

CHARACTERIZATION OF FUNCTIONS WHOSE  
SECOND DIFFERENCES APPROACH ZERO

by

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## I. INTRODUCTION

Of all the classes of functions which one encounters in the study of mathematics, the most prominent undoubtedly is the class of continuous functions. The concept of continuous functions is so basic and so useful that most other classes of functions are either defined in terms of this concept, or characterized by comparing or contrasting the new class to the class of continuous functions. In short, the class of continuous functions is as important to a functional analyst as a yardstick is to a surveyor.

In what is to follow, we encounter the above class of functions twice. First of all, our entire problem can be viewed as an outgrowth or a generalization of the concept of continuity. Namely, continuity at a point  $x$  is defined in terms of the first differences  $\Delta f(x, h)$  at  $x$ , where

$$\Delta f(x, h) = f(x + h) - f(x),$$

in the following manner. If  $x$  is a point of the domain of definition, and if for each  $\epsilon > 0$  there exists a  $\delta > 0$ , such that whenever  $0 < h < \delta$  then

$$|\Delta f(x, h)| < \epsilon,$$

we say that  $f(x)$  is continuous at  $x$ .

Our problem concerns itself with the characterization of functions whose second differences, rather than the first

differences, are arbitrarily small.

We define the second difference of a function  $f$  at a point  $x$  as the difference of the first difference, namely,

$$\Delta^2 f(x, h) = \Delta [\Delta f(x, h)] = f(x + h) - 2f(x) + f(x - h).$$

The second differences are said to approach zero at  $x$  iff for every  $\epsilon > 0$  there exists a  $\delta > 0$ , such that whenever  $0 < h < \delta$  then

$$|\Delta^2 f(x, h)| < \epsilon .$$

In this paper a particular type of approach to zero of the second differences is considered. If for every  $\epsilon > 0$  there exists a  $\delta > 0$ , such that whenever  $0 < h < \delta$  then

$$|\Delta^2 f(x, h)| < \epsilon$$

for all  $x$  in some interval  $I$ , we say that the second differences approach zero uniformly on  $I$ . It is at this point that we encounter the class of continuous functions for the second time. Here we use this class as a yardstick. That is, we compare and contrast the class of functions whose second differences approach zero uniformly to the class of continuous functions.

For convenience let us denote the class of functions whose second differences approach zero uniformly on some interval  $I$  by the letter  $F$ . There exists a well known class

of functions, namely, the class of convex functions, whose members have very many properties which are quite similar to the properties of the members of  $F^1$ . In particular, the functions which satisfy the functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2} [f(x) + f(y)]$$

and form a subclass of convex functions share all their properties investigated in this paper with the properties of the functions of the class  $F$ .

In the second chapter of this dissertation definitions and preliminary theorems about boundedness and pointwise continuity of the functions belonging to  $F$  are given. In the third chapter a theorem is proved which states the necessary and sufficient conditions that a subclass of  $F$  be identical to the class of uniformly continuous functions. This theorem allows us to break the class  $F$  into two subclasses. One of these subclasses is the class of uniformly continuous functions, whereas the other is characterized by the extremely discontinuous behavior of its members.

In the fourth chapter the properties of the members belonging to the discontinuous subclass are considered. All such functions are shown to be nonmeasurable in the sense of

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<sup>1</sup>Sengenhorst, Paul. "Über konvexe Funktionen. Math.-Phys. Semesterbericht 2: 217-230. 1952.

Lebesgue. In a recent publication<sup>2</sup>, Sindalovsky proved a similar result. His theorem, however, failed to point out that every Lebesgue measurable function whose second differences approach zero uniformly is uniformly continuous.

Chapter 5 contains some of the extensions of the theorems in the previous chapters. The strong restriction that the second differences approach zero uniformly on the entire interval  $I$  is weakened. An example is given to illustrate that a discontinuous function need not be unbounded if the second differences approach zero uniformly on an everywhere dense set of measure less than the measure of the interval of definition.

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<sup>2</sup>Sindalovsky, G. H. On certain questions of continuity of measurable functions. Moskov. Gos. Univ. Uč. Zap. 181. Mat 8: 175-182. 1956. (Russian).

## II. DEFINITIONS AND PRELIMINARY THEOREMS

Throughout this paper only real functions of one real variable are considered. Whenever it is practicable, the notation used conforms to the notation found in standard textbooks on the theory of functions. Deviations from this practice will be noted and the symbols used will be given carefully defined meanings in order to avoid confusion.

Definition 1.1. The first difference of a function  $f(x)$  at an interior point  $x_0$  is defined by the equation

$$\Delta f(x_0, h) = f(x_0 + h) - f(x_0).$$

Definition 1.2. An open interval  $(a, b)$  is denoted by  $I$ . The closed interval  $[a, b]$  is denoted by  $\bar{I}$ . Here  $a$  and  $b$  are real numbers  $a < b$ .

Definition 1.3. The second difference of a function  $f(x)$  at an interior point  $x_0$  is defined by the equation

$$\Delta^2 f(x_0, h) = f(x_0 + h) - 2f(x_0) + f(x_0 - h),$$

where both  $x_0 + h$  and  $x_0 - h$  belong to the domain of definition.

Definition 1.4. The second differences of a function  $f(x)$  which is defined in the interval  $I$  are said to approach zero at an interior point  $x_0$  of  $I$ , iff for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that whenever  $0 < h < \delta$  then

$$|\Delta^2 f(x_0, h)| < \epsilon.$$

Certainly, the second differences of all continuous functions approach zero at all points interior to the interval in which  $f(x)$  is defined. However, the functions need not be continuous in order that the second differences approach zero. Any function for which the functional value at a point of discontinuity is defined to be the arithmetic average of the limit values from the right and from the left is a function for which the second differences approach zero at the point of discontinuity. It is clear that the class of functions for which the second differences approach zero is much more inclusive than the class of continuous functions. In order to characterize these functions we need to know a little more about how the second differences approach zero. It is at this point that we introduce the notion "uniform approach to zero of the second differences".

Definition 1.5. The second differences of a function  $f(x)$  defined on  $I$  are said to approach zero uniformly in the interval  $I$  iff for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that whenever  $0 < h < \delta$  then

$$|\Delta^2 f(x, h)| < \epsilon$$

for all  $x \in I$  ( $x$  belonging to  $I$ ). If the above statement holds only for  $x$  belonging to some subset  $A$  of  $I$ , then we say that the second differences approach zero uniformly on  $A$ .

It is easy to see that there exist continuous functions

for which the second differences approach zero uniformly. In fact, for all uniformly continuous functions the second differences approach zero uniformly. Let us recall the definition of uniform continuity of a function.

Definition 1.6. A function  $f(x)$  is uniformly continuous on an interval  $I$  iff for each  $\epsilon > 0$  there exists a  $\delta > 0$  such that whenever  $0 < |h| < \delta$

$$|f(x+h) - f(x)| < \epsilon$$

for all  $x \in I$ .

Writing  $\Delta^2 f(x_0, h)$  in the following way

$$(0.1) \quad \Delta^2 f(x_0, h) = f(x_0 + h) - f(x_0) + f(x_0 - h) - f(x_0),$$

it then follows from the triangle inequality that

$$(0.2) \quad |\Delta^2 f(x_0, h)| \leq |f(x_0 + h) - f(x_0)| + |f(x_0 - h) - f(x_0)|.$$

According to Definition 1.6 there exists a  $\delta > 0$  such that

$$|f(x+h) - f(x)| < \frac{\epsilon}{2}$$

and

$$|f(x-h) - f(x)| < \frac{\epsilon}{2}$$

for all  $0 < |h| < \delta$  and  $x \in I$ .

Hence, using the above inequalities in Equation 0.2 we get  $|\Delta^2 f(x, h)| < \epsilon$  for all  $x \in I$  and  $0 < |h| < \delta$ . This,

however, is the definition for the uniform approach to zero of the second differences.

One might now suppose that the uniform approach to zero of the second differences holds for all continuous functions. The following example shows that the supposition above was somewhat hasty.

Example 1.1. Consider the function  $f(x) = \frac{1}{x}$  in the interval  $(0, 1)$ . We demonstrate the existence of a point  $x$  such that no matter how small we choose  $h$ , the value of

$$|\Delta^2 f(x, h)|$$

is arbitrarily large instead of arbitrarily small. Let  $0 < h < \frac{1}{2}$  be given. We define

$$x = h + \frac{1}{N}$$

where  $N$  is an arbitrarily large positive number and such that  $h + \frac{1}{N} < \frac{1}{2}$ .

$$\Delta^2 f(x, h) = \frac{N}{2Nh + 1} - \frac{2N}{Nh + 1} + N.$$

We see that the above expression can be made as large as we please by choosing  $N$  sufficiently large.

Although the unbounded function above is perhaps the simplest example of a function which is continuous, but whose second differences approach zero nonuniformly, there exist bounded functions which have the same property.

Example 1.2. Consider the following function defined on  $(0, 1)$ .

$$f(x) = \begin{cases} \sin \frac{\pi}{x} & \text{whenever } \sin \frac{\pi}{x} > 0 \\ 0 & \text{whenever } \sin \frac{\pi}{x} \leq 0 \end{cases}$$

$$f(x) = 1 \quad \text{when } x = \frac{2}{4n+1} \quad n = 1, 2, 3, \dots$$

$$f(x) = 0 \quad \text{when } x = \frac{1}{2n} \quad n = 1, 2, 3, \dots$$

$$\text{Let } h_n = \frac{1}{2n} - \frac{2}{4n+1} = \frac{1}{2n(4n+1)}.$$

Given any  $\delta > 0$ , we can find  $h_n < \delta$  and an  $x \in (0, 1)$ , namely  $x = \frac{1}{2n}$ , such that

$$\Delta^2 f\left(\frac{1}{2n}, h_n\right) = f\left(\frac{1}{2n} + h_n\right) - 2f\left(\frac{1}{2n}\right) + f\left(\frac{1}{2n} - h_n\right).$$

$$\sin\left(\frac{\pi}{\frac{1}{2n} + h_n}\right) = \sin\left(2\pi n - \frac{\pi n}{2n+1}\right) = -\sin\frac{\pi}{2}\left(\frac{n}{n + \frac{1}{2}}\right)$$

since  $0 < \frac{n}{n + \frac{1}{2}} < 1$  the argument of the sine function above

lies always in the first quadrant. Hence,

$$f\left(\frac{1}{2n} + h_n\right) = 0.$$

We established previously that

$$f\left(\frac{1}{2n}\right) = 0$$

and

$$f\left(\frac{1}{2n} - h_n\right) = f\left(\frac{2}{4n+1}\right) = 1.$$

Hence,

$$\Delta^2 f\left(\frac{1}{2n}, h_n\right) = 1$$

and the second differences of the above function do not approach zero uniformly.

At this point it seems logical to ask the question "Is it necessary that a function be uniformly continuous in order that its second differences approach zero uniformly?" The answer to this question is contained in Theorem 1.3, which is one of the key theorems of this paper. Before we state and prove Theorem 1.3 it is convenient to state and prove some preliminary theorems.

We need the following definition for the statement of the first theorem.

Definition 1.7. A function  $f(x)$  is said to be bounded at a point  $x_0$  iff there exists a constant  $M > 0$  and an interval  $I$  containing  $x_0$  such that whenever  $x \in I$   $|f(x)| < M$ . If for each  $M > 0$  and each  $I$  containing  $x_0$   $|f(\bar{x})| > M$  at some point  $\bar{x} \in I$ , then  $f(x)$  is said to be unbounded at  $x_0$ .

Theorem 1.1. Let a function  $f(x)$  be defined at all points within some open interval  $I$ . If the second differences of the function  $f(x)$  approach zero uniformly in  $I$  and if  $f(x)$  is discontinuous at  $x_0 \in I$ , then  $f(x)$  is unbounded at  $x_0$ .

Proof: For the proof it is sufficient to consider the case when  $f(x_0) = 0$ . For if  $f(x_0) = A \neq 0$  then the function

$$g(x) = f(x) - A$$

is still discontinuous at  $x_0$ ,

$$\Delta^2 g(x, h) \equiv \Delta^2 f(x, h) ,$$

but

$$g(x_0) = 0.$$

Let us assume that  $f(x)$  is bounded at  $x_0$ . That is, there exists a positive number  $M$  such that

$$|f(x)| < M$$

for some  $\bar{\delta}$  and all  $x$  belonging to the interval  $(x_0 - \bar{\delta}, x_0 + \bar{\delta})$ . We now construct a ladder of functional values whose arguments lie in the interval  $(x_0 - \bar{\delta}, x_0 + \bar{\delta})$  and such that the top-most rung of this ladder reaches through the roof given by the value  $M$ . This of course would give us the contradiction which proves the theorem.

Since  $f(x)$  is discontinuous at  $x_0$ , there exists an  $\epsilon > 0$ , and a sequence  $\{h_1\}$  converging to zero, such that

$$(1.1) \quad |f(x_0 + h_1)| > \epsilon .$$

Without any loss of generality we may assume  $h_1 > 0$  and  $f(x_0 + h_1) > \epsilon$ .

From the definition of the uniform approach to zero of the second differences it follows that there exists a  $\delta'$  such that for all  $0 < h < \delta'$  and all  $x \in I$

$$(1.2) \quad |\Delta^2 f(x, h)| < \frac{\epsilon}{2} .$$

Consider  $\delta = \min \{ \delta', \bar{\delta} \}$ . For any positive integer  $n$  there exists at least one member  $h_k$  of the sequence  $\{h_1\}$  such that

$$(1.3) \quad h_k < \frac{\delta}{2^n} .$$

The second difference

$$\begin{aligned} \Delta^2 f(x_0 + h_k, h_k) &= f(x_0) - 2f(x_0 + h_k) + f(x_0 + 2h_k) \\ &= f(x_0 + 2h_k) - 2f(x_0 + h_k). \end{aligned}$$

From (1.2) it follows that

$$f(x_0 + 2h_k) - 2f(x_0 + h_k) > - \frac{\epsilon}{2} .$$

Hence, using (1.1) we get

$$(1.4) \quad f(x_0 + 2h_k) > 2\epsilon - \frac{\epsilon}{2} .$$

This gives us our bottommost rung of the ladder. We continue to build our ladder by next considering

$$(1.5) \quad \Delta^2 f(x_0 + 2h_k, 2h_k) = f(x_0 + 2^2 h_k) - 2f(x_0 + 2h_k) .$$

Since  $2^2 h_k < \frac{\delta}{2^{n-2}} < \delta$  if  $n > 2$  we again can use (1.2) to obtain

$$f(x_0 + 2^2 h_k) - 2f(x_0 + 2h_k) > -\frac{\epsilon}{2}$$

or

$$(1.6) \quad f(x_0 + 2^2 h_k) > 2(2\epsilon - \frac{\epsilon}{2}) - \frac{\epsilon}{2} = 4\epsilon - \frac{3}{2}\epsilon.$$

Continuing in this fashion we obtain in general for  $m \leq n$

$$f(x_0 + 2^m h_k) > 2f(x_0 + 2^{m-1} h_k) - \frac{\epsilon}{2} > \left[2^m - \frac{(2^m - 1)}{2}\right] \epsilon$$

and thus we arrive at

$$(1.7) \quad f(x_0 + 2^m h_k) > \frac{1}{2} (2^m + 1) \epsilon.$$

Since  $n$  was arbitrary and  $\epsilon$  a fixed quantity, we may choose  $n$  large enough so that

$$\frac{1}{2} (2^n + 1) \epsilon > M.$$

It follows from (1.7) that

$$(1.8) \quad f(x_0 + 2^n h_k) > M$$

and our ladder is now high enough.

Since  $h_k$  was chosen so that  $2^n h_k < \delta$  then we have succeeded in finding a point  $\bar{x} = x_0 + 2^n h_k$  which belongs to the interval  $(x_0 - \bar{\delta}, x_0 + \bar{\delta})$  and for which the functional value exceeds the assumed bound  $M$ .

We have just proved that if a function whose second differences approach zero uniformly on an interval  $I$  is bounded at a point  $x_0$ , then the function is continuous at this point. It is reasonable now to investigate whether or not boundedness of the function at a point implies boundedness of the function in the entire interval. If this is the case then we immediately conclude that a function whose second differences approach zero uniformly and which is continuous even at one point in the interval must be continuous in the entire interval. We now state the following theorem.

Theorem 1.2. Let the function  $f(x)$  be defined at all points in an interval  $I$ . If the second differences of  $f(x)$  approach zero uniformly in  $I$  and if  $f(x)$  is bounded at one point  $x_0 \in I$ , then  $f(x)$  is bounded throughout the interval  $I$ .

Proof: Let the interval  $I$  have the endpoints  $a$  and  $b$ ,  $a < b$ . Let  $x_0$  be the point at which  $f(x)$  is bounded. Then there exists an  $M > 0$  and a  $\delta > 0$ , such that for all  $x \in [x_0 - \delta, x_0 + \delta]$

$$(2.1) \quad |f(x)| < M.$$

$$\text{Let } d = \max \{ b - x_0, x_0 - a \}$$

By hypothesis, for every  $\epsilon > 0$  there exists a  $\delta' > 0$  such that

$$(2.2) \quad |\Delta^2 f(x, h)| < \epsilon$$

for all  $x \in I$  and  $0 < h < \delta'$

Let  $\delta = \min \{ \bar{\delta}, \delta' \}$ .

Choose  $N$  to be the first positive integer

$$N > \frac{d}{\delta} .$$

We shall show that

$$(2.3) \quad |f(x)| < 3^n M + \frac{\epsilon}{2}(3^n - 1),$$

when  $x \in [x_0 - (2n+1)\delta, x_0 + (2n+1)\delta]$   $n = 0, 1, \dots, N$   
and wherever  $f(x)$  is defined in this interval.

Consider any point in the interval  $[x_0 - 3\delta, x_0 + 3\delta]$ .  
If  $x \in [x_0 - \delta, x_0 + \delta]$  then  $|f(x)| < M$  by hypothesis. Let  
 $x_1 \in [x_0 + \delta, x_0 + 3\delta]$ . The symmetric point  $\bar{x}_1$  with respect  
to  $x_0 + \delta$  lies in the interval  $[x_0 - \delta, x_0 + \delta]$ . The  
second difference is explicitly

$$(2.4) \quad \Delta^2 f(x_0 + \delta, \bar{x}_1 - (x_0 + \delta)) = f(\bar{x}_1) - 2f(x_0 + \delta) + f(x_1).$$

If  $x_1$  lies in  $I$ , then by (2.2)

$$(2.5) \quad -\epsilon < f(\bar{x}_1) - 2f(x_0 + \delta) + f(x_1) < \epsilon$$

or

$$(2.6) \quad -\epsilon - f(\bar{x}_1) + 2f(x_0 + \delta) < f(x_1) < \epsilon + 2f(x_0 + \delta) - f(\bar{x}_1)$$

Using (2.1), (2.6) becomes

$$(2.7) \quad -\epsilon - M - 2M < f(x_1) < \epsilon + 2M + M.$$

We can use the same reasoning to show that (2.7) holds

also if  $x_1 \in [x_0 - 3\delta, x_0 - \delta]$ . Hence rewriting (2.7) we get

$$(2.8) \quad |f(x_1)| < 3M + \epsilon,$$

whenever  $x_1 \in (x_0 - 3\delta, x_0 + 3\delta)$  and  $x_1 \in I$ .

We now prove (2.3) by induction. (2.8) establishes the validity of (2.3) for  $n = 1$ .

Let  $n - 1$  be any integer less than  $N$ . We assume that

$$(2.9) \quad |f(x)| < 3^{n-1} M + \frac{\epsilon}{2} (3^{n-1} - 1)$$

holds for all  $x \in [x_0 - (2n - 1)\delta, x_0 + (2n - 1)\delta]$  and  $x \in I$ . Suppose that the point  $x_1$  lies both in  $I$  and in the interval  $[x_0 + (2n - 1)\delta, x_0 + (2n + 1)\delta]$ . The symmetric point  $\bar{x}_1$  with respect to  $x_0 + (2n - 1)\delta$  must certainly lie in  $[x_0 - (2n - 1)\delta, x_0 + (2n - 1)\delta]$ . Hence,

$$(2.10) \quad -\epsilon - f(\bar{x}_1) + 2f(x_0 + (2n-1)\delta) < f(x_1) < \epsilon + 2f(x_0 + (2n-1)\delta) - f(\bar{x}_1).$$

Since  $\bar{x}_1$  and  $x_0 + (2n - 1)\delta$  both lie in the interval  $[x_0 - (2n - 1)\delta, x_0 + (2n - 1)\delta]$  and if  $x_1$  lies in  $I$  we have,

$$(2.11) \quad -\epsilon - 3 \cdot 3^{n-1} M + \frac{\epsilon}{2} (3^{n-1} - 1) < f(x_1) < \epsilon + 3 \cdot 3^{n-1} M + \frac{\epsilon}{2} (3^{n-1} - 1)$$

or

$$|f(x_1)| < 3^n M + \frac{\epsilon}{2} (3^n - 3) + \epsilon$$

$$(2.12) \quad |f(x_1)| < 3^N M + \frac{\epsilon}{2} (3^n - 1).$$

Similar reasoning for a point  $x_1 \in I$  and  $x_1 \in [x_0 - (2n+1)\delta, x_0 - (2n-1)\delta]$  would show that (2.12) also holds in this case. We have thus established inequality (2.3).

Since the interval

$$[x_0 - (2N+1)\delta, x_0 + (2N+1)\delta]$$

covers the interval  $I$  then we can conclude from (2.3) that  $f(x)$  is bounded in the interval  $I$ .

## III. UNIFORMLY CONTINUOUS FUNCTIONS

In this section we shall state and prove a result which shows that the class of uniformly continuous functions defined on an interval  $I$  is identical with the class of functions defined by the following two properties: (1) the second differences of the functions approach zero uniformly on  $I$  and (2) each function is continuous at least at one point in  $I$ . As we shall see, the functions whose second differences approach zero uniformly in an interval  $I$  subdivide into two classes. One of these classes is the class of uniformly continuous functions, whereas the other is a class characterized by the extremely discontinuous behavior of its members.

Before we state and prove a theorem which affirms the first statement in the above paragraph, let us recall a well known result from advanced calculus. A function which is continuous on a closed interval is also uniformly continuous on that interval<sup>3</sup>. Clearly, if a function is uniformly continuous on an interval, then it is also uniformly continuous on any subinterval of that interval. With the help of these two results we shall prove the following theorem.

Theorem 1.3. Let  $f(x)$  be defined in an open finite interval  $I$ . The necessary and sufficient conditions that  $f(x)$  be uniformly continuous in  $I$  are that (1) the second differences of

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<sup>3</sup>Widder, David V. Advanced calculus. New York. Prentice-Hall, Inc. 1947. P. 147.

$f(x)$  approach zero uniformly in  $I$  and (2)  $f(x)$  is continuous at least at one point in  $I$ .

Proof: We saw earlier that if a function is uniformly continuous in some interval  $I$  then the second differences approached zero uniformly in  $I$ . This proves the necessity part of the above theorem.

The sufficiency part of the proof is quite similar to the proof of Theorem 1.1. From hypothesis we have that  $f(x)$  is continuous at least at one point in  $I$ . From Theorem 1.2 it follows that  $f(x)$  is bounded in the entire interval  $I$ . Theorem 1.1 in turn implies that  $f(x)$  is continuous at every point in  $I$ . Therefore, in order to prove that  $f(x)$  is uniformly continuous in  $I$ , we only need to prove the existence of limits of the function  $f(x)$  at the endpoints of the interval  $I$ . For if both of these limits exist, then the function  $f(x)$  can be extended to a continuous function in the closure  $\bar{I}$  of  $I$ . From the remarks made previous to the statement of Theorem 1.3 it follows that  $f(x)$  is uniformly continuous in  $I$ .

We now proceed to prove the existence of limits by contradiction. Let  $I = (a, b)$ . Suppose  $\lim_{x \rightarrow a} f(x)$  does not exist. Since  $f(x)$  is bounded in  $I$  there must exist two null sequences  $\{h_1\}$  and  $\{\bar{h}_1\}$  ( $h_1 > 0, \bar{h}_1 > 0$   $i = 1, 2, 3 \dots$ ), and such that

$$\lim_{h_1 \rightarrow 0} f(a + h_1) = B \quad \text{and}$$

$$\lim_{\bar{h}_1 \rightarrow 0} f(a + \bar{h}_1) = C \quad B \neq C.$$

There is no loss of generality if we assume  $C = 0$  and  $B > 0$ .

We now choose an  $\epsilon$   $0 < \epsilon < \frac{B}{4}$ .

Let  $\{h'_i\}$  and  $\{\bar{h}'_i\}$  be the subsequences of  $\{h_i\}$  and  $\{\bar{h}_i\}$  for which

$$(3.1) \quad |f(a + h'_i) - B| < \epsilon \quad \text{and}$$

$$(3.2) \quad |f(a + \bar{h}'_i)| < \epsilon.$$

Since the second differences approach zero uniformly, there exists a  $\delta > 0$  such that for all  $x \in I$  and  $0 < h < \delta$

$$(3.3) \quad |\Delta^2 f(x, h)| < \frac{\epsilon}{2}.$$

Because  $\{h'_i\}$  and  $\{\bar{h}'_i\}$  form null sequences, it follows that for any positive integer  $n$  there exists at least one member of each sequence such that

$$h'_k < \frac{\delta}{2^n} \quad \text{and}$$

$$\bar{h}'_\ell < h'_k.$$

Let  $h'_k - \bar{h}'_\ell = h$ ,  $a + h'_k = \bar{x}$ , then  $\bar{x} - h = a + \bar{h}'_\ell$ .

Now let us consider

$$\Delta^2 f(\bar{x}, h) = f(\bar{x} + h) - 2f(\bar{x}) + f(\bar{x} - h).$$

From (3.3) we get

$$-\frac{\epsilon}{2} < f(\bar{x} + h) - 2f(\bar{x}) + f(\bar{x} - h)$$

or

$$(3.4) \quad 2f(\bar{x}) - f(\bar{x} - h) - \frac{\epsilon}{2} < f(\bar{x} + h).$$

Using (3.1) and (3.2) to reduce the left hand side of (3.4) we get

$$(3.5) \quad 2(B - \epsilon) - \epsilon - \frac{\epsilon}{2} < f(\bar{x} + h).$$

Since we chose  $4\epsilon < B$ , (3.5) can be written as

$$(3.6) \quad 5\epsilon - \frac{\epsilon}{2} < f(\bar{x} + h).$$

We shall now show that

$$(3.7) \quad \frac{3}{2}\epsilon(1 + 2^m) < f(\bar{x} + (2^m - 1)h) \quad \text{for all } m \leq n.$$

For  $m = 1$  (3.7) reduces to (3.6) and therefore (3.7) holds for  $m = 1$ . Assume (3.7) to hold for  $m = 1$ . Assume (3.7) to hold for  $m - 1$ , that is,

$$(3.8) \quad \frac{3\epsilon}{2}(1 + 2^{m-1}) < f(\bar{x} + (2^{m-1} - 1)h).$$

From (3.3) for  $m \leq n$

$$-\frac{\epsilon}{2} < f(\bar{x} + (2^m - 1)h) - 2f(\bar{x} + (2^{m-1} - 1)h) + f(\bar{x} - h)$$

or

$$(3.9) \quad 2f(\bar{x} + (2^{m-1}-1)h) - f(\bar{x} - h) - \frac{\epsilon}{2} < f(\bar{x} + (2^m - 1)h)$$

(3.8) and (3.2) reduce (3.9) to the following

$$(3.10) \quad \frac{6}{2} \epsilon (1 + 2^{m-1}) - \epsilon - \frac{\epsilon}{2} < f(\bar{x} + (2^m - 1)h) .$$

Simplification of the left hand side produces

$$(3.11) \quad \frac{3}{2} \epsilon (1 + 2^m) < f(\bar{x} + (2^m - 1)h) .$$

Since the choice of  $n$  was arbitrary, the left hand member of (3.11) can be made arbitrarily large. This, however, contradicts our hypothesis, namely that  $f(x)$  is bounded in  $I$ . We are thus forced to conclude that  $\lim_{x \rightarrow a} f(x)$  must exist.

Analogous reasoning would prove that  $\lim_{x \rightarrow b} f(x)$  exists also. This is sufficient to prove the theorem.

Theorem 1.3 subdivides the class of functions whose second differences approach zero into the class of uniformly continuous functions and into a class of functions, each member of which must not only be discontinuous at each point but also unbounded at every point in the interval of definition. The class of uniformly continuous functions is in this paper considered as a measuring stick and there is therefore no need to go further into the properties of this class. We thus turn our attention to the functions which seem to have highly unusual properties.

IV. PROPERTIES OF DISCONTINUOUS FUNCTIONS WHOSE SECOND  
DIFFERENCES APPROACH ZERO UNIFORMLY

As mentioned previously, the discontinuous functions whose second differences approach zero uniformly must be discontinuous and unbounded at every point in the interval of definition. An example of such a function has been in the mathematical literature for quite some time.

Hamel<sup>4</sup>, in his well known paper, found all the solutions of the following functional equation.

$$f\left(\frac{u+v}{2}\right) = \frac{1}{2} [f(u) + f(v)] \quad .$$

If the second differences of  $f(x)$  are identically zero for all  $h$  and  $x$ , i.e.

$$f(x+h) - 2f(x) + f(x-h) = 0 \quad ,$$

then letting

$$u = x + h$$

$$v = x - h$$

we obtain

$$\frac{1}{2} [f(u) + f(v)] = f\left(\frac{u+v}{2}\right) \quad .$$

Any solution of the above functional equation clearly belongs to the class of functions whose second differences

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<sup>4</sup>Hamel, G. Eine Basis aller Zahlen und die unstetige Lösung der Funktionalgleichung:  $f(x+y) = f(x) + f(y)$ . Math Annalen 60: 459-462. 1905.

approach zero uniformly. Its discontinuous solutions are to us of particular interest because they furnish examples of functions which are discontinuous, but whose second differences are, in fact, identically zero.

To obtain all of the solutions of the above functional equation, Hamel constructed a basis for the set of all real numbers, now called the Hamel's basis. He proved that any real number can be expressed uniquely as a finite rational-linear combination of the basis elements. That is, any real number  $x$ .

$$x = \sum_{\alpha} p_{\alpha} a_{\alpha}$$

where  $p_{\alpha}$  are rational numbers and  $a_{\alpha}$  are the basis elements, and where  $p_{\alpha} \neq 0$  for only a finite number of values of index  $\alpha$ .

Hamel then defined the class of all functions which satisfy the functional equation

$$f\left(\frac{u+v}{2}\right) = \frac{1}{2} [f(u) + f(v)]$$

by defining

$$H(x) = \sum_{\alpha} p_{\alpha} H(a_{\alpha})$$

and leaving  $H(a_{\alpha})$  arbitrary. Defining the function  $H$  at all the basis elements gives us an element of the class of all solutions of the functional equation above.

Two of the most interesting properties of a Hamel's function  $H(x)$  which is discontinuous are: (1) the functional values of  $H(x)$  form an everywhere dense set in the real continuum in every subinterval of the interval  $I$  of definition of  $H(x)$ ; (2)  $H(x)$  is nonmeasurable in the sense of Lebesgue.

As the following theorem shows, all discontinuous functions whose second differences approach zero uniformly have property (1) of the previous paragraph.

Theorem 1.4. If the second differences of a discontinuous function approach zero uniformly, then in any subinterval of the interval of definition the functional values form an everywhere dense set in the real continuum.

Proof: Let  $(a, b)$  denote the interval in which  $f(x)$  is defined. Since  $f(x)$  is discontinuous in  $(a, b)$  then according to Theorems 1.1, 1.2, the function  $f(x)$  is also unbounded at every point in  $(a, b)$ . In fact, with only a slight modification of the proof of Theorem 1.1, it can be shown that  $f(x)$  is unbounded from both above and below at each point of discontinuity. Hence, Theorem 1.2 would extend the unboundedness from both above and below to all points in the interval  $(a, b)$ .

Let us assume that there exists a subinterval  $(c, d)$  of  $(a, b)$  in which the functional values do not form an everywhere dense set in the real continuum, i.e. there exists a real number  $A$  and an  $\epsilon > 0$ , such that

$$(4.1) \quad |f(x) - A| > \epsilon$$

for all  $x \in (c, d)$ .

As in previous proofs, we may assume without any loss of generality that  $A = 0$ .

From hypothesis it follows that there exists a  $\delta > 0$ , such that

$$(4.2) \quad |\Delta^2 f(x, h)| < \epsilon$$

for all  $x \in (a, b)$  and  $0 < h < \delta$ .

Denote the midpoint of  $(c, d)$  by  $\bar{x}_1$  and let  $I_1$  denote the smaller of the two intervals  $(c, d)$  and  $(\bar{x}_1 - \delta, \bar{x}_1 + \delta)$ . Let  $B$  denote the set of functional values  $f(x) \geq \epsilon$ ,  $x$  in the interval  $I_1$ . Also let  $C$  denote the set of  $f(x) \leq -\epsilon$ ,  $x$  in  $I_1$ . Since both  $B$  and  $C$  are nonvoid sets because  $f(x)$  is unbounded from both above and below at each point in  $(a, b)$ , and since  $B$  is bounded from below and  $C$  from above, there exists a greatest lower bound  $g$  for  $B$  and a least upper bound  $\ell$  for  $C$ . Also,

$$(4.3) \quad \begin{aligned} g &\geq \epsilon \\ \ell &\leq -\epsilon. \end{aligned}$$

Hence, there exist two points  $x_1$  and  $x_2$  in  $I_1$ , such that

$$(4.4) \quad g \leq f(x_1) < g + \epsilon \quad \text{and}$$

$$(4.5) \quad \ell - \epsilon < f(x_2) \leq \ell.$$

In the interval  $I_1$  according to (4.2)

$$- \epsilon < f(x_1) - 2f\left(\frac{x_1 + x_2}{2}\right) + f(x_2) < \epsilon$$

or

$$(4.6) \quad \frac{f(x_1) + f(x_2) - \epsilon}{2} < f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2) + \epsilon}{2}.$$

Making use of (4.4) and (4.5) we obtain

$$(4.7) \quad \frac{g + l}{2} - \epsilon < f\left(\frac{x_1 + x_2}{2}\right) < \frac{g + l}{2} + \epsilon.$$

Now because  $l \leq -\epsilon$

$$\frac{g + l}{2} + \epsilon \leq \frac{g}{2} + \frac{\epsilon}{2}$$

and by using (4.3)

$$\frac{g + \epsilon}{2} \leq g.$$

Also

$$\frac{g + l}{2} - \epsilon \geq \frac{l}{2} - \frac{\epsilon}{2} \geq l.$$

Hence (4.7) becomes

$$(4.8) \quad l < f\left(\frac{x_1 + x_2}{2}\right) < g.$$

Now  $\left|f\left(\frac{x_1 + x_2}{2}\right)\right| \geq \epsilon$  by assumption.

Hence, either

$$(4.9) \quad \epsilon < f\left(\frac{x_1 + x_2}{2}\right) < g \quad \text{or}$$

$$(4.10) \quad l < f\left(\frac{x_1 + x_2}{2}\right) < -\epsilon.$$

Since  $g$  is the greatest lower bound of  $B$ , there cannot exist a point  $\frac{x_1 + x_2}{2}$  in  $I_1$  whose functional value is less than  $g$  and greater than  $\epsilon$ .

Analogously, since  $\ell$  is the least upper bound of  $C$ , there cannot exist a point  $\frac{x_1 + x_2}{2}$  in  $I_1$  whose functional value is greater than  $\ell$  and less than  $-\epsilon$ . Hence, we are forced to conclude that the functional values of  $f(x)$  form an everywhere dense set in the real continuum in every sub-interval of  $(a, b)$ .

We have thus established one of the properties which is common to both: (1) the functions which are discontinuous solutions of equation

$$(4.11) \quad f\left(\frac{x+y}{2}\right) = \frac{1}{2} [f(x) + f(y)] \quad ,$$

and (2) discontinuous functions whose second differences approach zero uniformly. Let us turn our attention to the other property of the discontinuous solutions of (4.11), namely, the Lebesgue nonmeasurability of the discontinuous solutions of (4.11). As we shall see, also this property is shared by the discontinuous functions whose second differences approach zero uniformly. Before we prove the above statement we will define the terms and prove a lemma.

**Definition 1.8.** The Lebesgue exterior measure of a set  $S$  in the set of all real numbers  $E_1$  is the greatest lower bound of the sum of the measures of all denumerable sequences of open

intervals which cover  $S$ . We denote the Lebesgue exterior measure of  $S$  by  $m_e(S)$ .

Definition 1.9. The Lebesgue inner measure  $m_i(S)$  of a set  $S$  is the least upper bound of the measures of all closed sets  $F$  which are contained in  $S$ .

Definition 1.10. A bounded set  $S$  is said to be Lebesgue measurable iff

$$m_i(S) = m_e(S) = m(S) ,$$

where  $m(S)$  is the Lebesgue measure of the set  $S$ .

Definition 1.11. A function  $f(x)$  is said to be measurable (Lebesgue) iff for every pair of real numbers  $h$  and  $k$ ,  $h < k$ , the set  $E = \{ x \mid h < f(x) < k \}$  is Lebesgue measurable.

We shall list some of the elementary properties which are immediate consequences of the above definitions and which we shall use without proving them.

Property 1.1. If  $S_1$  is a measurable set then so is the complement  $c(S_1)$  of  $S_1$ .

Property 1.2. If  $S_1$  and  $S_2$  are measurable sets  $S_1 \supset S_2$ , then  $m(S_1) \geq m(S_2)$ .

Property 1.3. If  $f(x)$  is measurable then the sets

$$E_1 = \{ x \mid h \leq f(x) \leq k \}$$

$$E_2 = \{ x \mid f(x) > h \}$$

$$E_3 = \{ x \mid f(x) < h \}$$

are all measurable. Here  $h$  and  $k$  are any real numbers  $h < k$ .

A property which is also elementary but not as often found in standard textbooks on the theory of variables is the following<sup>5</sup>.

Property 1.4. If  $S$  is a measurable set of positive finite measure, and if  $0 \leq a < 1$ , then there exists an open interval  $I$  such that

$$m(S \cap I) \geq a m(I)$$

Making use of the above properties we are now ready to prove the following lemma.

Lemma 1.1. Let  $S$  be a measurable set in  $I = (a, b)$ . If for every subinterval  $I'$  of  $I$  and every  $0 < \epsilon < \frac{m(I')}{2}$   $m(S \cap I') \geq \frac{1}{2} [m(I') - \epsilon]$ , then the measure of the set  $S$

$$m(S) = m(I) = b - a .$$

Proof: Suppose the measure of the set  $S$

$$m(S) < b - a .$$

Then since the complement  $c(S)$  is measurable by Property 1.1

$$m(c(S)) > 0 .$$

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<sup>5</sup>Halmos, Paul R. Measure theory. New York. D. Van Nostrand Company, Inc. 1950. P. 68.

Hence by Property 1.4 there exists a subinterval  $I_1$  of  $I$  such that

$$m(c(S) \cap I_1) \geq \frac{3}{4} m(I_1).$$

However, from the hypothesis above

$$m(S \cap I') \geq \frac{1}{2} [m(I') - \epsilon],$$

which implies that

$$m(c(S) \cap I') \leq \frac{1}{2} [m(I') + \epsilon]$$

for every  $I'$  and  $\epsilon > 0$ .

Therefore,

$$\frac{1}{2}m(I_1) + \frac{\epsilon}{2} \geq m(c(S) \cap I_1) \geq \frac{3}{4} m(I_1),$$

which is a contradiction, since  $\epsilon$  is arbitrarily small.

**Theorem 1.5.** A function  $f(x)$  whose second differences approach zero uniformly and which is discontinuous at one point in an interval  $I$  of definition is nonmeasurable (Lebesgue).

**Proof:** As we saw in the proof of Theorem 1.4, the discontinuity of  $f(x)$  at a point in  $I$  implies that  $f(x)$  is unbounded both from above and below at every point in  $I$ .

Assume that  $f(x)$  is measurable in  $I = (a, b)$ . Let  $S_n$  be a subset of  $I$

$$(5.1) \quad S_n = \{ x \in I \mid f(x) > n \}$$

where  $n$  is any positive integer. According to Property 1.3  $S_n$  is measurable.

By hypothesis, for  $\epsilon > 0$  there exists a  $\delta > 0$  such that  $|\Delta^2 f(x, h)| < \epsilon$  for all  $0 < h < \delta$  and  $x \in I$ .

Let  $(c, d) = I_0$  be any subinterval of  $I$ . If  $(d - c) > 2\delta$ , we break  $(c, d)$  into a finite number of subintervals  $I_1 = (c_1, d_1)$  such that  $d_1 - c_1 < 2\delta$ ,  $i = 1, 2, 3 \dots N$ ,  $c_1 = c$ ,  $d_1 = c_{i+1}$ ,  $d_n = d$ . The midpoints  $x_1 = \frac{c_1 + d_1}{2}$  of each interval. Since  $f(x)$  is unbounded above at every point of  $I$ , then for any  $\bar{\epsilon} > 0$  we can find a point  $\bar{x}_1$  such that  $x_1 - \bar{x}_1 < \frac{\bar{\epsilon}}{2}$  and such that

$$f(\bar{x}_1) > n + \epsilon.$$

By hypothesis

$$(5.2) \quad -\epsilon < f(\bar{x}_1 + h) - 2f(\bar{x}_1) + f(\bar{x}_1 - h)$$

for all  $0 < h < \delta$

or

$$(5.3) \quad 2n + \epsilon < f(\bar{x}_1 + h) + f(\bar{x}_1 - h).$$

Hence at least one of the values  $\bar{x}_1 + h$  or  $\bar{x}_1 - h$  must belong to  $S_n$  for each  $h$ . Let  $h$  vary between

$$0 < h < \min \{ \bar{x}_1 - c_1, d_1 - \bar{x}_1 \}.$$

Then

$$(5.4) \quad m(I_1 \cap S_n) \geq \frac{1}{2} m(I_1) - \frac{\bar{\epsilon}}{2^1}$$

and

$$(5.5) \quad m(I_0 \cap S_n) = m\left(\sum_{i=1}^N I_i \cap S_n\right) = \sum_{i=1}^N m(I_i \cap S_n) \\ \geq \sum_{i=1}^N \left(\frac{1}{2} m(I_i) - \frac{\bar{\epsilon}}{2^1}\right) \\ \geq \frac{1}{2} m(I_0) - \bar{\epsilon}.$$

From Lemma 1.1 we conclude that

$$(5.6) \quad m(S_n) = (b - a).$$

In exactly the same way we can show that

$$(5.7) \quad m(S_{-n}) = (b - a)$$

where  $S_{-n} = \{x \in I \mid f(x) \leq -n\}$  and where  $n$  is any positive integer. Therefore, since for any measurable set in  $I$  and in particular for  $S_n$

$$(5.8) \quad m(S_n \cap I) + m(c(S_n) \cap I) = m(I),$$

and since

$$m(c(S_n) \cap I) \geq m(S_{-n} \cap I) = (b - a)$$

by Property 1.2, we have from (5.8)

$$(5.9) \quad (b - a) + (b - a) \leq m(I) = b - a,$$

which is a contradiction. Therefore  $S_n$  cannot be measurable and hence  $f(x)$  cannot be measurable by Property 1.3. This completes the proof.

Before we continue with additional properties of the functions under consideration, let us outline the relationships between the classes of functions thus far considered. In order to simplify the discussion we make the following definitions.

Definition 1.12. The class of functions whose second differences approach zero uniformly on an interval  $I$  will be denoted by  $F$ . Any member of the class  $F$  will be denoted by  $f$ .

Definition 1.13. The class of uniformly continuous functions in  $I$  will be denoted by  $G$  and any member of  $G$  will be denoted by  $g$ .

Definition 1.14. The class of all discontinuous solutions of the functional equation

$$f\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2} [f(x_1) + f(x_2)]$$

will be denoted by  $H$  and sometimes referred to as the class of Hamel functions. Its members will again be denoted by  $h$ .

We have thus far determined the following class relationships.

$$F \supset G$$

$$F \supset H .$$

Also, if we consider the complement  $c(G)$  of  $G$  with respect to  $F$ , then

$$c(G) \supset H .$$

Now let us consider a new class of functions.

Definition 1.15. A function  $k$  is an element of the class  $K$  iff  $k = ag + bh$ , where  $g \in G$ ,  $h \in H$  and  $a, b$  are any two real numbers.

Clearly  $F \supset K$ . For,

$$|\Delta^2 k| = |\Delta^2(ag + bh)| = |\Delta^2(ag)| = |a| |\Delta^2 g| .$$

Since  $|\Delta^2 g|$  is arbitrarily small, then so is  $|a| |\Delta^2 g|$  and therefore  $k \in F$ .

It is equally obvious that

$$K \supset G \quad \text{and}$$

$$K \supset H$$

and that there exists at least one element  $k' = g' + h'$  such that  $k' \notin G$  and  $k' \notin H$ . Hence,

$$K \neq G \cup H ,$$

although

$$K \supset G \cup H .$$

The author has as yet not been able to prove or disprove that

$$K = F .$$

Figure 1 illustrates the relationships between the various classes.

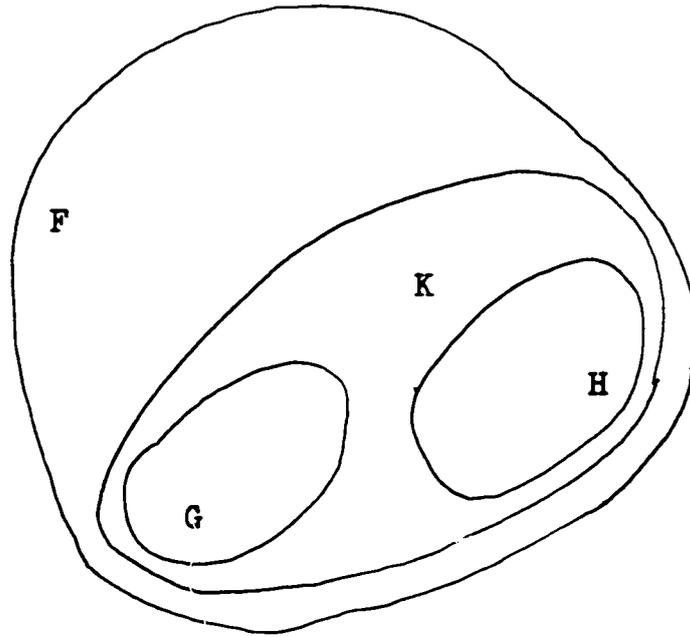


Figure 1. Relationship between classes

## V. EXTENSIONS OF THE PREVIOUS THEOREMS

In this section we shall consider functions whose second differences approach zero uniformly not on the entire interval  $I$ , but some subset  $A$  of  $I$ . We shall study the subsets  $A$  of  $I$  which allow us to prove the same theorems as those in the previous sections. We shall, in fact, state and prove extensions of the Theorems 1.1, 1.2, 1.3 and 1.5 for the case when  $A$  is any subset of  $I$  such that  $m(A) = m(I)$ . In other words, if the second differences of the function  $f(x)$  approach zero uniformly almost everywhere on  $I$ , then we can extend all the theorems which we have proved until now.

At the end of this section we shall construct an example to show that Theorems 1.1, and 1.3 need not hold if  $m(A) < m(I)$ .

Let us now state the Theorem 1.2E (E is used to denote the extension of Theorem 1.2).

Theorem 1.2E. Let the function  $f(x)$  be defined at all points in a finite interval  $I$ . If the second differences of  $f(x)$  approach zero uniformly on an everywhere dense set  $D$  in  $I$  and if  $f(x)$  is bounded at one point  $x_0 \in I$ , then  $f(x)$  is bounded throughout the interval  $I$ .

Proof: Since the proof follows almost exactly the same line of reasoning as the proof of Theorem 1.2 we shall not include all the details.

By hypothesis, set  $D$  is everywhere dense in  $I$ . Hence,

for every point  $x \in I$  there exists a point  $y \in D$  which is arbitrarily close to  $x$ .

In the proof of Theorem 1.2 we took the interval about  $x_0$ , namely  $[x_0 - \delta, x_0 + \delta]$  in which  $f(x)$  was bounded and the length  $2\delta$  of which was such that for all  $0 < h < \delta$

$$|\Delta^2 f(x, h)| < \epsilon .$$

Then we constructed a set of pivotal points  $x_0 \pm \delta$ ,  $x_0 \pm 3\delta$ ,  $\dots$ ,  $x_0 \pm (2n - 1)\delta$ , and expanded the original interval, step by step, showing successively that the function was bounded in each of the expanded intervals. We finally covered the entire finite interval by these extensions.

Instead of using  $x_0 \pm \delta$  for the pivotal points we may choose a set of points arbitrarily close to  $x_0 \pm \delta$ ,  $x_0 \pm 3\delta$ ,  $\dots$ ,  $x_0 \pm (2n - 1)\delta$ .

For example, the first pivotal point may be chosen to be

$$x_0 \pm \delta \left(1 - \frac{\epsilon'}{2n}\right),$$

where  $\epsilon' > 0$  and arbitrarily small, and where  $n$  refers to the number of intervals necessary to cover  $I$ . It is clear then that we can use exactly the same process as we did in Theorem 1.2 to show boundedness of  $f(x)$  in  $I$ .

It is convenient at this time to prove a lemma which is similar to Theorem 1.3 and which uses essentially the same method of proof. In the proof use is made of the following

result in measure theory<sup>6</sup>.

Let  $T$  be the one to one transformation of the entire real line onto itself, defined by  $T(x) = \alpha x$ ,  $\alpha \neq 0$ . If, for every Lebesgue measurable subset  $E$ ,  $T(E)$  denotes the set of all points of the form  $T(x)$  with  $x$  in  $E$ , then

$$m [T(E)] = |\alpha| m(E) .$$

Lemma 1.3E: Let a function  $f(x)$  be defined at every point of  $B$ . Let  $B$  be any subset of  $I$ , such that  $m(B) = m(I)$ . If the second differences of  $f(x)$  approach zero uniformly on the set  $B$  and if  $f(x)$  is continuous on  $B$ , then the extension of  $f(x)$  is uniformly continuous on  $\bar{I}$ .

Proof: Since  $f(x)$  is continuous on  $B$ , then we must show that  $f(x)$  is extensible to a continuous function on the closure of  $I$ , namely  $\bar{I}$ . That is, we will show that if  $x \in B$   $\lim_{x \rightarrow a} f(x)$

always exists for  $a \in c(B)$ , where the complement of  $B$  is taken with respect to  $\bar{I}$ .

From Theorem 1.2E it follows that  $f(x)$  is bounded on  $B$ .

Let  $a$  be any point of  $c(B)$ . We shall show that

$$\lim_{x \rightarrow a} f(x)$$

exists.

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<sup>6</sup>Halmos, Paul R. Measure theory. New York. D. Van Nostrand Company, Inc. 1950. Pp. 64-65.

Since  $f(x)$  is bounded on  $B$

$$\lim_{x \rightarrow a} f(x)$$

cannot be infinite.

Assume that the above limit does not exist, then there must exist two null sequences  $\{h_1\}$  and  $\{\bar{h}_1\}$  (assume  $h_1 > 0$   $\bar{h}_1 > 0$  in case the point  $a$  is not the right hand endpoint of  $I$ ), such that

$$\lim_{h_1 \rightarrow 0} f(a + h_1) = b \quad \text{and}$$

$$\lim_{\bar{h}_1 \rightarrow 0} f(a + \bar{h}_1) = c \quad b \neq c.$$

There is no loss of generality if we assume  $c = 0$  and  $b > 0$ .

We now choose an  $\epsilon$   $0 < \epsilon < \frac{b}{4}$ . Since the second differences approach zero uniformly on  $B$ , then there exists a  $\delta > 0$  such that for all  $x \in B$  and  $0 < h < \delta$ ,  $x \pm h \in B$

$$|\Delta^2 f(x, h)| < \frac{\epsilon}{2}.$$

Since  $\{h_1\}$   $\{\bar{h}_1\}$  are both null sequences there exist two members respectively such that

$$h_K < \frac{\delta}{2^n} \quad \bar{h}_\ell < h_K$$

and

$$|f(a + h_K) - b| < \frac{9\epsilon}{10}$$

$$|f(a + \bar{h}_\ell)| < \epsilon$$

for any positive integer  $n$ .

Since, however,  $f(x)$  is continuous on  $B$ , for  $\frac{\epsilon}{10}$  there exists an interval  $I'$  length  $2\delta'$ , such that

$$|f(\bar{x}) - f(\bar{x} + h)| < \frac{\epsilon}{10}$$

for all  $0 < h < \delta'$  where  $\bar{x} + h \in B$  and where  $\bar{x} = a + h_k$ . Denoting  $a + \bar{h}_\ell = \bar{x}_0$ , we may write

$$-\frac{\epsilon}{2} < f(\bar{x}_0) - 2f(y) + f(2y - x_0) \quad y \in B \cap I'$$

or

$$2f(y) - f(\bar{x}_0) - \frac{\epsilon}{2} < f(2y - x_0)$$

which reduces to

$$2(b - \epsilon) - \epsilon - \frac{\epsilon}{2} < f(2y - x_0)$$

and since  $b > 4\epsilon$

$$\frac{9}{2}\epsilon < f(2y - x_0) \quad \text{for all } y \in B \cap I'.$$

Since  $m(B \cap I') > 0$  it follows from the result stated immediately preceding this lemma that

$$m(B \cap I^2) > 0 ,$$

where  $I^2 = (\bar{x}, \bar{x} + 4\delta)$ .

This would give the first rung of the ladder analogous to the one constructed in the proof of Theorem 1.3.

Continuing, we again consider the point  $\bar{x}_0$  and  $y \in I^2 \cap B$ .

We obtain

$$-\frac{\epsilon}{2} < f(\bar{x}_0) - 2f(y) + f(2y - \bar{x}_0)$$

from which the following inequality is obtained

$$\frac{14}{2}\epsilon < f(2y - \bar{x}_0) \quad y \in I^2 \cap B.$$

Continuing in this fashion as we did in the proof of Theorem 1.3, we can show that  $f(x)$  is unbounded on the set  $B$ . This is the contradiction which establishes the lemma. We are now ready to state and prove the main result.

Theorem 1.5E. Let  $f(x)$  be defined at every point in some interval  $I$ . Let the second differences of  $f(x)$  approach zero uniformly almost everywhere in  $I$ . If  $f(x)$  is discontinuous at one point in  $I$ , then  $f(x)$  is nonmeasurable in the Lebesgue sense.

Proof: First of all we shall show that  $f(x)$  cannot be discontinuous at only one point. In fact, we shall show that if  $f(x)$  is discontinuous at all, it must be discontinuous either on a set of positive measure or on a nonmeasurable set.

Let  $H$  be the set of all discontinuities of  $f(x)$ . Suppose that

$$m(H) = 0 .$$

Let  $B$  be the set on which the second differences of  $f(x)$  approach zero uniformly. By hypothesis,

$$m(B) = m(I).$$

Consider the set

$$C = c(H) \cap B.$$

On  $C$   $f(x)$  is continuous and the second differences approach zero uniformly on this set.

$$m [c(C)] = m [H \cup c(B)] \leq m(H) + m [c(B)] = 0.$$

Therefore,

$$m(C) = m(I).$$

From Lemma 1.3E  $f(x)$  is uniformly continuous on  $C$ .

For convenience we denote the set of all  $y$  such that

$$|(y - x)| < \delta$$

by  $I(x, \delta)$ .

Since  $H$  is not a null set, there exists an element  $x \in H$ , an  $\epsilon > 0$ , and for every  $\delta > 0$  an element  $y \in I(x, \delta)$  such that

$$(E.1) \quad |f(y) - f(x)| > \frac{11}{10} \epsilon.$$

We define the set  $C(x, \delta)$  in the following way:

$y \in C(x, \delta)$  iff

$$y \in C \cap I(x, \delta) \text{ and}$$

$$|f(y) - f(x)| > \frac{11}{10} \epsilon.$$

Now suppose that there exists an  $\epsilon > 0$  such that for

every  $\delta > 0$  the set  $C(x, \delta)$  is now null. Since  $f(x)$  is uniformly continuous on  $C$ , there exists for  $\frac{\epsilon}{10}$  a  $\delta' > 0$  such that

$$(E.2) \quad |f(z_1) - f(z_2)| < \frac{\epsilon}{10}$$

for all  $z_1$  and  $z_2$  belonging to  $C$  and for which  $|z_1 - z_2| < \delta'$

Choose  $\delta$  in Equation E.2 such that  $\delta < 2\delta'$ . Then for all  $z \in C \cap I(x, \delta)$

$$|f(x) - f(z)| > \epsilon.$$

The set  $C \cap I(x, \delta)$  was obtained on the assumption that there exists an  $\epsilon > 0$  such that for every  $\delta > 0$  the set  $C(x, \delta)$  was not empty. Suppose that  $C(x, \delta)$  is empty for every  $\epsilon > 0$  and some  $\delta > 0$ . Then for  $\frac{\epsilon}{10}$  there exists a  $\delta' > 0$  such that whenever  $z \in C \cap I(x, \delta')$

$$|f(x) - f(z)| < \frac{\epsilon}{10}.$$

From Equation E.1 it now follows by choosing  $\delta < 2\delta'$  that for  $y$  instead of  $x$

$$|f(y) - f(z)| > \epsilon$$

for all  $z \in C \cap I(y, \delta)$ .

Since  $f(x)$  is uniformly continuous, then either  $f(y) - f(z) > \epsilon$  or  $f(z) - f(y) > \epsilon$ . In either case we have separated the functional values on  $C$  from the functional value at  $x$  or  $y$  belonging to  $I$ . Assume that  $f(x) = 0$  and

$$(E.3) \quad \frac{11}{10} \epsilon > f(z) > \epsilon \quad z \in C \cap I(x, \delta).$$

Since the second differences approach zero uniformly on  $C$ , then for  $\frac{\epsilon}{2}$  there exists a  $\delta' > 0$  such that

$$|f(z+h) - 2f(z) + f(z-h)| < \frac{\epsilon}{2}$$

for all  $z \in C$  and  $0 < h < \delta'$ . Let

$$\bar{\delta} = \min \{ \delta', \delta \}.$$

Then

$$-\frac{\epsilon}{2} < f(x) - 2f(z) + f(2z - y)$$

for all  $z \in C \cap I(x, \bar{\delta})$ . Using E.3 and  $f(x) = 0$  we get

$$\frac{3}{2} \epsilon < f(2z - y).$$

Since  $m[C \cap I(x, \bar{\delta})] = 2\bar{\delta}$ , it follows from the result stated immediately preceding Lemma 1.3E that the set of points at which the functional values are greater than  $\frac{3}{2} \epsilon$  is of measure  $4\bar{\delta}$ . This, however, is a contradiction, since  $f(x)$  is now discontinuous on a set of positive measure.

We have thus proved the first part of the theorem, i.e. the function cannot be discontinuous only on a set of measure zero.

Let us assume that  $f(x)$  is Lebesgue measurable. We define a set  $B(x, \delta, \epsilon)$ ,

$$B(x, \delta, \epsilon) \subset B \cap I(x, \delta), \text{ and such that}$$

$$(E.4) \quad |f(x) - f(y)| > 2\epsilon \quad \text{whenever } y \in B(x, \delta, \epsilon).$$

If  $f(x)$  is Lebesgue measurable then so is  $B(x, \delta, \epsilon)$ .

Since  $f(x)$  cannot be discontinuous only on a set of measure zero, then it follows that there exists an element  $x \in B$  and an  $\epsilon > 0$  such that for all  $\delta' > 0$

$$m [ B(x, \delta', \epsilon) ] > 0 .$$

If this were not true then  $f(x)$  would be discontinuous at most on a set of measure zero.

For  $\frac{\epsilon}{2}$  there exists a  $\delta$  such that

$$(E.5) \quad |\Delta^2 f(x, h)| < \frac{\epsilon}{2}$$

for all  $x \in B$  and  $0 < h < \delta$ .

Let us again assume that  $f(x) = 0$  then  $|f(y)| > 2\epsilon$  whenever  $y \in B(x, \delta, \epsilon)$ . Let  $\bar{B}(x, \delta, \epsilon)$  consist of all the values of  $y$  for which  $f(y) > \epsilon$  and  $\underline{B}(x, \delta, \epsilon)$  consist of  $y$  for which  $f(y) < -\epsilon$ . One of the sets  $\bar{B}(x, \delta, \epsilon)$  or  $\underline{B}(x, \delta, \epsilon)$  must be such that the measure of it is greater than zero, because the measure of  $B(x, \delta, \epsilon)$  is greater than zero. Suppose

$$m [ \bar{B}(x, \delta, \epsilon) ] > 0 .$$

Then it follows from E.5 that also

$$m \underline{B}(x, \delta, \epsilon) > 0 .$$

We have thus constructed the lowest rungs of the ladders both up and down, which are now easy to complete and thus show that  $f(x)$  is unbounded both from above and below on the set  $B$ .

Now we must show that  $f(x)$  is unbounded on  $B$  on every subinterval of  $I$ . Clearly  $f(x)$  cannot be continuous in any subinterval. For if it were then  $f(x)$  would be bounded in the entire interval  $I$  by Theorem 1.2E. We have already shown that this is not the case. We have also shown that  $f(x)$  could not be discontinuous on only a set of measure zero in any subinterval where there exists at least one discontinuity. The only case left is the case we just considered. Hence,  $f(x)$  is unbounded from both above and below in every subinterval of  $I$  on the set  $B$ . This, however, was all that was required in the proof of Theorem 1.5 to show that a function  $f(x)$  is not measurable.

We have thus extended the Theorems 1.1, 1.2, 1.3 and 1.5 to include a larger class of functions, namely, the class of functions whose second differences approach zero uniformly almost everywhere in the interval of definition. To complete the analysis, we shall show by an example that if the second differences approach zero uniformly on some subset, which, although everywhere dense, does not have a measure equal to the measure of the interval, then the conclusions of Theorem 1.1 and 1.3 would be false.

Let  $\{a_\alpha\}$  be the Hamel's basis for the set of real numbers. Assume that  $a_1 = 1$ . All the basis elements are linearly independent with respect to the rational coefficients  $p_\alpha$ . Every real number may be uniquely represented by a linear-rational combination of basis elements.

$$x = p_1 a_1 + \sum_{\alpha \in A} p_\alpha a_\alpha .$$

We define  $f(x)$  in the following way

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \frac{\pi}{2} \\ 0 & x = p_1 a_1 \quad \frac{\pi}{2} \leq x \leq \pi \\ \operatorname{sgn}(p_{\bar{\alpha}})(1 - \sin x) & x = \sum_{\alpha \in A} p_\alpha a_\alpha \quad \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

where  $\bar{\alpha} \neq 1$  is the first element of  $A$

and  $\operatorname{sgn}(p_{\bar{\alpha}})$  is the sign of  $p_{\bar{\alpha}}$ .

$f(x)$  is a function whose second differences approach zero uniformly on the set  $A$ .

$$A = (0, \frac{\pi}{2}) \cup \left\{ \text{all rationals } p \mid \frac{\pi}{2} < p < \pi \right\}$$

$$m(A) = \frac{\pi}{2}$$

$f(x)$  is discontinuous at every point in  $(\frac{\pi}{2}, \pi)$ , but  $f(x)$  is bounded in  $(0, \pi)$ , and continuous in  $(0, \frac{\pi}{2})$ .

## VI. SUMMARY

In this dissertation the functions whose second differences approach zero uniformly were characterized. In case the second differences approach zero uniformly on an entire interval  $I$ , it was found that such functions fall into two categories. If the functions are Lebesgue measurable, then they must be uniformly continuous in the interval of definition. If, however, the functions are not Lebesgue measurable then they must be unbounded at every point, and the functional values must form an everywhere dense set in the real continuum in every subinterval of the interval of definition.

The above results were extended to functions whose second differences approach zero almost everywhere in the interval of definition. Such functions were found to fall exactly into the same two categories as the functions whose second differences approach zero uniformly at all points of the interval.

Finally, an example was given of a discontinuous function whose second differences approach zero uniformly on an everywhere dense subset  $A$  of the interval  $I$ ,  $m(A) < m(I)$ . This example shows that all the results of this paper cannot be extended to everywhere dense sets of measure less than the measure of the interval.

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