. Second, it broadens the scope of individuals' preference domain, enabling a more nuanced expression of preferences over both institutions and position categories. This dual extension enhances the analytical robustness and practical applicability of the model.

Hatfield and Kominers (2019) identify a *hidden substitutability* in institutions' priority structures that makes stable and strategy proof matching possible. The rules in GL family have *substitutable completion* that satisfy the size monotonicity condition. Therefore, the GL choice rules satisfy the necessary conditions defined by Hatfield and Kominers (2019). In a related paper, Hatfield et al. (2021) characterize the unique stable and strategy-proof mechanism with three conditions on instituons' choice rules in many-to-one matching models with contracts: *observable substitutability, observable size monotonicity*, and *non-manipulability via contractual terms*. The GL choice rules satisfy these conditions, as well. Therefore, our main application—AA implementations in India—provides an important real-world market design setting for the theory developed in both Hatfield and Kominers (2019) and Hatfield et al. (2021).

This paper also contributes to a strand of literature that study drawbacks of providing agents a limited strategy space to express their preferences. Aygün and Turhan (2017) and Aygün and Turhan (2020) discuss the restricted preference domain issue for resource allocation problems in India. In the cadet-branch matching problem, *a proxy preference relation* over branch-service time pairs is *constructed* to implement the DA algorithm, which becomes an *indirect* mechanism due to the limited strategy space. Sönmez and Switzer (2013) show that the DA mechanism loses all of its desirable properties due to this restriction. Greenberg et al. (2023) report that the USMA recently adopted the expanded strategy space. We argue that a similar expansion can and must be implemented in India.

To incorporate diversity Nguyen and Vohra (2019) take a different approach from the rest of the literature by relaxing the integrality constraints of the matching problem so that individuals can be fractionally allocated to schools. Then, the authors introduce a rounding algorithm to round the fractional matching into an integral stable matching. While doing so they only violate the proportionality constraints in a limited way, but not the capacity constraint. Indian AA implementations are strictly regulated by the legal framework. As discussed earlier, certain aspects of institutions' choice rules must obey the Supreme Court judgments. Therefore, the approach taken by Nguyen and Vohra (2019) may not be suitable for object allocation problems in India.

This manuscript adopts a priority design methodology, proposing generalized DA mechanisms that operate under the stipulated choice rules. Doğan et al. (2022) furnish a comprehensive methodology for scrutinizing the robust interconnection between axioms characterizing the designed choice rules and axioms that characterizes the DA mechanism coupled with these rules.

Furthermore, the present study contributes to the existing body of literature concerning lexicographic choice rules. Noteworthy papers investigating the theoretical properties of these rules include Chambers and Yenmez (2017), Chambers and Yenmez (2018b), Chambers and Yenmez (2018a), Doğan et al. (2021a), and Doğan et al. (2021b), among others.

Finally, this paper contributes to the existing body of literature concerning diversity constraints in matching market design. The idea of reserve was introduced in the seminal work by Hafalir et al. (2013) after Kojima (2012) pointed out the drawbacks of the quota policy. Ehlers et al. (2014) show that, with hard bounds, there might not exist assignments that satisfy standard fairness and non-wastefulness properties. To achieve fair and non-wasteful assignments, the authors propose to interpret constraints as soft bounds. There is now a rich literature in market design on diversity implementations in matching markets. Important work in the literature include Abdulkadiroğlu (2005), Biró et al. (2010), Budish et al. (2013), Kamada and Kojima (2015), Doğan (2016), Fragiadakis and Troyan (2017), Kojima et al. (2018), Tomoeda (2018) Imamura (2020), Erdil and Kumano (2019), Delacrétaz (2021), Aygün and Bó (2021), Correa et al. (2021), Doğan and Yildiz (2022), Aygün and Turhan (2023), and Kamada and Kojima (2023), among many others.

2 Model

There is a finite set of *individuals* $\mathcal{I} = \{i_1, ..., i_n\}$ and a finite set of *institutions* $\mathcal{S} = \{s_1, ..., s_m\}$. Let $\overline{q}_s$ denote the capacity of institution s. There is a finite set of *contracts* $\mathcal{X}$. Each contract $x \in \mathcal{X}$ is associated with an individual $\mathbf{i}(x)$ and an institution $\mathbf{s}(x)$. There may be multiple contracts for each individual-institution pair.

An *outcome* (or a *matching*) is a set of contracts $X \subseteq \mathcal{X}$ with $\mathbf{i}(X) \equiv \bigcup_{x \in X} \{\mathbf{i}(x)\}$ and $\mathbf{s}(X) \equiv \bigcup_{x \in X} \{\mathbf{s}(x)\}$. For any $j \in \mathcal{I} \cup \mathcal{S}$, we let $X_j \equiv \{x \in X \mid j \in \{\mathbf{i}(x), \mathbf{s}(x)\}\}$. An outcome $X \subseteq \mathcal{X}$ is *feasible* if $\mid X_i \mid \leq 1$ for all $i \in \mathcal{I}$. Individual i is *unmatched under* X if $X_i = \emptyset$.

Each individual $i \in \mathcal{I}$ has *unit demand* over contracts in $\mathcal{X}_i$ and an outside option $\emptyset$. Let P_i denote the strict preference of individual i over $\mathcal{X}_i \cup \{\emptyset\}$. A contract $x \in \mathcal{X}_i$ is *acceptable for* i with respect to P_i if $xP_i\emptyset$. Individuals' preferences over contracts are extended to preferences over outcomes naturally.

Each institution $s \in \mathcal{S}$ has *multi-unit demand* and is endowed with a choice rule C_s that describes how s would choose from any set of contracts. For all $X \subseteq \mathcal{X}$ and $s \in \mathcal{S}$, the choice rule C_s is such that $C_s(X) \subseteq X_s$ and $\mid C_s(X) \mid \leq \overline{q}_s$. We denote $R_s(X) \equiv X \setminus C_s(X)$ the set of contracts that s rejects from X.

DEFINITION. A feasible outcome $X \subseteq \mathcal{X}$ is stable if

1. $X_i R_i \emptyset$, for all $i \in \mathcal{I}$,
2. $C_s(X) = X_s$, for all $s \in \mathcal{S}$, and
3. there is no $Z \subseteq (\mathcal{X} \setminus Y)$, such that $Z_s \subseteq C_s(Y \cup Z)$ for all $s \in \mathbf{s}(Z)$ and ZP_iY for all $i \in \mathbf{i}(Z)$.

Stability presents natural desiderata for an allocation: an individual will only be matched to a less desirable institution if, by following institutions' selection criteria, she would not be accepted given the individuals who have been matched to these institutions. Condition (2) requires that institutions' selection procedures are respected in the sense that when an institution is offered the set of contracts assigned to it under a particular matching, it selects all of them. Stability crucially depends on how institutions' selection procedures are defined. If each individual applies to only one institution, stability requires that the rules and regulations encoded in institutions' choice rules determine which contracts are selected.

Given a profile of choice rules $C = (C_s)_{s \in \mathcal{S}}$, a *mechanism* $\psi(\cdot; C)$ maps preference profiles $P = (P_i)_{i \in \mathcal{I}}$ to outcomes. Unless otherwise stated, we assume that institutions' choice rules are fixed and write $\psi(P)$ in place of $\psi(P; C)$. A mechanism ψ is *stable* if $\psi(P)$ is a stable outcome for every preference profile P. A mechanism ψ is *strategy-proof* if for every preference profile P and for each agent $i \in \mathcal{I}$, there is no $\widetilde{P}_i$, such that $\psi(\widetilde{P}_i, P_{-i}) P_i \psi(P)$.

2.1 Generalized Lexicographic Priorities

Each institution $s \in S$ has a set of *categories* $\mathcal{K}_s = \{1, ..., K_s\}$. Each category $k \in \mathcal{K}_s$ has an associated *sub-choice* rule $C_s^k : 2^{\mathcal{X}} \times \mathbb{Z}_{\geq 0} \longrightarrow 2^{\mathcal{X}_s}$ that specifies the contracts category k chooses given a set of offers and a *dynamic capacity* that varies as a function of the number of unused slots in the preceding categories. We require that, for a set of contracts $Y \subseteq \mathcal{X}_s$ and a capacity of κ, $\mid C_s^k(Y; \kappa) \mid \leq \kappa$.

Given a set contracts $Y \equiv Y^1 \subseteq \mathcal{X}$, a capacity $\overline{q}_s$ for institution s, and a capacity $\overline{q}_s^1$ for category 1, we compute the chosen set $C_s(Y, \overline{q}_s)$ in K_s steps, where category k chooses in step k, for $k = 1, ..., K_s$, as follows:

Step 1. Given Y^1 and $\overline{q}_s^1$, category 1 chooses $C_s^1(Y^1; \overline{q}_s^1)$. Let

$$r_1 = \overline{q}_s^1 - \mid C_s^1(Y^1; \overline{q}_s^1) \mid$$

be the number of vacant positions in category 1 and

$$Y^2 \equiv Y^1 \setminus \{x \in Y^1 \mid \mathbf{i}(x) \in \mathbf{i}[C_s^1(Y^1; \overline{q}_s^1)]\}$$

be the set of contracts of individuals who were not selected by category 1.

Step k ($2 \leq k \leq K_s$)**.** Given the set of remaining contracts Y^k and category k's dynamic capacity $q_s^k(r_1, ..., r_{k-1})$, category k chooses $C_s^k(Y^k; q_s^k(r_1, ..., r_{k-1}))$. Let

$$r_k = q_s^k(r_1, ..., r_{k-1}) - \mid C_s^k(Y^k; q_s^k(r_1, ..., r_{k-1})) \mid$$

be the number of unfilled positions in category k and

$$Y^{k+1} \equiv Y^k \setminus \{x \in Y^k \mid \mathbf{i}(x) \in \mathbf{i}[C_s^k(Y^k; q_s^k(r_1, ..., r_{k-1}))]\}$$

be the set of contracts of individuals who were not selected by any category in the first k step.

The union of categories' choices is the institution's chosen set. That is,

$$C^s(Y, \overline{q}_s) \equiv C_s^1(Y^1; \overline{q}_s^1) \cup \bigcup_{k=2}^{K_s} C_s^k(Y^k; q_s^k(r_1, ..., r_{k-1})).$$

Transfer Policy

Given an initial capacity of the first category $\overline{q}_s^1$, a *transfer policy* of institution s is a sequence of capacity functions $q_s = (\overline{q}_s^1, (q_s^k)_{k=2}^{K_s})$, where $q_s^k : \mathbb{Z}_+^{k-1} \longrightarrow \mathbb{Z}_+$ for all $k \in \mathcal{K}_s$ and such that

$$\overline{q}_s^1 + q_s^2(0) + q_s^3(0,0) + \cdots + q_s^K(0, ..., 0) = \overline{q}_s.$$

A transfer policy q_s is **monotonic**[13] if, for all $j \in \{2, ..., K_s\}$ and all pairs of sequences $(r_l, \widetilde{r}_l)$, such that $\widetilde{r}_l \geq r_l$ for all $l \leq j-1$,

$$q_s^j(\widetilde{r}_1, ..., \widetilde{r}_{j-1}) \geq q_s^j(r_1, ..., r_{j-1}), \ \textit{and}$$

$$\sum_{m=2}^{j} [q_s^m(\tilde{r}_1, ..., \widetilde{r}_{m-1}) - q_s^m(r_1, ..., r_{m-1})] \leq \sum_{m=1}^{j-1} [\widetilde{r}_m - r_m].$$

Conditions on Categories' Sub-Choice Rules

We impose three conditions on categories' sub-choice rules: *Substitutability, size monotonicity,* and *quota monotonicity.*

1. A choice rule C satisfies *substitutability* if for all $x, y \in \mathcal{X}$ and $X \subseteq \mathcal{X}$,

$$y \notin C(X \cup \{y\}) \implies y \notin C(X \cup \{x, y\}).$$

2. A choice rule C is *size monotonic* if for all contracts $x \in \mathcal{X}$ and sets of contracts $X \subseteq \mathcal{X}$, we have

$$\mid C(X) \mid \leq \mid C(X \cup \{x\} \mid.$$

3. A choice rule C is *quota monotonic* if for any $q, q' \in \mathbb{Z}_+$ such that $q < q'$, for all $Y \subseteq \mathcal{X}$

[13] Monotonicity of a transfer policy, first introduced in Westkamp (2013).

$$C(Y,q) \subseteq C\left(Y,q'\right), \ and$$
$$\mid C\left(Y,q'\right) \mid - \mid C(Y,q) \mid \leq q' - q.$$

Substitutability and size monotonicity are standard properties.[14] Quota monotonicity is a new property and requires that (1) if there is an increase in the capacity, we require the choice rule to select every contract it was choosing before increasing its capacity from any given set of contracts, and (2) if the capacity increases by κ, then the difference between the number of contracts chosen with the increased capacity, and the initial capacity cannot exceed κ.

DEFINITION. A choice rule C^s is *generalized lexicographic* (GL) if it can be represented as a pair $\left(\left(C_s^k\right)_{k\in\mathcal{K}_s}, q_s\right)$ of a collection of sub-choice functions and a transfer policy, where each sub-choice function C_s^k satisfies substitutability, size monotonicity, quota monotonicity, and the transfer function q^s is monotonic.

2.2 Cumulative Offer Mechanism

Given individuals' preferences and institutions' overall choice functions, the outcome of the COM is computed by the *cumulative offer process* (COP) as follows:

Step 1. Some individual $i^1 \in \mathcal{I}$ proposes her most-preferred contract, $x^1 \in \mathcal{X}_{i^1}$. Institution $\mathrm{s}(x^1)$ holds x^1 if $x^1 \in C_{\mathbf{s}(x^1)}(\{x^1\})$, and rejects x^1 otherwise. Set $A^2_{\mathbf{s}(x^1)} = \{x^1\}$, and set $A^2_{s'} = \emptyset$ for each $s' \neq \mathrm{s}(x^1)$; these are the sets of contracts available to institutions at Step 2.

Step l. Some individual $i^l \in \mathcal{I}$, for whom no institution currently holds a contract, proposes her most-preferred contact that has not yet been rejected, $x^l \in X_{i^l} \setminus \left(\bigcup_{j\in S} A^l_j\right)$. Institution $\mathrm{s}\left(x^l\right)$ holds the set of contracts in $C_{\mathbf{s}\left(x^l\right)}\left(A^l_{\mathbf{s}\left(x^l\right)} \cup \left\{x^l\right\}\right)$ and rejects all other contracts in $A^l_{s\left(x^l\right)} \cup \left\{x^l\right\}$; institutions $s' \neq \mathrm{s}\left(x^l\right)$ continue to hold all contracts they held at the end of Step $l{-}1$. Set $A^{l+1}_{\mathbf{s}\left(x^l\right)} = A^l_{\mathbf{s}\left(x^l\right)} \cup \left\{x^l\right\}$, and set $A^{l+1}_{s'} = A^l_{s'}$ for each $s' \neq \mathrm{s}\left(x^l\right)$.

If no individual can propose a new contract at any time—that is, if all individuals for whom no contracts are on hold have proposed all contracts they find acceptable—then the algorithm terminates. The outcome of the COP is the set of contracts held by institutions at the end of the last step. In the COP, individuals propose contracts sequentially. Institutions accumulate offers, choosing at each step (according to their choice rules) a set of contracts to hold from the set of all previous offers. The process terminates when no individual wishes to propose a contract.

Given a preference profile of students $P = (P_i)_{i\in I}$ and a profile of choice functions for institutions $C = (C_s)_{s\in S}$, let $\Phi(P,C)$ denote the outcome of the COM. Let $\Phi_i(P,C)$ denote the assignment of individual $i \in \mathcal{I}$ and $\Phi_s(P,C)$ denote the assignment of institution $s \in \mathcal{S}$.

[14]Substitutable and size monotonic choice rules satisfy the following irrelevance of rejected contracts (IRC) condition: A choice rule C^s satisfies the IRC if for all $X \subset \mathcal{X}$ and $x \in \mathcal{X} \setminus X$, $x \notin C^s(X \cup \{x\})$ implies $C^s(X) = C^s(X \cup \{x\})$.

We are now ready to present our first main result.

Theorem 1. *The COM under GL choice rules is stable and strategy-proof.*

Proof. See Appendices 5.2 and 5.3. □

To prove Theorem 1, we first define a *completion*—introduced in Hatfield and Kominers (2019)—for a given GL choice rule. Then, we show that the completion satisfies the substitutability and size monotonicity.[15]

3 Affirmative Action in India

In this section, we employ GL choice functions as a tool to proffer pragmatic solutions to resource allocation problems in India with intricate diversity constraints. To do so, we first formulate resource allocation problems in India via a many-to-one matching framework with contracts. Subsequently, we construct GL choice functions for institutions, meticulously incorporating both vertical and horizontal reservations as well as de-reservations within the allocation schema.

Vertical Reservations. Let $\mathcal{R} = \{SC, ST, OBC, EWS\}$ be the set of reserved categories. In each institution $s \in \mathcal{S}$, $\overline{q}_s^{SC}$, $\overline{q}_s^{ST}$, $\overline{q}_s^{OBC}$, and $\overline{q}_s^{EWS}$ positions are earmarked for the reserved categories SC, ST, OBC, and EWS respectively. The residual $\overline{q}_s^{o,} = \overline{q}_s - \left(\overline{q}_s^{SC} + \overline{q}_s^{ST} + \overline{q}_s^{OBC} + \overline{q}_s^{EWS}\right)$ positions are called the open category positions. We denote the set of vertical categories $\mathcal{V} = \{o, SC, ST, OBC, EWS\}$, where o denotes the open category. Let $\overline{q}_s = (\overline{q}_s^o, \overline{q}_s^{SC}, \overline{q}_s^{ST}, \overline{q}_s^{OBC}, \overline{q}_s^{EWS})$ be the vector of vertical reservations at institution s.

The function $t : \mathcal{I} \rightarrow \mathcal{R} \cup \{GC\}$ denotes individuals' category membership. For every individual $i \in \mathcal{I}$, $t(i)$, or t_i, denotes the category individual i belongs to. We denote a profile of reserved category membership by $T = (t_i)_{i \in \mathcal{I}}$, and let $\mathcal{T}$ be the set of all possible reserved category membership profiles.

Horizontal Reservations. Let $H = \{h_1, \ldots, h_L\}$ be the set of horizontal types. The correspondence $\rho : I \rightrightarrows H \cup \{\emptyset\}$ represents individuals' horizontal type eligibility. That is, $\rho(i) \subseteq H \cup \{\emptyset\}$ denotes the set of horizontal types individual i can claim. $\rho(i) = \emptyset$ means that i does not have any horizontal type. The SCI's judgment in Indra Shawney (1992) mandates that horizontal reservations must be implemented within each vertical category on a minimum guarantee basis. We denote by κ_v^j the number of positions reserved for horizontal type $h_j \in H$ at vertical category $v \in \mathcal{V}$. The vector $\kappa_v^s \equiv (\kappa_v^1, ..., \kappa_v^L)$ denotes the horizontal reservations in vertical category $v \in \mathcal{V}$ at institution s. Let $\kappa_s \equiv \{\kappa_v^s\}_{v \in \mathcal{V}}$ denote the horizontal reservations at institution s.

[15] We also show that the completion satisfies the irrelevance of rejected contracts (IRC) condition of Aygün and Sönmez (2013).

Priority Orders. Each institution s has a strict priority ordering $\succ_s$ over $\mathcal{I} \cup \{\emptyset\}$. We write $i \succ_s j$ to mean that applicant i has a higher priority than j at s. Similarly, we write $i \succ_s \emptyset$ to say that applicant i is acceptable for s. $\emptyset \succ_s i$ means that applicant i is unacceptable for s. The profile of institutions' priorities is denoted $\succ = (\succ_{s_1}, ..., \succ_{s_m})$. For each institution $s \in \mathcal{S}$, the merit ordering for individuals of type $r \in \mathcal{R}$, denoted by $\succ_s^r$, is obtained from $\succ_s$ in a straightforward manner as follows:

- for $i, j \in \mathcal{I}$ such that $t_i = r$, $t_j \neq r$, $i \succ_s \emptyset$, and $j \succ_s \emptyset$, we have $i \succ_s^r \emptyset \succ_s^r j$, where $\emptyset \succ_t^r j$ means individual j is unacceptable for category r at institution s.
- for any other $i, j \in \mathcal{I}$, $i \succ_s^r j$ if and only if $i \succ_s j$.

Contracts. We define $\mathcal{X} \equiv \bigcup\limits_{t(i)=GC} \{i\} \times \mathcal{S} \times \{O\} \bigcup\limits_{t(i)\in\mathcal{R}} \{i\} \times \mathcal{S} \times \{r, O\}$ as the set of all contracts. Each contract $x \in \mathcal{X}$ is between an individual $\mathbf{i}(x)$ and an institution $\mathbf{s}(x)$ and specifies a vertical category $\mathbf{t}(x)$ under which individual $\mathbf{i}(x)$ is admitted.

Individuals' Preferences. Each GC individual has a preference over $\mathcal{S}\times \{GC\} \cup \{\emptyset\}$. Each reserve category r member has a preference over $\mathcal{S} \times \{r, GC\} \cup \{\emptyset\}$, where $\emptyset$ denotes the outside option.[16] We write $(s, v)P_i(s', v')$ to mean that individual i strictly prefers admission to institution s via vertical category v to admission to institution s' via vertical category v'. The *at-least-as-well* relation R_i is obtained from P_i as follows: $(s, v)R_i(s', v')$ if and only if either $(s, v)P_i(s', v')$ or $(s, v) = (s', v')$. An institution and vertical category pair (s, v) is *acceptable* to individual i if it is at least as good as the outside option $\emptyset$ and is *unacceptable* to her if it is worse than the outside option $\emptyset$.

Legal mandates. Certain attributes of institutional choice rules are explicitly delineated through pivotal decisions rendered by the SCI.[17] Specifically, positions in the open category are allocated prior to those designated for Scheduled Castes (SC), Scheduled Tribes (ST), Other Backward Classes (OBC), and Economically Weaker Sections (EWS). This implies that vertical reservations function as an *over-and-above* policy. Furthermore, horizontal reservations are enacted within each of these vertical categories. That is, horizontal reservations are *compartmentalized*. Within each such category, the allocation of positions is mandated to adhere to applicants' merit scores, albeit subject to the constraints imposed by horizontal reservations. The SCI mandates that horizontal reservations are implemented as *minimum guarantees*. That is, horizontally reserved positions are allocated before horizontally unreserved positions within each vertical category.

[16]We assume that individuals do not have preferences for horizontal types as there is no evidence that individuals care about the horizontal types under which they are admitted. All discussion regarding the reservation policy in India centers around vertical reservations.

[17]See Aygün and Turhan (2022) and Sönmez and Yenmez (2022) for details.

Notably, although it was not explicitly mandated, the SCI intimated in its landmark judgment in the case of Ashoka Kumar Thakur vs. Union of India (2008) that unfilled positions allocated for OBC candidates should be transitioned to open category positions in the context of admissions to publicly-funded universities.[18]

The implementation of horizontal reservations within vertical categories remains ambiguous when applicants possess multiple horizontal types, as underscored in Sönmez and Yenmez (2022). This absence of prescriptive guidance affords considerable latitude in the formulation of institutional selection procedures, calibrated to align with specific policy objectives set forth by authorities. A pivotal consideration in this context is the counting of an admitted applicant against specific horizontal reservation types when multiple such types could be invoked. Two primary approaches to reconciling horizontal reservations with merit scores within a given vertical category have been proposed:

1. *One-to-All Horizontal Matching*: An admitted applicant with multiple eligible horizontal types is counted against *all* of them, as detailed in Aygün (2014) and Aygün (2017).

2. *One-to-One Horizontal Matching*: An admitted applicant with multiple eligible horizontal types is counted against *only one* of these types, as elucidated in Sönmez and Yenmez (2022).

Additionally, de-reservation policies introduce another layer of flexibility in the design of institutional selection procedures, owing to the absence of explicit directives. In certain allocation contexts in India, such as technical university admissions, unfilled OBC positions are reallocated to the open category. Conversely, in government job recruitments, OBC reservations are treated as *hard reserves*, meaning they are not subject to reallocation. To accommodate these divergent practices, we introduce two distinct transfer policies: (i) a *No Transfer Policy*, and (ii) an *OBC De-Reservation Policy.*

The selection protocols employed by institutions in India are obligated to adhere to legally stipulated vertical and horizontal reservations, as well as transfer (de-reservation) policies. In order to articulate institutions' *overall choice rules*, our approach unfolds in two stages. Initially, we introduce *sub-choice rules*, which amalgamate horizontal reservations with merit scores within each distinct vertical category. Subsequently, these vertical-specific sub-choice rules are interconnected through an overarching transfer policy.

First delineated in Aygün and Bó (2021), the notion of fairness in a choice function serves as a pivotal criterion in design considerations. According to this principle, if an individual is not selected—i.e., none of her contracts are accepted—then those who are chosen must either possess superior merit scores or be beneficiaries of vertical or horizontal reservations that the unchosen individual lacks. In the specific context of India, a choice rule's fairness mandates that merit scores must be accorded due consideration, subject to legally imposed vertical and

[18]The judgment can be accessed at https://indiankanoon.org/doc/63489929/.

horizontal reservations. When comparing two individuals with identical vertical and horizontal types, precedence is given to the individual boasting the higher merit score.

DEFINITION. An overall choice rule C^s is **fair** if, for any given set of contracts $X \subseteq \mathcal{X}$, the chosen set $C^s(X)$ satisfies the following: If $\mathbf{i}(x) \notin \mathbf{i}[C^s(X)]$, then for every $y \in C^s(X)$ one of the followings hold:

(1) $\mathbf{i}(y) \succ_{\mathbf{s}(y)} \mathbf{i}(x)$,

(2) $\mathbf{t}(x) \neq \mathbf{t}(y)$ or $\rho(\mathbf{i}(y)) \setminus \rho(\mathbf{i}(x)) \neq \emptyset$.

In the context of merit-based object allocation subject to AA constraints in India, we are interested in mechanisms that *eliminate justified envy* in the following sense:

DEFINITION. A feasible matching $X \subseteq \mathcal{X}$ *eliminates justified envy* if, for any given pair contracts $x, y \in X$, $(\mathbf{s}(y), \mathbf{t}(y)) P_{\mathbf{i}(x)} (\mathbf{s}(x), \mathbf{t}(x))$ implies either $\mathbf{i}(y) \succ_{\mathbf{s}(y)} \mathbf{i}(x)$ or $\rho(\mathbf{i}(x)) \not\supseteq \rho(\mathbf{i}(y))$. A mechanism ψ *eliminates justified envy,* if for every preference profile P, $\psi(P)$ eliminates justified envy.

The elimination of justified envy criterion can be interpreted as respecting merit scores subject to vertical and horizontal reservations and de-reservations. A feasible matching eliminates justified envy if an individual envies the assignment of another individual, then either the former individual has a lower merit score at that institution or the latter individual has a horizontal reservation type that the former does not. Note that if $(\mathbf{s}(y), \mathbf{t}(y)) P_{\mathbf{i}(x)} (\mathbf{s}(x), \mathbf{t}(x))$ is possible only when the individual $\mathbf{i}(x)$ is eligible for vertical category $\mathbf{t}(y)$.

Stability with respect to fair choice rules does not imply elimination of justified envy. Our first result establishes the independence of two properties of a given matching: (i) elimination of justified envy and (2)) stability with respect to fair choice rules.[19]

Proposition 1. *Elimination of justified envy and stability with respect to fair choice rules are independent.*

Proof. Consider the following problem with a single institution $\mathcal{S} = \{s\}$, and two individuals $\mathcal{I} = \{i, j\}$. There are two vertical categories, r and o. Institution s has two positions. One of them is reserved for r applicants and the other one is an open category. If open category position remains empty, its capacity is not transferred to category r. Suppose that both i and j are eligible for vertical category r. Let $X = \{x_1, x_2, y_1, y_2\}$, where $\mathbf{i}(x_1) = \mathbf{i}(x_2) = i$, $\mathbf{i}(y_1) = \mathbf{i}(y_2) = j$, $\mathbf{t}(x_1) = \mathbf{t}(y_1) = o$, and $\mathbf{t}(x_2) = \mathbf{t}(y_2) = r$. Suppose that individuals have the following preferences: $x_2 P_i x_1$ and $y_2 P_j y_1$. Individual i has higher merit score than j, i.e., $i \succ_s j$.

[19] Romm et al. (2021) generalize the definition of justified envy in matching with contracts environments and show that stable allocations might have justified envy. Our fairness definition for choice rules is different than their definition of strong priority, but the two notions are related.

Let C^s be the following fair choice rule: The open category position is filled before the r position following the merit scores. The SC position is filled following the merit score of the agents associated with the remaining contracts.

The allocation $Y = \{x_1, y_1\}$ eliminates justified envy, but not stable with respect to C^s because Y is blocked via $\{y_2\}$. That is, $C^s\{x_1, y_1, y_2\} = \{x_1, y_2\}$. On the other hand, the allocation $Z = \{x_1, y_2\}$ is stable with respect to C^s but does not eliminate justified envy because individual i envies j's assignment. $\square$

In Subsections 3.1 and 3.2, we initially formulate fair GL choice rules for institutions that are in compliance with legal requisites, focusing on both one-to-one and one-to-many horizontal matching arrangements. Subsequently, we introduce allocation mechanisms that are strategy-proof, devoid of justified envy, and stable in accordance with the aforementioned fair GL choice rules of institutions.

3.1 Direct Mechanism Design via One-to-All Horizontal Matching

In scenarios where a selected applicant with multiple horizontal categories is counted against all such categories she is eligible for, as discussed in Aygün (2014) and Aygün (2017), complementarities among applicants may arise, thereby obviating the existence of stable matchings. However, when horizontal reservations adhere to a specific structure, termed as "*hierarchical*" (alternatively known as "nested" or "laminar"), stable matchings can indeed be realized.[20] In a considerable number of resource allocation problems in India, horizontal reservations conform to this hierarchical structure.

DEFINITION. We say that horizontal reservations are *hierarchical*, if for any pair of types $h, h^{'} \in H$ such that $\rho^{-1}(h) \cap \rho^{-1}(h^{'}) \neq \emptyset$, either $\rho^{-1}(h) \subset \rho^{-1}(h^{'})$ or $\rho^{-1}(h^{'}) \subset \rho^{-1}(h)$. When $\rho^{-1}(h^{'}) \subset \rho^{-1}(h)$, then we say that the horizontal type h *contains* $h^{'}$.

In the absence of horizontal reservations, candidate selection within each vertical category is predicated exclusively on merit scores, resulting in an outcome that can be termed "*fully meritorious*." However, the introduction of horizontal reservations necessitates a departure from this idealized, merit-centric outcome. The "one-to-all" horizontal type matching framework is optimally aligned for minimizing such deviations. This observation is formally encapsulated through the introduction of a "*merit-based comparison*" criterion.

DEFINITION. We say that a set of individuals I *merit-based dominates* a set of individuals J with $\mid I \mid = \mid J \mid = n$ at institution s if there exists a bijection $g : I \rightarrow J$ such that

1. for all $i \in I$, $i \succeq_s g(i)$, and,
2. there exists $j \in I$ such that $j \succ_s g(j)$.

[20] In a related work, Kamada and Kojima (2018) identify the hierarchy as necessary and sufficient condition on constraint structures for the existence of a stable and strategy-proof mechanism.

Specifically, the merit score of the highest-ranking applicant in set I is equal to or greater than that of the highest-ranking applicant in set J. This holds true for the second highest-ranking applicant in both sets, and continues in a similar manner down the ranking hierarchy. Furthermore, there exists an integer k such that the merit score of the k^{th}-highest-ranking applicant in I is greater than the score of the k^{th}-highest-ranking applicant in J.

We extend the merit-based comparison criterion to compare q-acceptant choice rules. Let $\widetilde{C}$ and C be q-acceptant rules. We say that $\widetilde{C}$ *merit-based dominates* C if, for any set of contracts $X \subseteq \mathcal{X}$, the set of individuals $\mathbf{i}(\widetilde{C}(X))$ merit-based dominates the set of individuals $\mathbf{i}(C(X))$.[21]

DEFINITION. A sub-choice rule C_v^s for vertical category v at institution s is *merit-based undominated subject to horizontal reservations* if, for every set of contracts $X \subseteq \mathcal{X}$,

(1) $C_v^s(X)$ satisfies horizontal reservations κ_v^s, and

(2) $C_v^s(X)$ is not merit-based dominated by any set $Y \subseteq X_v$ that satisfies horizontal reservations.

Hierarchical sub-choice rule. Fix vertical category $v \in \mathcal{V}$. Let $X \subseteq \mathcal{X}_s$. Remove all contracts in $X \setminus X_v$. Let $\kappa^s = (\kappa_v^s)_{v \in \mathcal{V}}$ denote the horizontal reservation of institution s, where

$$(\kappa_v^s)^1 \equiv ((\kappa_v^{s,j})^1)_{h_j \in H}$$

is the vector of horizontal reservations at vertical category $v \in \mathcal{V}$. In each step of the hierarchical sub-choice rule $C_v^{ⓗ}$, the number of horizontal reservations is updated.

Let H^1 be the set of horizontal types that do not contain another horizontal type.

Step 1: If no individual has any horizontal type, then choose the contracts of individuals with the highest merit scores for all positions. Otherwise, for every horizontal type $h_j \in H^1$, if there are less than $(\kappa_v^{s,j})^1$ individuals, choose contracts of all of them. Otherwise, choose the contracts of the $(\kappa_v^{s,j})^1$ highest-scoring individuals with type h_j. Reduce the number of available positions and number of horizontally reserved positions for any type that contains h_j by the number of chosen contracts. Eliminate h_j from the horizontal types to be considered.

If there are no individual or positions left, then end the process and return the chosen set of contracts. Let H^2 be the set of remaining horizontal types that does not contain another horizontal type. Let $\kappa_s^2 = (\kappa_v^{s,j})^2)_{v \in \mathcal{V}}$ denote the updated numbers of horizontal reservations for Step 2, where $(\kappa_v^{s,j})^2 \equiv (\kappa_v^{s,j})_{h_j \in H \setminus H^1}$ is the updated number of horizontally reserved positions for types that have not yet been considered.

[21]This domination relation does not induce a complete binary relation. It might be the case that two choice rules are incomparable with regard to this domination criterion. Consider the following example: Let $q = 2$ and $I = \{i_1, i_2, i_3, i_4\}$. Suppose that individuals' test scores are ordered from highest to lowest as $i_1 - i_2 - i_3 - i_4$. Let $\widetilde{C}$ and C be q-responsive choice rules such that $\widetilde{C}(I) = \{i_1, i_4\}$ and $C(I) = \{i_2, i_3\}$. According to our domination criterion, the sets $\{i_1, i_4\}$ and $\{i_2, i_3\}$ do not dominate each other.

Step n (n≥2): If there is no horizontal type left to be considered, then choose contracts of individuals following the merit score ranking for the remaining positions. Otherwise, for every type $h_j \in H^n$, if there are less than $(\kappa_v^{s,j})^n$ individuals in the remaining set, choose contracts of all. Otherwise, choose the contracts of the $(\kappa_v^{s,j})^n$ highest-scoring individuals with type h_j. Reduce the total number of available positions and number of horizontally reserved positions for any type that contains h_j by the number of chosen contracts. Eliminate h_j from the set of types to be considered.

If no individuals or positions are left, end the process and return the chosen set of individuals. Let H^{n+1} be the set of remaining types that do not contain another horizontal type. Let $\kappa_s^{n+1} = ((\kappa_v^{s,j})^{n+1})_{v\in\mathcal{V}}$ denote the updated numbers of horizontal reservations for Step n+1, where $(\kappa_v^{s,j})^{n+1} \equiv (\kappa_v^{s,j})_{h_j\in H\setminus(\bigcup_{r=1}^{n} H^r)}$ is the updated number of horizontally reserved for types that have not yet been considered.

Theorem 2. $\mathbf{C}_v^{\textcircled{h}}$ *is the unique choice rule that is merit-based undominated subject to horizontal reservations.*

Overall Choice Rule with No Transfer $C^{\textcircled{h}NT}$

Each vertical category chooses contracts via the hierarchical sub-choice rule $C^{\textcircled{h}}$. The overall choice rule with no transfers, $C^{\textcircled{h}NT}$, is written as

$$C_s^{\textcircled{h}NT} \equiv \left(\left(C_v^{\textcircled{h}}\right)_{v\in\mathcal{V}}, q_s\right),$$

where $q_s = \left(q_s^o, q_s^{SC}, q_s^{ST}, q_s^{OBC}, , , q_s^{EWS}\right)$ such that $q_s^v = \overline{q}_s^v$ for all $v \in \{o, SC, ST, OBC, EWS\}$.

Overall Choice Rule with Transfer $C^{\textcircled{h}T}$

According to overall choice rule with transfer, $C^{\textcircled{h}T}$, unfilled OBC positions are made open category and filled following merit scores. $C^{\textcircled{h}T}$ is written as

$$C^{\textcircled{h}T} \equiv \left(\left(C_v^{\textcircled{h}}\right)_{v\in\mathcal{V}}, q_s\right),$$

where

- the precedence sequence is $o - SC - ST - OBC - EWS - D$, where D denotes "*de-reserved positions*" that are reverted from OBC, and
- $q_s = \left(q_s^o, q_s^{SC}, q_s^{ST}, q_s^{OBC}, q_s^{EWS}, q_s^D\right)$ such that $q_s^v = \overline{q}_s^v$ for all $v \in \{o, SC, ST, OBC, EWS\}$, and $q_s^D = r_{OBC}$ is the number of vacant OBC positions.

Theorem 3. $C^{\textcircled{h}NT}$ *and* $C^{\textcircled{h}T}$ *are fair and GL choice rules.*

Proof. See Appendix 5.3. □

To prove that $C^{ⓗNT}$ and $C^{ⓗT}$ are in the GL family we show that the hierarchical choice rule $C_v^{ⓗ}$ is substitutable, size monotonic, and quota monotonic.

Next, we propose the COM as a direct mechanism coupled with institutions' overall choice rules $C^{ⓗNT}$ and $C^{ⓗT}$.

Let $\Phi^{ⓗNT}$ and $\Phi^{ⓗT}$ denote the COM under profile of choice rules $C^{ⓗNT} = \left(C_s^{ⓗNT}\right)_{s\in\mathcal{S}}$ and $C^{ⓗT} = \left(C_s^{ⓗT}\right)_{s\in\mathcal{S}}$, respectively.

Theorem 4. *$\Phi^{ⓗNT}$ and $\Phi^{ⓗT}$ are stable with respect to $C^{ⓗNT}$ and $C^{ⓗT}$, respectively. Moreover, $\Phi^{ⓗNT}$ and $\Phi^{ⓗT}$ are strategy-proof for individuals and eliminate justified envy.*

Proof. See Appendix 5.3. □

When confronting hierarchical horizontal constraints and an objective on the part of policy-makers to identify the most meritorious outcomes, we introduce two distinct allocation mechanisms. For the context of government employment allocation, we propose the mechanism $\Phi^{ⓗNT}$, and for the allocation of seats in public educational institutions, we introduce the mechanism $\Phi^{ⓗT}$.

3.2 Direct Mechanism Design via One-to-One Horizontal Matching

One salient merit of the one-to-one horizontal matching approach lies in its capacity to facilitate the design of stable and strategy-proof mechanisms, irrespective of the underlying structure of horizontal reservations. This framework was introduced in the context of Indian resource allocation by Sönmez and Yenmez (2022). To orchestrate the allocation of individuals to horizontally reserved positions, they formulated the *Meritorious Horizontal* choice rule, denoted $C^{ⓜ}$.

Within each vertical category v, Sönmez and Yenmez (2022) consider the assignment of horizontally reserved positions $H^v = \bigcup_{j=1}^{L} H_j^v$ to a set of individuals I as a one-to-one matching problem, where H_j^v denotes the set of positions that are reserved for horizontal type $h_j \in H$ at vertical category v. Denote by κ_v^j the number of positions in H_j^v.

Given a vertical category $v \in \mathcal{V}$ and a set of individuals I such that $t_i = v$ for all $i \in I$, Sönmez and Yenmez (2022) define the one-to-one matching between the individuals and horizontally reserved positions as a mapping $\mu : I \to H^v \bigcup \{\emptyset\}$ such that for any $i \in I$ and $h_j \in H$,

- $\mu(i) \in H_j^v$ implies $h_j \in \rho(i)$, and for any $i, k \in I$,
- $\mu(i) = \mu(k) \neq \emptyset$ implies $i = k$.

To align with the terminology used in Sönmez and Yenmez (2022), we let $n^v(I)$ represent the maximum number of horizontally reserved positions that can be allocated to individuals within a specified vertical category v. A matching between individuals and these horizontally reserved positions is said to have "*maximum cardinality*" if no alternative assignment exists that would strictly increase the total number of such positions allocated to eligible individuals. The authors introduce the concept of *increasing horizontal reservation utility* as follows: For a given vertical category v and a set of individuals I belonging to category v, an individual i is said to "*increase horizontal reservation utilization*" if $n^v(I \cup \{i\}) = n^v(I) + 1$.

Meritorious Horizontal Sub-choice Rule. Given a vertical category v, a set of applicants A such that $\mathbf{v}(i) = v$ for all $i \in A$, a profile of applicants' horizontal types $(\rho(i))_{i \in A}$, meritorious horizontal sub-choice rule $C^{ⓜ}$ selects applicants as follows:

$\mathbf{Step\ 1.1}$: *Choose the highest merit score individual in A with a horizontal type. Denote this individual by i_1 and let $A_1 = \{i_1\}$. If no such individual exists, proceed to Step 2.*

$\mathbf{Step\ 1.n\ (2, ..., \sum_{h_j \in \mathcal{H}} \kappa_v^j)}$: *Choose the highest merit score individual in $A \setminus A_{n-1}$ who increase the horizontal reservation utilization given A_{n-1}, if any. Denote this individual by i_n and let $A_n = A_{n-1} \cup \{i_n\}$. If no such individual exists, proceed to Step 2.*

$\mathbf{Step\ 2}$: *Fill remaining positions following the merit score with individuals among the unassigned individuals until all positions are assigned or all individuals are selected.*

Overall Choice Rule with No Transfers $C^{ⓜNT}$

Under ona-to-all horizontal matching arrangement, we propose the overall choice rule with no transfer, $C^{ⓜNT}$, for allocating government jobs in India. According to this rule, each vertical category chooses applicant via the sub-choice rule $C^{ⓜ}$. The rule $C^{ⓜNT}$ is an GL choice rule and can be written as

$$C_s^{ⓜNT} \equiv \left(\left(C_v^{ⓜ}\right)_{v \in \mathcal{V}}, q_s \right)$$

such that $q_s = \left(q_s^o, q_s^{SC}, q_s^{ST}, q_s^{OBC}, q_s^{EWS}\right)$, where $q_s^v = \overline{q}_s^v$ for all $v \in \{o, SC, ST, OBC, EWS\}$.

Overall Choice Rule with Transfer $C^{ⓜT}$

According to overall choice rule with transfer, $C^{ⓜT}$, unfilled OBC positions are reverted to open categoryand filled following merit scores. The rule $C^{ⓜT}$ is an GL choice rule and can be written as

$$C_s^{ⓜT} \equiv \left(\left(C_v^{ⓜ}\right)_{v \in \mathcal{V}}, q_s \right),$$

where

- the precedence sequence is $o - SC - ST - OBC - EWS - D$, where D denotes "*de-reserved positions*" that are reverted from OBC, and

- $q_s = \left(q_s^o, q_s^{SC}, q_s^{ST}, q_s^{OBC}, q_s^{EWS}, q_s^D\right)$ such that $q_s^v = \overline{q}_s^v$ for all $v \in \{o, SC, ST, OBC, EWS\}$, and $q_s^D = r_{OBC}$ is the number of vacant OBC positions.

Theorem 5. *$C^{\textcircled{m}NT}$ and $C^{\textcircled{m}T}$ are fair and GL choice rules.*

Proof. See Appendix 5.3. □

When each vertical category $v \in \mathcal{V}$ chooses applicants via $C_v^{\textcircled{m}}$, then under any monotonic transfer policy, institutions' overall choice rules are in the GL family. To show this, we only need to prove that $C_s^{\textcircled{m}}$ satisfies quota monotonicity.[22]

Next, we introduce two direct mechanism via the COM. The first one, $\Phi^{\textcircled{m}NT}$, is the COM under the profile of institutional choice rules $C^{\textcircled{m}NT} = \left(C_s^{\textcircled{m}NT}\right)_{s\in\mathcal{S}}$. The second one, $\Phi^{\textcircled{m}T}$, is the COM under profile of institutional choice rules $C^{\textcircled{m}T} = \left(C_s^{\textcircled{m}T}\right)_{s\in\mathcal{S}}$.

Theorem 6. *$\Phi^{\textcircled{m}NT}$ and $\Phi^{\textcircled{m}T}$ are stable with respect to $C^{\textcircled{m}NT}$ and $C^{\textcircled{m}T}$, respectively. Moreover, $\Phi^{\textcircled{m}NT}$ and $\Phi^{\textcircled{m}T}$ are strategy-proof for individuals and eliminate justified envy.*

Proof. See Appendix 5.3. □

When authorities decide to implement one-to-one horizontal matching convention, we propose $\Phi^{\textcircled{m}NT}$ for government recruitments and $\Phi^{\textcircled{m}T}$ for admissions to public schools.

4 Conclusion

This paper furnishes strong evidence of constrained strategy spaces in multiple matching markets within India, specifically impacting reserved-category applicants. These individuals are precluded from articulating their preferences concerning the nature of their allocations in public school admissions and government job placements. We identify the deficiencies ensuing from this limitation. Utilizing an expansive matching model, we formalize institutional selection processes through Generalized Lexicographic (GL) choice rules. The GL framework serves to generalize well-known choice rules in market design literature, allowing for the nuanced handling of diversity considerations. We introduce a priority schema for institutions and advance a mechanism design tailored for centralized clearinghouses, contextualized within the landscape of comprehensive affirmative action policies in India. Our proposal fulfills critical criteria requisite for the effective functioning of matching markets. The framework we establish is versatile, enabling researchers to address resource allocation problems that integrate diversity concerns, such as multidimensional reserves in Brazil or the 2015 inclusionary education legislation in Chile.

[22] Fleiner (2001) proved that $C_s^{\textcircled{m}}$ satisfies the substitutes condition. Moreover, Sönmez and Yenmez (2022) proved that it satisfies size monotonicity.

5 APPENDICES

5.1 Evidence of Individuals' Preferences on Position Categories

5.1.1 *Evidence from Court Cases*

In a recent Bombay High Court case—*Shilpa Sahebrao Kadam And Another vs. The State of Maharastra* (2019)[23]—the following quote highlights that the petitioners (reserve category members) do not report their membership to reserve categories and seek assignment via open category positions.

> *"The petitioners contend that though they belong to reserved category, they have filled in their application forms as a general category candidates and have not claimed any benefit as a member belonging to reserved category."*

In another case from Bombay High Court—*Vinod Kadubal Rathod vs. Maharashtra State Electricity* (2017)[24]—two petitioners applied to technician positions without reporting their OBC membership.

> *"The petitioners belong to VJ (A) Category. However, they applied for the said post from General Category and as such, did not claim the benefits of reservation meant for VJ (A) Category. Further, they mentioned in their applications that they were meritorious sports persons and as such, they claimed benefit of the reservation meant for the said category.*"

Many other court cases report that reserved category members did not report their memberships to their eligible categories and sought admission via open category positions, which decreases their admission chances. It suggests that applicants do so knowingly and reflects their preferences.

5.1.2 *Evidence from the Literature*

Khanna (2020) discuss the incentive effects of affirmative action in India as follows:

> *"In the Indian context, affirmative action programs are more salient and larger in magnitude than in most other countries. Reserving a very large fraction of seats may allow low-ability, low-caste students to get into college and into public sector jobs, exacerbating employers negative stereotypes, leading to more discrimination and less human capital accumulation."*

[23]The judgment is available at https://indiankanoon.org/doc/89017459/.
[24]The judgment is available at https://indiankanoon.org/doc/162611497/.

Gille (2013) analyzes the determinants of reserved category members' applications for the reservation policy in education, particularly the role of social stigma attached to the reservation policy. The author focuses on the OBC applicants and analyzes the impact of the social positions of the individuals' reference group on their choice of applying for reservation. She discusses that the cost of social stigma is higher for people with higher social status. She further argues that according to the literature on the Indian caste system, status is even more important for the OBC. She finds that, for a given wealth level, individuals from socially higher subcastes at the village level are less prone to apply for reservations in education than individuals with lower subcastes.

Pandey and Pandey (2018) argue that students from reserve categories are likely to face humiliation and harassment from their teachers and fellow students. From a survey at an IIT campus, the authors report that 13 percent of students in SC/ST caste categories felt teacher attitudes toward them were hostile. Moreover, they report that 21 percent of students in SC/ST categories found fellow students' attitudes hostile compared to zero percent in the GC.

Gudavarthi (2012) explains the stigma as follows:

> *"The singularly debilitating limitation of the system of reservations in India has been to increasingly produce a large number of social groups that suffer forms of public humiliation, resentment and insult. The purpose of reservations to provide the disadvantaged social groups a head start in realizing their potential remains arrested and minimal, due to their inability to overcome the stigma that is attached to such policies."*

A recent article series in *Nature* examines the data on racial and ethnic diversity in Science in different countries. The article on India[25] makes a strong case regarding the stigma attached to reserved category positions. The article's author interviews a student who is a member of the ST category. The student indicates that he prefers obtaining his position via open category rather than ST reservation. The author reports that

> *"...he does not want to publicize his last name or institution because the student fears that doing so would bring his social status to the attention of a wider group of Indian scientists. "They'd know that I am from a lower category and will think that I have progressed because of the quota," he says."*

5.1.3 *Evidence from Discussion Forums*

A cursory search on the internet reveals how pervasive it is that many reserved category candidates apply to positions without claiming the benefit of their vertical category. In a popular

[25]The article can be accessed at https://www.nature.com/articles/d41586-023-00015-2.pdf.

online discussion forum Quora, users exchanged ideas in response to the question: *"What happens if an SC applicant fills an application form as a GC member in India?"* [26]

The following quote from a user indicates the prevalence of the practice:

> *"I know several persons from SC community who applied in general category and got admissions and government positions. There were several cases of SC candidates contesting and winning from general constituencies in Kerala... In short, availing a reservation facility is an option and not compulsory."*

It is important to note that reserved category candidates know as a matter of fact that claiming the benefit of reservation policy help them obtaining positions. The following quote from another user in the same discussion puts forth the dilemma reserve category applicants face:

> *"Reserved people bear the brunt of the heat from both side. If they don't use reservation, then you say they are blocking general category people. If they are using reservation then they are being blamed too. What do they do!!!"*

Another user responded in the same discussion as follows:

> *"The thing is I'd have got a General seat even if I've filled up the form as an SC because I secured All India Rank 02. I've done that intentionally, it's not that I lost my caste certificate or anything which made me doing that. I've done that because I knew I could do it in one go; confidence you name it probably... I want to lead life of a general candidate professionally, I don't want any extra bucks/promotions. I want what I deserve as a general candidate."*

Numerous comments on the web indicate the type of positions they matched under matters for many reserved category members.

5.2 Substitutable Completion of GL Choice Rules

The following definitions are from Hatfield and Kominers (2019).

"*A* **completion** *of a choice function C^s of institution $s \in S$ is a choice function $\overline{C}^s$, such that for all $X \subseteq \mathcal{X}$, either $\overline{C}^s(X) = C^s(X)$ or there exists a distinct $x, x' \in \overline{C}^s(X)$ such that $\mathbf{i}(x) = \mathbf{i}(x')$.*

If a choice function C^s has a completion that satisfies the substitutability and IRC condition, then C^s is said to be substitutably completable.

If every choice function in a profile $C = (C^s)_{s \in S}$ is substitutably completable, then we say that C is substitutably completable."

Let $C^s(\cdot; q^s)$ be a GL choice rule given the *capacity transfer scheme* q^s. We define a related choice function $\overline{C}^s(\cdot; q^s)$ as follows: Given a set contracts $Y \equiv Y^1 \subseteq \mathcal{X}$, a capacity $\overline{q}^s$ for

[26]This discussion is available at https://www.quora.com/What-happens-if-an-SC-fills-a-form-as-a-general-category-member-in-India (last accessed 05/05/2020).

institution s, and a capacity $\overline{q}_1^s$ for category 1, we compute the chosen set $\overline{C}^s(Y;q^s)$ in K_s steps, where categories choose in a sequential order.

Step 1. Given Y^1 and $\overline{q}_1^s$, category 1 chooses $\overline{C}_1^s(Y^1;\overline{q}_1^s)$. Let $r_1 = \overline{q}_1^s - \mid \overline{C}_1^s(Y^1;\overline{q}_1^s) \mid$. Let $Y^2 \equiv Y^1 \setminus \overline{C}_1^s(Y^1;\overline{q}_1^s)$.

Step k. $(2 \leq k \leq K_s)$**:** Given Y^k and its dynamic capacity $q_k^s(r_1,...,r_{k-1})$, category k chooses $\overline{C}_k^s(Y^k;q_k^s(r_1,...,r_{k-1}))$. Let $r_k = q_k^s(r_1,...,r_{k-1}) - \mid \overline{C}_k^s(Y^k;q_k^s(r_1,...,r_{k-1})) \mid$. Let $Y^{k+1} \equiv Y^k \setminus \overline{C}_k^s\left(Y^k;q_k^s(r_1,\ldots,r_{k-1})\right)$.

The union of categories' choices is the institution's chosen set, i.e.,

$$\overline{C}^s(Y;q^s) \equiv \overline{C}_1^s(Y^1;\overline{q}_1^s) \cup \bigcup_{k=2}^{K_s} \overline{C}_k^s(Y^k;q_k^s(r_1,...,r_{k-1})).$$

The difference between C^s and $\overline{C}^s$ is as follows: In the computation of C^s, if a contract of an individual is chosen by some category, then her other contracts are removed for the rest of the procedure. According to $\overline{C}^s$, if an individual's contract is chosen by category k, then her other contracts will still be available for the remaining categories.

Proposition 2. *$\overline{C}^s$ is a completion of C^s.*

Proof. Consider an offer set $Y = Y^1 \subseteq X$ and assume there is no pair of contracts $z, z' \in Y^1$ such that $i(z) = i(z')$ and $z, z' \in \overline{C}^s(Y;q^s)$. We want to show that $\overline{C}^s(Y;q^s) = C^s(Y;q^s)$.

Let (Y_j, r_j, Y^{j+1}) and $(Z_j, \overline{r}_j, Z^{j+1})$ be sequences of a set of contracts chosen by category j, the number of vacant slots in category j, and the set of contracts that remains in the choice procedure after category j selects under C^s anf $\overline{C}^s$, respectively.

Given $\overline{q}_1^s$ and $Y^1 = Z^1$, we have $Z_1 = \overline{C}_1^s(Z^1;\overline{q}_1^s) = C_1^s(Y^1;\overline{q}_1^s) = Y_1$. Moreover, $\overline{r}_1 = r_1$ and $\overline{q}_2^s(\overline{r}_1) = q_2^s(r_1)$. Suppose, for all $j \in \{2,...,k-1\}$, we have $Y_j = Z_j$. We must show that $Y_k = Z_k$.

Since $Y_j = Z_j$ for all $j = 1,\ldots,k-1$, we obtain $r_j = \overline{r}_j$ for all $j = 1,\ldots,k-1$. Then, $q_k^s(r_1,...,r_{k-1}) = \overline{q}_k^s(\overline{r}_1,...,\overline{r}_{k-1})$. Since there are no two contracts of an individual chosen by $\overline{C}^s$, all of the remaining contracts of individuals—whose contracts were chosen by previous categories—are rejected by $C_k^s(\overline{Y}^k;\overline{q}_k^s(\overline{r}_1,...,\overline{r}_{k-1}))$. Therefore, the IRC[27] of C_k^s implies that $C_k^s(\overline{Y}^k;\overline{q}_k^s(\overline{r}_1,...,\overline{r}_{k-1})) = C_k^s(Y^k;q_k^s(r_1,...,r_{k-1}))$. Hence, we obtain $\overline{Y}_k = Y_k$, $\overline{r}_k = r_k$, and $\overline{q}_{k+1}^s(\overline{r}_1,...,\overline{r}_k) = q_{k+1}^s(r_1,...,r_k)$. Since each category selects the same set of contracts under C^s and $\overline{C}^s$, the result follows. □

Proposition 3. *$\overline{C}^s$ satisfies the IRC condition.*

[27] Substitutability and size monotonicity of a sub-choice function imply that it satisfies the irrelevance of rejected contracts (IRC), as shown by Aygün and Sönmez (2013)

Proof. Consider $Y = Y^1 \subseteq X$ such that $Y \neq \overline{C}^s(Y; q^s)$. Take $x \in Y \setminus \overline{C}^s(Y; q^s)$. We need to show that $\overline{C}^s(Y; q^s) = \overline{C}^s(Y \setminus \{x\}; q^s)$.

Let $Z^1 = Y^1 \setminus \{x\}$. Let (Y_j, r_j, Y^{j+1}) and $(Z_j, \tilde{r}_j, Z^{j+1})$ be sequences of the set of chosen contracts, the number of vacant slots, and the remaining set of contracts for category $j = 1, ..., K_s$ under $\overline{C}^s$ from Y and Z, respectively.

For category 1, since the sub-choice functions satisfy the IRC, we have $Y_1 = Z_1$. Moreover, $r_1 = \tilde{r}_1$ and $Y^2 \setminus \{x\} = Z^2$. By induction, for each $j = 2, ..., k-1$, assume that

$$Y_j = Z_j,\ r_j = \tilde{r}_j,\ and\ Y^j \setminus \{x\} = Z^j.$$

We need to show that the above equalities hold for $j = k$. Since $x \notin \overline{C}^s(Y; q^s)$ and sub-choice functions satisfy the IRC condition, we obtain

$$\overline{C}^s_k(Y^k; q^s_k(\overline{r}_1, ..., \overline{r}_{k-1})) = \overline{C}^s_k(Z^k; q^s_k(\tilde{r}_1, ..., \tilde{r}_{k-1})).$$

By our inductive assumption that $r_j = \tilde{r}_j$ for each $j = 2, ..., k-1$, category k's capacity is the same under both choice processes. The number of remaining slots is the same as well, i.e., $r_k = \tilde{r}_k$. Finally, we know that x is not chosen from the set $Z^k \cup \{x\}$, then we have $Y^{k+1} = Z^{k+1} \cup \{x\}$.

Since for all $j \in \{1, ..., K_s\}$, $Y_j = Z_j$, we have $\overline{C}^s(Y; q^s) = \overline{C}^s(Z; q^s)$. Hence, $\overline{C}^s$ satisfies the IRC. □

Proposition 4. *$\overline{C}^s$ is substitutable.*

Proof. Consider $Y \subseteq X$ such that $Y \neq \overline{C}^s(Y; q^s)$. Let x be such that $x \in Y \setminus \overline{C}^s(Y; q^s)$, and let z be an arbitrary contract in $X \setminus Y$. Let $Z = Y \cup \{z\}$. We need to show that $x \notin \overline{C}^s(Z; q^s)$.

Let (Y_j, r_j, Y^j) and $(Z_j, \tilde{r}_j, Z^j)$ be sequences of the set chosen contracts, the number of vacant slots, and the set of remaining contracts for categories $j = 1, ..., K_s$ from Y and Z, respectively, under $\overline{C}$.

If $z \in Z \setminus \overline{C}^s(Z; q^s)$, then IRC of $\overline{C}^s$ implies $\overline{C}^s(Z; q^s) = \overline{C}^s(Y; q^s)$. Therefore, $x \notin \overline{C}^s(Z; q^s)$.

Suppose $z \in \overline{C}^s(Z; q^s)$. Consider the first category. If $z \notin Z_1$, then by IRC we have $Y_1 = Z_1$ and, thus, $r_1 = \widetilde{r}_1$. Moreover, we obtain $Z^2 = Y^2 \cup \{z\}$ and $q^s_2(r_1) = q^s_2(\widetilde{r}_1)$. Now, consider the case where $z \in Z_1$. Note that by substitutability all contracts that are rejected from Y^1 is rejected from Z^1. By size monotonicity $\mid Z_1 \mid \geq \mid Y_1 \mid$. There are two possibilities: (1) $Z_1 = Y_1 \cup \{z\}$, or (2) $Z_1 = Y_1 \cup \{z\} \setminus \{w\}$ for some $w \in Y_1$. In the former case, we obtain $Y^2 = Z^2$ and $r_1 = 1 + \widetilde{r}_1$. Moreover, we have $q^s_2(\widetilde{r}_1) \leq q^s_2(r_1) \leq 1 + q^s_2(\widetilde{r}_1)$ by monotonicity of capacity transfers. In the latter case, we obtain $Z^2 = Y^2 \cup \{w\}$ and $r_1 = \widetilde{r}_1$. Therefore, we have $q^s_2(r_1) = q^s_2(\widetilde{r}_1)$.

Suppose now that for all $\gamma = j, ..., k-1$ we have that either

$$\left[Z^\gamma = Y^\gamma \cup \{w\} \; for \; some \; w \; and \; q^s_{\gamma+1}(\tilde{r}_1, ..., \tilde{r}_\gamma) = q^s_{\gamma+1}(r_1, ..., r_\gamma)\right]$$

or

$$\left[Z^\gamma = Y^\gamma \; and \; q^s_{\gamma+1}(\tilde{r}_1, ..., \tilde{r}_\gamma) \leq q^s_{\gamma+1}(r_1, ..., r_\gamma) \leq 1 + q^s_{\gamma+1}(\tilde{r}_1, ..., \tilde{r}_\gamma)\right].$$

We have already shown that it holds for $\gamma = 1$ and we will now show that it also holds for $\gamma = k$.

In the former case, by inductive assumption, we have $Z^{k-1} = Y^{k-1} \cup \{w\}$ for some w. If w is not chosen from the set Z^{k-1}, then exactly the same set of contracts will be chosen from Y^{k-1} and Z^{k-1}. It is because category k's capacity is the same under both choice processes and the sub-choice function satisfies the IRC condition. We, then, conclude $Z^k = Y^k \cup \{w\}$. Moreover, since the number of vacant slots in category k will be the same under both processes, we obtain $q^s_{k+1}(r_1, ..., r_j) = q^s_{k+1}(\tilde{r}_1, ..., \tilde{r}_j)$. If w is chosen from Z^{k-1}, we have two possibilities, depending on whether category k's capacity is exhausted under the choice process beginning with Y^0 or not.

If it is not exhausted, then we will have

$$C^s_k(Z^{k-1}; q^s_k(\tilde{r}_1, ..., \tilde{r}_{k-1})) = \{w\} \cup C^s_k(Y^{k-1}; q^s_k(r_1, ..., r_{k-1})),$$

which implies that $Z^k = Y^k$. Moreover, we will have $r_k = \tilde{r}_k + 1$. The monotonicity of capacity transfer scheme implies[28]

$$q^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_k) \leq q^{k+1}_s(r_1, ..., r_k) \leq 1 + q^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_k).$$

If category k's capacity is exhausted, then choosing w from Z^{k-1} implies that there exists a contract υ that is chosen from Y^{k-1} but rejected from Z^{k-1}. Then, we obtain $Z^k = Y^k \cup \{\upsilon\}$ since the sub-choice function is substitutable and satisfies the IRC condition. Also, category k's capacity is the same under both processes. In this case, we have $r_k = \tilde{r}_k = 0$. Since $\tilde{r}_i \leq r_i$ for all $i = 1, ..., k$, we will have $q^s_{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \leq q^s_{k+1}(r_1, ..., r_k)$ from the first monotonicity condition. Since $q^s_k(\tilde{r}_1, ..., \tilde{r}_{k-1}) = q^s_k(r_1, ..., r_{k-1})$ and $\tilde{r}_k = r_k$, we obtain $q^s_{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \geq q^s_{k+1}(r_1, ..., r_k)$ by the second monotonicity condition.

We now analyze the latter case, in which we have $Z^{k-1} = Y^{k-1}$ and either $q^s_k(r_1, ..., r_{k-1}) = q^s_k(\tilde{r}_1, ..., \tilde{r}_{k-1})$ or $q^s_k(r_1, ..., r_{k-1}) = 1 + q^s_k(\tilde{r}_1, ..., \tilde{r}_{k-1})$. If $q^k_s(r_1, ..., r_{k-1}) = q^k_s(\tilde{r}_1, ..., \tilde{r}_{k-1})$, then given that $Z^{k-1} = Y^{k-1}$, we will have $Z^k = Y^k$. This also implies $r_k = \tilde{r}_k$. Moreover, we obtain $q^s_{k+1}(r_1, ..., r_k) = q^s_{k+1}(\tilde{r}_1, ..., \tilde{r}_k)$ by monotonicity of the capacity transfer scheme. Note that $\tilde{r}_i \leq r_i$ for all $i = 1, ..., k$ implies $q^s_{k+1}(r_1, ..., r_k) \geq q^s_{k+1}(\tilde{r}_1, ..., \tilde{r}_k)$ by the first monotonic-

[28]The first inequality follows from the fact that $\tilde{r}_i \leq r_i$ for all $i = 1, ..., k$. The second inequality follows from the second condition of the monotonicity of the capacity transfer schemes.

ity condition. The second condition of monotonicity implies $q^s_{k+1}(r_1,...,r_k) \leq q^s_{k+1}(\tilde{r}_1,...,\tilde{r}_k)$.

If $q^s_k(r_1,...,r_{k-1}) = 1 + q^s_k(\tilde{r}_1,...,\tilde{r}_{k-1})$, then given $Z^{k-1} = Y^{k-1}$, we have two cases to consider.

Case 1: If $C^s_k(Z^{k-1}; q^s_k(\tilde{r}_1,...,\tilde{r}_{k-1})) = C^s_k(Y^{k-1}; q^s_k(r_1,...,r_{k-1}))$, then we have $Z^k = Y^k$. Also, the monotonicity of capacity transfers implies that

$$q^s_{k+1}(\tilde{r}_1,...,\tilde{r}_k) \leq q^s_{k+1}(r_1,...,r_k) \leq 1 + q^s_{k+1}(\tilde{r}_1,...,\tilde{r}_k).$$

Case 2: If $C^s_k(Z^{k-1}; q^s_k(\tilde{r}_1,...,\tilde{r}_{k-1})) \cup \{\vartheta\} = C^s_k(Y^{k-1}, q^s_k(r_1,...,r_{k-1}))$ for some ϑ, then we will have $Z^k = Y^k \cup \{\vartheta\}$. Moreover, the monotonicity of capacity transfers implies

$$q^{k+1}_s(r_1,...,r_k) = q^{k+1}_s(\tilde{r}_1,...,\tilde{r}_k).$$

Since $x \notin Y_k$, we have $x \notin Z_k$ for all $k = 1,...,K_s$. We conclude that $x \notin \overline{C}^s(Y \cup \{z\}; q^s)$. $\square$

Proposition 5. *$\overline{C}^s$ satisfies size monotonicity.*

Proof. Take Y and Z such that $Y \subseteq Z \subseteq X$. Let q^s the capacity transfer scheme of institution s. We want to show that

$$\mid \overline{C}^s(Y; q^s) \mid \leq \mid \overline{C}^s(Z; q^s) \mid .$$

Let (Y_j, r_j, Y^j) and $(Z_j, \tilde{r}_j, Z^j)$ be the sequences of sets of chosen contracts, numbers of unfilled slots and sets of remaining contracts for categories $j = 1,...,K_s$ under choice processes beginning with $Y = Y^0$ and $Z = Z^0$, respectively.

For the first category with capacity $\overline{q}^1_s$, since the sub-choice function is size monotonic, we have

$$\mid Y_1 \mid = \mid C^s_1(Y^0; \overline{q}^1_s) \mid \leq \mid C^s_1(Z^0; \overline{q}^1_s) \mid = \mid Z_1 \mid .$$

This, in turn, implies that $r_1 = \overline{q}^1_s - \mid Y_1 \mid \geq \tilde{r}_1 = \overline{q}^1_s - \mid Z_1 \mid$. Moreover, we have $Y^1 \subseteq Z^1$. To see this, consider a $y \in Y^1$. It means that $y \notin Y_1$. If y is not chosen from a smaller set Y^0, then it cannot be chosen from a larger set Z^0 because the sub-choice function is substitutable.

Suppose that $\tilde{r}_j \leq r_j$ and $Y^j \subseteq Z^j$ hold for all $j = 1,...,k-1$. We need to show that (1) $\tilde{r}_k \leq r_k$ and (2) $Y^k \subseteq Z^k$.

We first show (1). Given that $\tilde{r}_j \leq r_j$ for all $j = 1,...,k-1$, the first condition of the monotonicity implies that $q^s_k(r_1,...,r_{k-1}) \geq q^s_k(\tilde{r}_1,...,\tilde{r}_{k-1})$. By size monotonicity of sub-choice functions, we have

$$\mid Z_k \mid = \mid C^s_k(Z^{k-1}, q^k_s(\tilde{r}_1,...,\tilde{r}_{k-1})) \mid \geq \mid C^s_k(Y^{k-1}, q^k_s(\tilde{r}_1,...,\tilde{r}_{k-1})) \mid$$

Moreover, quota monotonicity of sub-choice rules, then, implies

$$\mid Y_k \mid - \mid C^s_k(Y^{k-1}, q^s_k(\tilde{r}_1,...,\tilde{r}_{k-1})) \mid \leq q^s_k(r_1,...,r_{k-1}) - q^s_k(\tilde{r}_1,...,\tilde{r}_{k-1}).$$

The difference on the left-hand side, $\mid Y_k \mid - \mid C_k^s(Y^{k-1}, q_k^s(\tilde{r}_1, ..., \tilde{r}_{k-1})) \mid$, is the difference between the number of chosen contracts when the capacity is (weakly) increased from $q_k^s(\tilde{r}_1, ..., \tilde{r}_{k-1})$ to $q_k^s(r_1, ..., r_{k-1})$. Hence, it cannot exceed the increase in the capacity, which is $q_k^s(r_1, ..., r_{k-1}) - q_k^s(\tilde{r}_1, ..., \tilde{r}_{k-1})$. Therefore, we now have

$$\mid Y_k \mid - \mid Z_k \mid \leq q_k^s(r_1, ..., r_{k-1}) - q_k^s(\tilde{r}_1, ..., \tilde{r}_{k-1}).$$

Rearranging gives us

$$q_k^s(\tilde{r}_1, ..., \tilde{r}_{k-1}) - \mid Z_k \mid \leq q_k^s(r_1, ..., r_{k-1}) - \mid Y_k \mid .$$

That is $\tilde{r}_k \leq r_k$.

We now prove (2). Consider a contract $x \in Y^k$. That is $x \notin C_k^s(Y^{k-1}, q_k^s(r_1, ..., r_{k-1}))$. When the set Y^{k-1} is expanded to $\tilde{Y}^{k-1}$, x cannot be chosen since the sub-choice function is substitutable. That is,

$$x \notin C_k^s(Z^{k-1}, q_k^s(r_1, ..., r_{k-1})).$$

When the capacity is reduced to $q_k^s(\tilde{r}_1, ..., \tilde{r}_{k-1})$, quota monotonicity of the sub-choice function implies that

$$x \notin C_k^s(Z^{k-1}, q_k^s(\widetilde{r}_1, ..., \widetilde{r}_{k-1})).$$

That is, x cannot be chosen in category k. Hence, $x \in Z^k$. We conclude that $Y^k \subseteq Z^k$.

Now let $\tau_j = r_j - \tilde{r}_j$. As shown above, $\tau_j \geq 0$ for all $j = 1, ..., K_s$. Plugging $r_j = q_j^s(r_1, ..., r_{j-1}) - \mid Y_j \mid$ and $\tilde{r}_j = q_j^s(\tilde{r}_1, ..., \tilde{r}_{k-1}) - \mid Z_j \mid$ gives us

$$\mid Z_j \mid = q_j^s(r_1, ..., r_{j-1}) - q_j^s(\tilde{r}_1, ..., \tilde{r}_{j-1}) + \mid Y_j \mid + \tau_j.$$

Summing both the right and left hand sides for $j = 1, ..., K_s$ yields

$$\sum_{j=1}^{K_s} \mid Z_j \mid = \sum_{j=1}^{K_s} \mid Y_j \mid + \sum_{j=2}^{K_s} [q_j^s(r_1, ..., r_{j-1}) - q_j^s(\tilde{r}_1, ..., \tilde{r}_{j-1})] + \sum_{j=1}^{K_s} \tau_j.$$

Since each $\tau_j \geq 0$, we have

$$\sum_{j=1}^{K_s} \mid Z_j \mid \geq \sum_{j=1}^{K_s} \mid Y_j \mid + \sum_{j=2}^{K_s} [q_j^s(r_1, ..., r_{j-1}) - q_j^s(\tilde{r}_1, ..., \tilde{r}_{j-1})].$$

Since $q_j^s(r_1, ..., r_{j-1}) \geq q_j^s(\tilde{r}_1, ..., \tilde{r}_{j-1})$ for all $j = 2, ..., K_s$, we have

$$\sum_{[j=1}^{K_s} \mid Z_j \mid \geq \sum_{j=1}^{K_s} \mid Y_j \mid.$$

That is $\mid \overline{C}^s(Y; q^s) \mid \leq \mid \overline{C}^s(Z; q^s) \mid$, the desired conclusion. □

5.3 Omitted Proofs

Proof of Theorem 1. We showed above that GL choice rules have substitutable completions that satisfy size monotonicity. Then, Theorem 1 follows from Hatfield and Kominers (2019).

Proof of Theorem 2. We first show that $C^{ⓗ}$ is merit-based undominated. Since each individual can have at most one contract with a given vertical category, we consider a set of individuals rather than a set of contracts for a given vertical category.

Toward a contradiction, suppose that $C^{ⓗ}$ is merit-based dominated. Then, for some $X \subseteq \mathcal{X}$, there exists $Y \subseteq X$, such that $\mathbf{i}(Y)$ merit-based dominates $\mathbf{i}(C^{ⓗ}(X))$. Let $\widehat{I}_1 = \mathbf{i}(C^{ⓗ}(X)) \setminus \mathbf{i}(Y)$ and $\widehat{J} = \mathbf{i}(Y) \setminus \mathbf{i}(C^{ⓗ}(X))$. For each $i \in \widehat{I}_1$, let n_i be the step in $C^{ⓗ}$ at which i is chosen. Define $\widehat{n}_1 = \underset{i \in \widehat{I}_1}{min}\ n_i$. Let $\hat{i}_1$ be the highest-scoring individual among individuals $i \in \widehat{I}_1$ with $n_i = \widehat{n}_1$. Consider the horizontal type $\widehat{h}_1$ in which $\widehat{i}_1$ is chosen. Since $\mathbf{i}(Y)$ merit-based dominates $\mathbf{i}(C^{ⓗ}(X))$, it must be the case that $\mathbf{i}(Y)$ satisfies horizontal reservations. Since the set of chosen individuals before Step $\widehat{n}_1$ is also in the set $\mathbf{i}(Y)$, and $C^{ⓗ}$ selects top-scoring individuals within each horizontal type, to fill the remaining positions, given that $\mathbf{i}(Y)$ does not contain $\widehat{i}_1$, there exists at least one individual in $\mathbf{i}(Y)$, who has the horizontal type $\widehat{h}_1$ and whose score is lower than that of $\widehat{i}_1$. Among those, let $\widehat{j}_1$ be the highest-scoring individual. The set $\mathbf{i}(Y) \cup \{\hat{i}_1\} \setminus \{\hat{j}_1\} = \widetilde{I}^1$ merit-based dominates $\mathbf{i}(Y)$.

Let $\widehat{I}_2 = \widehat{I}_1 \setminus \{\hat{i}_1\}$. For each $i \in \widehat{I}_2$, let n_i denote the step in $C^{ⓗ}$ at which i is chosen. Define $\widehat{n}_2 = \underset{i \in \widehat{I}_2}{min}\ n_i$. Let $\widehat{i}_2$ be the highest-scoring individual among individuals $i \in \widehat{I}_2$ with $n_i = \widehat{n}_2$. Consider the horizontal type $\widehat{h}_2$ in which $\widehat{i}_2$ is chosen. Since the set of chosen individuals before Step $\widehat{n}_2$ is also in the set $\widetilde{I}^1$, and $C^{ⓗ}$ selects top-scoring individuals within each horizontal type, to fill the remaining positions, given that $\widetilde{I}^1$ does not contain $\widehat{i}_2$, there exists at least one individual in $\widetilde{I}^1$, who has the horizontal type $\widehat{h}_2$ and whose score is lower than that of $\widehat{i}_2$. Among those, let $\widehat{j}_2$ be the highest-scoring individual. The set $\widetilde{I}^1 \cup \{\hat{i}_2\} \setminus \{\hat{j}_2\} = \widetilde{I}^2$ merit-based dominates $\widetilde{I}^1$.

We continue in the same fashion. The set $\mathbf{i}(Y) \cup \{\widehat{i}_1, ..., \widehat{i}_l\} \setminus \{\hat{j}_1, ..., \hat{j}_l\} = \widetilde{I}^l$ merit-based dominates the set $\mathbf{i}(Y) \cup \{\widehat{i}_1, ..., \widehat{i}_{l-1}\} \setminus \{\hat{j}_1, ..., \hat{j}_{l-1}\} = \widetilde{I}^{l-1}$. Since the set $\widehat{I}_1$ is finite, in finitely many steps, call it m, we reach

$$\mathbf{i}(Y) \cup \{\widehat{i}_1, ..., \widehat{i}_m\} \setminus \{\hat{j}_1, ..., \hat{j}_m\} = \widetilde{I}^m = \mathbf{i}(C^{ⓗ}(X)).$$

Hence, $\mathbf{i}(C^{ⓗ}(X))$ merit-based dominates $\mathbf{i}(Y)$. This contradicts our supposition. Thus, $C^{ⓗ}$ is merit-based undominated.

We next show uniqueness. Toward a contradiction, suppose that there is a merit-based undominated choice rule C , such that, for a given set of contracts $X \subseteq \mathcal{X}_s$, $C^{ⓗ}(X) \neq C(X)$. Define $\widehat{I}_1 \equiv \mathbf{i}(C^{ⓗ}(X)) \setminus \mathbf{i}(C(X))$. Let n_i be the step in $C^{ⓗ}$ at which individual i is chosen.

Let $\widehat{n}_1 = \underset{i \in \widehat{I}_1}{min}\ n_i$. Among all individuals with $n_i = \widehat{n}_1$, call the individual with the highest merit score $\widehat{i}_1$. Consider the horizontal type $\widehat{h}_1$ in which $\widehat{i}_1$ is chosen. We know that $\mathbf{i}(C(X))$ satisfies horizontal reservations. Since the set of chosen individuals before Step $\widehat{n}_1$ is also in the set $\mathbf{i}(C(X))$, and $C^{ⓗ}$ selects top-scoring individuals within each horizontal type, to fill the remaining positions, given that $\mathbf{i}(C(X))$ does not contain $\widehat{i}_1$, there exists at least one individual in $\mathbf{i}(C(X))$, who has the horizontal type $\widehat{h}_1$ and whose score is lower than that of $\widehat{i}_1$. Among those, let $\widehat{j}_1$ be the highest-scoring individual. The set $\mathbf{i}(C(X)) \cup \{\hat{i}_1\} \setminus \{\hat{j}_1\} = \widetilde{I}^1$ merit-based dominates $\mathbf{i}(C(X))$. This contradict with our supposition that $\mathbf{i}(CX))$ is merit-based undomianted. Thus, $C^{ⓗ}$ is the only merit-based undominated rule.

Proof of Theorem 3. We first show that $C_s^{ⓗNT} \equiv \left(\left(C_v^{ⓗ}\right)_{v\in\mathcal{V}}, q_s\right)$ and $C_s^{ⓗT} \equiv \left(\left(C_v^{ⓗ}\right)_{v\in\mathcal{V}}, q_s\right)$ are in the GL family. Note that under $C_s^{ⓗNT}$ we have $q_s^v = \overline{q}_s^v$ for all $v \in \{o, SC, ST, OBC, EWS\}$. That is, there is no transfer of capacity. No transfer scheme trivially satisfy monotonicity. Under $C_s^{ⓗT}$, we have $q_s^v = \overline{q}_s^v$ for all $v \in \{o, SC, ST, OBC, EWS\}$ and $q_s^D = r_{OBC}$ is the number of vacant OBC positions, where $q_s = \left(q_s^O, q_s^{SC}, q_s^{ST}, q_s^{OBC}, q_s^{EWS}, q_s^D\right)$. Monotonicity of capacity transfer functions is straightforwardly satisfied under $C_s^{ⓗT}$. We need to show that $C_v^{ⓗ}$ is substitutable, size monotonic, and quota monotonic.

Substitutability. Consider a vertical category v and a set of contracts $X \subseteq \mathcal{X}_v$. Note that each individual in $\mathbf{i}(X)$ has only one contract in X. Suppose that contracts $x, y \in \mathcal{X}_v \setminus X$ are such that $\mathbf{i}(x) = i$ and $\mathbf{i}(y) = j$. Suppose that $x \in C_v^{ⓗ}(X \cup \{x, y\})$. Let m be the step in $C_v^{ⓗ}$ at which x is chosen from the set $X \cup \{x, y\}$ when horizontal type h is considered. We must show that $x \in C_v^{ⓗ}(X \cup \{x\})$.

Consider $y \notin C_v^{ⓗ}(X \cup \{x, y\})$. It must be the case that at each horizontal type that individual j has, individuals who have higher scores than j fill the capacity. Removing individual j does not change the set of chosen individuals and updated reservations. Then, contract x will be chosen at Step m in $C_v^{ⓗ}$ when horizontal type h is considered from the set $X \cup \{x\}$.

Now suppose that $y \in C_v^{ⓗ}(X \cup \{x, y\})$. There are two cases to consider:

(1) y is chosen when horizontal type $h^{'}$ is considered, where h does not contain $h^{'}$ and $h^{'}$ does not contain h. In this case, x will still be chosen at Step m in $C_v^{ⓗ}$ since removing individual j from the applicant pool does not change the set of chosen individuals by horizontal types that are contained by h and their updated capacities.

(2) y is chosen when horizontal type $h^{'}$ is considered, where either $h^{'}$ contains h or h contains $h^{'}$. Suppose that y is considered and chosen at some Step l in $C_v^{ⓗ}$ where $l \geq m$. Then, x is still be chosen at Step m in $C_v^{ⓗ}$ from $X \cup \{x\}$, because considering and choosing y at the same or a later stage in $C_v^{ⓗ}$ from $X \cup \{x, y\}$ does not affect the chosen sets prior to Step m from $X \cup \{x\}$. Moreover, the updated capacities of the horizontal types remain unchanged.

Also, the number of individuals with higher scores than i and considered at Step m does not increase.

Now consider the case where y is chosen from $X \cup \{x, y\}$ when horizontal type $h^{'}$ is considered, where h contains $h^{'}$. That is, y is considered and chosen from the set $X \cup \{x, y\}$ at some Step l in $C_v^{\textcircled{h}}$, where $l < m$. When horizontal type h is considered at Step m in $C_v^{\textcircled{h}}$ for $X \cup \{x\}$, the updated number of horizontal reserves at which individual i is considered is either the same or one more than the updated number of the same horizontal types in the choice process beginning with $X \cup \{x, y\}$. Moreover, the number of individuals considered for the same horizontal types and have higher scores than individual i does not increase. Hence, x is be chosen at Step m in $C_v^{\textcircled{h}}$ from $X \cup \{x\}$.

Size monotonicity. $C_v^{\textcircled{h}}$ is q-acceptant since, in the last step of $C_v^{\textcircled{h}}$, if there is no horizontal type left to be considered, contracts are chosen following the merit score ranking for remaining positions. Since q-acceptance implies size monotonicity, $C_v^{\textcircled{h}}$ is size monotonic.

Quota monotonicity. Consider a set of contracts $X \subseteq \mathcal{X}_v$. The capacity of vertical category v is $\overline{q}_s^v$. We first show that $C_v^{\textcircled{h}}(X, \overline{q}_s^v) \subseteq C_v^{\textcircled{h}}(X, 1 + \overline{q}_s^v)$. In the computations of $C_v^{\textcircled{h}}$ with capacity $1 + \overline{q}_s^v$, the updated capacities of horizontal types at each stage is not lower than the corresponding updated capacities of horizontal types at each stage in the computation of $C_v^{\textcircled{h}}$ with capacity $\overline{q}_s^v$. Moreover, the sets of individuals competing for positions at each horizontal type at each stage do not expand when the total capacity is increased. Thus, if x is chosen in Step m of $C_v^{\textcircled{h}}$ when the capacity is $\overline{q}_s^v$, then it must be chosen in Stem m (or at an earlier step) of $C_v^{\textcircled{h}}$ when the capacity is $1 + \overline{q}_s^v$.

We now show that $\mid C_v^{\textcircled{h}}(X, 1+\overline{q}_s^v) \mid - \mid C_v^{\textcircled{h}}(X, \overline{q}_s^v) \mid \leq 1$. Recall that $C_v^{\textcircled{h}}$ is q-acceptant. There are two cases to consider: (1) Suppose that all $X = C_v^{\textcircled{h}}(X, \overline{q}_s^v)$. Then, if the total capacity is increased from $\overline{q}_s^v$ to $1 + \overline{q}_s^v$, by q-acceptance, we have $X = C_v^{\textcircled{h}}(X, 1 + \overline{q}_s^v)$. Therefore, we have $\mid C_v^{\textcircled{h}}(X, 1 + \overline{q}_s^v) \mid - \mid C_v^{\textcircled{h}}(X, \overline{q}_s^v) \mid = 0$. (2) Suppose that $C_v^{\textcircled{h}}(X, \overline{q}_s^v) \subset X$, which implies $\mid C_v^{\textcircled{h}}(X, \overline{q}_s^v) \mid = \overline{q}_s^v$. Since all contracts in X are associated with vertical type v, when the capacity is increase to $1 + \overline{q}_s^v$, then we have $\mid C_v^{\textcircled{h}}(X, 1 + \overline{q}_s^v) \mid = 1 + \overline{q}_s^v$. Hence, we have $\mid C_v^{\textcircled{h}}(X, 1 + \overline{q}_s^v) \mid - \mid C_v^{\textcircled{h}}(X, \overline{q}_s^v) \mid = 1$.

This ends the first part of the proof. We now prove that $C_s^{\textcircled{h}NT}$ and $C_s^{\textcircled{h}T}$ are fair.

We first show that $C_v^{\textcircled{h}}$ is fair. Toward a contradiction, suppose that $C_v^{\textcircled{h}}$ is not fair. Then, there exists a set of contracts $X \subseteq \mathcal{X}_s$ and a pair of contract $x, y \in X$ such that $x \notin C_v^{\textcircled{h}}(X)$, $y \in C_v^{\textcircled{h}}(X)$, $\mathbf{i}(x) \succ_s \mathbf{i}(y)$, and $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$. Let $Y = (C_v^{\textcircled{h}}(X) \setminus \{y\}) \cup \{x\}$. Note that $\mathbf{i}(Y)$ merit-based dominates $\mathbf{i}(C_v^{\textcircled{h}}(X))$, which contradicts $C_v^{\textcircled{h}}$ being merit-based undominated. Hence, $C_v^{\textcircled{h}}$ is fair.

We now show that $C^{\textcircled{h}NT}$ is fair. Toward a contradiction, suppose that $C^{\textcircled{h}NT}$ is not fair. Then, there exists a set of contracts $X \subseteq \mathcal{X}_s$ and a pair of contract $x, y \in X$ such that $x \notin C^{\textcircled{h}NT}(X)$, $y \in C^{\textcircled{h}NT}(X)$, $\mathbf{i}(x) \succ_s \mathbf{i}(y)$, and $\mathbf{t}(x) = \mathbf{t}(y) = v$, and $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$.

$x \notin C^{ⓗNT}$ implies $x \notin C_{v'}^{ⓗ}$ for all $v' \in \mathcal{V}$. Let $Y = (C_v^{ⓗ}(X') \setminus \{y\}) \cup \{x\}$, where X' is the set of contracts considered by $C_v^{ⓗ}$. Note that $\mathbf{i}(Y)$ merit-based dominates $\mathbf{i}(C_v^{ⓗ}(X'))$. This contradicts $C_v^{ⓗ}$ being merit-based undominated. Thus, $C^{ⓗNT}$ is fair.

Next, we show that $C^{ⓗT}$ is fair. Toward a contradiction, suppose not. Then, there exists a set of contracts $X \subseteq \mathcal{X}_s$ and a pair of contracts $x, y \in X$, such that $x \notin C^{ⓗT}(X)$, $y \in C^{ⓗT}(X)$, $\mathbf{i}(x) \succ_s \mathbf{i}(y)$, $\mathbf{t}(x) = \mathbf{t}(y) = v$, and $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$. $x \notin C^{ⓗT}$ implies $x \notin C_{v'}^{ⓗ}$ for all $v' \in \mathcal{V}$. However, y cannot be chosen while x is rejected in the category EO (extra open category positions reverted from OBC) because $\mathbf{i}(x) \succ_s \mathbf{i}(y)$. So, there must be a vertical category $v \in \{o, SC, ST, OBC\}$ such that $x, y \in X' \subseteq X$, $x \notin C_v^{ⓗ}(X')$ and $y \in C_v^{ⓗ}(X')$, where X' is the contract offers category v receives. Let $Y = (C_v^{ⓗ}(X') \setminus \{y\}) \cup \{x\}$. Note that $\mathbf{i}(Y)$ merit-based dominates $\mathbf{i}(C_v^{ⓗ}(X'))$. This contradicts with $C_v^{ⓗ}$ being merit-based undominated. Therefore, $C^{ⓗT}$ is fair.

Proof of Theorem 4. We already showed that $C_s^{ⓗNT}$ and $C_s^{ⓗNT}$ are in the GL family. Our Theorem 1 then implies that $\Phi^{ⓗNT}$ and $\Phi^{ⓗT}$ are stable with respect to $C_s^{ⓗNT}$ and $C_s^{ⓗNT}$, respectively, and that $\Phi^{ⓗNT}$ and $\Phi^{ⓗT}$ are both strategy-proof.

We need to show that $\Phi^{ⓗNT}$ and $\Phi^{ⓗT}$ eliminate justified envy. Toward a contradiction, suppose that they are not. Then, there must exist a preference profile P such that the outcome of the COP under preference profile P does not eliminate justified envy. Let X be the outcome of the COP under preference profile P. Since X does not eliminate justified envy, there must exist contracts $x, y \in X$ such that $(\mathbf{s}(y), \mathbf{t}(y)) P_{\mathbf{i}(x)} (\mathbf{s}(x), \mathbf{t}(x))$. Since $(\mathbf{s}(y), \mathbf{t}(y))$ is acceptable to individual $\mathbf{i}(x)$, she is eligible for vertical category $\mathbf{t}(y)$. There are two cases to consider:

Case 1: $\mathbf{s}(x) \neq \mathbf{s}(y)$**.** If $(\mathbf{s}(y), \mathbf{t}(y)) P_{\mathbf{i}(x)} (\mathbf{s}(x), \mathbf{t}(x))$, then individual $\mathbf{i}(x)$ must have offered the contract $(\mathbf{i}(x), \mathbf{s}(y), \mathbf{t}(y))$ before she offered contract x. The contract $(\mathbf{i}(x), \mathbf{s}(y), \mathbf{t}(y))$ must have rejected by institution $\mathbf{s}(y)$. Moreover, individual $\mathbf{i}(x)$'s all contracts that are associated with institution $\mathbf{s}(y)$ must be rejected. Since choice rules $C_s^{ⓗNT}$ and $C_s^{ⓗNT}$ are fair, we have either $\mathbf{i}(y) \succ_{\mathbf{s}(y)} (\mathbf{i}(x)$ or $\rho(\mathbf{i}(x)) \not\supseteq \rho(\mathbf{i}(y))$.

Case 2: $\mathbf{s}(x) = \mathbf{s}(y)$**.** Let $\mathbf{s}(x) = \mathbf{s}(y) = s$. There are two sub-cases to consider:

Sub-case 2.1. $\mathbf{t}(y)$ **precedes** $\mathbf{t}(x)$**.** Consider the last step of the COP. At this step, the contract $(\mathbf{i}(x), s, \mathbf{t}(y))$ is available to institution s, but is rejected. Then, it must be the case that either $\Gamma(\mathbf{i}(y), s) > \Gamma(\mathbf{i}(x), s)$ or $\rho(\mathbf{i}(x)) \not\supseteq \rho(\mathbf{i}(y))$ hold. To see why consider the case where $\Gamma(\mathbf{i}(y), s) < \Gamma(\mathbf{i}(x), s)$ and $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$. Lex X be the outcome of the COP and X_s be the set of contracts assigned to institution s. If we replace the contract y with $(\mathbf{i}(x), s, \mathbf{t}(y))$, the resulting set of contracts $(X_s \setminus \{y\}) \cup \{(\mathbf{i}(x), s, \mathbf{t}(x)\}$ merit-based dominates X_s,[29] which is a

[29] Note that $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$ ensures that $(X_s \setminus \{y\}) \cup \{(\mathbf{i}(x), s, \mathbf{t}(x)\}$ satisfies horizontal reservations.

contradiction because for every vertical type in C^{NT} and C^{T}, contracts are chosen according to C^{H} that is merit-based undominated.

Sub-case 2.2. $\mathbf{t}(x)$ precedes $\mathbf{t}(y)$. Consider the last step of the COP. At this step, the contract $(\mathbf{i}(x), s, \mathbf{t}(y))$ is available to institution s, but is rejected. Since the contract $(\mathbf{i}(x), s, \mathbf{t}(y))$ was offered at some earlier step, we have

$$(\mathbf{i}(x), s, \mathbf{t}(y)) \in F_{\mathbf{t}(y)}(X^M).$$

Since $C_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)}) = C_{\mathbf{t}(y)}(F_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)})$, we have

$$(\mathbf{i}(x), s, \mathbf{t}(y)) \notin C_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M), q^M_{\mathbf{t}(y)}).$$

If the COP outcome X is not fair, then it must be the case that

$$(\mathbf{i}(x), s, \mathbf{t}(y)) \in C^H_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M) \cup \{(\mathbf{i}(x), s, \mathbf{t}(y))\}.$$

Because otherwise,

$$C^H_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M), q^M_{\mathbf{t}(y)}) \setminus \{y\} \cup \{(\mathbf{i}(x), s, \mathbf{t}(y))\}$$

merit-based dominates $C^H_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M), q^M_{\mathbf{t}(y)})$. But, we know that $C^{hier}_{\mathbf{t}(y)}$ is merit-based undominated.

We know that

$$C^H_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)}) = C^M_{\mathbf{t}(y)}(F_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)}) \subseteq H_{\mathbf{t}(y)}(X^M) \subseteq F_{\mathbf{t}(y)}(X^M).$$

Therefore, by the IRC of $C^H_{\mathbf{t}(y)}$, we must have

$$C^H_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)}) = C^M_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)}) \cup \{\mathbf{i}(x), s, \mathbf{t}(y)).$$

This contradicts with $(\mathbf{i}(x), s, \mathbf{t}(y)) \in C^H_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M) \cup \{(\mathbf{i}(x), s, \mathbf{t}(y))\}$.

Proof of Theorem 5. We first show that $C_s^{ⓜ NT} \equiv \left(\left(C_v^{ⓜ}\right)_{v\in\mathcal{V}}, q_s\right)$ and $C_s^{ⓜ T} \equiv \left(\left(C_v^{ⓜ}\right)_{v\in\mathcal{V}}, q_s\right)$ are in the GL family. Capacity transfer functions under $C_s^{ⓜ NT}$ and $C_s^{ⓜ NT}$ are same as the those of $C_s^{ⓗ NT}$ and $C_s^{ⓗ T}$, respectively. In Proposition 1, we showed that both are monotonic. We need to show that $C_v^{ⓜ}$ is substitutable, size monotonic, and quota monotonic. Sönmez and Yenmez (2022) prove that $C_v^{ⓜ}$ is substitutable and size monotonic. We only need to show that $C_v^{ⓜ}$ is quota-monotonic. That is, for any given set of applicants $A \subseteq \mathcal{I}$ and $q \in \mathbb{N}$, we need to show that (1) $C_v^{ⓜ}(A, q) \subseteq C_v^{ⓜ}(A, q+1)$ and (2) $\mid C_v^{ⓜ}(A, q+1) \mid - \mid C_v^{ⓜ}(A, q) \mid \leq 1$.

We consider two cases:

(1) The additional position is *not* horizontally reserved. In this case, since the set of individuals $A \subseteq \mathcal{I}$ and the set of *horizontally reserved positions* $H^v = \bigcup_{j=1}^{L} H_j^v$ remains unchanged, when capacity is increased the horizontal type-matching remains the same in Step 1 of $C_v^{ⓜ}$. Moreover, since in Step 2 remaining positions are filled by following the merit scores ranking with individuals among the unassigned individuals until all positions are assigned or all individuals are selected, every individual who were chosen when capacity was q will be selected when capacity is increased to $q+1$. Therefore, both conditions of quota monotonicity are trivially satisfied.

(2) The additional position *is horizontally reserved.* Without loss of generality, let h_j be the horizontal type whose capacity is increased by 1. For each $i \in C_v^{ⓜ}(A,q)$, given the applicants chosen before i in the first step of $C_v^{ⓜ}$, i is chosen because she increases the utilization of horizontally reserved positions, which means there are available horizontally reserved positions for her. Adding one more type h_j position (weakly) increases the horizontally reserved positions available to her. Thus, i continues to be chosen. That is, $C_v^{ⓜ}(A,q) \subseteq C_v^{ⓜ}(A,q+1)$. Since each applicant who are chosen when capacity was q continues to be chosen when capacity is increased by one, the number of chosen applicants in Step 1 of $C_v^{ⓜ}$ either stays the same or increases by one. By Step 2 of $C_v^{ⓜ}$, we can conclude that $\mid C_v^{ⓜ}(A,q+1) \mid - \mid C_v^{ⓜ}(A,q) \mid \leq 1$.

Next, we show that $C^{ⓜT}$ is fair. Toward a contradiction, suppose not. Then, there exists a set of contracts $X \subseteq \mathcal{X}_s$ and a pair of contracts $x, y \in X$, such that $x \notin C^{ⓜT}(X)$, $y \in C^{ⓜT}(X)$, $\mathbf{i}(x) \succ_s \mathbf{i}(y)$, $\mathbf{t}(x) = \mathbf{t}(y) = v$, and $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$. $x \notin C^{ⓜT}$ implies $x \notin C_{v'}^{ⓜ}$ for all $v' \in \mathcal{V}$. However, y cannot be chosen while x is rejected in the category D, because $\mathbf{i}(x) \succ_s \mathbf{i}(y)$. So, there must be a vertical category $v \in \{o, SC, ST, OBC, EWS\}$ such that $x, y \in X' \subseteq X$, $x \notin C_v^{ⓜ}(X')$ and $y \in C_v^{ⓜ}(X')$, where X' is the contract offers category v receives. Step 2 of $C_v^{ⓜ}(X')$ fills remaining positions following the merit score with individuals among the unassigned individuals until all positions are assigned or all individuals are selected. Since $\mathbf{i}(x) \succ_s \mathbf{i}(y)$, then it cannot be the case that y is chosen in Step 2. Thus, y must be chosen in some sub-step k of Step 1, i.e., Step $1.k$. Since $x \notin C_v^{ⓜ}(X')$, it means x was rejected in every sub-step of Step 1. $\mathbf{i}(x) \succ_s \mathbf{i}(y)$ implies that x was considered before $y$and $x$could not increase the horizontal reservation utilization. However, y increased the horizontal reservation utilization. This contradicts with $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$. Hence, we conclude that $C^{ⓜT}$ is fair.

Proof of Theorem 6. We already showed that $C_s^{ⓜNT}$ and $C_s^{ⓜNT}$ are in the GL family (Proposition 2). Theorem 1 then implies that $\Phi^{ⓜNT}$ and $\Phi^{ⓜT}$ are stable with respect to $C_s^{ⓜNT}$ and $C_s^{ⓜNT}$, respectively, and that $\Phi^{ⓜNT}$ and $\Phi^{ⓜT}$ are both strategy-proof. We are left to show that $\Phi^{ⓜNT}$ and $\Phi^{ⓜT}$ eliminate justified envy. Toward a contradiction, suppose that they are not. Then, there must exist a preference profile P such that the outcome of the COP under preference profile P does not eliminate justified envy. Let X be the outcome of the COP under preference profile P. Since X does not eliminate justified envy, there must exist contracts $x, y \in X$ such that $(\mathbf{s}(y), \mathbf{t}(y)) P_{\mathbf{i}(x)} (\mathbf{s}(x), \mathbf{t}(x))$. Since $(\mathbf{s}(y), \mathbf{t}(y))$ is acceptable to individual $\mathbf{i}(x)$, she is eligible for vertical category $\mathbf{t}(y)$.

References

ABDULKADIROĞLU, A. (2005): "College admissions with affirmative action," *International Journal of Game Theory*, 33, 535–549.

ALKAN, A. AND D. GALE (2003): "Stable schedule matching under revealed preference," *Journal of Economic Theory*, 112, 289–306.

AYGÜN, O. (2014): "Three essays on matching with contracts," Ph.D. thesis, Boston College.

——— (2017): "Verimli pozitif ayrımcılık tasarımı," .

AYGÜN, O. AND I. BÓ (2021): "College admission with multidimensional privileges: The Brazilian affirmative action case," *American Economic Journal: Microeconomics*, 13, 1–28.

AYGÜN, O. AND T. SÖNMEZ (2013): "Matching with contracts: Comment," *American Economic Review*, 103, 2050–2051.

AYGÜN, O. AND B. TURHAN (2017): "Large-scale affirmative action in school choice: Admissions to IITs in India," *American Economic Review*, 107, 210–213.

——— (2020): "Dynamic reserves in matching markets," *Journal of Economic Theory*, 188.

——— (2022): "How to de-reserve reserves: Admissions to technical colleges in India," *Management Science*.

——— (2023): "Priority design for engineering colleges in India," *Indian Economic Review*, 1–11.

BASWANA, S., P. P. CHAKRABARTI, S. CHANDRAN, Y. KANORIA, AND U. PATANGE (2019): "Centralized admissions for engineering colleges in India," *INFORMS Journal on Applied Analytics*, 45, 338–354.

BIRÓ, P., T. FLEINER, R. W. IRVING, AND D. F. MANLOVE (2010): "The college admissions problem with lower and common quotas," *Theoretical Computer Science*, 411, 3136–3153.

BUDISH, E., Y.-K. CHE, F. KOJIMA, AND P. MILGROM (2013): "Designing random allocation mechanisms: Theory and applications," *American Economic Review*, 103, 585–623.

CHAMBERS, C. P. AND M. B. YENMEZ (2017): "Choice and matching," *American Economic Journal: Microeconomics*, 9, 126–147.

——— (2018a): "On lexicographic choice," *Economics Letters*, 171, 222–224.

——— (2018b): "A simple characterization of responsive choice," *Games and Economic Behavior*, 111, 217–221.

CORREA, J., R. EPSTEIN, J. ESCOBAR, I. RIOS, B. BAHAMONDES, C. BONET, N. EPSTEIN, N. ARAMAYO, M. CASTILLO, A. CRISTI, ET AL. (2021): "School choice in Chile," *Operations Research*, 70, 1066–1087.

DELACRÉTAZ, D. (2021): "Processing reserves simultaneously," in *Proceedings of the 22nd ACM Conference on Economics and Computation*, 345–346.

DOĞAN, B. (2016): "Responsive affirmative action in school choice," *Journal of Economic Theory*, 165, 69–105.

DOĞAN, B., S. DOĞAN, AND K. YILDIZ (2021a): "Lexicographic choice under variable capacity constraints," *Journal of Public Economic Theory*, 23, 172–196.

——— (2021b): "On capacity-filling and substitutable choice rules," *Mathematics of Operations Research*, 46, 856–868.

DOĞAN, B., K. IMAMURA, AND M. B. YENMEZ (2022): "Market Design with Deferred Acceptance: A Recipe for Policymaking," *arXiv preprint arXiv:2209.06777*.

DOĞAN, B. AND K. YILDIZ (2022): "Choice with affirmative action," *Management Science*.

ECHENIQUE, F. AND M. B. YENMEZ (2015): "How to control controlled school choice," *American Economic Review*, 105, 2679–94.

EHLERS, L., I. E. HAFALIR, M. B. YENMEZ, AND M. A. YILDIRIM (2014): "School choice with controlled choice constraints: Hard bounds versus soft bounds," *Journal of Economic Theory*, 153, 648–683.

ERDIL, A. AND T. KUMANO (2019): "Efficiency and stability under substitutable priorities with ties," *Journal of Economic Theory*, 184, 104950.

FLEINER, T. (2001): "A matroid generalization of the stable matching polytope," in *International Conference on Integer Programming and Combinatorial Optimization*, Springer, 105–114.

FRAGIADAKIS, D. AND P. TROYAN (2017): "Improving matching under hard distributional constraints," *Theoretical Economics*, 12, 863–908.

GREENBERG, K., P. A. PATHAK, AND T. SÖNMEZ (2023): "Redesigning the US Army's Branching Progress: A Case Study in Minimalist Market Design," *arXiv preprint arXiv:2303.06564*.

HAFALIR, I. E., M. B. YENMEZ, AND M. A. YILDIRIM (2013): "Effective affirmative action in school choice," *Theoretical Economics*, 8, 325–363.

HATFIELD, J. W. AND S. D. KOMINERS (2019): "Hidden substitutes," *Working Paper*.

HATFIELD, J. W., S. D. KOMINERS, AND A. WESTKAMP (2021): "Stability, strategy-proofness, and cumulative offer mechanisms," in *Review of Economic Studies*, vol. 88, 1457–1502.

HATFIELD, J. W. AND P. R. MILGROM (2005): "Matching with contracts," *American Economic Review*, 95, 913–935.

IMAMURA, K. (2020): "Meritocracy versus diversity," *Working Paper*.

KAMADA, Y. AND F. KOJIMA (2015): "Efficient matching under distributional constraints: Theory and applications," *American Economic Review*, 105, 67–99.

——— (2018): "Stability and strategy-proofness for matching with constraints: A necessary and sufficient condition," *Theoretical Economics*, 13, 761–793.

——— (2023): "Fair matching under constraints: Theory and applications," *Review of Economic Studies*, rdad046.

KELSO, A. S. AND V. P. CRAWFORD (1982): "Job matching, coalition formation, and gross substitutes," *Econometrica*, 1483–1504.

KOJIMA, F. (2012): "School choice: Impossibilities for affirmative action," *Games and Economic Behavior*, 75, 685–693.

KOJIMA, F., A. TAMURA, AND M. YOKOO (2018): "Designing matching mechanisms under constraints: An approach from discrete convex analysis," *Journal of Economic Theory*, 176, 803–833.

KOMINERS, S. D. AND T. SÖNMEZ (2016): "Matching with slot-specific priorities: Theory," *Theoretical Economics*, 11, 683–710.

NGUYEN, T. AND R. VOHRA (2019): "Stable matching with proportionality constraints," *Operations Research*, 67, 1503–1519.

ROMM, A., A. E. ROTH, AND R. I. SHORRER (2021): "Stability vs. no justified envy," *Working Paper*.

SÖNMEZ, T. AND T. B. SWITZER (2013): “Matching With (Branch-of-Choice) Contracts at the United States Military Academy,” *Econometrica*, 81, 451–488.

SÖNMEZ, T. AND M. B. YENMEZ (2022): “Affirmative action in India via vertical, horizontal, and overlapping reservations,” *Econometrica*, 90, 1143–1176.

TOMOEDA, K. (2018): “Finding a stable matching under type-specific minimum quotas,” *Journal of Economic Theory*, 176, 81–117.

WESTKAMP, A. (2013): “An analysis of the German university admissions system,” *Economic Theory*, 53, 561–589.