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HARMONIC FUNCTIONS ASSOCIATED WITH THE COMPLEX ROTATION GROUP

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I. INTRODUCTION

A. Physical Motivation

The significance of symmetry in physics has long been recognized. The usual procedure in exploiting the symmetry possessed by a particular physical system is to find an invariance group of operations which preserve the symmetry possessed by the system. This leads to conservation laws such as those of energy, linear momentum, and angular momentum. The operations of the invariance group carry the dynamical system from one dynamically possible state of motion to another; thus the possible quantum states of the system are in exact correspondence with the irreducible representations of the group. The irreducible representations of the invariance group do not mix, and as a result the equations of motion for a state specified by a particular irreducible representation are mathematically uncoupled from any other irreducible representation. An example of this is partial wave analysis as used in scattering theory, whereby the scattering amplitude is expressed as a superposition of irreducible representations of the three-dimensional rotation group.

Although symmetry and invariance groups have been quite useful in physics in various ways, they have failed to uncover much of the dynamics of physical problems. In particular, for the case of scattering theory it has been the property of analyticity in scattering amplitudes which shows the relationships between scattering at different energies and which gives the connection between different energy channels. Analyticity may be considered either as a dynamical postulate or as a consequence of some deeper postulates. Hence symmetry alone gives no connection between different energies and therefore very little information about the dynamics of the problem. In order to correct this, symmetry must be enlarged to embrace some of the aspects described by dynamics, e.g. the potential, or analyticity postulates. This has led to the concept of "dynamical groups."

Symmetry and analyticity can be combined together in the theory of the analytic scattering matrix (S-matrix). Analytic continuation of the components of energy-momentum four-vectors is used in order to prove crossing symmetry. Clearly any S-matrix element depending on Lorentz invariant scalar products of four-vectors is invariant under complex as well as real Lorentz transformations. Thus it seems as if analysis under the complex Lorentz group should have the merits of symmetry analysis and at the same time include the very important dynamical principle of crossing symmetry.

The first step, rather than studying the complex Lorentz group, should be to study a simpler group as a model. Since the complex rotation group in three dimensions $O(3,C)$ is isomorphic to the real, restricted Lorentz group (Macfarlane 1962), this is a good choice. Furthermore, if the program works out, there should be an application to non-relativistic scattering theory. The S-matrix can be shown to be invariant to the group of complex rotations, and hence a partial wave analysis should be possible in terms of complex rotation group irreducible representations. The relationships to conventional partial waves and to the conventional analytic properties of the S-matrix would be of interest.

This thesis concentrates on developing the mathematical tools for the above-mentioned program. No applications are included in this work. The stage of development is discussed in the last chapter.

Because of the isomorphism of the two groups, the harmonic functions of the complex rotation group would also carry the irreducible representations of the real, restricted Lorentz group. Thus this connection serves as an additional motivation for working with the complex rotation group. A further understanding of the Lorentz group is a possible result of this work. The infinite-dimensional unitary representations of the Lorentz group have received considerable attention in past years. Dirac (1945), Gel'fand and Naĭmark (1946), and Harish-Chandra (1947) did the original work and classified all of the unitary irreducible representations. Representations on the space of complex valued functions were constructed by Gel'fand and Naĭmark (1947). Dolginov, et al. (1956, 1960) found unitary representations of the Lorentz group by analytic continuation of the unitary representations of the four-dimensional rotation group, and Ström (1965) derived matrix elements of unitary representations by a method of integrating the infinitesimal relations. Unitary representations of the Lorentz group on four-vector manifolds for the case of zero spin were obtained by Hussain (1965) and Zmuidzinas (1966). Recently Chang and O'Raifeartaigh (1968) and Greiman (1969) have derived unitary representations on a conformal group basis instead of the usual rotation group basis. In this thesis functions which carry the unitary representations of the Lorentz group are obtained by working directly with the complex rotation group.

In this thesis the term "harmonic functions" of a group is used in reference to the mathematical functions which carry the irreducible representations of that group. The term "harmonics" arose originally in the description of the modes of vibration of a string fastened at both ends.

In Fourier analysis the complete orthonormal set of functions for the interval of study became known as the harmonic functions. This term was further generalized with the introduction of the spherical harmonic functions as the carriers of the irreducible representations of the three-dimensional rotation group. It is with the latter meaning that "harmonic functions" is used in this thesis. The functions are used in the harmonic analysis associated with the complex rotation group. Harmonic analysis is the theory of expressing functions defined on a manifold in terms of the irreducible representations of a given group.

B. Work Done in this Thesis

Chapter 2 describes the complex rotation group with both infinitesimal and finite transformations being considered. The infinitesimal generators are written in terms of the components of the real and imaginary parts of a complex three-vector. Chapter 3 contains a treatment of the irreducible representations of the complex rotation group. The invariants of the group are derived, and the labeling of the basis states in a representation is developed.

In Chapter 4 a parametrization for the complex three-vector is chosen, and the generators of the group are constructed in terms of four parameters. The invariant volume element of the complex rotation group is found. The results are original.

Chapter 5 contains the R(4) analogy. The rotation group in four dimensions is studied in a manner exactly parallel to that used for the complex rotation group. The group structure is analyzed; the generators of the group are constructed in a chosen parametrization; and the eigenvalue equations are developed. Three different but equivalent forms of the R(4) solutions are found by use of the fact that R(4) is locally isomorphic to the direct product of two R(3) groups. Because of the close relationship of the two groups, the solutions to the R(4) eigenvalue equations are useful in solving similar equations for the complex rotation group. The results of Chapter 5 are original.

Three different sets of solutions to the eigenvalue equations of the complex rotation group are found in Chapter 6. They are obtained via analytic continuation of the R(4) solutions. A method of solution exactly parallel to that for the R(4) equations could not be rigorously proven because of the presence of rotation matrices with complex angular momentum but is presented in the Appendix as a matter of interest. Chapter 6 and the Appendix contains original work.

In Chapter 7 a study is made of the coupling coefficient contained in the solutions to the complex rotation group eigenvalue equations. The coefficient is the complex generalization of the R(3) Clebsch-Gordan coefficient as introduced by Andrews and Gunson (1964). Recurrence relationships satisfied by the coupling coefficient are derived and are identical with the R(3) case; the results of this work are original.

The properties of the solutions to the eigenvalue equations of the complex rotation group are examined closely in Chapter 8. The solutions are seen to contain the associated Legendre function of degree $-\frac{1}{2} \pm i\nu$, known as the associated conical function. For the associated conical functions an orthogonality property is proven, and an expansion theorem is established by use of the Titchmarsh eigenfunction expansion theory (1962). The results of the expansion theorem agree with the generalized Mehler formulas as listed in Magnus, et al. (1966). A non-conventional orthogonality property is derived for the complex rotation group harmonic functions. It is shown that certain expansions can be made in terms of the harmonic functions. The results of Chapter 8 are original.

In Chapter 9 the conclusions of this thesis are listed and discussed. Possible future work is suggested.

II. THE COMPLEX ROTATION GROUP

The complex rotation group O(3,C) is the group of complex transformations with typical group element r which carry the complex three-vector $\vec{z} = \vec{x} + i\vec{y}$, where $\vec{x}$ and $\vec{y}$ are real three-vectors, into the vector $\vec{z}'$ whose components are given by

$$z_i' = r_{ij} z_j \qquad (i, j = 1, 2, 3) \tag{2.1}$$

and whereby the norm $\vec{z}\cdot\vec{z} = z_i z_i$ is preserved. In other words, complex rotations are simply complex orthogonal transformations in a complex space of three dimensions. The vectors which span the three-dimensional complex space can be thought of as being formed from the space and time components of an antisymmetric tensor field (Kurşunoğlu 1961). The complex rotation group has six essential parameters.

In matrix notation Eq. 2.1 can be written as

$$\mathcal{Z}' = R\mathcal{Z} \quad , \tag{2.2}$$

where $\mathcal{Z}'$ and $\mathcal{Z}$ are complex three-component column matrices and R is the 3×3 complex rotation matrix with elements r_{ij}. The preservation of the norm

$$\mathcal{Z}'^{T}\mathcal{Z}' = \mathcal{Z}^{T}\mathcal{Z} \quad , \tag{2.3}$$

where $\mathcal{Z}^{T}$ represents the row matrix which is the transpose of the column matrix $\mathcal{Z}$, places the following condition on the transformation matrices:

$$R^{T}R = I_3 \quad , \tag{2.4}$$

where I_3 is the 3×3 unit matrix. This is equivalent to the condition that the determinant of R, $\det R$, equals ± 1. Transformation matrices R with $\det R = +1\ (-1)$ describe proper (improper) complex rotations. Only proper complex rotations will be considered in this work.

A. Infinitesimal Generators

The complex rotation matrices form a group, with the group operation being matrix multiplication. The elements of the infinitesimal complex rotation matrix R can be written as

$$r_{ij} = \delta_{ij} + i\epsilon_{ij} \quad , \tag{2.5}$$

where the δ_{ij} are the elements of the unit matrix I_3 and the ϵ_{ij} are complex infinitesimals. From the conditions imposed on the r_{ij} by Eq. 2.4, it follows that

$$\epsilon_{ij} = -\epsilon_{ji} \tag{2.6}$$

or, in other words, the infinitesimal transformation matrix $\mathcal{E}$ with elements ϵ_{ij} is antisymmetric. Thus with the matrix R in the form

$$R = I_3 + i\mathcal{E} \tag{2.7}$$

the matrix $i\mathcal{E}$ can be expressed as

$$i\mathcal{E} = \begin{bmatrix} 0 & -(\alpha_3+i\beta_3) & \alpha_2+i\beta_2 \\ \alpha_3+i\beta_3 & 0 & -(\alpha_1+i\beta_1) \\ -(\alpha_2+i\beta_2) & \alpha_1+i\beta_1 & 0 \end{bmatrix} = i\left(\alpha_j J_j + i\beta_j J_j\right) \quad , \tag{2.8}$$

where α_j and β_j are real infinitesimals and the J_j are defined by the matrices

$$J_j = i\begin{bmatrix} 0 & \delta_{j3} & -\delta_{j2} \\ -\delta_{j3} & 0 & \delta_{j1} \\ \delta_{j2} & -\delta_{j1} & 0 \end{bmatrix} \quad . \tag{2.9}$$

The six generators of complex infinitesimal rotations in the complex space are J_j and iJ_j. The former generates real rotations (conventional rotations), and the latter generates pure imaginary rotations (velocity transformations).

The infinitesimal generators can be found in operator form by considering the transformation properties of a scalar function

$f(z_1, z_2, z_3) \equiv f(x_1, y_1, x_2, y_2, x_3, y_3)$ defined in the complex three-dimensional space. The treatment is similar to that of Rose (1957) for real rotations. The complex rotation R carries the function $f(z_1, z_2, z_3)$ into the original function of the rotated coordinates:

$$f(z_1', z_2', z_3') = R\, f(z_1, z_2, z_3) \quad . \tag{2.10}$$

For the special case that R is an infinitesimal real rotation of $d\Psi$ about the three-axis, one can use Eqs. 2.7 and 2.8 to write the transformation equations:

$$\begin{aligned} z_1' &= z_1 - z_2\, d\Psi \quad , \\ z_2' &= z_2 + z_1\, d\Psi \quad , \\ z_3' &= z_3 \quad . \end{aligned} \tag{2.11}$$

Separating the real and imaginary parts, one obtains

$$\begin{aligned} x_1' &= x_1 - x_2\, d\Psi \quad , & y_1' &= y_1 - y_2\, d\Psi \quad , \\ x_2' &= x_2 + x_1\, d\Psi \quad , \quad \text{and} & y_2' &= y_2 + y_1\, d\Psi \quad , \\ x_3' &= x_3 \quad , & y_3' &= y_3 \quad . \end{aligned} \tag{2.12}$$

The left side of Eq. 2.10 can be expanded to first order as

$$f(z_1', z_2', z_3') = \left[1 + d\Psi\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_1}\right)\right] f(z_1, z_2, z_3) \quad . \tag{2.13}$$

The right side of Eq. 2.10 can be expressed in terms of the infinitesimal generator J_3 :

$$R_3(d\Psi)\, f(z_1, z_2, z_3) = \left[1 + i\, d\Psi\, J_3\right] f(z_1, z_2, z_3) \quad . \tag{2.14}$$

Combination of the two previous equations gives

$$J_3 = -i\left[(\vec{x} \times \vec{\nabla}_x)_3 + (\vec{y} \times \vec{\nabla}_y)_3\right] \quad , \tag{2.15}$$

which is the expected form since the real and imaginary parts of the complex three-vector transform separately under a real rotation. In a similar manner the generators J_1 and J_2 of infinitesimal real rotations about the 1-axis and 2-axis are found. The exact forms of J_1 and J_2 are seen by making a cyclic permutation of 1, 2, 3 in the expression for J_3. Thus the operator $\vec{J}$ is written in the vector form:

$$\vec{J} = -i\left[(\vec{x} \times \vec{\nabla}_x) + (\vec{y} \times \vec{\nabla}_y)\right] . \tag{2.16}$$

Now for the special case of an infinitesimal pure imaginary rotation of $i\,d\nu$ about the 3-axis, one finds the coordinate transformation equations to be

$$\begin{aligned} x_1' &= x_1 + y_2\,d\nu , & \qquad y_1' &= y_1 - x_2\,d\nu , \\ x_2' &= x_2 - y_1\,d\nu , \quad \text{and} & y_2' &= y_2 + x_1\,d\nu , \\ x_3' &= x_3 , & y_3' &= y_3 . \end{aligned} \tag{2.17}$$

In this case the left side of Eq. 2.10 can be expanded to first order as

$$f(z_1', z_2', z_3') = \left[1 + d\nu\left(y_2 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial y_2}\right)\right] f(z_1, z_2, z_3) . \tag{2.18}$$

The right side of Eq. 2.10 can be written in terms of the infinitesimal generator K_3 (iJ_3 in previous matrix formulation):

$$R_3(i\,d\nu)\, f(z_1, z_2, z_3) = \left[1 + i\,(i\,d\nu)(i K_3)\right] f(z_1, z_2, z_3) . \tag{2.19}$$

Comparison of the two previous equations yields

$$K_3 = -i\left[y_1 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_2} + x_2 \frac{\partial}{\partial y_1}\right] . \tag{2.20}$$

This form is expected since a pure imaginary rotation mixes the real and imaginary parts of the complex three-vector. Similarly derived are the generators K_1 and K_2 of infinitesimal pure imaginary rotations about the 1-axis and 2-axis respectively. Their forms are obtained by making a cyclic permutation of 1, 2, 3 in Eq. 2.20. In vector form the operator $\vec{K}$ is expressed as

$$\vec{K} = -i\left[(\vec{y} \times \vec{\nabla}_x) - (\vec{x} \times \vec{\nabla}_y)\right] . \tag{2.21}$$

The commutation relations of the six generators $\vec{J}$ and $\vec{K}$ are found to be

$$
\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k \quad , \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k \quad , \\
[J_i, K_j] &= i\epsilon_{ijk} K_k \quad ,
\end{aligned}
\tag{2.22}
$$

where ϵ_{ijk} is the Lévi-Civita symbol defined by

$$
\epsilon_{ijk} = \begin{cases} +1 \text{ if even permutation of } 1, 2, 3 \\ -1 \text{ if odd permutation of } 1, 2, 3 \\ 0 \text{ otherwise} \quad . \end{cases}
\tag{2.23}
$$

Thus it is seen that the generators of the complex rotation group O(3,C) obey the same commutation rules as do the six generators of the real, restricted Lorentz group. The two groups have the same number of independent parameters, and their Lie algebras have identical structures. These similarities suggest that the two groups might be isomorphic. This has in fact been proven by Macfarlane (1962) who obtained explicit formulas that realize the isomorphism of the complex rotation group and the Lorentz group. Earlier Kurşunoğlu (1961) had discussed a complex orthogonal representation of the Lorentz group.

B. Finite Transformations

A finite complex rotation can be built up from a succession of infinitesimal ones; this is done by exponentiation of an infinitesimal complex rotation. Typical complex rotations are denoted by

$$
\begin{aligned}
R_1 &= \exp[i(\phi + i\lambda) J_1] \quad , \\
R_2 &= \exp[i(\theta + i\mu) J_2] \quad , \\
R_3 &= \exp[i(\psi + i\nu) J_3] \quad ,
\end{aligned}
\tag{2.24}
$$

where the subscripts of R give the axis of rotation about which the complex rotation is to be made.

A complex rotation of magnitude $(\psi + i\nu)$ about the 3-axis is chosen to be an example. The transformation matrix has the form:

$$R_3(\psi+i\nu) = \begin{bmatrix} \cos(\psi+i\nu) & -\sin(\psi+i\nu) & 0 \\ \sin(\psi+i\nu) & \cos(\psi+i\nu) & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{2.25}$$

where

$$\begin{aligned} \cos(\psi+i\nu) &= \cos\psi \cosh\nu - i \sin\psi \sinh\nu , \\ \sin(\psi+i\nu) &= \sin\psi \cosh\nu + i \cos\psi \sinh\nu . \end{aligned} \tag{2.26}$$

From the matrix transformation equation

$$Z' = R_3 Z \tag{2.27}$$

one obtains the following equations:

$$\begin{aligned} z_1' &= (z_1 \cos\psi - z_2 \sin\psi)\cosh\nu - i\,(z_1 \sin\psi + z_2 \cos\psi)\sinh\nu , \\ z_2' &= (z_2 \cos\psi + z_1 \sin\psi)\cosh\nu - i\,(z_2 \sin\psi - z_1 \cos\psi)\sinh\nu , \\ z_3' &= z_3 . \end{aligned} \tag{2.28}$$

By equating the real and imaginary parts of the above equations, one gets transformation equations for each component of the complex three-vector. The complex rotation by $R_3(\psi + i\nu)$ corresponds to a real rotation of magnitude ψ about the 3-axis plus a uniform motion with velocity v (where $\tanh\nu = \frac{v}{c}$) along the 3-axis. This is made apparent by looking at the two special cases of first $\nu = 0$ and then $\psi = 0$.

For the special case of a real rotation, i.e. $\nu = 0$, one obtains the transformation equations

$$\begin{aligned} x_1' &= x_1 \cos\psi - x_2 \sin\psi , & \qquad y_1' &= y_1 \cos\psi - y_2 \sin\psi , \\ x_2' &= x_2 \cos\psi + x_1 \sin\psi , \quad \text{and} & y_2' &= y_2 \cos\psi + y_1 \sin\psi , \\ x_3' &= x_3 , & y_3' &= y_3 \end{aligned} \tag{2.29}$$

for a real rotation of magnitude Ψ about the 3-axis. The real and imaginary parts of the complex three-vector transform separately as real three-vectors.

For the special case of a pure imaginary rotation, i.e. $\Psi = 0$, one gets the transformation equations

$$\begin{aligned} x_1' &= x_1 \cosh\nu + y_2 \sinh\nu , & \qquad y_1' &= y_1 \cosh\nu - x_2 \sinh\nu , \\ x_2' &= x_2 \cosh\nu - y_1 \sinh\nu , & \text{and} \qquad y_2' &= y_2 \cosh\nu + x_1 \sinh\nu , \\ x_3' &= x_3 , & y_3' &= y_3 \end{aligned} \tag{2.30}$$

for a rotation of magnitude $i\nu$ about the 3-axis. The transformation mixes the real and imaginary parts of the complex three-vector. If the identification $\tanh\nu = \frac{v}{c}$ is made, the above equations correspond to a velocity transformation (pure Lorentz transformation) along the 3-axis for the components of an antisymmetric second rank tensor with the x_i and y_i being identified as the spatial and time parts of the tensor respectively. Probably the most familiar example of a transformation of this type is a Lorentz transformation of the electromagnetic field vectors written as the complex three-vector $\vec{E} + i\vec{H}$.

III. IRREDUCIBLE REPRESENTATIONS OF THE COMPLEX ROTATION GROUP

The complex rotation group has two invariant operators which commute with each of the six generators $\vec{J}$ and $\vec{K}$ of the group. These so-called Casimir operators are $J^2 - K^2$ and $\vec{J}\cdot\vec{K}$, and their eigenvalues are used to label the irreducible unitary representations of O(3,C). The solutions of the $J^2 - K^2$ and $\vec{J}\cdot\vec{K}$ eigenvalue equations will be called the harmonic functions of the complex rotation group, and these functions carry the irreducible representations of the group. Since the complex rotation group and the real, restricted Lorentz group have been shown to be isomorphic (Macfarlane 1962), the harmonic functions of O(3,C) also carry the irreducible representations of the Lorentz group. The labeling of the unitary representations of the homogeneous Lorentz group is well known (Harish-Chandra 1947). For unitary representations the two invariants of the Lorentz group have the values

$$J^2 - K^2 = \ell^2 - \nu^2 - 1 \quad , \qquad \vec{J}\cdot\vec{K} = \ell\nu \quad , \tag{3.1}$$

where ℓ is positive integral or half-integral and ν is real. For the case $\ell = 0$, then $\nu^2 > -1$. The irreducible representations of the complex rotation group will be characterized in the same manner.

Since the complex rotation group contains the group of real rotations in three dimensions, R(3), as a subgroup, the basis states of an O(3,C) representation may also be labeled by the irreducible representations $|jm\rangle$ of R(3). The diagonal operators of this R(3) basis are J^2 and J_3 :

$$\begin{aligned} J^2\,|jm\rangle &= j(j+1)\,|jm\rangle \quad , \\ J_3\,|jm\rangle &= m\,|jm\rangle \quad , \end{aligned} \tag{3.2}$$

where j is equal to positive integral or half-integral values or zero and m goes in integral steps from -j to +j. The two labels j and m plus the scalar values of the Casimir operators J^2-K^2 and $\vec{J}\cdot\vec{K}$ are sufficient to label completely the irreducible representations of O(3,C). The eigenvalues of J^2-K^2 and $\vec{J}\cdot\vec{K}$ will later be shown to equal $\ell^2-\nu^2-1$ and $\ell\nu$ respectively, where ℓ is the minimum value of j (hence zero or positive integer or half-integer) and ν is any real number. The O(3,C) harmonic functions are thus defined by the following four eigenvalue equations:

$$\left(J^2-K^2,\ \vec{J}\cdot\vec{K},\ J^2,\ J_3\right)|\ell\nu jm\rangle = \left(\ell^2-\nu^2-1,\ \ell\nu,\ j(j+1),\ m\right)|\ell\nu jm\rangle \quad (3.3)$$

By choice of a suitable coordinate system in the R(3) subspace, the matrix elements of the operators $\vec{J}$ and $\vec{K}$ are given by the following relations (suppressing the labels ℓ,ν in $|\ell\nu jm\rangle$):

$$\begin{aligned} J_3\,|jm\rangle &= m\,|jm\rangle\ , \\ J_\pm\,|jm\rangle &= \left[(j\mp m)(j\pm m+1)\right]^{\frac{1}{2}}\,|j\ m\pm1\rangle\ , \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} K_3\,|jm\rangle &= -i\left[(j+1)^2-m^2\right]^{\frac{1}{2}} A^{\ell\nu}_{j+1}\,|j+1\ m\rangle + \frac{m\,\ell\nu}{j(j+1)}\,|jm\rangle \\ &\quad + i\,(j^2-m^2)^{\frac{1}{2}}\,A^{\ell\nu}_{j}\,|j-1\ m\rangle\ , \\ K_\pm\,|jm\rangle &= \pm i\left[(j+1\pm m)(j+2\pm m)\right]^{\frac{1}{2}} A^{\ell\nu}_{j+1}\,|j+1\ m\pm1\rangle \\ &\quad + \frac{\ell\nu}{j(j+1)}\left[(j\mp m)(j+1\pm m)\right]^{\frac{1}{2}}\,|j\ m\pm1\rangle \\ &\quad \pm i\left[(j\mp m)(j-1\mp m)\right]^{\frac{1}{2}} A^{\ell\nu}_{j}\,|j-1\ m\pm1\rangle\ , \end{aligned} \quad (3.5)$$

where

$$
\begin{aligned}
J_{\pm} &\equiv J_1 \pm i J_2 \quad , \\
K_{\pm} &\equiv K_1 \pm i K_2 \quad , \\
A_j^{\ell\nu} &\equiv \frac{1}{j}\left[\frac{(j^2-\ell^2)(j^2+\nu^2)}{4j^2-1}\right]^{\frac{1}{2}} \quad .
\end{aligned} \tag{3.6}
$$

The preceding matrix elements of $\vec{J}$ will not be explicitly derived in this work. The derivation can be found in most of the standard text-books on angular momentum theory (see for example, Rose 1957).

Before finding the matrix elements of $\vec{K}$ and the eigenvalues of the Casimir operators, one must first derive the so-called reduced matrix elements of $\vec{K}$. The tensor properties of the operator $\vec{K}$ and the Wigner-Eckart Theorem will be used. The treatment will be that of Pursey (1965)*.

$\vec{K}$ is a vector operator and hence transforms like an irreducible first rank tensor, $K_\mu^{(1)}$ $(\mu = -1, 0, +1)$, under rotations:

$$
R K_\mu^{(1)} R^{-1} = \sum_{\nu} K_\nu^{(1)} D_{\nu\mu}^{(1)}(R) \quad , \tag{3.7}
$$

where $D_{\nu\mu}^{(1)}(R)$ is the R(3) rotation matrix of angular momentum unity. The tensor operator $K_\mu^{(1)}$ is defined in terms of the components of $\vec{K}$:

$$
\begin{aligned}
K_0^{(1)} &= i K_3 \quad , \\
K_{\pm 1}^{(1)} &= \frac{\mp i K_1 + K_2}{(2)^{\frac{1}{2}}}
\end{aligned} \tag{3.8}
$$

and obeys the hermiticity property

$$
K_\mu^{(1)\dagger} = (-1)^{1-\mu} K_{-\mu}^{(1)} \quad . \tag{3.9}
$$

---

*Pursey, D. L., Ames, Iowa. Group theory classroom notes. Private communication. 1965.

The following irreducible tensor commutator is defined:

$$\left[K^{(1)}, K^{(1)}\right]^{(k)}_{\nu} \equiv \sum_{\lambda,\mu} (1\lambda 1\mu|11k\nu)\left[K^{(1)}_{\lambda}, K^{(1)}_{\mu}\right] \quad , \tag{3.10}$$

where $(1\lambda 1\mu|11k\nu)$ is a Clebsch-Gordan (C-G) coefficient. By interchanging λ and μ in the C-G coefficient and shifting the commutator on the right side of the above equation, one can show that

$$\left[K^{(1)}, K^{(1)}\right]^{(k)}_{\nu} = (-1)^{1+k}\left[K^{(1)}, K^{(1)}\right]^{(k)}_{\nu} \quad , \tag{3.11}$$

and hence

$$\left[K^{(1)}, K^{(1)}\right]^{(k)}_{\nu} = \begin{cases} 0 & \text{if } k \neq 1 \\ i\,2^{\frac{1}{2}} J^{(1)}_{\nu} & \text{if } k = 1 \end{cases} \quad , \tag{3.12}$$

where the components of the tensor operator $J^{(1)}_{\nu}$ are given by

$$\begin{aligned} J^{(1)}_{0} &= i J_3 \quad , \\ J^{(1)}_{\pm 1} &= \frac{\mp i J_1 + J_2}{2^{\frac{1}{2}}} \quad . \end{aligned} \tag{3.13}$$

Eq. 3.10 can be rewritten in terms of the irreducible tensor product $\left(K^{(1)}K^{(1)}\right)^{(1)}_{\nu}$:

$$\left[K^{(1)}, K^{(1)}\right]^{(1)}_{\nu} \equiv 2\left(K^{(1)}K^{(1)}\right)^{(1)}_{\nu} \tag{3.14}$$

Combining Eqs. 3.12 and 3.14, one gets

$$\left(K^{(1)}K^{(1)}\right)^{(1)}_{\nu} = \frac{i}{2^{\frac{1}{2}}} J^{(1)}_{\nu} \quad . \tag{3.15}$$

Taking the matrix elements of the preceding equation with respect to the states $\langle jm|$ and $|j'm'\rangle$ and using the Wigner-Eckart Theorem, one obtains

$$(j'm'1\nu|j'1jm)\langle j\|(K^{(1)}K^{(1)})^{(1)}\|j'\rangle = \frac{i}{2^{1/2}}(j'm'1\nu|j'1jm)\langle j\|J^{(1)}\|j'\rangle \quad , \tag{3.16}$$

where the $\langle j \| J^{(1)} \| j' \rangle$ denote reduced matrix elements. The C-G coefficient on each side of the equation can be divided out if its value is non-zero. C-G coefficients do not vanish on the condition that the three angular momenta (j', 1, and j in this case) obey the triangle rule; this yields the selection rule

$$j' = j,\ j \pm 1 \ . \tag{3.17}$$

By cancelling out the two C-G coefficients and taking the reduced matrix element $\langle j \| J^{(1)} \| j' \rangle$ to have the value $+i\,[j(j+1)\,(2j+1)]^{\frac{1}{2}}\ \delta_{jj'}$ (Fano and Racah 1959) in Eq. 3.16, one gets

$$\langle j \| (K^{(1)} K^{(1)})^{(1)} \| j' \rangle = -\delta_{jj'} \left[\frac{j(j+1)(2j+1)}{2} \right]^{\frac{1}{2}} . \tag{3.18}$$

The general formula for the reduced matrix element of an irreducible tensor product is given by Fano and Racah (1959):

$$\langle j \| (A^{(k_1)} B^{(k_2)})^{k} \| j' \rangle = (-1)^{j+j'+k}\, [2k+1]^{\frac{1}{2}} \sum_{j''} \overline{W} \begin{pmatrix} j & j' & k \\ k_2 & k_1 & j'' \end{pmatrix} \langle j \| A^{(k_1)} \| j'' \rangle \langle j'' \| B^{(k_2)} \| j' \rangle , \tag{3.19}$$

where $\overline{W}$ is the Wigner 6-j symbol. By use of the preceding equation, one gets Eq. 3.18 in the form:

$$(-1)^{j+j'+1}\, 3^{\frac{1}{2}} \sum_{j''} \overline{W} \begin{pmatrix} j & j' & 1 \\ 1 & 1 & j'' \end{pmatrix} \langle j \| K^{(1)} \| j'' \rangle \langle j'' \| K^{(1)} \| j' \rangle = -\delta_{jj'} \left[\frac{j(j+1)(2j+1)}{2} \right]^{\frac{1}{2}} . \tag{3.20}$$

According to the selection rule in Eq. 3.17, there are three cases to be considered: (1) $j' = j+1$, (2) $j' = j-1$, (3) $j' = j$. Since cases (1) and (2) give equivalent equations, only case (1) need be examined here. It reduces to the following equation:

$$\left[\frac{j(j+1)}{2j+1} \right]^{\frac{1}{2}} \langle j \| K^{(1)} \| j \rangle = \left[\frac{(j+1)(j+2)}{2j+3} \right]^{\frac{1}{2}} \langle j+1 \| K^{(1)} \| j+1 \rangle . \tag{3.21}$$

Obviously each side of the equation is independent of j so that one can set

$$\left[\frac{j(j+1)}{2j+1}\right]^{\frac{1}{2}} \langle j \| K^{(1)} \| j \rangle = iB \quad , \tag{3.22}$$

where B is a constant independent of j. Case (3) reduces to the equation

$$\frac{1}{j+1} \langle j+1 \| K^{(1)} \| j \rangle \langle j \| K^{(1)} \| j+1 \rangle - \frac{1}{j} \langle j \| K^{(1)} \| j-1 \rangle \langle j-1 \| K^{(1)} \| j \rangle$$

$$= \left[(j+1)^2 - \frac{B^2}{(j+1)^2}\right] - \left[j^2 - \frac{B^2}{j^2}\right] \quad , \tag{3.23}$$

which implies that one can identify

$$\frac{1}{j} \langle j \| K^{(1)} \| j-1 \rangle \langle j-1 \| K^{(1)} \| j \rangle = A + j^2 - \frac{B^2}{j^2} \quad , \tag{3.24}$$

where A is a constant independent of j. The angular momentum j is a positive integer or half-integer or zero. There must exist a minimum value of j, say ℓ , for which the ladder of reduced matrix elements terminates:

$$\langle \ell \| K^{(1)} \| \ell-1 \rangle = 0 \quad . \tag{3.25}$$

For this case Eq. 3.24 gives for A :

$$A = \frac{B^2}{\ell^2} - \ell^2 \quad . \tag{3.26}$$

It is convenient to write

$$B = \ell \nu \tag{3.27}$$

so that then A becomes

$$A = \nu^2 - \ell^2 \tag{3.28}$$

where ν is a real number and ℓ is the minimum value of j. Substitution into Eqs. 3.22 and 3.24 gives for the reduced matrix elements of $K^{(1)}$:

$$\begin{aligned} \langle j \| K^{(1)} \| j \rangle &= i \ell \nu \left[\frac{2j+1}{j(j+1)} \right] , \\ \left| \langle j \| K^{(1)} \| j-1 \rangle \right|^2 &= \frac{1}{j} (j^2 - \ell^2)(j^2 + \nu^2) . \end{aligned} \tag{3.29}$$

Now that the task of calculating the reduced matrix elements is completed, the matrix elements of $\vec{K}$ can be found. Previously selection rules for the j-values were established in Eq. 3.17. Selection rules for the m-values follow from the commutation relations

$$\begin{aligned} \left[J_3 , K_3 \right] &= 0 , \\ \left[J_3 , K_\pm \right] &= \pm K_\pm . \end{aligned} \tag{3.30}$$

The preceding relations imply that

$$\begin{aligned} K_3 |jm\rangle &= \alpha |jm\rangle , \\ K_\pm |jm\rangle &= \alpha_\pm |j\, m\pm 1\rangle , \end{aligned} \tag{3.31}$$

where $\alpha_\pm$ and α are coefficients to be determined. Thus matrix elements of K_+, K_-, and K_3 with reference to $\langle j'm'|$ and $|jm\rangle$ must have $m' = m+1$, $m-1$, m respectively in addition to $j' = j$, $j\pm1$ in order to be non-zero. The relationships of $K_\pm$ and K_3 to the components of the tensor operator $K^{(1)}$ are

$$K_{\pm} = \pm i\, 2^{1/2} K^{(1)}_{\pm 1} \quad , \qquad K_3 = -i K^{(1)}_0 \quad . \tag{3.32}$$

By using the preceding equations, the Wigner-Eckart Theorem, a set of tables of C-G coefficients, the reduced matrix elements of $K^{(1)}$ from Eq. 3.29, and the selection rules for j' and m', one can easily calculate the non-zero matrix elements of $K_{\pm}$ and K_3 with reference to the states $\langle j'm'|$ and $|jm\rangle$. The results are seen to coincide exactly with the relations given earlier in Eq. 3.5.

Next, the eigenvalues of the Casimir operators J^2-K^2 and $\vec{J}\cdot\vec{K}$ in an O(3,C) irreducible representation can be found in terms of ℓ and ν . The operators are written in the form

$$\begin{aligned} J^2-K^2 &= J^2 - \left(K_+K_- + K_3^2 + J_3\right) \quad , \\ \vec{J}\cdot\vec{K} &= \tfrac{1}{2}\left(J_+K_- + J_-K_+\right) + J_3K_3 \quad . \end{aligned} \tag{3.33}$$

Taking the expectation values of the operators with reference to the states $\langle jm|$ and $|jm\rangle$, one obtains

$$\begin{aligned} \langle J^2-K^2\rangle &= \langle jm|J^2-J_3|jm\rangle - \sum_{j''} \langle jm|K_+|j''\,m-1\rangle\langle j''\,m-1|K_-|jm\rangle \\ &\quad - \sum_{j''} \langle jm|K_3|j''m\rangle\langle j''m|K_3|jm\rangle \quad , \end{aligned} \tag{3.34}$$

$$\begin{aligned} \langle \vec{J}\cdot\vec{K}\rangle &= \tfrac{1}{2}\langle jm|J_+|j\,m-1\rangle\langle j\,m-1|K_-|jm\rangle + \tfrac{1}{2}\langle jm|J_-|j\,m+1\rangle\langle j\,m+1|K_+|jm\rangle \\ &\quad + \langle jm|J_3|jm\rangle\langle jm|K_3|jm\rangle \quad . \end{aligned}$$

By use of the matrix elements of $\vec{J}$ and $\vec{K}$ as given in Eqs. 3.4 and 3.5, one finally gets after some work:

$$\langle J^2 - K^2 \rangle = \ell^2 - \nu^2 - 1 \quad ,$$
$$\langle \vec{J}\cdot\vec{K} \rangle = \ell\nu \qquad (3.35)$$

for the eigenvalues of the two Casimir operators. The label ℓ is the minimum value of the angular momentum j and hence is positive integral or half-integral or zero whereas ν is any real number.

There is no upper limit for j in a unitary irreducible representation; so that means unitary irreducible representations of the complex rotation group are infinite-dimensional (except for the one-dimensional trivial representation). The four numbers ℓ , ν , j, m are used to characterize the O(3,C) harmonic functions which carry the unitary irreducible representations of the group. These four numbers give complete and unique labeling for the O(3,C) irreducible representations.

IV. DEVELOPMENT OF THE GENERATORS IN A CHOSEN PARAMETRIZATION

In Chapter 2 the infinitesimal generators $\vec{J}$ and $\vec{K}$ of the complex rotation group are expressed in terms of the components of the real and imaginary parts of the complex three-vector. It is the purpose of this chapter to choose a parametrization for the complex three-vector and then to develop the generators in terms of the chosen parametrization.

A. Choice of Complex Three-Vector of Unit Magnitude

In Eqs. 2.16 and 2.21 the generators $\vec{J}$ and $\vec{K}$ are written as functions of the two three-vectors $\vec{x}$ and $\vec{y}$, where the complex three-vector $\vec{z} = \vec{x} + i\vec{y}$, and derivatives with respect to $\vec{x}$ and $\vec{y}$. Of the six coordinates of $\vec{x}$ and $\vec{y}$, only four need be regarded as independent since the requirement that the norm $\vec{z}\cdot\vec{z}$ be preserved yields the two relationships

$$\begin{aligned} x^2 - y^2 &= a^2 - b^2 \;, \\ \vec{x}\cdot\vec{y} &= ab \;, \end{aligned} \tag{4.1}$$

where a and b are real constants such that both are not zero simultaneously.

In this work the choice of a complex three-vector $\hat{z}$ of unit magnitude is made:

$$\hat{z} = \frac{\vec{z}}{(\vec{z}\cdot\vec{z})^{\frac{1}{2}}} \equiv \vec{\lambda} + i\vec{\mu} \;, \tag{4.2}$$

where $\vec{\lambda}$ and $\vec{\mu}$ are three-vectors such that

$$\begin{aligned} \lambda^2 - \mu^2 &= 1 \;, \\ \vec{\lambda}\cdot\vec{\mu} &= 0 \;. \end{aligned} \tag{4.3}$$

Any other complex three-vector $\vec{z} = \vec{x} + i\vec{y}$ with invariants given by Eq. 4.1 can be expressed in terms of $\hat{z}$:

$$\vec{z} = (\vec{x} + i\vec{y}) = (a+ib)\hat{z} = (a+ib)(\vec{\lambda} + i\vec{\mu}) \quad , \tag{4.4}$$

whereupon the identification

$$\begin{aligned} \vec{x} &= a\vec{\lambda} - b\vec{\mu} \quad , \\ \vec{y} &= b\vec{\lambda} + a\vec{\mu} \end{aligned} \tag{4.5}$$

is made. Hence the general case of a complex three-vector as represented by $\vec{z} = \vec{x} + i\vec{y}$ can always be reduced to the special case of the complex three-vector $\hat{z} = \vec{\lambda} + i\vec{\mu}$ with $\vec{\lambda}$ and $\vec{\mu}$ defined by Eq. 4.3. All work on the complex rotation group in this thesis will be done in terms of the complex three-vector of unit magnitude.

B. Coordinate Axes in the Complex Three-Space

The complex three-dimensional space is spanned by three mutually perpendicular coordinate axes with the unit vector along each axis being real. The components of the complex three-vector $\hat{z}$ are given by

$$\hat{z} = z_1\hat{i}_1 + z_2\hat{i}_2 + z_3\hat{i}_3 \quad , \tag{4.6}$$

where z_1, z_2, z_3 are complex numbers and $\hat{i}_1$, $\hat{i}_2$, $\hat{i}_3$ are real unit vectors. Then by definition the complex conjugate of the complex three-vector is

$$\hat{z}^* \equiv z_1^*\hat{i}_1 + z_2^*\hat{i}_2 + z_3^*\hat{i}_3 \quad . \tag{4.7}$$

The real part of the complex three-vector $\hat{z}$ is thus defined by

$$\mathrm{Re}\,\hat{z} \equiv \tfrac{1}{2}(\hat{z}^* + \hat{z}) \quad , \tag{4.8}$$

and this is what is meant by the three-vector $\vec{\lambda}$ which constitutes the real part of $\hat{z}$. Likewise the imaginary part of the complex three-vector is defined by

$$\mathrm{Im}\,\hat{z} = \frac{1}{2i}\left(\hat{z} - \hat{z}^*\right) \quad , \qquad (4.9)$$

and this is the three-vector $\vec{\mu}$.

At this stage one adopts the convention that all rotations in the complex space are treated as "active," i.e. vectors are rotated whereas coordinate axes remain fixed. Thus the definitions of $\hat{z}^*$, $\mathrm{Re}\,\hat{z}$, and $\mathrm{Im}\,\hat{z}$ are invariant. The vector values of $\hat{z}^*$, $\mathrm{Re}\,\hat{z}$, and $\mathrm{Im}\,\hat{z}$ get transformed by the active complex rotations.

C. Parametrization of the Complex Three-Vector

In the original orientation of the complex three-vector the six parameters are the coordinates of $\vec{\lambda}$ and $\vec{\mu}$, namely λ_1, λ_2 , λ_3 , μ_1 , μ_2 , μ_3 . Only four of these need be considered as independent because of the conditions imposed in Eq. 4.3. An active real rotation R of the complex three-vector is made so as to have $\vec{\lambda}$ lie along the 3-axis and $\vec{\mu}$ to lie in the first quadrant of the 3-1 plane. This is shown in Figure 4.1. The new variables chosen are the three Euler angles ψ, θ, ϕ

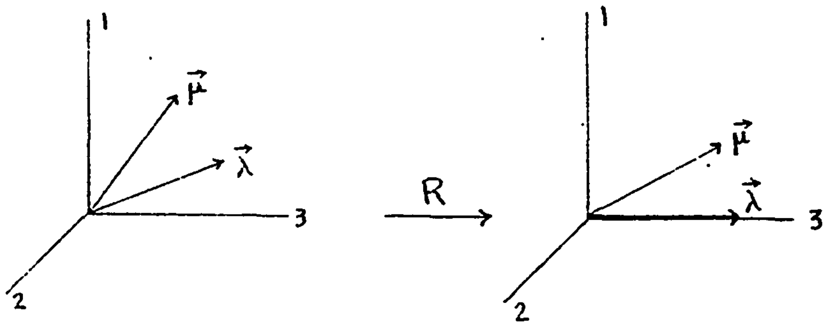

Figure 4.1. Active rotation of the complex three-vector $\hat{z} = \vec{\lambda} + i\,\vec{\mu}$

which define the active rotation, the magnitude of $\vec{\lambda}$ denoted by λ, and the quantities $\lambda^2 - \mu^2 \equiv c$ and $\vec{\lambda} \cdot \vec{\mu} \equiv d$. After the transformation is completed, the variables c and d can be set equal to 1 and 0 respectively so as to arrive at four independent parameters. Having d equal to zero is equivalent to having the three-vectors $\vec{\lambda}$ and $\vec{\mu}$ perpendicular to each other.

The active rotation which carries the complex three-vector $\hat{z}$ from its original orientation to its new orientation of $\vec{\lambda}$ along the 3-axis and $\vec{\mu}$ in the 3-1 plane is defined by the successive real rotations: (1) ϕ about the 3-axis, (2) θ about the 2-axis, and (3) ψ about the 3-axis. This definition of Euler angles is consistent with that used by Fano and Racah (1959). The matrix form of the rotation is given by

$$R = \begin{bmatrix} \cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi & \cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi & -\cos\psi\sin\theta \\ -\sin\psi\cos\theta\cos\phi - \cos\psi\sin\phi & -\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix} \tag{4.10}$$

The three-vectors $\vec{\lambda}$ and $\vec{\mu}$ represented by the column matrices Λ and M are transformed in the following way:

$$\Lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \longrightarrow R\Lambda \equiv \Lambda' = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix},$$

$$M = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \longrightarrow RM \equiv M' = \begin{pmatrix} f \\ 0 \\ d/\lambda \end{pmatrix}, \tag{4.11}$$

where $c \equiv \lambda^2 - \mu^2$, $d \equiv \vec{\lambda}\cdot\vec{\mu}$,

$$f \equiv \left[\mu^2 - \left(\frac{\vec{\lambda}\cdot\vec{\mu}}{\lambda}\right)^2\right]^{\frac{1}{2}} = \left[\lambda^2 - c - \frac{d^2}{\lambda^2}\right]^{\frac{1}{2}} .$$

The inverse matrix R^{-1} of R , which is simply the transpose R^T , is used to express the three-vectors $\vec{\lambda}$ and $\vec{\mu}$ in terms of the new parameters:

$$\Lambda = R^T \Lambda' ,$$

$$M = R^T M' . \tag{4.12}$$

In particular, this gives

$$\begin{aligned}
\vec{\lambda} &= \hat{\imath}_1 \lambda_1 + \hat{\imath}_2 \lambda_2 + \hat{\imath}_3 \lambda_3 \\
&= \hat{\imath}_1 \lambda \sin\theta \cos\phi + \hat{\imath}_2 \lambda \sin\theta \sin\phi + \hat{\imath}_3 \lambda \cos\theta , \\
\vec{\mu} &= \hat{\imath}_1 \mu_1 + \hat{\imath}_2 \mu_2 + \hat{\imath}_3 \mu_3 \\
&= \hat{\imath}_1 \left[f(\cos\psi \cos\theta \cos\phi - \sin\psi \sin\phi) + \frac{d}{\lambda}\sin\theta \cos\phi\right] \\
&\quad + \hat{\imath}_2 \left[f(\cos\psi \cos\theta \sin\phi + \sin\psi \cos\phi) + \frac{d}{\lambda}\sin\theta \sin\phi\right] \\
&\quad + \hat{\imath}_3 \left[- f \cos\psi \sin\theta + \frac{d}{\lambda}\cos\theta\right] .
\end{aligned} \tag{4.13}$$

D. Development of the Generators

By use of Eqs. 2.16, 2.21, and 4.5 the generators $\vec{J}$ and $\vec{K}$ are expressed in terms of the three-vectors $\vec{\lambda}$ and $\vec{\mu}$ and derivatives $\vec{\nabla}_\lambda$ and $\vec{\nabla}_\mu$. The vectors $\vec{\lambda}$ and $\vec{\mu}$ have already been found in terms of the new parameters ψ , θ , ϕ , λ , c, d in Eq. 4.13. Next, the derivatives $\vec{\nabla}_\lambda$ and $\vec{\nabla}_\mu$ must be written as functions of the new parameters and derivatives with respect to them before one can calculate the generators.

For $\vec{\nabla}_\lambda$ and $\vec{\nabla}_\mu$ one uses

$$\begin{aligned}
\vec{\nabla}_\lambda &= \hat{\imath}_1 \frac{\partial}{\partial\lambda_1} + \hat{\imath}_2 \frac{\partial}{\partial\lambda_2} + \hat{\imath}_3 \frac{\partial}{\partial\lambda_3} , \\
\vec{\nabla}_\mu &= \hat{\imath}_1 \frac{\partial}{\partial\mu_1} + \hat{\imath}_2 \frac{\partial}{\partial\mu_2} + \hat{\imath}_3 \frac{\partial}{\partial\mu_3} .
\end{aligned} \tag{4.14}$$

By combining the first of the above expressions with the values of λ_1, λ_2, and λ_3 as given in Eq. 4.13, one can write

$$
\begin{aligned}
\hat{\iota}_1 = \vec{\nabla}_\lambda(\lambda_1) &= \sin\theta\cos\phi\ (\vec{\nabla}_\lambda \lambda) + \lambda\cos\theta\cos\phi(\vec{\nabla}_\lambda\theta) - \lambda\sin\theta\sin\phi(\vec{\nabla}_\lambda\phi), \\
\hat{\iota}_2 = \vec{\nabla}_\lambda(\lambda_2) &= \sin\theta\sin\phi\ (\vec{\nabla}_\lambda \lambda) + \lambda\cos\theta\sin\phi(\vec{\nabla}_\lambda\theta) + \lambda\sin\theta\cos\phi(\vec{\nabla}_\lambda\phi), \\
\hat{\iota}_3 = \vec{\nabla}_\lambda(\lambda_3) &= \cos\theta\ (\vec{\nabla}_\lambda\lambda) - \lambda\sin\theta\ (\vec{\nabla}_\lambda\theta) \ .
\end{aligned}
\tag{4.15}
$$

The preceding three equations can be solved to yield

$$
\begin{aligned}
\vec{\nabla}_\lambda \lambda &= \hat{\iota}_1 \sin\theta\cos\phi + \hat{\iota}_2 \sin\theta\sin\phi + \hat{\iota}_3\cos\theta \ , \\
\vec{\nabla}_\lambda \theta &= \hat{\iota}_1 \frac{\cos\theta\cos\phi}{\lambda} + \hat{\iota}_2 \frac{\cos\theta\sin\phi}{\lambda} + \hat{\iota}_3\left(\frac{-\sin\theta}{\lambda}\right), \\
\vec{\nabla}_\lambda \phi &= \hat{\iota}_1\left(\frac{-\sin\phi}{\lambda\sin\theta}\right) + \hat{\iota}_2 \frac{\cos\phi}{\lambda\sin\theta} \ .
\end{aligned}
\tag{4.16}
$$

Similarly by combining the expression for $\vec{\nabla}_\lambda$ in Eq. 4.14 with the values of μ_1, μ_2, and μ_3 as given in Eq. 4.13, one can obtain three additional equations whose solutions give the following relations:

$$
\begin{aligned}
\vec{\nabla}_\lambda c &= 2\lambda\,(\hat{\iota}_1 \sin\theta\cos\phi + \hat{\iota}_2\sin\theta\sin\phi + \hat{\iota}_3\cos\theta) \ , \\
\vec{\nabla}_\lambda d &= \hat{\iota}_1 (f\cos\psi\cos\theta\cos\phi - f\sin\psi\sin\phi + \frac{d}{\lambda}\sin\theta\cos\phi) \\
&\quad + \hat{\iota}_2 (f\cos\psi\cos\theta\sin\phi + f\sin\psi\cos\phi + \frac{d}{\lambda}\sin\theta\sin\phi) \\
&\quad + \hat{\iota}_3 (-f\cos\psi\sin\theta + \frac{d}{\lambda}\cos\theta) \ , \\
\vec{\nabla}_\lambda \psi &= \hat{\iota}_1\left(\frac{d}{\lambda^2 f}\sin\psi\cos\theta\cos\phi + \frac{d}{\lambda^2 f}\cos\psi\sin\phi + \frac{\cot\theta\sin\phi}{\lambda}\right) \\
&\quad + \hat{\iota}_2\left(\frac{d}{\lambda^2 f}\sin\psi\cos\theta\sin\phi - \frac{d}{\lambda^2 f}\cos\psi\cos\phi - \frac{\cot\theta\cos\phi}{\lambda}\right) \\
&\quad + \hat{\iota}_3\left(-\frac{d}{\lambda^2 f}\sin\psi\sin\theta\right) \ .
\end{aligned}
\tag{4.17}
$$

Then one uses

$$
\begin{aligned}
\vec{\nabla}_\lambda &= (\vec{\nabla}_\lambda c)\frac{\partial}{\partial c} + (\vec{\nabla}_\lambda d)\frac{\partial}{\partial d} + (\vec{\nabla}_\lambda \lambda)\frac{\partial}{\partial \lambda} \\
&\quad + (\vec{\nabla}_\lambda \psi)\frac{\partial}{\partial \psi} + (\vec{\nabla}_\lambda \theta)\frac{\partial}{\partial \theta} + (\vec{\nabla}_\lambda \phi)\frac{\partial}{\partial \phi} \ ,
\end{aligned}
\tag{4.18}
$$

where the $(\vec{\nabla}_\lambda c)$, $(\vec{\nabla}_\lambda d)$, $(\vec{\nabla}_\lambda \lambda)$, $(\vec{\nabla}_\lambda \Psi)$, $(\vec{\nabla}_\lambda \theta)$, and $(\vec{\nabla}_\lambda \phi)$ are given by Eqs. 4.16 and 4.17, to express $\vec{\nabla}_\lambda$ in terms of the new parameters and derivatives with respect to each of them.

Using the same procedure on $\vec{\nabla}_\mu$ as above for $\vec{\nabla}_\lambda$, one obtains

$$\begin{aligned}\vec{\nabla}_\mu = {} & (\vec{\nabla}_\mu c)\frac{\partial}{\partial c} + (\vec{\nabla}_\mu d)\frac{\partial}{\partial d} + (\vec{\nabla}_\mu \lambda)\frac{\partial}{\partial \lambda} \\ & + (\vec{\nabla}_\mu \Psi)\frac{\partial}{\partial \Psi} + (\vec{\nabla}_\mu \theta)\frac{\partial}{\partial \theta} + (\vec{\nabla}_\mu \phi)\frac{\partial}{\partial \phi} \quad ,\end{aligned} \tag{4.19}$$

which can be expressed as

$$\begin{aligned}\vec{\nabla}_\mu = {} & \hat{\imath}_1\left(-2f\cos\Psi\cos\theta\cos\phi + 2f\sin\Psi\sin\phi - \frac{2d}{\lambda}\sin\theta\cos\phi\right)\frac{\partial}{\partial c} \\ & + \hat{\imath}_2\left(-2f\cos\Psi\cos\theta\sin\phi - 2f\sin\Psi\cos\phi - \frac{2d}{\lambda}\sin\theta\sin\phi\right)\frac{\partial}{\partial c} \\ & + \hat{\imath}_3\left(2f\cos\Psi\sin\theta - \frac{2d}{\lambda}\cos\theta\right)\frac{\partial}{\partial c} \\ & + \lambda\left(\hat{\imath}_1\sin\theta\cos\phi + \hat{\imath}_2\sin\theta\sin\phi + \hat{\imath}_3\cos\theta\right)\frac{\partial}{\partial d} \\ & - \hat{\imath}_1\frac{1}{f}\left(\sin\Psi\cos\theta\cos\phi + \cos\Psi\sin\phi\right)\frac{\partial}{\partial \Psi} \\ & - \hat{\imath}_2\frac{1}{f}\left(\sin\Psi\cos\theta\sin\phi - \cos\Psi\cos\phi\right)\frac{\partial}{\partial \Psi} + \hat{\imath}_3\frac{\sin\Psi\sin\theta}{f}\frac{\partial}{\partial \Psi} \quad .\end{aligned} \tag{4.20}$$

Now that $\vec{\lambda}$, $\vec{\mu}$, $\vec{\nabla}_\lambda$, and $\vec{\nabla}_\mu$ have all been found in terms of the new parameters and derivatives with respect to them the generators $\vec{J}$ and $\vec{K}$ can be calculated. By use of Eqs. 2.16 and 4.5 one writes

$$\vec{J} = -i\left[(\vec{\lambda}\times\vec{\nabla}_\lambda) + (\vec{\mu}\times\vec{\nabla}_\mu)\right] \quad , \tag{4.21}$$

which, by the use of Eqs. 4.13, 4.18, and 4.20, takes the form

$$\vec{J} = -i\left[\begin{aligned} & \hat{\imath}_1\left(\csc\theta\cos\phi\frac{\partial}{\partial\Psi} - \sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \\ & + \hat{\imath}_2\left(\csc\theta\sin\phi\frac{\partial}{\partial\Psi} + \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \\ & + \hat{\imath}_3\frac{\partial}{\partial\phi}\end{aligned}\right] \quad . \tag{4.22}$$

This is the expected expression for $\vec{J}$, the so-called angular momentum operator which generates real rotations. If one sets the parameters c and d equal to 1 and 0 respectively so as to reduce to four independent parameters, there is no effect on the generator $\vec{J}$, which remains as given above.

By use of Eqs. 2.21 and 4.5 one writes

$$\vec{K} = -i\left[(\vec{\mu}\times\vec{\nabla}_\lambda) - (\vec{\lambda}\times\vec{\nabla}_\mu)\right] , \tag{4.23}$$

and then one can employ Eqs. 4.13, 4.18, and 4.20 to express $\vec{K}$ in terms of the new parameters. In addition, the parameters c and d are set equal to 1 and 0 respectively in order to reduce to the four parameters λ , ψ, θ , and ϕ . The operator $\vec{K}$ can be further simplified by making the change of variable λ to u , where $\lambda = \cosh u$. The new variable u has physical meaning since the angle between $\vec{\lambda}$ and the complex three-vector $\hat{z}$ is simply iu. Finally one obtains

$$\begin{aligned} i\vec{K} = & \left[\begin{array}{l} \hat{\imath}_1(\sin\psi\cos\theta\cos\phi + \cos\psi\sin\phi) \\ + \hat{\imath}_2(\sin\psi\cos\theta\sin\phi - \cos\psi\cos\phi) - \hat{\imath}_3\sin\psi\sin\theta \end{array}\right]\frac{\partial}{\partial u} \\ & + \tanh u\,(\hat{\imath}_1\sin\theta\cos\phi + \hat{\imath}_2\sin\theta\sin\phi + \hat{\imath}_3\cos\theta)\,(-iJ_1') \\ & + \coth u\left[\begin{array}{l} \hat{\imath}_1(\cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi) \\ + \hat{\imath}_2(\cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi) \\ - \hat{\imath}_3\cos\psi\sin\theta \end{array}\right](iJ_3') \end{aligned} \tag{4.24}$$

where the "body" angular momentum operators J_1', J_2', and J_3' have been identified as

$$\begin{aligned} iJ_1' &= -\csc\theta\cos\psi\frac{\partial}{\partial\phi} + \sin\psi\frac{\partial}{\partial\theta} + \cot\theta\cos\psi\frac{\partial}{\partial\psi} , \\ iJ_2' &= \csc\theta\sin\psi\frac{\partial}{\partial\phi} + \cos\psi\frac{\partial}{\partial\theta} - \cot\theta\sin\psi\frac{\partial}{\partial\psi} , \\ iJ_3' &= \frac{\partial}{\partial\psi} . \end{aligned} \tag{4.25}$$

As mentioned previously, the operator $\vec{K}$ generates pure imaginary rotations (velocity transformations).

E. Invariant Volume Element

For later use in orthogonality integrals the invariant volume element of the complex rotation group parameter space is found. In the original six-parameter space the volume element is

$$d\tau = d\lambda_1\, d\lambda_2\, d\lambda_3\, d\mu_1\, d\mu_2\, d\mu_3 \quad . \tag{4.26}$$

In terms of the new parameters the volume element is

$$d\tau = \left| J\left(\frac{\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3}{c, d, \lambda, \Psi, \theta, \phi}\right)\right| dc\, d(d)\, d\lambda\, d\Psi\, d\theta\, d\phi \ , \tag{4.27}$$

where $J\left(\frac{\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3}{c, d, \lambda, \Psi, \theta, \phi}\right)$ is the Jacobian of the transformation from old to new parameters. By use of Eq. 4.13 it is found that

$$\left| J\left(\frac{\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3}{c, d, \lambda, \Psi, \theta, \phi}\right)\right| = \frac{\lambda \sin\theta}{2} \ , \tag{4.28}$$

and hence

$$d\tau = \frac{\lambda \sin\theta}{2}\, dc\, d(d)\, d\lambda\, d\Psi\, d\theta\, d\phi \quad . \tag{4.29}$$

For the subspace, in particular the four-parameter subspace of the complex rotation group, one uses the Dirac delta functions $\delta(c-1)$ and $\delta(d)$ and integrates over c and d to find the invariant volume element of the four-parameter subspace:

$$d\tau' = \frac{\lambda \sin\theta}{2}\, d\lambda\, d\Psi\, d\theta\, d\phi \quad , \tag{4.30}$$

or alternatively in terms of the parameter u, where $\cosh u = \lambda$, one gets

$$d\tau' = \frac{\sinh 2u}{4} \sin\theta\, du\, d\Psi\, d\theta\, d\phi \quad . \tag{4.31}$$

V. THE R(4) ANALOGY

Before calculating the Casimir operators $J^2 - K^2$ and $\vec{J}\cdot\vec{K}$ of the complex rotation group and working with the two eigenvalue equations to find the harmonic function solutions, it is helpful and advantageous to work with analogous equations associated with the rotation group in four dimensions, R(4). The group R(4) is similar in structure to the complex rotation group and can be treated in a manner very parallel to that for O(3,C) as done in previous chapters. Two real three-vectors $\vec{u}$ and $\vec{v}$ can be used as a basis for developing the R(4) generators and hence the R(4) invariant operators. By use of the fact that R(4) is locally isomorphic to the direct product of two three-dimensional rotation groups, one can easily find the solutions to the R(4) eigenvalue equations. Additional solutions which are equivalent but simpler in form are obtained by certain rotations of the three-vectors to new orientations. The techniques used to solve the R(4) equations can be later applied to the O(3,C) equations.

A. The R(4) Group

The four-dimensional rotation group is the group of real, linear, orthogonal transformations in four-dimensional Euclidean space which leave invariant the quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2$, where x_1, x_2, x_3, and x_4 label a point in the space (Hamermesh 1962). There are six infinitesimal generators of the group rotations, and these generators in operator form can be expressed in terms of x_1, x_2, x_3, x_4 and derivatives with respect to them. The corresponding Lie algebra is described by the following commutation relations between the generators $\vec{M}$ and $\vec{N}$:

$$[M_i, M_j] = i\epsilon_{ijk} M_k \quad ,$$
$$[M_i, N_j] = i\epsilon_{ijk} N_k \quad , \qquad (5.1)$$
$$[N_i, N_j] = i\epsilon_{ijk} M_k \quad .$$

The R(4) Lie algebra can be realized by the construction of the generators in terms of two real three-vectors $\vec{u}$ and $\vec{v}$. This is suggested from the fact that R(4) is locally isomorphic to the direct product of two R(3) groups. Transformations of $\vec{u}$ and $\vec{v}$ in the three-dimensional space are real and linear and leave invariant the two quantities $u^2 + v^2$ and $\vec{u} \cdot \vec{v}$. In terms of $\vec{u}$ and $\vec{v}$ the infinitesimal generators in operator form become

$$\vec{M} = -i\left[(\vec{u} \times \vec{\nabla}_u) + (\vec{v} \times \vec{\nabla}_v)\right] \quad ,$$
$$\vec{N} = -i\left[(\vec{v} \times \vec{\nabla}_u) + (\vec{u} \times \vec{\nabla}_v)\right] \qquad (5.2)$$

and obey the commutation relations given by Eq. 5.1. The two invariant operators which commute with all of the infinitesimal generators are $\frac{1}{2}(M^2 + N^2)$ and $\vec{M} \cdot \vec{N}$. The solutions of the two corresponding eigenvalue equations may be considered to be R(4) harmonic functions. If one makes a linear transformation (Schweber 1961) to the basis consisting of the operators $\vec{P}$ and $\vec{Q}$, where

$$\vec{P} = \tfrac{1}{2}(\vec{M} + \vec{N}) \quad ,$$
$$\vec{Q} = \tfrac{1}{2}(\vec{M} - \vec{N}) \quad , \qquad (5.3)$$

one finds that $\vec{P}$ and $\vec{Q}$ commute with each other,

$$[P_i, Q_j] = 0 \quad , \tag{5.4}$$

and that the components of $\vec{P}$ and $\vec{Q}$ obey the angular momentum commutation rules:

$$\begin{aligned} [P_i, P_j] &= i\epsilon_{ijk} P_k \quad , \\ [Q_i, Q_j] &= i\epsilon_{ijk} Q_k \quad . \end{aligned} \tag{5.5}$$

The two invariant operators become

$$\begin{aligned} F &= \tfrac{1}{2}(M^2 + N^2) = P^2 + Q^2 \quad , \\ G &= \vec{M} \cdot \vec{N} = P^2 - Q^2 \quad . \end{aligned} \tag{5.6}$$

Thus each irreducible representation of R(4) can be labeled by the pair of indices (p,q) where $P^2 = p(p+1)$ and $Q^2 = q(q+1)$ for the irreducible representation labeled by p and q. The labels p and q can take on positive integral or half-integral values plus the value zero. Additional labels for the irreducible representations are provided by the indices m_1 and m_2, the values of the third components of the operators, P_3 and Q_3 respectively. Therefore one denotes by $|p m_1 q m_2\rangle$ the (2p+1)(2q+1) basis functions which span an R(4) irreducible representation space. These functions are defined by the following (Schweber 1961):

$$\begin{aligned} (P_1 \pm iP_2)|p m_1 q m_2\rangle &= [(p \mp m_1)(p \pm m_1 + 1)]^{\frac{1}{2}} |p\, m_1{\pm}1\, q\, m_2\rangle \quad , \\ P_3 |p m_1 q m_2\rangle &= m_1 |p m_1 q m_2\rangle \quad . \end{aligned} \tag{5.7}$$

and

$$
\begin{aligned}
(Q_1 \pm iQ_2)\,|pm_1 q m_2\rangle &= [(q \mp m_2)(q \pm m_2 + 1)]^{\frac{1}{2}}\,|pm_1 q m_2 \pm 1\rangle , \\
Q_3\,|pm_1 q m_2\rangle &= m_2\,|pm_1 q m_2\rangle \quad .
\end{aligned} \tag{5.8}
$$

B. Parametrization and Development of the Generators

The R(4) analogy to the O(3,C) work can be continued in the choice of parametrization and the expression of the generators in terms of new variables. Of the six position coordinates of the two three-vectors $\vec{u}$ and $\vec{v}$, only four need be considered as independent since

$$
\begin{aligned}
u^2 + v^2 &= a^2 + b^2 \quad , \\
\vec{u} \cdot \vec{v} &= ab \quad ,
\end{aligned} \tag{5.9}
$$

where a and b are real constants such that $a^2 \neq b^2$. In this work the special choice of three-vectors $\vec{\rho}$ and $\vec{\omega}$ is made, where

$$
\begin{aligned}
\rho^2 + \omega^2 &= 1 \quad , \\
\vec{\rho} \cdot \vec{\omega} &= 0 \quad .
\end{aligned} \tag{5.10}
$$

Any other pair of three-vectors, say $\vec{u}$ and $\vec{v}$, with invariants given by Eq. 5.9 can be expressed in terms of $\vec{\rho}$ and $\vec{\omega}$:

$$
\begin{aligned}
\vec{u} &= a\vec{\rho} + b\vec{\omega} \quad , \\
\vec{v} &= b\vec{\rho} + a\vec{\omega} \quad .
\end{aligned} \tag{5.11}
$$

So the general case of vectors $\vec{u}$ and $\vec{v}$ can always be reduced to the special case of vectors $\vec{\rho}$ and $\vec{\omega}$. All R(4) work will be done in terms of the special vectors $\vec{\rho}$ and $\vec{\omega}$.

The space is three-dimensional and is spanned by three mutually perpendicular coordinate axes. The real unit vectors along each of the axes are denoted by $\hat{\iota}_1$, $\hat{\iota}_2$, and $\hat{\iota}_3$, where the subscripts indicate the particular axis. All rotations in this space are active ones whereby the vectors, not the coordinate axes, are rotated. The three-vectors $\vec{\rho}$ and $\vec{\omega}$ are expressed as

$$\begin{aligned} \vec{\rho} &= \hat{\iota}_1 \rho_1 + \hat{\iota}_2 \rho_2 + \hat{\iota}_3 \rho_3 \quad , \\ \vec{\omega} &= \hat{\iota}_1 \omega_1 + \hat{\iota}_2 \omega_2 + \hat{\iota}_3 \omega_3 \quad . \end{aligned} \tag{5.12}$$

In the original orientation of $\vec{\rho}$ and $\vec{\omega}$ the infinitesimal generators $\vec{M}$ and $\vec{N}$ can be expressed in terms of the six variables ρ_1 , ρ_2 , ρ_3 , ω_1 , ω_2 , ω_3 and derivatives with respect to them. The change of variables to six other parameters (only four of which are considered as independent) is defined by the active rotation R of the pair of vectors $\vec{\rho}$ and $\vec{\omega}$ so as to have $\vec{\rho}$ lie along the 3-axis and $\vec{\omega}$ to lie in the first quadrant of the 3-1 plane. This is shown in Figure 5.1. The new parameters are the three Euler angles ψ , θ , ϕ which define the active rotation, the magnitude of $\vec{\rho}$ denoted by ρ , and the quantities $\rho^2 + \omega^2 \equiv c$ $\vec{\rho} \cdot \vec{\omega} \equiv d$. The operators $\vec{M}$ and $\vec{N}$ can be written in terms of the

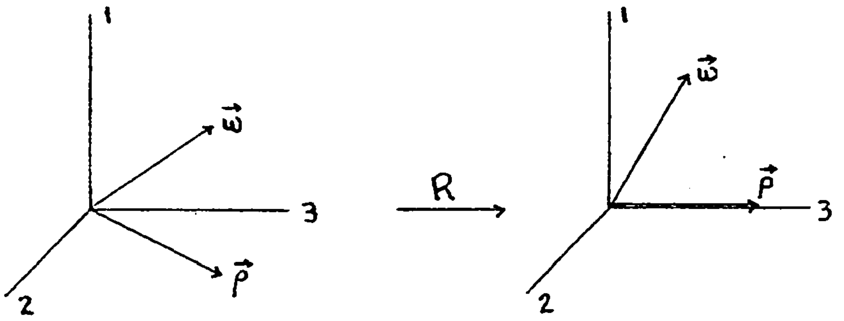

Figure 5.1. Active rotation of the three-vectors $\vec{\rho}$ and $\vec{\omega}$

new parameters and derivatives with respect to them. Finally application of the conditions $c = 1$ and $d = 0$ (i.e. vectors $\vec{\rho}$ and $\vec{\omega}$ are perpendicular) puts $\vec{M}$ and $\vec{N}$ in terms of the four parameters ρ, Ψ, θ, ϕ and derivatives with respect to each of them.

The active rotation which carries the three-vectors $\vec{\rho}$ and $\vec{\omega}$ from their original orientations to their new orientations of $\vec{\rho}$ along the 3-axis and $\vec{\omega}$ in the 3-1 plane is defined by the successive real rotations: (1) ϕ about the 3-axis, (2) θ about the 2-axis, and (3) Ψ about the 3-axis. This definition of Euler angles is consistent with that used in Fano and Racah (1959). The matrix form of the rotation is identical with that given in Eq. 4.10. The three-vectors $\vec{\rho}$ and $\vec{\omega}$ are expressed in terms of the new parameters:

$$\begin{aligned}
\vec{\rho} &= \hat{\iota}_1 \rho \sin\theta \cos\phi + \hat{\iota}_2 \rho \sin\theta \sin\phi + \hat{\iota}_3 \rho \cos\theta \quad , \\
\vec{\omega} &= \hat{\iota}_1 \left[f(\cos\Psi \cos\theta \cos\phi - \sin\Psi \sin\phi) + \frac{d}{\rho} \sin\theta \cos\phi \right] \\
&\quad + \hat{\iota}_2 \left[f(\cos\Psi \cos\theta \sin\phi + \sin\Psi \cos\phi) + \frac{d}{\rho} \sin\theta \sin\phi \right] \\
&\quad + \hat{\iota}_3 \left[- f \cos\Psi \sin\theta + \frac{d}{\rho} \cos\theta \right] \quad ,
\end{aligned} \tag{5.13}$$

where $c \equiv \rho^2 + \omega^2$, $d \equiv \vec{\rho} \cdot \vec{\omega}$, $f \equiv \left[c - \rho^2 - \frac{d}{\rho^2} \right]^{\frac{1}{2}}$. Similar to the development in Eqs. 4.14 through 4.18 for $\vec{\nabla}_\lambda$ in Chapter 4 one can find the operators $\vec{\nabla}_\rho$ and $\vec{\nabla}_\omega$ in terms of the new variables. The expression for $\vec{\nabla}_\rho$ is identical to that for $\vec{\nabla}_\lambda$ if one makes the change $\lambda \rightarrow \rho$ and remembers that c, d, and f are now defined slightly differently than before in Chapter 4. Also the operator $\vec{\nabla}_\omega$ can be obtained from $\vec{\nabla}_\mu$ in Eq. 4.20 if one changes the sign of the coefficient of derivative $\frac{\partial}{\partial c}$ and lets $\lambda \rightarrow \rho$. With $\vec{\rho}$, $\vec{\omega}$, $\vec{\nabla}_\rho$, and $\vec{\nabla}_\omega$ all in terms of the new parameters one can calculate the generators $\vec{M}$ and $\vec{N}$ by using Eq. 5.2

with $\vec{\rho}$ and $\vec{\omega}$ replacing $\vec{u}$ and $\vec{v}$ and then applying the conditions $c = 1$ and $d = 0$. The generator $\vec{M}$ turns out to be the angular momentum operator:

$$
\begin{aligned}
i\vec{M} = {} & \hat{\iota}_1(\csc\theta\cos\phi\frac{\partial}{\partial\psi} - \sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}) \\
& + \hat{\iota}_2(\csc\theta\sin\phi\frac{\partial}{\partial\psi} + \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}) \\
& + \hat{\iota}_3\frac{\partial}{\partial\phi} \quad .
\end{aligned}
\tag{5.14}
$$

The operator $\vec{N}$ is simplified by making the change of variable ρ to χ, where $\rho = \cos\chi$, and by identifying the "body" angular momentum operators J_1', J_2', J_3' as in Eq. 4.25:

$$
\begin{aligned}
i\vec{N} = {} & \begin{bmatrix} -\hat{\iota}_1(\sin\psi\cos\theta\cos\phi + \cos\psi\sin\phi) \\ -\hat{\iota}_2(\sin\psi\cos\theta\sin\phi - \cos\psi\cos\phi) + \hat{\iota}_3\sin\psi\sin\theta \end{bmatrix}\frac{\partial}{\partial\chi} \\
& + \tan\chi\left\{\hat{\iota}_1\sin\theta\cos\phi + \hat{\iota}_2\sin\theta\sin\phi + \hat{\iota}_3\cos\theta\right]\ (-iJ_1') \\
& - \cot\chi\begin{bmatrix} \hat{\iota}_1(\cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi) \\ +\hat{\iota}_2(\cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi) \\ -\hat{\iota}_3\cos\psi\sin\theta \end{bmatrix}\ (iJ_3')
\end{aligned}
\tag{5.15}
$$

The physical meaning of the new variable χ will be seen in the next section of this chapter.

C. Solutions of the Eigenvalue Equations

By use of the operators $\vec{M}$ and $\vec{N}$ given by Eqs. 5.14 and 5.15 one calculates the R(4) invariant operators to be

$$
\begin{aligned}
\frac{M^2+N^2}{2} &= \tfrac{1}{2}\left[M^2 + \cot^2\chi\ {J_3'}^2 + \tan^2\chi\ {J_1'}^2 - \frac{\partial^2}{\partial\chi^2} - 2\cot 2\chi\frac{\partial}{\partial\chi}\right], \\
\vec{M}\cdot\vec{N} &= (-\frac{\partial}{\partial\chi} + \tan\chi)\ iJ_2' - \frac{1}{\sin\chi\cos\chi}\ J_1'J_3' \quad .
\end{aligned}
\tag{5.16}
$$

The solutions of the two corresponding eigenvalue equations will be called the R(4) harmonic functions. The isomorphism of R(4) to the direct product of two R(3) groups is exploited by making the transformation of the operators $\vec{M}$ and $\vec{N}$ to the operators $\vec{P}$ and $\vec{Q}$ as defined by Eq. 5.3. As seen earlier, $\vec{P}$ and $\vec{Q}$ obey the commutation relations of two commuting angular momentum operators, and the operators ($P^2 + Q^2$) and ($P^2 - Q^2$) are equivalent to the invariants $\frac{1}{2}(M^2 + N^2)$ and $\vec{M} \cdot \vec{N}$ respectively. Thus instead of attempting to find direct solutions of the above equations in Eq. 5.16, one can seek solutions of the ($P^2 + Q^2$) and ($P^2 - Q^2$) equations. This is the method of attack used in the following work.

Suggested solutions to the R(4) eigenvalue equations can be obtained by finding the relationship between the operators $\vec{P}$ and $\vec{Q}$ and the three-vectors $\vec{\alpha}$ and $\vec{\beta}$, where

$$\begin{aligned} \vec{\alpha} &= \vec{\rho} + \vec{\omega} \quad , \\ \vec{\beta} &= \vec{\rho} - \vec{\omega} \quad . \end{aligned} \tag{5.17}$$

From the following relations involving $\vec{\alpha}$, $\vec{\beta}$, and $\vec{\rho}$:

$$\begin{aligned} \alpha^2 = \beta^2 &= 1 \quad , \\ \vec{\alpha} \cdot \vec{\rho} = \cos^2 \chi &= \vec{\beta} \cdot \vec{\rho} \quad , \\ \vec{\alpha} \cdot \vec{\beta} &= \cos 2\chi \quad , \end{aligned} \tag{5.18}$$

one concludes that $\vec{\alpha}$ and $\vec{\beta}$ are vectors of unit magnitude which lie in the 3-1 plane at angles of $\pm\chi$ with respect to the 3-axis. This is shown in Figure 5.2 :

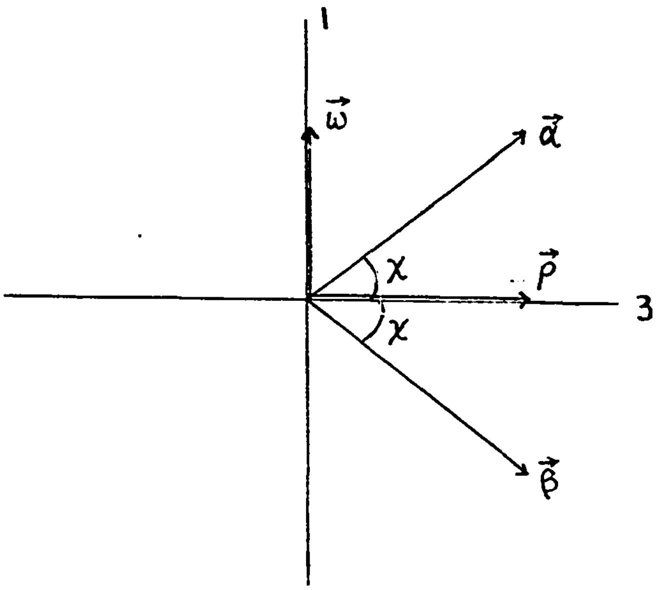

Figure 5.2. Orientation of the three-vectors $\vec{\alpha}$ and $\vec{\beta}$

By the use of Eq. 5.2 with $\vec{\rho}$ and $\vec{\omega}$ replacing $\vec{u}$ and $\vec{v}$ plus Eqs. 5.3 and 5.17 one can easily show that

$$\begin{aligned} \vec{P} &= -i(\,\vec{\alpha} \times \vec{\nabla}_{\alpha}\,) \quad , \\ \vec{Q} &= -i(\,\vec{\beta} \times \vec{\nabla}_{\beta}\,) \quad . \end{aligned} \tag{5.19}$$

This indicates that $\vec{P}$ and $\vec{Q}$ are the angular momentum operators corresponding to the three-vectors $\vec{\alpha}$ and $\vec{\beta}$ respectively. It is well-known (Rose 1957) that the eigenfunction solutions ψ to the eigenvalue equations corresponding to the angular momentum operator $\vec{L}$,

$$L^2\Psi = \ell(\ell+1)\Psi \quad ,$$
$$L_3\Psi = m\,\Psi \quad , \tag{5.20}$$

are simply the spherical harmonic functions $Y_\ell^m(\theta,\phi)$, where θ is the colatitude and ϕ is the azimuthal angle of the position vector. It follows that the solutions to the following R(4) eigenvalue equations

$$\left(P^2 \pm Q^2\right)|pm_1 q m_2\rangle = \left[p(p+1) \pm q(q+1)\right]|pm_1 q m_2\rangle \tag{5.21}$$

would consist of a product of the two spherical harmonic functions corresponding to operators $\vec{P}$ and $\vec{Q}$, namely $Y_p^{m_1}(\theta_\alpha,\phi_\alpha)\,Y_q^{m_2}(\theta_\beta,\phi_\beta)$. The angles θ_α , ϕ_α , θ_β , and ϕ_β refer to the original orientation of the three-vectors $\vec{\alpha}$ and $\vec{\beta}$ before the active rotation. The above spherical harmonic functions can be expressed in terms of the new orientation of $\vec{\alpha}$ and $\vec{\beta}$ as denoted by the angles θ_α' , ϕ_α' , θ_β' , and ϕ_β' .

In the coupled representation the R(4) solutions have the form

$$\langle\theta_\alpha,\phi_\alpha,\theta_\beta,\phi_\beta|pqjm\rangle = \sum_{m_1,m_2}(pm_1qm_2|pqjm)\,Y_p^{m_1}(\theta_\alpha,\phi_\alpha)\,Y_q^{m_2}(\theta_\beta,\phi_\beta) \tag{5.22}$$

which can be expressed as

$$\langle\theta_\alpha',\phi_\alpha',\theta_\beta',\phi_\beta'|pqjm\rangle = \sum_{m_1',m_2'}(pm_1'qm_2'|pqjm')\,Y_p^{m_1'}(\theta_\alpha',\phi_\alpha')\,Y_q^{m_2'}(\theta_\beta',\phi_\beta')\,D^{j*}_{m'm}(\psi,\theta,\phi) \ , \tag{5.23}$$

where the $(pm_1'qm_2'|pqjm')$ denote Clebsch-Gordan (C-G) coefficients, and $D^{j*}_{m'm}(\psi,\theta,\phi)$ denotes the matrix representation of the inverse rotation R^{-1}. The convention chosen for the rotation matrices is that of Fano and Racah (1959). From Figure 5.2 it is easily seen that $\theta_\alpha' = \theta_\beta' = \chi$,

$\Phi_\alpha' = 0$, and $\Phi_\beta' = \pi$. Using the definition of the spherical harmonic function,

$$Y_\ell^m(\theta,\phi) \equiv \left[\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{\frac{1}{2}} e^{im\phi} P_\ell^m(\cos\theta), \tag{5.24}$$

where $P_\ell^m(\cos\theta)$ is the associated Legendre function on the cut, $P_\ell^m(x)$ with $-1 < x < 1$, as defined by Erdélyi, et al. (1953), one can write the solutions as

$$\begin{aligned} S_{jm}^{pq}(\chi,\psi,\theta,\phi) &\equiv \langle \chi,\psi,\theta,\phi \,|\, pqjm\rangle \\ &= \left[\frac{(2p+1)(2q+1)}{(4\pi)^2}\right]^{\frac{1}{2}} \sum_{m_1',m_2'} (-1)^{m_2'} \left[\frac{(p-m_1')!\,(q-m_2')!}{(p+m_1')!\,(q+m_2')!}\right]^{\frac{1}{2}} (p m_1' q m_2' \,|\, pqjm') \\ &\quad \cdot P_p^{m_1'}(\cos\chi)\, P_q^{m_2'}(\cos\chi)\, D_{m'm}^{j*}(\psi,\theta,\phi) \; . \end{aligned} \tag{5.25}$$

The preceding functions can be shown to satisfy the R(4) eigenvalue equations as given by combining Eqs. 5.16 and 5.21:

$$\begin{aligned} \tfrac{1}{2}\left[M^2 + \cot^2\chi\, J_3'^2 + \tan^2\chi\, J_1'^2 - \frac{\partial^2}{\partial\chi^2} - 2\cot 2\chi \frac{\partial}{\partial\chi}\right] S_{jm}^{pq} \\ = \left[p(p+1) + q(q+1)\right] S_{jm}^{pq} \quad , \end{aligned}$$

$$\tag{5.26}$$

$$\begin{aligned} \left[\left(-\frac{\partial}{\partial\chi} + \tan\chi\right) iJ_2' - \frac{1}{\sin\chi\cos\chi} J_1' J_3'\right] S_{jm}^{pq} \\ = \left[p(p+1) - q(q+1)\right] S_{jm}^{pq} \quad . \end{aligned}$$

The verification of the solutions expressed by Eq. 5.25 will be discussed in more detail.

The suggested R(4) harmonic functions in Eq. 5.25 can be written in the convenient form (with constant factors dropped):

$$\underline{S}^{\,pq}_{jm}(\chi,\psi,\theta,\phi) = \sum_{m'} f^{pq}_{jm'}(\chi)\, D^{j\,*}_{m'm}(\psi,\theta,\phi) , \qquad (5.27)$$

where

$$f^{pq}_{jm'} = \sum_{m_1'} (-1)^{m'-m_1'} \left[\frac{(p-m_1')!\,(q-m'+m_1')!}{(p+m_1')!\,(q+m'-m_1')!}\right]^{\frac{1}{2}} (p\,m_1'\,q\,m'-m_1' | pq\,j\,m')$$
$$\cdot\, P_p^{m_1'}(\cos\chi)\; P_q^{m'-m_1'}(\cos\chi) . \qquad (5.28)$$

The angular momentum operators act on the rotation matrices in the following manner:

$$M^2\, D^{j\,*}_{m'm}(\psi,\theta,\phi) = j(j+1)\, D^{j\,*}_{m'm}(\psi,\theta,\phi) ,$$

$$J_3'\, D^{j\,*}_{m'm}(\psi,\theta,\phi) = -m'\, D^{j\,*}_{m'm}(\psi,\theta,\phi) ,$$

$$J_1'\, D^{j\,*}_{m'm} = -\tfrac{1}{2}\left\{[j(j+1)-m'(m'+1)]^{\frac{1}{2}} D^{j\,*}_{m'+1\,m} + [j(j+1)-m'(m'-1)]^{\frac{1}{2}} D^{j\,*}_{m'-1\,m}\right\}$$

$$(5.29)$$

$$iJ_2'\, D^{j\,*}_{m'm} = -\tfrac{1}{2}\left\{[j(j+1)-m'(m'+1)]^{\frac{1}{2}} D^{j\,*}_{m'+1\,m} - [j(j+1)-m'(m'-1)]^{\frac{1}{2}} D^{j\,*}_{m'-1\,m}\right\}$$

$$J_1'^{\,2}\, D^{j\,*}_{m'm} = \tfrac{1}{4}\,[j(j+1)-m'(m'+1)]^{\frac{1}{2}}\,[j(j+1)-(m'+1)(m'+2)]^{\frac{1}{2}}\, D^{j\,*}_{m'+2\,m}$$
$$+ \tfrac{1}{4}\,[j(j+1)-m'(m'-1)]^{\frac{1}{2}}\,[j(j+1)-(m'-1)(m'-2)]^{\frac{1}{2}}\, D^{j\,*}_{m'-2\,m}$$
$$+ \tfrac{1}{2}\left(j(j+1)-m'^{\,2}\right) D^{j\,*}_{m'm} .$$

By substituting the functions from Eq. 5.27 into the eigenvalue equations given by Eq. 5.26 and by using the preceding relations given by Eq. 5.29, one gets two differential equations involving $f^{pq}_{jm'}$ and $D^{j*}_{m'm}$. The $D^{j*}_{m'm}$ -dependence can be removed by collecting terms for $D^{j*}_{m'm}$, with m' and m fixed. One obtains two coupled differential equations for the $f_{m'}$ (suppressing all the labels except m' for $f^{pq}_{jm'}$):

$$\left[j(j+1) + m'^2 \cot^2\chi + \frac{j(j+1)-m'^2}{2}\tan^2\chi - \frac{\partial^2}{\partial\chi^2} - 2\cot 2\chi \frac{\partial}{\partial\chi} - 2p(p+1) - 2q(q+1) \right] f_{m'}$$
$$+ \frac{\tan^2\chi}{4}\left[j(j+1)-(m'-1)(m'-2)\right]^{\frac{1}{2}}\left[j(j+1)-m'(m'-1)\right]^{\frac{1}{2}} f_{m'-2}$$
$$+ \frac{\tan^2\chi}{4}\left[j(j+1)-(m'+1)(m'+2)\right]^{\frac{1}{2}}\left[j(j+1)-m'(m'+1)\right]^{\frac{1}{2}} f_{m'+2} = 0 \qquad (5.30)$$

and

$$\left[j(j+1)-m'(m'-1)\right]^{\frac{1}{2}}\left(\frac{\partial}{\partial\chi} - \tan\chi - \frac{m'-1}{\sin\chi\cos\chi}\right) f_{m'-1}$$
$$- \left[j(j+1)-m'(m'+1)\right]^{\frac{1}{2}}\left(\frac{\partial}{\partial\chi} - \tan\chi + \frac{m'+1}{\sin\chi\cos\chi}\right) f_{m'+1}$$
$$- \left[2p(p+1) - 2q(q+1)\right] f_{m'} = 0 \qquad (5.31)$$

By use of the associated Legendre function recursion relationships for $-1 < x < +1$ (Erdélyi, <u>et al.</u> 1953),

$$(1-x^2)\frac{d}{dx}P^\mu_\nu(x) = -\mu x\, P^\mu_\nu(x) - [1-x^2]^{\frac{1}{2}} P^{\mu+1}_\nu(x),$$
$$= \mu x\, P^\mu_\nu(x) + (\nu+\mu)(\nu-\mu+1)[1-x^2]^{\frac{1}{2}} P^{\mu-1}_\nu(x) \qquad (5.32)$$

and the Clebsch-Gordan recurrence formulas (Rose 1957),

$$[(j\mp m)(j\pm m+1)]^{\frac{1}{2}}\ (j_1\, m_1\, j_2\, m\pm 1-m_1 \,|\, j_1 j_2\, j\, m\pm 1)$$

$$= [(j_1\mp m_1+1)(j_1\pm m_1)]^{\frac{1}{2}}\ (j_1\, m_1\mp 1\, j_2\, m-m_1\pm 1 \,|\, j_1 j_2\, j m)$$

$$+ [(j_2\mp m\pm m_1)(j_2\pm m\mp m_1+1)]^{\frac{1}{2}}\ (j_1 m_1\, j_2\, m-m_1 \,|\, j_1 j_2\, j m) \quad ,$$

$$(5.33)$$

$$[J(J+1)-j_1(j_1+1)-j_2(j_2+1)-2m(M-m)]\ (j_1\, m\, j_2\, M-m \,|\, j_1 j_2\, J M)$$

$$= [(j_1-m+1)(j_1+m)(j_2+M-m+1)(j_2-M+m)]^{\frac{1}{2}}\ (j_1\, m-1\, j_2\, M-m+1 \,|\, j_1 j_2 J M)$$

$$+ [(j_1+m+1)(j_1-m)(j_2-M+m+1)(j_2+M-m)]^{\frac{1}{2}}\ (j_1\, m_1+1\, j_2\, M-m_1-1 \,|\, j_1 j_2 J M) \quad ,$$

one can show that after some labor that $f^{M}_{jm'}$ as given by Eq. 5.27 satisfies the two coupled differential equations expressed by Eqs. 5.30 and 5.31. This completes the discussion of the proof of the suggested solutions to the R(4) eigenvalue equations. Thus a set of R(4) harmonic functions is given by Eq. 5.27.

D. Additional R(4) Solutions

A simpler χ-dependence in the R(4) solutions can be obtained if the active rotation of the three-vectors $\vec{\rho}$ and $\vec{\omega}$ is such that either the vector $\vec{\alpha} = \vec{\rho} + \vec{\omega}$ or the vector $\vec{\beta} = \vec{\rho} - \vec{\omega}$ lies along the 3-axis in the new orientation. This is shown in Figure 5.3.

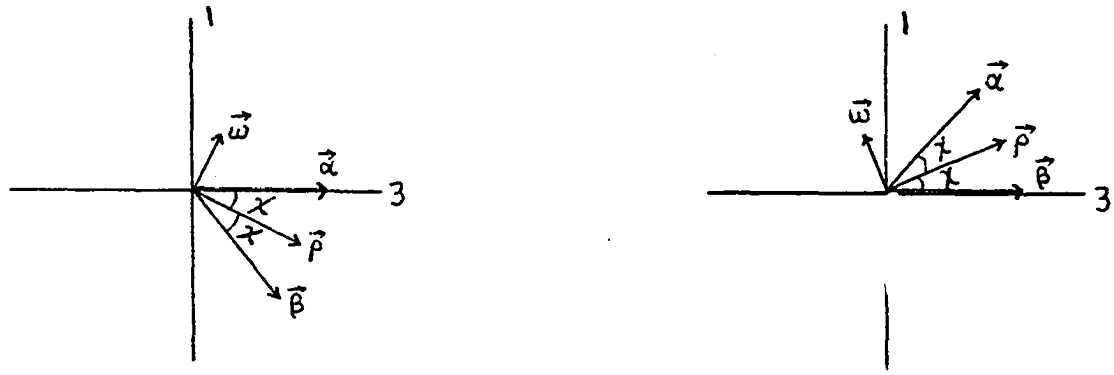

Figure 5.3. Other orientations of the three-vectors $\vec{\rho}$, $\vec{\omega}$, $\vec{\alpha}$, and $\vec{\beta}$

By using the same procedure that led up to the suggested solutions given by Eq. 5.25, one can find solutions corresponding to each of the above-mentioned cases:

(1) Vector $\vec{\alpha}$ along the 3-axis. The angles are now $\theta'_\alpha = 0$, $\theta'_\beta = 2\chi$, ϕ'_α, $\phi'_\beta = \pi$ and the product of spherical harmonic functions becomes

$$Y_p^{m_1'}(0,\phi_\alpha')\, Y_q^{m_2'}(2\chi,\pi) = (-1)^{m_2'}\left[\frac{(2p+1)(2q+1)}{(4\pi)^2}\frac{(q-m_2')!}{(q+m_2')!}\right]^{\frac{1}{2}} P_q^{m_2'}(\cos 2\chi)\,\delta_{m_1' 0} \quad (5.34)$$

The suggested R(4) solutions are (with constant factors dropped)

$$\langle\chi,\psi',\theta',\phi'|pqjm\rangle_1 = \sum_{m'} (-1)^{m'}\left[\frac{(q-m')!}{(q+m')!}\right]^{\frac{1}{2}} (p\,0\,q\,m'|pq\,j\,m') \cdot P_q^{m'}(\cos 2\chi)\, D^{j\,*}_{m'm}(\psi',\theta',\phi') , \quad (5.35)$$

where the Euler angles ψ', θ', ϕ' describe the active rotation of the vectors $\vec{\rho}$ and $\vec{\omega}$ to a new orientation such that $\vec{\alpha}$ lies along the 3-axis and $\vec{\beta}$ lies in the 3-1 plane.

(2) Vector $\vec{\beta}$ along the 3-axis. The angles are now $\theta'_\alpha = 2\chi$, $\theta'_\beta = 0$, $\phi'_\alpha = 0$, ϕ'_β and the product of spherical harmonic functions becomes

$$Y_p^{m_1'}(2\chi,0)\,Y_q^{m_2'}(0,\phi_\beta') = \left[\frac{(2p+1)(2q+1)}{(4\pi)^2}\frac{(p-m_1')!}{(p+m_1')!}\right]^{\frac{1}{2}} P_p^{m_1'}(\cos 2\chi)\,\delta_{m_2' 0} \quad . \qquad (5.36)$$

The suggested R(4) solutions are (with constant factors dropped)

$$\langle \chi, \psi'', \theta'', \phi'' | pqjm \rangle_2 = \sum_{m'} \left[\frac{(p-m')!}{(p+m')!}\right]^{\frac{1}{2}} (pm'q0|pqjm') \cdot P_p^{m'}(\cos 2\chi)\, D^{j\,*}_{m'm}(\psi'', \theta'', \phi'') \quad , \qquad (5.37)$$

where the Euler angles ψ'', θ'', ϕ'' describe the active rotation of the vectors $\vec{\rho}$ and $\vec{\omega}$ to a new orientation such that $\vec{\beta}$ lies along the 3-axis and $\vec{\alpha}$ lies in the 3-1 plane.

The two additional sets of suggested solutions given by Eqs. 5.35 and 5.37 can both be shown to satisfy the R(4) eigenvalue equations. The case of vector $\vec{\alpha}$ along the 3-axis will be treated first. The Euler angles ψ', θ', ϕ' for this case differ from the original Euler angles ψ, θ, ϕ as used earlier (case of $\vec{\rho}$ along 3-axis) by only a rotation of magnitude χ about the 2-axis. In terms of the rotation matrices this is written as

$$D^{j\,*}_{m'm}(\psi', \theta', \phi') = \sum_{m''} d^{j}_{m''m'}(-\chi)\, D^{j\,*}_{m''m}(\psi, \theta, \phi) \quad , \qquad (5.38)$$

where the d-matrix is defined by

$$d^{j}_{m''m'}(-\chi) \equiv D^{j}_{m''m'}(0,-\chi,0) = \langle jm'' | e^{-iJ_2\chi} | jm' \rangle \tag{5.39}$$

with χ varying from 0 to π . An explicit expression for the $d^{j}_{m''m'}$ in terms of the hypergeometric function is found in most standard angular momentum textbooks. The suggested R(4) solutions from Eq. 5.35 can be rewritten as

$$T^{pq}_{jm}(\chi,\psi,\theta,\phi) \equiv \langle \chi,\psi,\theta,\phi | pqjm \rangle_1 = \sum_{m''} f^{pq}_{jm''}(\chi)\, D^{j*}_{m''m}(\psi,\theta,\phi) \,, \tag{5.40}$$

where

$$f^{pq}_{jm''}(\chi) = \sum_{m'} (-1)^{m'}\, g^{pq}_{jm'}(\chi)\; d^{j}_{m''m'}(-\chi) \tag{5.41}$$

and

$$g^{pq}_{jm'}(\chi) = \left[\frac{(q-m')!}{(q+m')!}\right]^{\frac{1}{2}} (p0qm' | pqjm')\; P^{m'}_{q}(\cos 2\chi) \,. \tag{5.42}$$

The above form of the solutions is convenient because the same original Euler angle dependence is used, and much of the previous work from the case of $\vec{p}$ along the 3-axis can be retained. Substitution of Eq. 5.40 into the R(4) eigenvalue equations yields the same coupled equations for $f^{pq}_{jm''}$ as given before in Eqs. 5.30 and 5.31. By using the form of $f^{pq}_{jm''}$ as given by Eq. 5.41 and applying the differential relations (Fano and Racah 1959)

$$\pm \frac{d}{d\theta}\, d^{j}_{m'm}(-\theta) = \frac{m'\cos\theta - m}{\sin\theta}\, d^{j}_{m'm}(-\theta) + \left[j(j+1) - m'(m'\pm 1)\right]^{\frac{1}{2}} d^{j}_{m'\pm 1\, m}(-\theta) \tag{5.43}$$

one obtains the two differential equations in terms of both $d^j_{m''m'}$ and $g^{pq}_{jm'}$. The $d^j_{m''m'}$-dependence can be removed by use of the recursion formulas (Fano and Racah 1959):

$$[\ell(\ell+1)-\nu(\nu+1)]^{\frac{1}{2}}\, d^\ell_{\nu+1\,\mu}(\theta) + [\ell(\ell+1)-\nu(\nu-1)]^{\frac{1}{2}}\, d^\ell_{\nu-1\,\mu}(\theta) = \frac{2(\nu\cos\theta-\mu)}{\sin\theta}\, d^\ell_{\nu\mu}(\theta) ,$$

$$[\ell(\ell+1)-\nu(\nu+1)]^{\frac{1}{2}}\, d^\ell_{\nu+1\,\mu}(\theta) - [\ell(\ell+1)-\nu(\nu-1)]^{\frac{1}{2}}\, d^\ell_{\nu-1\,\mu}(\theta)$$
$$= [\ell(\ell+1)-\mu(\mu-1)]^{\frac{1}{2}}\, d^\ell_{\nu\,\mu-1}(\theta) - [\ell(\ell+1)-\mu(\mu+1)]^{\frac{1}{2}}\, d^\ell_{\nu\,\mu+1}(\theta) , \tag{5.44}$$

$$[\ell(\ell+1)-\nu(\nu+1)]^{\frac{1}{2}}\, d^\ell_{\nu+1\,\mu}(\theta) + [\ell(\ell+1)-\nu(\nu-1)]^{\frac{1}{2}}\, d^\ell_{\nu-1\,\mu}(\theta)$$
$$= \pm [\ell(\ell+1)-\mu(\mu+1)]^{\frac{1}{2}}\, d^\ell_{\nu\,\mu+1}(\theta) \pm [\ell(\ell+1)-\mu(\mu-1)]^{\frac{1}{2}}\, d^\ell_{\nu\,\mu-1}(\theta)$$
$$- \frac{2(\mu\pm\nu)\sin\theta}{(1\pm\cos\theta)}\, d^\ell_{\nu\mu}(\theta)$$

and by collecting terms for $(-1)^{m'}$ $d^j_{m''m'}(-\chi)$, with m'' and m' fixed. This yields two coupled differential equations for the $g^{pq}_{jm'}$ functions:

$$\left[2j(j+1) - 2p(p+1) - 2q(q+1) + 4m'^2\cot^2 2\chi - \frac{\partial^2}{\partial\chi^2} - 2\cot 2\chi\frac{\partial}{\partial\chi}\right] g^{pq}_{jm'}$$
$$+ [j(j+1)-m'(m'-1)]^{\frac{1}{2}}\left[-\frac{\partial}{\partial\chi} + 2(m'-1)\cot 2\chi\right] g^{pq}_{j\,m'-1} \tag{5.45}$$
$$+ [j(j+1)-m'(m'+1)]^{\frac{1}{2}}\left[\frac{\partial}{\partial\chi} + 2(m'+1)\cot 2\chi\right] g^{pq}_{j\,m'+1} = 0$$

and

$$2\left[j(j+1) - 2m'^2 - p(p+1) - q(q+1)\right] g^{pq}_{jm'}$$

$$+ \left[j(j+1) - m'(m'-1)\right]^{\frac{1}{2}} \left[-\frac{\partial}{\partial\chi} + 2(m'-1)\cot 2\chi\right] g^{pq}_{j\,m'-1} \tag{5.46}$$

$$+ \left[j(j+1) - m'(m'+1)\right]^{\frac{1}{2}} \left[\frac{\partial}{\partial\chi} + 2(m'+1)\cot 2\chi\right] g^{pq}_{j\,m'+1} = 0 .$$

With the use of the associated Legendre function recursion formulas given by Eq. 5.32 and the Clebsch-Gordan recurrence relations given by Eq. 5.33, one can easily prove that the functions $g^{pq}_{jm'}$ as defined by Eq. 5.42 satisfy both of the preceding equations. Thus the solutions $T^{pq}_{jm}(\chi,\psi,\theta,\phi)$ expressed by Eqs. 5.40, 5.41, and 5.42 are indeed harmonic function solutions of the R(4) group. This completes the case of the three-vector $\vec{\alpha}$ along the 3-axis.

The proof of the validity of the solutions arising from the case of the three-vector $\vec{\beta}$ along the 3-axis is parallel to the preceding work of $\vec{\alpha}$ along the 3-axis. The relationship between the Euler angles ψ'', θ'', ϕ'' and the original Euler angles ψ, θ, ϕ (for $\vec{\rho}$ along 3-axis) is found and written in terms of the rotation matrices:

$$D^{j\,*}_{m'm}(\psi'',\theta'',\phi'') = \sum_{m''} d^{j}_{m''m'}(\chi)\; D^{j\,*}_{m''m}(\psi,\theta,\phi) . \tag{5.47}$$

The suggested R(4) solutions from Eq. 5.37 are rewritten as

$$\tilde{T}^{pq}_{jm}(\chi,\psi,\theta,\phi) \equiv \langle \chi,\psi,\theta,\phi | pqjm \rangle_2 = \sum_{m''} \tilde{f}^{pq}_{jm''}(\chi)\; D^{j\,*}_{m''m}(\psi,\theta,\phi) , \tag{5.48}$$

where
$$\tilde{f}^{pq}_{jm''}(\chi) = \sum_{m'} h^{pq}_{jm'}(\chi)\; d^{j}_{m''m'}(\chi) \tag{5.49}$$

and
$$h^{pq}_{jm'}(\chi) = \left[\frac{(p-m')!}{(p+m')!}\right]^{\frac{1}{2}} (pm'q0|pqjm')\; P^{m'}_{p}(\cos 2\chi) \quad . \tag{5.50}$$

In a step-by-step process the preceding solutions can be shown to satisfy the R(4) eigenvalue equations. By use of Eq. 5.48 in the R(4) equations one obtains two equations for $\tilde{f}^{pq}_{jm''}$. Then Eq. 5.49 allows the two equations to be written in terms of $h^{pq}_{jm'}$ and $d^{j}_{m''m'}$, whereupon the $d^{j}_{m''m'}$-dependence can be removed to leave two coupled differential equations for $h^{pq}_{jm'}$. The functions $h^{pq}_{jm'}$ as given by Eq. 5.50 identically satisfy the two equations. Thus the $\tilde{T}^{pq}_{jm}(\chi,\psi,\theta,\phi)$ are also R(4) harmonic functions.

The three sets of R(4) solutions as expressed by Eqs. 5.27, 5.40, and 5.48 are completely equivalent. This can be shown by use of the inverse of the Clebsch-Gordan series for the d-functions as expressed by

$$\delta_{jj'}\; d^{j}_{m''m'}(\pm\chi) = \sum_{\substack{\mu_1,\mu_2,\\ \mu_3,\mu_4}} (j_1\mu_1 j_2\mu_2|j_1 j_2 j m'')(j_1\mu_3 j_2\mu_4|j_1 j_2 j' m') \cdot d^{j_1}_{\mu_1\mu_3}(\pm\chi)\; d^{j_2}_{\mu_2\mu_4}(\pm\chi) \tag{5.51}$$

and by use of the orthogonality relations of the Clebsch-Gordan coupling coefficients. The equivalence of the three sets of solutions comes as no surprise since they all satisfy the same pair of differential equations. It is convenient to display individually the three sets of solutions because the solutions to the eigenvalue equations of the complex rotation group will be found in an analogous manner to that used for the group R(4).

VI. SOLUTIONS OF THE EIGENVALUE EQUATIONS OF THE COMPLEX ROTATION GROUP

It would be elegant to proceed in a manner exactly parallel to Chapter 5 in order to find the harmonic functions of the complex rotation group. However one soon gets into problems involving rotation matrices corresponding to complex angular momentum and also complex projections of angular momentum along the 3-axis. While several authors (Beltrametti and Luzzatto 1963, Andrews and Gunson 1964, Gunson 1965) have considered such rotation matrices, their properties are still rather poorly known and it appears best to avoid these difficulties at the present time. A heuristic account of how such a treatment could go is presented in the Appendix.

A set of solutions to the O(3,C) eigenvalue equations is found by an analytic continuation of R(4) solutions. Two other sets of solutions are obtained by making additional pure imaginary rotations of the complex three-vector. Much of the work in this chapter is quite similar to that of Chapter 5.

A. The Eigenvalue Equations

The infinitesimal generators $\vec{J}$ and $\vec{K}$ of the complex rotation group were calculated in Chapter 4 and are given by Eqs. 4.22 and 4.24. The Casimir operators are found to be

$$\vec{J}\cdot\vec{K} = \left(\frac{\partial}{\partial u} + \tanh u\right) iJ_2' + \frac{1}{\sinh u \ \cosh u} J_1' J_3' \quad , \qquad (6.1)$$

$$J^2 - K^2 = J^2 - \coth^2 u \, J_3'^{\,2} - \tanh^2 u \, J_1'^{\,2} + \frac{\partial^2}{\partial u^2} + 2\coth 2u \frac{\partial}{\partial u} \quad . \qquad (6.2)$$

The solutions to the $\vec{J}\cdot\vec{K}$ and J^2-K^2 eigenvalue equations will be called the O(3,C) harmonic functions. These functions are very similar to the R(4) solutions as found in Chapter 5. The relationship between the R(4) and O(3,C) infinitesimal generators and hence also the Casimir operators is clearly evident: (1) the angular momentum operators $\vec{M}$ of R(4) and $\vec{J}$ of O(3,C) are identical; (2) via the complex substitutions $\vec{N} \longrightarrow i\vec{K}$ and $\chi \longrightarrow iu$ in the R(4) operators, one obtains the O(3,C) operators. The above relationships suggest that the O(3,C) solutions may be found by analytic continuation of the R(4) solutions. This will be investigated in Section B.

By introducing the two operators $\vec{P}$ and $\vec{Q}$, where

$$\vec{P} = \tfrac{1}{2}(\vec{J} + i\vec{K}) \quad , \tag{6.3}$$

$$\vec{Q} = \tfrac{1}{2}(\vec{J} - i\vec{K}) \quad , \tag{6.4}$$

one finds that they obey the following commutation relations:

$$\left[P_i, Q_j\right] = 0 \quad , \tag{6.5}$$

$$\left[P_i, P_j\right] = i\, \epsilon_{ijk}P_k \quad , \tag{6.6}$$

$$\left[Q_i, Q_j\right] = i\, \epsilon_{ijk}Q_k \quad . \tag{6.7}$$

The two Casimir operators can be written in the form

$$F = J^2 - K^2 = 2(P^2 + Q^2) \quad , \tag{6.8}$$

$$G = \vec{J} \cdot \vec{K} = -i\,(P^2 - Q^2) \quad . \tag{6.9}$$

Thus each O(3,C) irreducible representation can be labeled by the pair of indices (p,q), where $P^2 = p(p+1)$ and $Q^2 = q(q+1)$ for the (p,q) representation. In Chapter 3 it was shown that in general the irreducible representations of O(3,C) can be specified by the pair of labels (ℓ, ν), where $J^2 - K^2 = \ell^2 - \nu^2 - 1$ and $\vec{J} \cdot \vec{K} = \ell\nu$ for the (ℓ, ν) representation. Combining these two pairs of labels by use of Eqs. 6.8 and 6.9, one easily shows that

$$p = \frac{-1 \pm (\ell + i\nu)}{2} \quad , \tag{6.10}$$

$$q = \frac{-1 \pm (\ell - i\nu)}{2} \quad . \tag{6.11}$$

The non-unitary finite-dimensional irreducible representations are characterized by p and q being positive integral or positive half-integral or zero. Since for unitary infinite-dimensional irreducible representations of O(3,C) the index ℓ takes on positive integral or positive half-integral values or zero and the index ν is any real number, then p and q are complex numbers as given by Eqs. 6.10 and 6.11. Only the unitary irreducible representations are of interest in this thesis.

B. Solutions via Analytic Continuation of R(4) Solutions

The R(4) work and the O(3,C) work are completely parallel. In each case a special pair of three-vectors is used to write the infinitesimal generators. The two pairs of Casimir operators and hence the two sets of eigenvalue equations are found to be very much alike. It seems reasonable to expect that an analytic continuation of the R(4) solutions would yield solutions to the O(3,C) equations. By letting $\chi \longrightarrow iu$, $p \longrightarrow p = \frac{1}{2}\left[-1 \pm (\ell + i\nu)\right]$, $q \longrightarrow q = \frac{1}{2}\left[-1 \pm (\ell - i\nu)\right]$, factorials $\longrightarrow$ gamma functions, and Clebsch-Gordan coefficients $\longrightarrow$ complex generalization of Clebsch-Gordan coefficients (Andrews and Gunson 1964) in the R(4) solutions given by Eqs. 5.27 and 5.28, one obtains

$$\langle u, \psi, \theta, \phi | \ell \nu j m \rangle = \sum_{m'} f^{\ell\nu}_{jm'}(u)\, D^{j*}_{m'm}(\psi, \theta, \phi) \quad , \tag{6.12}$$

where

$$f^{\ell\nu}_{jm'}(u) = \sum_{m_1} A^{\ell\nu}_{jm_1\, m'-m_1} \left[\frac{\Gamma(p-m_1+1)\,\Gamma(q-m'+m_1+1)}{\Gamma(p+m_1+1)\,\Gamma(q+m'-m_1+1)}\right]^{\frac{1}{2}} (pm_1 q m'-m_1|pqjm')'$$

$$\cdot\, \mathcal{P}^{m_1}_{p}(\cosh u)\; \mathcal{P}^{m'-m_1}_{q}(\cosh u) \; . \qquad (6.13)$$

The quantum numbers m_1 and $m'-m_1$ take on integral or half-integral values that can range from $-\infty$ to $+\infty$; the $A^{\ell\nu}_{jm_1 m'-m_1}$ is a factor still to be determined; the $(pm_1 q m'-m_1|pqjm')'$ denote the coupling coefficient which is a complex generalization of the Clebsch-Gordan coefficient as introduced by Andrews and Gunson (1964). Also $\mathcal{P}$ is the associated Legendre function $\mathcal{P}^{\mu}_{\nu}(z)$ for μ , ν , z unrestricted (Erdélyi, et al. 1953) and is related to the associated Legendre function $P^{\mu}_{\nu}(x)$ defined on the cut, $-1 < x < +1$, by the following:

$$P^{\mu}_{\nu}(x) = \frac{1}{2}\left[e^{\frac{i\mu\pi}{2}}\, \mathcal{P}^{\mu}_{\nu}(x+i0) + e^{-\frac{i\mu\pi}{2}}\, \mathcal{P}^{\mu}_{\nu}(x-i0)\right] \; . \qquad (6.14)$$

The preceding suggested O(3,C) solutions can be substituted into the eigenvalue equations and shown to be valid solutions. This will be done in a step-by-step procedure. By inserting $\langle u,\psi,\theta,\phi|\ell\nu jm\rangle$ into the $\vec{J}\cdot\vec{K}$ and J^2-K^2 equations as given by Eqs. 6.1 and 6.2 and by using the relations from Eq. 5.29 which describe the effect of the angular momentum operators on the rotation matrices, one gets the following (suppressing all labels except m' for $f^{\ell\nu}_{jm'}$):

$$\sum_{m'} \left(\frac{\partial}{\partial u} + \tanh u\right) \frac{f_{m'}}{2} \left[-\{j(j+1) - m'(m'+1)\}^{\frac{1}{2}} D^{j*}_{m'+1\, m} + \{j(j+1) - m'(m'-1)\}^{\frac{1}{2}} D^{j*}_{m'-1\, m} \right]$$

$$+ \sum_{m'} \frac{f_{m'}}{\sinh u \cosh u} \frac{m'}{2} \Big[\{j(j+1) - m'(m'+1)\}^{\frac{1}{2}} D^{j*}_{m'+1\, m} + \{j(j+1) - m'(m'-1)\}^{\frac{1}{2}} D^{j*}_{m'-1\, m} \Big] \tag{6.15}$$

$$= -i \left[p(p+1) - q(q+1) \right] \sum_{m'} f_{m'} D^{j*}_{m'\, m}$$

and

$$\sum_{m'} \left[j(j+1) - m'^2 \coth^2 u - \frac{j(j+1) - m'^2}{2} \tanh^2 u + \frac{\partial^2}{\partial u^2} + 2 \coth 2u \frac{\partial}{\partial u} \right] f_{m'} D^{j*}_{m'\, m}$$

$$- \sum_{m'} \frac{\tanh^2 u}{4} f_{m'} \Big[\{j(j+1) - m'(m'+1)\}^{\frac{1}{2}} \{j(j+1) - (m'+1)(m'+2)\}^{\frac{1}{2}} D^{j*}_{m'+2\, m} + \{j(j+1) - m'(m'-1)\}^{\frac{1}{2}} \{j(j+1) - (m'-1)(m'-2)\}^{\frac{1}{2}} D^{j*}_{m'-2\, m} \Big] \tag{6.16}$$

$$= 2 \left[p(p+1) + q(q+1) \right] \sum_{m'} f_{m'} D^{j*}_{m'\, m} .$$

Collection of terms for $D^{j*}_{m'm}$, with m' and m fixed, gives coupled differential equations for the functions $f_{m'}$:

$$- \left(\frac{\partial}{\partial u} + \tanh u - \frac{m'-1}{\sinh u \cosh u} \right) f_{m'-1} \{j(j+1) - m'(m'-1)\}^{\frac{1}{2}}$$

$$+ \left(\frac{\partial}{\partial u} + \tanh u + \frac{m'+1}{\sinh u \cosh u} \right) f_{m'+1} \{j(j+1) - m'(m'+1)\}^{\frac{1}{2}} \tag{6.17}$$

$$+ 2i \left[p(p+1) - q(q+1) \right] f_{m'} = 0$$

and

$$\left[j(j+1) - m'^2 \coth^2 u - \frac{j(j+1)-m'^2}{2} \tanh^2 u + \frac{\partial^2}{\partial u^2} + 2\coth 2u \frac{\partial}{\partial u} - 2p(p+1) - 2q(q+1)\right] f_{m'}$$

$$- \frac{\tanh^2 u}{4} \left\{j(j+1)-(m'-1)(m'-2)\right\}^{\frac{1}{2}} \left\{j(j+1) - m'(m'-1)\right\}^{\frac{1}{2}} f_{m'-2} \tag{6.18}$$

$$- \frac{\tanh^2 u}{4} \left\{j(j+1)-(m'+1)(m'+2)\right\}^{\frac{1}{2}} \left\{j(j+1)-m'(m'+1)\right\}^{\frac{1}{2}} f_{m'+2} = 0 .$$

By use of the associated Legendre function recursion relationships for z unrestricted (Erdélyi, et al. 1953),

$$(z^2-1) \frac{d}{dz} \mathcal{P}_\nu^\mu(z) = -\mu z \mathcal{P}_\nu^\mu(z) + (\nu+\mu)(\nu-\mu+1)(z^2-1)^{\frac{1}{2}} \mathcal{P}_\nu^{\mu-1}(z),$$

$$= +\mu z \mathcal{P}_\nu^\mu(z) + (z^2-1)^{\frac{1}{2}} \mathcal{P}_\nu^{\mu+1}(z) , \tag{6.19}$$

and the assumption that the complex generalization of the Clebsch-Gordan coefficients obey the same recurrence formulas given by Eq. 5.33 as do the ordinary R(3) Clebsch-Gordan coefficients (shown to be a valid assumption in Chapter 7), one can show that $f_{m'}$ expressed by Eq. 6.13 with

$$A^{\ell\nu}_{jm_1m'-m_1} = (-i)^{m'} (-1)^{m'-m_1} = (i)^{m'} (-1)^{m_1} \tag{6.20}$$

does indeed satisfy the coupled differential equations as given by Eqs. 6.17 and 6.18. This completes the proof of the validity of the suggested O(3,C) solutions. Therefore a set of harmonic functions of the complex rotation group can be written as

$$\langle u,\Psi,\theta,\Phi \,|\, \ell\nu jm \rangle = \sum_{m'} f^{\ell\nu}_{jm'}(u)\, D^{j\,*}_{m'm}(\Psi,\theta,\Phi) \quad , \tag{6.21}$$

where

$$f^{\ell\nu}_{jm'}(u) = \sum_{m_1} i^{m'}(-1)^{m_1}\left[\frac{\Gamma(p-m_1+1)\,\Gamma(q-m'+m_1+1)}{\Gamma(p+m_1+1)\,\Gamma(q+m'-m_1+1)}\right]^{\frac{1}{2}} \cdot (pm_1 q\, m'-m_1 | pq\, jm')'\, \mathcal{P}^{m_1}_{p}(\cosh u)\, \mathcal{P}^{m'-m_1}_{q}(\cosh u) \quad . \tag{6.22}$$

C. Additional Sets of Solutions

Two other sets of O(3,C) solutions could be obtained by direct analytic continuation of the R(4) solutions given by Eqs. 5.40 and 5.48. However the physical interpretation of the new complex angle appearing in the solutions would be lost in the process. Therefore the technique of analytic continuation will not be used, but instead the method of Chapter 5 will be employed. Additional solutions are found by working with the original eigenvalue equations expressed by Eqs. 6.15 and 6.16 and making added rotations of the complex three-vector. The work will be parallel to that of the R(4) case.

In Section D of Chapter 5 two new sets of R(4) solutions were gotten by making additional active rotations of the three-vectors $\vec{\rho}$ and $\vec{\omega}$ through angles of $\pm\chi$ in the 3-1 plane. These added rotations carried the three-vectors from an orientation of $\vec{\rho}$ along the 3-axis to a final orientation of either $\vec{\alpha} = \vec{\rho} + \vec{\omega}$ or $\vec{\beta} = \vec{\rho} - \vec{\omega}$ along the 3-axis. The new solutions were seen to be obtainable from the original R(4) eigenvalue equations by making the following change:

$$f_{m''} \longrightarrow f_{m''} = \sum_{m'} g_{m'} \, d^{j}_{m'' m'}(\mp \chi) \quad , \tag{6.23}$$

where the $f_{m''}$ are defined by their presence in the R(4) solutions:

$$\langle \chi, \psi, \theta, \phi | pqjm \rangle = \sum_{m''} f^{pq}_{jm''}(\chi) \, D^{j*}_{m'' m}(\psi, \theta, \phi) \quad . \tag{6.24}$$

The $d^{j}_{m'' m'}(\mp \chi)$ are the rotation matrix elements for a real rotation of $\mp \chi$ about the 2-axis, and the functions $g_{m'}$ were found by solving the equations resulting from the substitution of Eq. 6.23 into the original coupled differential equations for $f_{m''}$. It was seen that the substitution as expressed by Eq. 6.23 effectively carries out the additional real rotations of $\pm \chi$ of the three-vectors.

A corresponding treatment can be carried out for the complex rotation group. Additional pure imaginary rotations of $\pm$ iu about the 2-axis can be made on the complex three-vector $\hat{z} = \vec{\lambda} + i\vec{\mu}$, whose orientation is such that $\vec{\lambda}$ lies along the 3-axis and $\vec{\mu}$ along the 1-axis, in order to lead one to other sets of O(3,C) harmonic functions. These added rotations are simply velocity transformations, where $\tanh u = \frac{v}{c}$, along the 2-axis. After a rotation of + iu about the 2-axis, the rotated vector $\hat{z}'$ is seen to lie along the 3-axis whereas $(\hat{z}^*)'$ is at an angle of 2iu with respect to the 3-axis:

$$\begin{aligned} \hat{z} = \hat{\imath}_1(i\mu) + \hat{\imath}_3\lambda &\longrightarrow \hat{z}' = \hat{\imath}_3 \quad , \\ \hat{z}^* = \hat{\imath}_1(-i\mu) + \hat{\imath}_3\lambda &\longrightarrow (\hat{z}^*)' = \hat{\imath}_1(-i \sinh 2u) + \hat{\imath}_3 \cosh 2u \quad . \end{aligned} \tag{6.25}$$

If the rotation is $-iu$ about the 2-axis, the rotated vector $(\hat{z}^*)'$ is seen to lie along the 3-axis whereas $\hat{z}'$ is at an angle of $2iu$ with respect to the 3-axis:

$$\begin{aligned} \hat{z} &= \hat{\imath}_1(i\mu) + \hat{\imath}_3\lambda \longrightarrow \hat{z}' = \hat{\imath}_1(i\sinh 2u) + \hat{\imath}_3\cosh 2u , \\ \hat{z}^* &= \hat{\imath}_1(-i\mu) + \hat{\imath}_3\lambda \longrightarrow (\hat{z}^*)' = \hat{\imath}_3 . \end{aligned} \tag{6.26}$$

The O(3,C) solutions corresponding to these new final orientations of the complex three-vector can be obtained by making the substitution

$$f_{m''} \longrightarrow f_{m''} = \sum_{m'} g_{m'}\, d^j_{m''m'}(\mp iu) \tag{6.27}$$

in the coupled differential equations given by Eqs. 6.17 and 6.18. The $f_{m''}$ are defined by their presence in the expression for the O(3,0) solutions:

$$\langle u,\psi,\theta,\phi|\ell\nu jm\rangle = \sum_{m''} f^{\ell\nu}_{jm''}(u)\, D^{j*}_{m''m}(\psi,\theta,\phi) . \tag{6.28}$$

The functions $g_{m'}$ are to be found by solving the new equations, and the $d^j_{m''m'}(\mp iu)$ denote the elements of the rotation matrix for rotations of $\mp iu$ about the 2-axis. In explicit form, one can write

$$d^j_{m'm}(iu) = e^{-i\frac{\pi}{2}(m'-m)}\left[\frac{(j-m)!\,(j+m')!}{(j+m)!\,(j-m')!}\right]^{\frac{1}{2}} \frac{\left(\cosh\frac{u}{2}\right)^{2j+m-m'}\left(\sinh\frac{u}{2}\right)^{m'-m}}{(m'-m)!}$$

$$\cdot\, {}_2F_1\left(m'-j, -m-j\,;\, m'-m+1\,;\, \tanh^2\frac{u}{2}\right) \;;\quad m' \geq m , \tag{6.29}$$

where the phase is chosen so that the relationship between the $d^j_{m'm}$ with real argument and the $\mathfrak{d}^j_{m'm}$ with unrestricted argument is

$$\begin{aligned} d^j_{m'm}(\theta) &= \mathfrak{d}^j_{m'm}(\theta + i0) \quad , \\ &= e^{i\pi(m'-m)}\, \mathfrak{d}^j_{m'm}(\theta - i0) \quad . \end{aligned} \tag{6.30}$$

The substitution expressed by Eq. 6.27 effectively carries out the additional pure imaginary rotations of $\pm$ iu of the complex three-vector. This approach enables one to avoid the use of complex Euler angles in describing the new final orientation of the complex three-vector $\hat{z}$. Instead, the real Euler angles ψ , θ , ϕ which denote the rotation of $\hat{z}$ from its original position to an orientation such that $\vec{\lambda}$ and $\vec{\mu}$ lie along the 3-axis and 1-axis respectively are retained, and the additional rotations of magnitude $\pm$ iu about the 2-axis are built into the solutions by use of Eq. 6.27.

First of all, for the case of an additional rotation of the complex three-vector $\hat{z}$ through angle + iu about the 2-axis one writes the solutions as

$$\langle u, \psi, \theta, \phi \,|\, \ell \nu j m \rangle_1 = \sum_{m''} f^{\ell\nu}_{jm''}(u)\, D^{j\,*}_{m''m}(\psi, \theta, \phi) \;, \tag{6.31}$$

where

$$f^{\ell\nu}_{jm''}(u) = \sum_{m'} (-1)^{m'}\, g^{\ell\nu}_{jm'}(u)\, \mathfrak{d}^j_{m''m'}(-iu) \quad . \tag{6.32}$$

By substituting the above form for $f_{m''}$ into the coupled differential equations given by Eqs. 6.17 and 6.18 and by using the following differential relations

$$\pm \frac{d}{du} d^{j}_{m'm}(-iu) = \frac{m' \cosh u - m}{\sinh u} d^{j}_{m'm}(-iu) + i\left[j(j+1) - m'(m'\pm 1)\right]^{\frac{1}{2}} d^{j}_{m'\pm 1\, m}(-iu) \quad , \tag{6.33}$$

one obtains two differential equations in terms of both $d^{j}_{m''m'}$ and $g_{m'}$. The $d^{j}_{m''m'}$ -dependence can be removed by use of the recursion formulas

$$\left[\ell(\ell+1)-\nu(\nu+1)\right]^{\frac{1}{2}} d^{\ell}_{\nu+1\,\mu}(iu) + \left[\ell(\ell+1)-\nu(\nu-1)\right]^{\frac{1}{2}} d^{\ell}_{\nu-1\,\mu}(iu) = \frac{2i(\mu-\nu\cosh u)}{\sinh u} d^{\ell}_{\nu\mu}(iu),$$

$$\left[\ell(\ell+1)-\nu(\nu+1)\right]^{\frac{1}{2}} d^{\ell}_{\nu+1\,\mu}(iu) - \left[\ell(\ell+1)-\nu(\nu-1)\right]^{\frac{1}{2}} d^{\ell}_{\nu-1\,\mu}(iu)$$
$$= \left[\ell(\ell+1)-\mu(\mu-1)\right]^{\frac{1}{2}} d^{\ell}_{\nu\,\mu-1}(iu) - \left[\ell(\ell+1)-\mu(\mu+1)\right]^{\frac{1}{2}} d^{\ell}_{\nu\,\mu+1}(iu) , \tag{6.34}$$

$$\left[\ell(\ell+1)-\nu(\nu+1)\right]^{\frac{1}{2}} d^{\ell}_{\nu+1\,\mu}(iu) + \left[\ell(\ell+1)-\nu(\nu-1)\right]^{\frac{1}{2}} d^{\ell}_{\nu-1\,\mu}(iu)$$
$$= \pm\left[\ell(\ell+1)-\mu(\mu+1)\right]^{\frac{1}{2}} d^{\ell}_{\nu\,\mu+1}(iu) \pm \left[\ell(\ell+1)-\mu(\mu-1)\right]^{\frac{1}{2}} d^{\ell}_{\nu\,\mu-1}(iu)$$
$$- \frac{2i(\mu\pm\nu)\sinh u}{(1\pm\cosh u)} d^{\ell}_{\nu\mu}(iu)$$

and by collecting terms for $(-1)^{m'} d^{j}_{m''m'}(-iu)$, with m'' and m' fixed. One obtains two coupled differential equations for the $g_{m'}$ functions:

$$-\left\{j(j+1)-m'(m'+1)\right\}^{\frac{1}{2}}\left[\frac{\partial}{\partial u} + 2(m'+1)\coth 2u\right] g_{m'+1}$$
$$+\left\{j(j+1)-m'(m'-1)\right\}^{\frac{1}{2}}\left[\frac{\partial}{\partial u} - 2(m'-1)\coth 2u\right] g_{m'-1} \tag{6.35}$$
$$-2i\left[j(j+1)-2m'^{2}-p(p+1)+q(q+1)\right] g_{m'} = 0$$

and

$$
\begin{aligned}
&\left[2j(j+1)-2p(p+1)-2q(q+1)-4m'^2\coth^2 2u+\frac{\partial^2}{\partial u^2}+2\coth 2u\frac{\partial}{\partial u}\right]g_{m'} \\
&+i\left\{j(j+1)-m'(m'-1)\right\}^{\frac{1}{2}}\left[\frac{\partial}{\partial u}-2(m'-1)\coth 2u\right]g_{m'-1} \qquad\qquad (6.36) \\
&-i\left\{j(j+1)-m'(m'+1)\right\}^{\frac{1}{2}}\left[\frac{\partial}{\partial u}+2(m'+1)\coth 2u\right]g_{m'+1} = 0 .
\end{aligned}
$$

The solutions $g_{m'}$ to the two preceding equations are found to be

$$
g^{\ell\nu}_{jm'}(u) = (-i)^{m'}\left[\frac{\Gamma(q-m'+1)}{\Gamma(q+m'+1)}\right]^{\frac{1}{2}}(p0qm'|pqjm')'\,\mathcal{P}^{m'}_q(\cosh 2u), \qquad (6.37)
$$

where p and q are given by Eqs. 6.10 and 6.11 and $(p0qm'|pqjm')'$ again denote the complex generalization of the Clebsch-Gordan coefficient. By combining Eqs. 6.31, 6.32, and 6.37 one can write in explicit form:

$$
\begin{aligned}
Z^{\ell\nu}_{jm}(u,\psi,\theta,\phi) \equiv \langle u,\psi,\theta,\phi|\ell\nu jm\rangle_1 &= \sum_{m',m''} i^{m'}\left[\frac{\Gamma(q-m'+1)}{\Gamma(q+m'+1)}\right]^{\frac{1}{2}}(p0qm'|pqjm')' \qquad (6.38) \\
&\cdot \mathcal{P}^{m'}_q(\cosh 2u)\, d^j_{m''m'}(-iu)\, D^{j*}_{m''m}(\psi,\theta,\phi)
\end{aligned}
$$

as a set of harmonic functions of the complex rotation group. This completes the work for the case of additional rotation of the complex three-vector $\hat{z}$ through angle $+iu$ about the 2-axis.

The case of additional rotation of $\hat{z}$ through an angle of $-iu$ about the 2-axis goes exactly parallel to the preceding development for rotation of $+iu$. One writes the corresponding solutions as:

$$
\langle u,\psi,\theta,\phi|\ell\nu jm\rangle_2 = \sum_{m''}\tilde{f}^{\ell\nu}_{jm''}(u)\, D^{j*}_{m''m}(\psi,\theta,\phi), \qquad (6.39)
$$

where

$$\tilde{f}^{\ell\nu}_{jm''}(u) = \sum_{m'} h^{\ell\nu}_{jm'}(u)\, d^{j}_{m''m'}(+iu) \quad . \tag{6.40}$$

By substitution of $\tilde{f}_{m''}$ into the coupled differential equations given by Eqs. 6.17 and 6.18 and by using the differential relationships for $d^{j}_{m''m'}$ (+iu) as given by Eq. 6.33, one obtains two differential equations in terms of both $h_{m'}$ and $d^{j}_{m''m'}$ (+iu). The $d^{j}_{m''m'}$ -dependence can be removed from the equations by use of the recursion formulas expressed by Eq. 6.34 and by collecting terms for a particular $d^{j}_{m''m'}$ with m'' and m' fixed. One is left with two coupled differential equations for $h_{m'}$ which are identical to Eqs. 6.35 and 6.36 if one makes the interchange $p \rightleftarrows q$. The solutions $h_{m'}$ are found to be:

$$h^{\ell\nu}_{jm'}(u) = (-i)^{m'}\left[\frac{\Gamma(p-m'+1)}{\Gamma(p+m'+1)}\right]^{\frac{1}{2}} (pm'q0|pqjm')'\, \mathcal{P}^{m'}_{p}(\cosh 2u)\, , \tag{6.41}$$

where all of the symbols are the same as defined previously. By combining Eqs. 6.39, 6.40, and 6.41 one has another set of O(3,C) harmonic functions:

$$\begin{aligned}\tilde{Z}^{\ell\nu}_{jm}(u,\psi,\theta,\phi) &\equiv \langle u,\psi,\theta,\phi|\ell\nu jm\rangle_2 \\ &= \sum_{m',m''} (-i)^{m'}\left[\frac{\Gamma(p-m'+1)}{\Gamma(p+m'+1)}\right]^{\frac{1}{2}} (pm'q0|pqjm')' \\ &\quad \cdot \mathcal{P}^{m'}_{p}(\cosh 2u)\, d^{j}_{m''m'}(+iu)\, D^{j*}_{m''m}(\psi,\theta,\phi)\, .\end{aligned} \tag{6.42}$$

This completes the work for the case of additional rotation of the complex three-vector through angle $-iu$ about the 2-axis.

Three sets of harmonic functions of the complex rotation group have been found. The first set, given by Eqs. 6.21 and 6.22, are obtained via an analytic continuation of R(4) solutions. They correspond to a final orientation of the complex three-vector $\hat{z}$ such that $\vec{\lambda}$ and $\vec{\mu}$ lie along the 3-axis and 1-axis respectively. The second and third sets of solutions, given by Eqs. 6.38 and 6.42, are gotten by making additional rotations of $\hat{z}$ through angles of $\pm iu$ about the 2-axis. They correspond to final orientations such that the rotated vector $\hat{z}'$ lies along the 3-axis for the first case and the rotated vector $(\hat{z}^*)'$ lies along the 3-axis in the second case. The three sets of O(3,C) harmonic functions appear to be nonequivalent. A Clebsch-Gordan series reduction comparable to Eq. 5.51 for rotation matrices with complex angular momentum has been derived by Andrews and Gunson (1964). However the series is in terms of "local representations" which are no longer unitary nor true representations. Also no orthogonality theorem has been established for the coupling coefficients corresponding to complex generalizations of the R(3) Clebsch-Gordan coefficients. Both of the preceding items, a Clebsch-Gordan series and orthogonality of the coupling coefficients, are needed before one can study more closely the relationship between the three sets of O(3,C) harmonic functions.

VII. STUDY OF THE COUPLING COEFFICIENT

A coupling coefficient in the form of a complex generalization of the R(3) Clebsch-Gordan coefficient appears in the solutions to the eigenvalue equations of the complex rotation group as found in the preceding chapter. In this chapter the origin of the coupling coefficient will be discussed and a formal definition will be made. The properties of the coefficient, namely the recurrence relationships which it satisfies, will be developed.

A. Origin and Definition of the Coupling Coefficient

The Clebsch-Gordan (C-G) coefficients of the three-dimensional rotation group are simply elements of the unitary transformation connecting the coupled representation $|JM\rangle$ with the uncoupled representation $|j_1 m_1\rangle\,|j_2 m_2\rangle$ of angular momentum eigenfunctions:

$$|JM\rangle = \sum_{m_1, m_2} (j_1 m_1 j_2 m_2 | j_1 j_2 JM)\,|j_1 m_1\rangle\,|j_2 m_2\rangle\,, \tag{7.1}$$

where j_1, j_2, and J are positive integers, half-integers, or zero. The orthonormality of the eigenfunctions require that the coefficients obey orthogonality relations. Recurrence relations for the coefficients are obtained by application of the appropriate angular momentum operators (for detailed treatment, see Brink and Satchler 1962), and these are sufficient for an explicit determination up to a phase factor of the C-G coefficients. Explicit expressions for these coefficients have been derived by Wigner (1931) and Racah (1942).

Eq. 7.1 can be thought of from a group theoretical standpoint. The eigenfunctions $|j_1 m_1\rangle\,|j_2 m_2\rangle$ span a $(2j_1 + 1)\cdot(2j_2 + 1)$ manifold. If the coordinate system is given a rotation, the eigenfunctions of the uncoupled basis transform according to a $(2j_1 + 1)\cdot(2j_2 + 1)$ -dimensional representation of the rotation group, namely $D^{j1} \times D^{j2}$ a direct product of rotation matrices. This direct product is reducible and the transformation of Eq. 7.1 can be used to reduce it to irreducible components D^J. The eigenfunctions $|JM\rangle$ form a basis for the reduced representation. The explicit reduction is written as

$$D^{j_1}_{\mu_1 m_1} D^{j_2}_{\mu_2 m_2} = \sum_J (j_1\mu_1 j_2\mu_2 | j_1 j_2 J\mu)(j_1 m_1 j_2 m_2 | j_1 j_2 JM) D^J_{\mu M} \qquad (7.2)$$

and is known as the Clebsch-Gordan series. Thus the C-G coefficients appear in the coupling rule for the rotation matrices.

The coupling coefficient used in Chapter 6 is essentially a complex generalization of the R(3) Clebsch-Gordan coefficient. It couples together complex values of the angular momentum quantum numbers and appears in the reduction of products of rotation matrices with complex angular momentum (the complex generalization of the Clebsch-Gordan series reduction). Rotation matrices corresponding to complex angular momentum arose in the study of stable particle scattering amplitudes. It was found that in order to make a complete investigation of the amplitudes an analytic continuation of the physical real values of angular momentum must be made to complex values. The extension of the unitary representations of R(3), i.e. the rotation matrices, to complex angular momentum leads one to matrices which are no longer unitary or true representations since not all of the usual properties of a representation are retained. Gunson (1965) developed the problem in terms of "local representations" of the R(3) group whereby the association between the products of operators and the products of corresponding group elements can be made only in a neighborhood of the identity. Beltrametti and Luzzatto (1963) built up a set of infinite matrices corresponding to the R(3) group elements; however these matrices do not obey the product rules that true representations follow. Further relations for the "local representations" were derived by Andrews and Gunson (1964) who found a complex generalization of the Clebsch-Gordan reduction formula for tensor products of "local representations." The coupling coefficients for the "local representation" matrices of the first kind are essentially just complex generalizations of the R(3) C-G coefficients:

$$
\begin{aligned}
(j_1 m_1 j_2 m_2 | j_1 j_2 j m)' \\
= \left\{ \frac{\Gamma(j_1+m_1+1)\,\Gamma(j_2-m_2+1)}{\Gamma(j_1-m_1+1)\,\Gamma(j_2+m_2+1)} \, \Gamma(j-m+1)\,\Gamma(j+m+1)\,(2j+1) \right\}^{\frac{1}{2}} \\
\cdot \left\{ \frac{\Gamma(j_1-j_2+j+1)\,\Gamma(-j_1+j_2+j+1)}{\Gamma(j_1+j_2-j+1)\,\Gamma(j_1+j_2+j+2)} \right\}^{\frac{1}{2}} \frac{1}{\Gamma(j-j_2+m_1+1)\,\Gamma(j-j_1-m_2+1)} \qquad (7.3) \\
\cdot \, {}_3F_2 \left[\begin{matrix} -j_1+m_1 \,, & -j_2-m_2 \,, & -j_1-j_2+j \\ j-j_1-m_2+1 \,, & j-j_2+m_1+1 & \end{matrix} \right] ,
\end{aligned}
$$

where ${}_3F_2$ is the generalized hypergeometric function of unit argument. The above expression reduces to the Wigner form for the C-G coefficients if $(j_1 - m_1)$ and $(j_2 - m_2)$ are integral. Since the rotation matrices with complex angular momentum are neither unitary nor true representations, the corresponding coupling coefficients no longer automatically obey orthogonality relations, recurrence formulas, and other properties possessed by the R(3) C-G coefficients. This is a consequence of the absence of the complete group structure.

In this thesis the coupling coefficients appearing in the solutions to the eigenvalue equations of the complex rotation group are defined to be that given by Eq. 7.3. Properties of the coefficients, in particular recurrence formulas, will be derived from the mathematical definition. This treatment then is reverse to the development of the R(3) C-G coefficients, whereby the orthogonality and recurrence properties of the coefficients are used to find an explicit expression.

B. Properties of the Coupling Coefficient

The coupling coefficient defined by Eq. 7.3 in terms of the ${}_3F_2$ hypergeometric function can be shown to satisfy the usual R(3) Clebsch-Gordan recurrence relations as expressed by Eq. 5.33. The proofs are carried out by the use of Rainville's (1945) contiguous relations for the hypergeometric function and the application of Bailey's (1935) transformation involving two ${}_3F_2$ series.

An abbreviated notation for the ${}_pF_q$ functions will be used:

$$F \equiv {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; x), \tag{7.4}$$

$$F(\alpha_1+) \equiv {}_pF_q(\alpha_1+1, \alpha_2, \ldots \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; x), \tag{7.5}$$

$$F(\beta_1-) \equiv {}_pF_q(\alpha_1, \alpha_2, \ldots \alpha_p; \beta_1-1, \beta_2, \ldots, \beta_q; x), \tag{7.6}$$

and similarly for other functions. The useful contiguous function relationships from Rainville (1945) are as follows:

$$(\alpha_1 - \beta_k + 1)\, F = \alpha_1 F(\alpha_1 +) - (\beta_k - 1)\, F(\beta_k -) \quad ; \; k = 1, 2, \ldots q \; ; \tag{7.7}$$

$$[(1-x)\alpha_1 + (A-B)\,x]\, F = (1-x)\alpha_1 F(\alpha_1 +) - x \sum_{j=1}^{q} U_j \, F(\beta_j +) \; ; \quad p = q+1 \; ; \tag{7.8}$$

$$(1-x)\, F = F(\alpha_k -) + x \sum_{j=1}^{q} W_{jk} \, F(\beta_j +) \quad ; \; k = 1, 2, \ldots p \quad , \tag{7.9}$$

where the notations are

$$A \equiv \sum_{s=1}^{p} \alpha_s \quad , \tag{7.10}$$

$$B \equiv \sum_{s=1}^{q} \beta_s \quad , \tag{7.11}$$

$$U_j = \frac{\prod_{s=1}^{p} (\alpha_s - \beta_j)}{\beta_j \prod_{s=1\,(\neq j)}^{q} (\beta_s - \beta_j)} \quad , \tag{7.12}$$

$$W_{jk} = \frac{\prod_{s=1\,(\neq k)}^{p} (\alpha_s - \beta_j)}{\beta_j \prod_{s=1\,(\neq j)}^{q} (\beta_s - \beta_j)} = \frac{U_j}{\alpha_k - \beta_j} \quad . \tag{7.13}$$

The first recurrence relation to be proved is

$$\begin{aligned} &[(J+M+1)(J-M)]^{\frac{1}{2}} (j_1 m_1 j_2 m_2 | j_1 j_2 J\, M+1)' \\ &\quad = [(j_1 - m_1 + 1)(j_1 + m_1)]^{\frac{1}{2}} (j_1\, m_1 - 1\, j_2 m_2 | j_1 j_2 J M)' \\ &\quad + [(j_2 - m_2 + 1)(j_2 + m_2)]^{\frac{1}{2}} (j_1 m_1\, j_2\, m_2 - 1 | j_1 j_2 J M)' \, . \end{aligned} \tag{7.14}$$

A special equation connecting $F(\alpha_1 -)$, $F(\beta_1 +)$, and F can be derived from Rainville's contiguous relations. For the ${}_3F_2$ functions in Eq. 7.9 with $x=1$ and $k=1$ one writes:

$$F(\alpha_1-) = -W_{11}F(\beta_1+) - W_{21}F(\beta_2+) \quad . \tag{7.15}$$

By setting $x=1$ in Eq. 7.8 for ${}_3F_2$ functions one gets

$$(A-B)F = -U_1 F(\beta_1+) - U_2 F(\beta_2+) \quad . \tag{7.16}$$

Elimination of $F(\beta_2 +)$ from Eqs. 7.15 and 7.16 and the use of the definitions given in Eqs. 7.10, 7.11, 7.12, and 7.13 yields the equation:

$$(\beta_2-\alpha_1)F(\alpha_1-) = (\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)F - \frac{(\alpha_2-\beta_1)(\alpha_3-\beta_1)}{\beta_1}F(\beta_1+) \, . \tag{7.17}$$

One defines the parameters to be

$$\begin{aligned}
\alpha_1 &= J - M + 1 \quad , \\
\alpha_2 &= J + j_1 - j_2 + 1 \quad , \\
\alpha_3 &= J + j_1 + j_2 + 2 \quad , \\
\beta_1 &= J + j_1 - m_2 + 2 \quad , \\
\beta_2 &= 2J + 2 \quad ,
\end{aligned} \tag{7.18}$$

and then rewrites Eq. 7.17. A more identifiable form for the equation can be obtained by the use of Bailey's transformation for ${}_3F_2$ with unit argument:

$${}_3F_2\left[\begin{matrix} a, b, c; \\ e, f \end{matrix}\right] = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\,{}_3F_2\left[\begin{matrix} e-a, f-a, s; \\ s+b, s+c \end{matrix}\right] , \tag{7.19}$$

where $s = e + f - a - b - c$.

It is desirable to use the inverse transformation, that is to identify

$$
\begin{aligned}
e - a &= \alpha_1 - 1 = J - M \quad , \\
f - a &= \alpha_2 = J + j_1 - j_2 + 1 \quad , \\
s &= \alpha_3 = J + j_1 + j_2 + 2 \quad , \\
s + b &= \beta_1 = J + j_1 - m_2 + 2 \quad , \\
s + c &= \beta_2 = 2J + 2 \quad ,
\end{aligned}
\tag{7.20}
$$

where $M + 1 = m_1 + m_2$.

This yields the new parameters:

$$
\begin{aligned}
a &= -j_1 + m_1 \quad , \\
b &= -j_2 - m_2 \quad , \\
c &= J - j_1 - j_2 \quad , \\
e &= J - j_1 - m_2 + 1 \quad , \\
f &= J - j_2 + m_1 + 1 \quad .
\end{aligned}
\tag{7.21}
$$

In terms of the new parameters Eq. 7.17 can be written in the form:

$$
\begin{aligned}
(f-b-1)\, F\left[\begin{matrix} a, b, c\, ; \\ e, f \end{matrix}\right] &= (f-1)\, F\left[\begin{matrix} a-1, b, c\, ; \\ e, f-1 \end{matrix}\right] \\
&\quad + \frac{b(c-e)}{e}\, F\left[\begin{matrix} a, b+1, c\, ; \\ e+1, f \end{matrix}\right] ,
\end{aligned}
\tag{7.22}
$$

which becomes

$$
\begin{aligned}
&(J+M+1)\; {}_3F_2\left[\begin{matrix} -j_1+m_1, -j_2-m_2, -j_1-j_2+J\, ; \\ J-j_1-m_2+1, J-j_2+m_1+1 \end{matrix}\right] \\
&= (J-j_2+m_1)\; {}_3F_2\left[\begin{matrix} -j_1+m_1-1, -j_2-m_2, -j_1-j_2+J\, ; \\ J-j_1+m_2+1, J-j_2+m_1 \end{matrix}\right] \\
&\quad + \frac{(j_2-m_2+1)(j_2+m_2)}{(J-j_1-m_2+1)}\; {}_3F_2\left[\begin{matrix} -j_1+m_1, -j_2-m_2+1, -j_1-j_2+J\, ; \\ J-j_1-m_2+2, J-j_2+m_1+1 \end{matrix}\right] .
\end{aligned}
\tag{7.23}
$$

Multiplying Eq. 7.23 by the factor

$$\left\{\frac{\Gamma(j_1+m_1+1)\Gamma(j_2-m_2+1)}{\Gamma(j_1-m_1+1)\Gamma(j_2+m_2+1)}\,\Gamma(J-M+1)\Gamma(J+M+1)(2J+1)\,\frac{\Gamma(j_1-j_2+J+1)}{\Gamma(j_1+j_2-J+1)}\right\}^{\frac{1}{2}}$$

$$\cdot\left\{\frac{\Gamma(-j_1+j_2+J+1)}{\Gamma(j_1+j_2+J+2)}\right\}^{\frac{1}{2}}\frac{1}{\Gamma(J-j_2+m_1+1)\,\Gamma(J-j_1-m_2+1)}$$

collecting terms, and comparing with the coupling coefficient definition given by Eq. 7.3, one finally gets the recurrence relation as stated in Eq. 7.14.

In a very similar manner as above the recurrence relation

$$\begin{aligned}\left[(J-M+1)(J+M)\right]^{\frac{1}{2}}&(j_1m_1j_2m_2|j_1j_2J\,M-1)'\\ &=\left[(j_1+m_1+1)(j_1-m_1)\right]^{\frac{1}{2}}(j_1m_1+1\,j_2m_2|j_1j_2JM)' \qquad (7.24)\\ &\quad+\left[(j_2+m_2+1)(j_2-m_2)\right]^{\frac{1}{2}}(j_1m_1\,j_2m_2+1|j_1j_2JM)'\end{aligned}$$

can be proved. A special equation linking $F(\alpha_2 -)$, $F(\beta_1 +)$, and F can be derived from Rainville's contiguous relations. For ${}_3F_2$ in Eq. 7.9 with x=1 and k=2 one has

$$F(\alpha_2 -) = -W_{12}\,F(\beta_1 +) - W_{22}\,F(\beta_2 +)\ . \qquad (7.25)$$

Again for ${}_3F_2$ in Eq. 7.8 with x=1 one gets

$$(A - B)\,F = -U_1\,F(\beta_1+) - U_2\,F(\beta_2+)\ . \qquad (7.26)$$

Elimination of $F(\beta_2 +)$ from the preceding two equations and use of the definitions given in Eqs. 7.10, 7.11, 7.12, 7.13 gives the equation

$$(\beta_2-\alpha_2)\,F(\alpha_2-) = -\frac{(\alpha_1-\beta_1)(\alpha_3-\beta_1)}{\beta_1}\,F(\beta_1+) + (\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)\,F \quad . \tag{7.27}$$

Since

$${}_3F_2\left[\begin{matrix} a,\ b,\ c\ ; \\ e,\ f \end{matrix}\ x\right] = {}_3F_2\left[\begin{matrix} b,\ a,\ c; \\ e,\ f \end{matrix}\ x\right] , \tag{7.28}$$

one can write Bailey's transformation for ${}_3F_2$ with unit argument as

$${}_3F_2\left[\begin{matrix} a,b,c; \\ e,f \end{matrix}\right] = \frac{\Gamma(e)\,\Gamma(f)\,\Gamma(s)}{\Gamma(b)\,\Gamma(s+a)\,\Gamma(s+c)}\ {}_3F_2\left[\begin{matrix} e-b,f-b,s; \\ s+a\ ,\ s+c \end{matrix}\right] , \tag{7.29}$$

where $s = e + f - a - b - c$. Again the inverse transformation is used with the identification

$$\begin{aligned} e - b &= \alpha_1 = J + j_2 - j_1 + 1 \quad , \\ f - b &= \alpha_2 - 1 = J + M \quad , \\ s &= \alpha_3 = J + j_1 + j_2 + 2 \quad , \\ s + a &= \beta_1 = J + j_2 + m_1 + 2 \quad , \\ s + c &= \beta_2 = 2J + 2 \quad , \end{aligned} \tag{7.30}$$

where $m_1 + m_2 = M - 1$. This yields the new parameters:

$$\begin{aligned} a &= -j_1 + m_1 \quad , \\ b &= -j_2 - m_2 \quad , \\ c &= -j_1 - j_2 + J \quad , \\ e &= J - j_1 - m_2 + 1 \quad , \\ f &= J - j_2 + m_1 + 1 \quad . \end{aligned} \tag{7.31}$$

In terms of the new parameters one can express Eq. 7.27 as

$$(e-a-1)\, F\begin{bmatrix} a, b, c; \\ e, f \end{bmatrix} = \frac{a(c-f)}{f}\, F\begin{bmatrix} a+1, b, c; \\ e, f+1 \end{bmatrix} + (e-1)\, F\begin{bmatrix} a, b-1, c; \\ e-1, f \end{bmatrix} . \tag{7.32}$$

This equation can easily be shown to be equivalent to the recurrence relation given in Eq. 7.24.

The last recurrence relation to be proved is

$$\begin{aligned} &\left[J(J+1) - j_1(j_1+1) - j_2(j_2+1) - 2m_1m_2\right] (j_1 m_1 j_2 m_2 | j_1 j_2 J M)' \\ &= \left\{(j_1-m_1+1)(j_1+m_1)(j_2+m_2+1)(j_2-m_2)\right\}^{\frac{1}{2}} \\ &\qquad \cdot (j_1 m_1-1\; j_2 m_2+1 | j_1 j_2 J M)' \\ &+ \left\{(j_1+m_1+1)(j_1-m_1)(j_2-m_2+1)(j_2+m_2)\right\}^{\frac{1}{2}} \\ &\qquad \cdot (j_1 m_1+1\; j_2 m_2-1 | j_1 j_2 J M)' . \end{aligned} \tag{7.33}$$

An equation connecting the functions F, F(β_1 -), and F(β_1 +) can be derived from Rainville's contiguous relations. With x=1 and k=1 Eqs. 7.9 and 7.8 become Eqs. 7.15 and 7.16 respectively as written previously. Elimination of F(β_2 +) yields

$$(A-B)F - (\alpha_1-\beta_2)F(\alpha_1-) = \left[(\alpha_1-\beta_2)W_{11} - U_1\right] F(\beta_1+) . \tag{7.34}$$

For $\alpha_1 \longrightarrow \alpha_1-1$ and k=1, Eq. 7.7 for ${}_3F_2$ functions can be written as

$$(\alpha_1-\beta_1)F(\alpha_1-) = (\alpha_1-1)F - (\beta_1-1)F(\alpha_1-, \beta_1-) . \tag{7.35}$$

Also for $x=1$, $k=1$, and $\beta_1 \longrightarrow \beta_1 - 1$, Eq. 7.9 in terms of $_3F_2$ can be expressed in the form

$$F(\alpha_1-, \beta_1-) = -W_{11}'F - W_{21}'F(\beta_1-; \beta_2+) \quad , \tag{7.36}$$

where the primes on W_{11}' and W_{21}' denote $\beta_1 \longrightarrow \beta_1-1$ in the expressions. Eqs. 7.34, 7.35, and 7.36 can be combined to give the equation

$$\begin{aligned}(A-B)F - \frac{(\alpha_1-\beta_2)(\alpha_1-1)}{(\alpha_1-\beta_1)}F + \frac{(\alpha_1-\beta_2)(\beta_1-1)}{(\alpha_1-\beta_1)}\left[-W_{11}'F - W_{21}'F(\beta_1-,\beta_2+)\right] \\ = \left[(\alpha_1-\beta_2)W_{11} - U_1\right]F(\beta_1+) \quad .\end{aligned} \tag{7.37}$$

By setting $x=1$ and $\beta_1 \longrightarrow \beta_1 - 1$ for $_3F_2$ functions in Eq. 7.8 one obtains

$$(A-B+1)\,F(\beta_1-) = -U_1'F - U_2'F(\beta_1-,\beta_2+) \quad , \tag{7.38}$$

where the primes again denote $\beta_1 \longrightarrow \beta_1 - 1$ in the particular expression. Elimination of $F(\beta_1 -, \beta_2 +)$ from Eqs. 7.37 and 7.38 plus use of the definitions in Eqs. 7.10, 7.11, 7.12, and 7.13 gives finally

$$\begin{aligned}\left[-(\alpha_1-\beta_2)(\alpha_1-1) + (\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2)(\alpha_1-\beta_1) + (\alpha_2-\beta_1+1)(\alpha_3-\beta_1+1)\right]F \\ = -(\beta_1-1)(\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2+1)F(\beta_1-) \\ - \frac{(\alpha_1-\beta_1)(\alpha_2-\beta_1)(\alpha_3-\beta_1)}{\beta_1}F(\beta_1+) \quad .\end{aligned} \tag{7.39}$$

Again the inverse of Bailey's transformation, Eq. 7.19, is used with the identification of new parameters being

$$
\begin{aligned}
e - a &= \alpha_1 = J - M + 1 \quad , \\
f - a &= \alpha_2 = J + j_1 - j_2 + 1 \quad , \\
s &= \alpha_3 = J + j_1 + j_2 + 2 \quad , \\
s + b &= \beta_1 = J + j_1 - m_2 + 2 \quad , \\
s + c &= \beta_2 = 2J + 2 \quad ,
\end{aligned}
\tag{7.40}
$$

where $M = m_1 + m_2$. This gives the following relations:

$$
\begin{aligned}
a &= -j_1 + m_1 \quad , \\
b &= -j_2 - m_2 \quad , \\
c &= J - j_1 - j_2 \quad , \\
e &= J - j_1 - m_2 + 1 \quad , \\
f &= J - j_2 + m_1 + 1 \quad .
\end{aligned}
\tag{7.41}
$$

After application of the inverse Bailey transformation to Eq. 7.39, one has the following:

$$
\begin{aligned}
&\left[(f-b)(e-a-1) - a(c-f) - (1-b)(e-c-1)\right] \, {}_3F_2\left[\begin{matrix} a, b, c \; ; \\ e, f \end{matrix}\right] \\
&= (e-1)(f-1) \, {}_3F_2\left[\begin{matrix} a-1, b-1, c \; ; \\ e-1, f-1 \end{matrix}\right] \\
&+ \frac{(a)(c-f)(e-c)(-b)}{ef} \, {}_3F_2\left[\begin{matrix} a+1, b+1, c \; ; \\ e+1, f+1 \end{matrix}\right] ,
\end{aligned}
\tag{7.42}
$$

which can be written as

$$\left[J(J+1)-j_1(j_1+1)-j_2(j_2+1)-2m_1m_2\right]\ {}_3F_2\left[\begin{matrix}-j_1+m_1\ ,-j_2-m_2\ ,-j_1-j_2+J\ ;\\ J-j_1-m_2+1\ ,\ J-j_2+m_1+1\end{matrix}\right]$$

$$= (J-j_2+m_1)(J-j_1-m_2)\ {}_3F_2\left[\begin{matrix}-j_1+m_1-1\ ,-j_2-m_2-1\ ,\ -j_1-j_2+J\ ;\\ J-j_1-m_2\ ,\ J-j_2+m_1\end{matrix}\right] \tag{7.43}$$

$$+\ \frac{(j_1+m_1+1)(j_1-m_1)(j_2-m_2+1)(j_2+m_2)}{(J-j_2+m_1+1)(J-j_1-m_2+1)}\ {}_3F_2\left[\begin{matrix}-j_1+m_1+1\ ,-j_2-m_2+1\ ,-j_1-j_2+J\ ;\\ J-j_1-m_2+2\ ,\ J-j_2+m_1+2\end{matrix}\right].$$

Multiplying through Eq. 7.43 by the factor

$$\left\{\frac{\Gamma(j_1+m_1+1)\,\Gamma(j_2-m_2+1)}{\Gamma(j_1-m_1+1)\,\Gamma(j_2+m_2+1)}\,\Gamma(J-M+1)\,\Gamma(J+M+1)\,(2J+1)\,\frac{\Gamma(j_1-j_2+J+1)}{\Gamma(j_1+j_2-J+1)}\right\}^{\frac{1}{2}}$$

$$\cdot\left\{\frac{\Gamma(-j_1+j_2+J+1)}{\Gamma(j_1+j_2+J+2)}\right\}^{\frac{1}{2}}\ \frac{1}{\Gamma(J-j_2+m_1+1)\ \Gamma(J-j_1-m_2+1)}\ ,$$

collecting terms, and comparing with the coupling coefficient definition given by Eq. 7.3 one finally obtains the recurrence relation as written in Eq. 7.33.

The definition of the coupling coefficient expressed by Eq. 7.3 and the three recurrence relations given by Eqs. 7.14, 7.24, and 7.33 all hold for complex angular momenta j_1 and j_2 such that the sum $j_1 + j_2$

is still complex. However if $j_1 + j_2 = \ell - 1 \equiv J_{min} - 1$ then the factor $\{\Gamma(j_1+j_2-J)\}^{-\frac{1}{2}}$ must be removed from the original definition of the coupling coefficient as given by Eq. 7.3. This slightly altered coefficient can be shown to obey the same recurrence relations as proven for the original coefficient.

For later use in the next chapter the sum $\sum_{m_1} (p o q m_1 | p q j m_1)'^2$ need be examined at this point. The corresponding summation for the ordinary R(3) Clebsch-Gordan coefficients can be shown to equal $(-1)^q \frac{(2j+1)}{(2p+1)}$ by use of the symmetry and unitarity properties of the coefficients. Since the complex coupling coefficients used in this work do not possess the unitarity property, a similar proof cannot be made for them. The coefficients must be studied from the standpoint of mathematical functions, not as coupling coefficients for group representations. The reason is that the usual group structure is not present for the rotation matrices with complex angular momentum. However one can show that the coefficient $(p o q m_1 | p q j m_1)'$ converges as $m_1 \longrightarrow \infty$; the proof is analogous to the method used by Holman and Biedenharn (1966) for the generalized Wigner coefficient. It appears safe to say that the above-mentioned sum is equal to a finite constant whose exact value cannot be determined until the properties of the rotation matrices with complex angular momentum are better understood. Also, once the angular momentum quantum number j is known, the range of m_1-values in the sum is set. Similarly, the sum $\sum_{m_1} \left| (p o q m_1 | p q j m_1)' \right|^2$ should be equal to a finite constant.

VIII. PROPERTIES OF THE SOLUTIONS

The solutions to the eigenvalue equations of the complex rotation group are examined in detail in this chapter. The requirement that the solutions be convergent at $u=0$ and $u=\infty$ restricts the quantum numbers $\hat{\ell}$ and m to values of zero and integer respectively. The associated Legendre functions of degree $-\frac{1}{2} \pm \frac{i\nu}{2}$ appearing in the harmonic solutions are known as associated conical functions; orthogonality and expansion theorems are established for the functions. The $O(3,C)$ harmonic functions are found to have a non-conventional orthogonality property. Even though certain expansions can be made in terms of the harmonics, it is seen that they do not form a complete set over the four-parameter space.

A. Convergence of the Solutions

Three different sets of solutions to the eigenvalue equations of the complex rotation group are given by Eqs. 6.21, 6.38, and 6.42. The condition that they be convergent throughout the parameter space is imposed on the solutions; that is, the solutions must be square-integrable over the u, ψ, θ, ϕ -parameter space whose volume element has been found to be:

$$d\tau = \frac{\sinh 2u}{4} \sin\theta \, du \, d\psi \, d\theta \, d\phi \quad . \tag{8.1}$$

It is well-known (see Rose 1957, for example) that the rotation matrices $D^{j*}_{mm'}(\psi, \theta, \phi)$ are square-integrable in the ψ, θ, ϕ -parameter space and in fact possess an orthogonality property on the surface of a unit sphere. So the convergence of the u-dependent part of the solutions remains to be examined. Certain conditions on the quantum numbers result from this examination.

The u-dependence of the three sets of solutions given by Eqs. 6.21, 6.38, and 6.42 are (1) $\mathcal{P}_p^{m_1}(\cosh u)\, \mathcal{P}_q^{m'-m_1}(\cosh u)$, (2) $\mathcal{P}_q^{m'}(\cosh 2u)\, d^j_{m''m'}(-iu)$, and (3) $\mathcal{P}_p^{m'}(\cosh 2u)\, d^j_{m''m'}(+iu)$ respectively, where $p = \frac{-1 \pm (\ell + i\nu)}{2}$, $q = \frac{-1 \pm (\ell - i\nu)}{2}$, $\ell \equiv j_{min}$, ν = any real number, and the m-values are either integral or half-integral. Care must be taken at the points $u=0$ and $u=\infty$ because the associated Legendre function (of the first kind $\mathcal{P}$ and of the second kind $\mathcal{Q}$) has singularities at values of 1 and ∞ for its argument. At $z \sim 1$, this behavior is (Erdélyi, et al. 1953)

$$\mathcal{P}_\nu^\mu(z) \sim \begin{cases} (z-1)^{\mu/2} & , \quad \mu \neq 1,2,3,\ldots \\ (z-1)^{m/2} & , \quad \mu = m = 0,1,2,3,\ldots \end{cases} \tag{8.2}$$

,

$$\mathcal{Q}_\nu^\mu(z) \sim \begin{cases} (z-1)^{-\mu/2} & , \quad \mathrm{Re}\,\mu > 0 \\ (z-1)^{\mu/2} & , \quad \mathrm{Re}\,\mu < 0 \end{cases} \tag{8.3}$$

,

and also

$$d^j_{mm'}(z) \sim (z-1)^{\frac{m-m'}{2}} \quad , \quad m \geq m' \quad . \tag{8.4}$$

As $z \longrightarrow \infty$, the behavior is (Erdélyi, et al. 1953)

$$\mathcal{P}_\nu^\mu(z) \sim \begin{cases} z^{\nu} & , \quad \mathrm{Re}\,\nu > -\frac{1}{2} \\ z^{-\nu-1} & , \quad \mathrm{Re}\,\nu < -\frac{1}{2} \end{cases} \tag{8.5}$$

,

$$\mathcal{Q}_\nu^\mu(z) \sim z^{-\nu-1} \tag{8.6}$$

and also

$$d^j_{mm'}(z) \sim z^j \quad . \tag{8.7}$$

For case (1) it is obvious that for proper behavior at cosh u = 1 the $\mathcal{P}$'s instead of the $\mathcal{Q}$'s must be chosen and also that m_1 and m' must be integral. As cosh u $\longrightarrow \infty$, $\mathcal{P}_p^{m_1}\mathcal{P}_q^{m'-m_1} \sim (\cosh u)^{\ell-1}$ and hence one must set $\ell=0$ for the u-dependent part of the solution to be square-integrable. For cases (2) and (3) it might appear that the solutions are not convergent as u $\longrightarrow \infty$. However in certain orthogonality integrals, as will be shown later, the $d^j_{mm'}(\pm iu)$ part is removed completely by summation over m-values. Thus the u-dependencies in cases (2) and (3) will be studied for the circumstance that the $d^j_{mm'}(\pm iu)$ can be removed. For proper behavior at cosh 2u = 1, one must require that m' be integral and that $\mathcal{P}$ associated Legendre functions be used instead of $\mathcal{Q}$'s. Square-integrability of $\mathcal{P}_q^{m'}(\cosh 2u)$ or $\mathcal{P}_p^{m'}(\cosh 2u)$ over the u-space is obtained only if one sets $\ell = j_{min} = 0$.

From the requirement that the solutions be convergent over the parameter space one obtains the conditions that the m-values be integral and that $\ell = j_{min} = 0$. Hence the quantum numbers p and q become

$$p = \frac{-1 \pm i\nu}{2} \quad ,$$
$$q = \frac{-1 \pm i\nu}{2} \quad . \tag{8.8}$$

The u-dependence (besides the $d^j_{mm'}$ part) is carried by the functions $\mathcal{P}^{m'}_{-\frac{1}{2}\pm\frac{i\nu}{2}}(\cosh 2u)$, which are also known as associated conical functions. In order to develop an orthogonality theorem for the entire solutions one must first establish an orthogonality property for the associated conical functions. This will be considered in the next section.

B. Orthogonality of the Associated Conical Functions

In this section orthogonality of the associated conical functions will be proven in the sense that

$$\lim_{N\to\infty} \int_0^\infty d\nu' f(\nu') \int_0^N \mathcal{P}^m_{-\frac{1}{2}-\frac{i\nu}{2}}(\cosh u)\ \mathcal{P}^m_{-\frac{1}{2}-\frac{i\nu'}{2}}(\cosh u)\ \sinh u\ du = \frac{4\pi\ f(\nu)}{\nu \sinh(\frac{\pi\nu}{2})\ \left|\Gamma(\frac{1}{2}-m+\frac{i\nu}{2})\right|^2} \tag{8.9}$$

for functions $f(\nu')$ which obey Dirichlet's Conditions in the ν'-interval of $(0, \infty)$ and which are analytic in the interval $\nu' = \nu - \epsilon$ to $\nu' = \nu + \epsilon$, ϵ small. The orthogonality property expressed by Eq. 8.9 can be rewritten in the form

$$\int_0^\infty \mathcal{P}^m_{-\frac{1}{2}-\frac{i\nu}{2}}(\cosh u)\ \mathcal{P}^m_{-\frac{1}{2}-\frac{i\nu'}{2}}(\cosh u)\ \sinh u\ du = \frac{4\pi\ \delta(\nu-\nu')}{\nu \sinh(\frac{\pi\nu}{2})\ \left|\Gamma(\frac{1}{2}-m+\frac{i\nu}{2})\right|^2}, \tag{8.10}$$

where $\delta(\nu-\nu')$ is the Dirac delta function.

Consider the eigenfunction solutions

$$\begin{aligned} F_1 &= \{\sinh u\}^{\frac{1}{2}}\ \mathcal{P}^m_{-\frac{1}{2}-i\sigma}(\cosh u) \\ F_2 &= \{\sinh u\}^{\frac{1}{2}}\ \mathcal{P}^m_{-\frac{1}{2}-i\sigma'}(\cosh u) \end{aligned} \tag{8.11}$$

to the two eigenvalue equations

$$LF_1 = \lambda F_1 \quad ,$$
$$LF_2 = \lambda' F_2 \quad , \tag{8.12}$$

where $L = -\frac{d^2}{du^2} + \frac{m^2 - \frac{1}{4}}{\sinh^2 u}$, $\lambda = \sigma^2$, $\lambda' = \sigma'^2$ (with σ, σ' ranging from 0 to ∞). One can then write

$$(\lambda - \lambda') \int_0^\infty F_1 F_2 \, du = \int_0^\infty \left[(LF_1) F_2 - F_1 (LF_2) \right] du \quad , \tag{8.13}$$

which becomes

$$(\lambda - \lambda') \int_0^\infty F_1 F_2 \, du = \left(F_1 \frac{dF_2}{du} - F_2 \frac{dF_1}{du} \right) \Big|_{u=0}^{u=\infty} \quad . \tag{8.14}$$

By using Eqs. 8.11 and 8.12 and also $z \equiv \cosh u$, one gets

$$\begin{aligned}(\sigma^2 - \sigma'^2) \int_1^\infty \mathcal{P}^m_{-\frac{1}{2}-i\sigma}(z)\, \mathcal{P}^m_{-\frac{1}{2}-i\sigma'}(z)\, dz \\ = (z^2-1) \left(\mathcal{P}^m_{-\frac{1}{2}-i\sigma} \frac{d}{dz} \mathcal{P}^m_{-\frac{1}{2}-i\sigma'} - \mathcal{P}^m_{-\frac{1}{2}-i\sigma'} \frac{d}{dz} \mathcal{P}^m_{-\frac{1}{2}-i\sigma} \right) \Big|_{z=1}^{z=\infty} \quad .\end{aligned} \tag{8.15}$$

The associated Legendre recursion relation

$$(z^2-1) \frac{d}{dz} \mathcal{P}^\mu_\nu(z) = \mu z \mathcal{P}^\mu_\nu(z) + \{z^2-1\}^{\frac{1}{2}} \mathcal{P}^{\mu+1}_\nu(z) \tag{8.16}$$

can be used to write the preceding equation in the form:

$$\begin{aligned}(\sigma^2 - \sigma'^2) \int_1^\infty \mathcal{P}^m_{-\frac{1}{2}-i\sigma}(z)\, \mathcal{P}^m_{-\frac{1}{2}-i\sigma'}(z)\, dz \\ = \{z^2-1\}^{\frac{1}{2}} \left(\mathcal{P}^m_{-\frac{1}{2}-i\sigma} \mathcal{P}^{m+1}_{-\frac{1}{2}-i\sigma'} - \mathcal{P}^m_{-\frac{1}{2}-i\sigma'} \mathcal{P}^{m+1}_{-\frac{1}{2}-i\sigma} \right) \Big|_{z=1}^{z=\infty} \quad .\end{aligned} \tag{8.17}$$

There is no contribution from the bottom limit of the right side of the above equation since at $z=1$ the functions $\mathcal{P}^m_\nu$ go as $(z-1)^m$ and hence vanish. From Erdélyi, et al. (1953) the leading terms of $\mathcal{P}^{m}_{-\frac{1}{2}-i\sigma}$ as $z \longrightarrow \infty$ are given by

$$\mathcal{P}^{m}_{-\frac{1}{2}-i\sigma} \sim \frac{2^{-\frac{1}{2}-i\sigma}\pi^{-\frac{1}{2}}\Gamma(-i\sigma)\, z^{-\frac{1}{2}-i\sigma}}{\Gamma(\frac{1}{2}-i\sigma-m)} + \frac{2^{-\frac{1}{2}+i\sigma}\pi^{-\frac{1}{2}}\Gamma(i\sigma)\, z^{-\frac{1}{2}+i\sigma}}{\Gamma(\frac{1}{2}+i\sigma-m)}$$

$$= \left[\frac{2}{\pi}\right]^{\frac{1}{2}} z^{-\frac{1}{2}} \operatorname{Re}\left\{\frac{2^{i\sigma}\Gamma(i\sigma)\, z^{i\sigma}}{\Gamma(\frac{1}{2}+i\sigma-m)}\right\}. \tag{8.18}$$

The non-leading terms of $\mathcal{P}^{m}_{-\frac{1}{2}-i\sigma}$ as $z \longrightarrow \infty$ depend upon $z^{-\frac{3}{2} \mp i\sigma}$, $z^{-\frac{5}{2} \mp i\sigma}$, and increasingly negative powers and can easily be seen to give no contribution to the right side of Eq. 8.17. Then the use of Eq. 8.18 yields

$$\left[z^2-1\right]^{\frac{1}{2}} \left(\mathcal{P}^{m}_{-\frac{1}{2}-i\sigma}\mathcal{P}^{m+1}_{-\frac{1}{2}-i\sigma'} - \mathcal{P}^{m}_{-\frac{1}{2}-i\sigma'}\mathcal{P}^{m+1}_{-\frac{1}{2}-i\sigma}\right)\Bigg|^{z=\infty}$$

$$= \frac{\sigma-\sigma'}{\pi} \operatorname{Re}\left\{\frac{2^{i(\sigma+\sigma')}\Gamma(i\sigma)\Gamma(+i\sigma')\exp[i(\sigma+\sigma')\ln z]}{i\,\Gamma(\frac{1}{2}+i\sigma-m)\,\Gamma(\frac{1}{2}+i\sigma'-m)}\right\} \tag{8.19}$$

$$+ \frac{\sigma+\sigma'}{\pi} \operatorname{Re}\left\{\frac{2^{i(\sigma-\sigma')}\Gamma(i\sigma)\Gamma(-i\sigma')\exp[i(\sigma-\sigma')\ln z]}{i\,\Gamma(\frac{1}{2}+i\sigma-m)\,\Gamma(\frac{1}{2}-i\sigma'-m)}\right\}$$

after some manipulation of terms.

Thus Eq. 8.17 can be written in the form

$$\int_1^\infty \mathcal{P}^m_{-\frac{1}{2}-i\sigma}(z)\,\mathcal{P}^m_{-\frac{1}{2}-i\sigma'}(z)\,dz$$

$$\begin{aligned} &= \mathrm{Im}\left\{A(\sigma,\sigma',m)\right\} \lim_{z\to\infty} \frac{\cos[(\sigma+\sigma')\ln 2z]}{\pi(\sigma+\sigma')} \\ &+ \mathrm{Re}\left\{A(\sigma,\sigma',m)\right\} \lim_{z\to\infty} \frac{\sin[(\sigma+\sigma')\ln 2z]}{\pi(\sigma+\sigma')} \\ &+ \mathrm{Im}\left\{A(\sigma,-\sigma',m)\right\} \lim_{z\to\infty} \frac{\cos[(\sigma-\sigma')\ln 2z]}{\pi(\sigma-\sigma')} \\ &+ \mathrm{Re}\left\{A(\sigma,-\sigma',m)\right\} \lim_{z\to\infty} \frac{\sin[(\sigma-\sigma')\ln 2z]}{\pi(\sigma-\sigma')} \quad , \end{aligned} \tag{8.20}$$

where $$A(\sigma,\sigma',m) \equiv \frac{\Gamma(i\sigma)\,\Gamma(i\sigma')}{\Gamma(\frac{1}{2}+i\sigma-m)\,\Gamma(\frac{1}{2}+i\sigma'-m)} \quad .$$

The orthogonality integral with function $f(\sigma')$ is now considered in the following manner:

$$\lim_{N\to\infty} \int_0^\infty d\sigma'\, f(\sigma') \int_1^N \mathcal{P}^m_{-\frac{1}{2}-i\sigma}(z)\,\mathcal{P}^m_{-\frac{1}{2}-i\sigma'}(z)\,dz$$

$$\begin{aligned} &= \lim_{x\to\infty} \int_0^\infty d\sigma' f(\sigma')\, \mathrm{Im}\left\{A(\sigma,\sigma',m)\right\} \frac{\cos[(\sigma+\sigma')x]}{\pi(\sigma+\sigma')} \\ &+ \lim_{x\to\infty} \int_0^\infty d\sigma' f(\sigma')\, \mathrm{Re}\left\{A(\sigma,\sigma',m)\right\} \frac{\sin[(\sigma+\sigma')x]}{\pi(\sigma+\sigma')} \\ &+ \lim_{x\to\infty} \int_0^\infty d\sigma' f(\sigma')\, \mathrm{Im}\left\{A(\sigma,-\sigma',m)\right\} \frac{\cos[(\sigma-\sigma')x]}{\pi(\sigma-\sigma')} \\ &+ \lim_{x\to\infty} \int_0^\infty d\sigma' f(\sigma')\, \mathrm{Re}\left\{A(\sigma,-\sigma',m)\right\} \frac{\sin[(\sigma-\sigma')x]}{\pi(\sigma-\sigma')} \quad , \end{aligned} \tag{8.21}$$

where the substitution $x \equiv \ln 2z$ has been made. Each of the four terms on the right side of the preceding equation will be examined closely.

By defining $\sigma + \sigma' \equiv \rho$,

$$\frac{f(\sigma')\ \mathrm{Im}\{A(\sigma,\sigma',m)\}}{\pi(\sigma+\sigma')} \equiv g(\rho) \quad ,$$

and $$\frac{f(\sigma')\ \mathrm{Re}\{A(\sigma,\sigma',m)\}}{\pi(\sigma+\sigma')} \equiv h(\rho) \quad , \tag{8.22}$$

one can write for the first two terms of the right side of Eq. 8.21:

$$\lim_{x\to\infty} \int_{\sigma}^{\infty} d\rho\ g(\rho) \cos\rho x \quad + \quad \lim_{x\to\infty} \int_{\sigma}^{\infty} d\rho\ h(\rho) \sin\rho x \quad = 0 \tag{8.23}$$

according to MacRobert (1947). He states that each of the terms in Eq. 8.23 equals zero if the functions g(ρ) and h(ρ) satisfy Dirichlet's Conditions: (1) the function is continuous in the interval under consideration except perhaps for a finite number of finite discontinuities; (2) the function has only a finite number of turning points in the interval. Hence the requirement for the validity of Eq. 8.23 is that the function f(σ') must obey Dirichlet's Conditions in the interval (0, ∞).

The third term on the right side of Eq. 8.21 is examined next. By defining $$\rho \equiv \sigma' - \sigma$$

and $$g(\rho) \equiv \frac{1}{\pi} f(\sigma')\ \mathrm{Im}\left\{A(\sigma,-\sigma',m)\right\} \quad , \tag{8.24}$$

one can express the term as

$$-\lim_{x\to\infty} \int_{-\sigma}^{\infty} d\rho\ \frac{g(\rho)}{\rho} \cos\rho x \quad = \quad -\lim_{x\to\infty} \left(\int_{-\sigma}^{-\epsilon} + \int_{-\epsilon}^{+\epsilon} + \int_{+\epsilon}^{\infty} \right) d\rho\ \frac{g(\rho)}{\rho} \cos\rho x \quad . \tag{8.25}$$

Again if $f(\sigma')$ is assumed to satisfy Dirichlet's Conditions in the interval $(0, \infty)$, then obviously the function $\frac{g(\rho)}{\rho}$ obeys the conditions in the intervals $(-\sigma, -\epsilon)$ and $(+\epsilon, \infty)$ and one has

$$\lim_{x\to\infty} \left(\int_{-\sigma}^{-\epsilon} + \int_{+\epsilon}^{\infty} \right) d\rho \, \frac{g(\rho)}{\rho} \cos\rho x = 0 \tag{8.26}$$

according to MacRobert (1947). If the function $f(\sigma')$ is assumed to be analytic in the interval $(\sigma - \epsilon, \sigma + \epsilon)$, ϵ small, the function $g(\rho)$ is then analytic in the interval $(-\epsilon, +\epsilon)$. By making the Taylor expansion

$$g(\rho) = g(0) + \rho\, g'(0) + \frac{\rho^2}{2!} g''(0) + \ldots , \tag{8.27}$$

one can write the remaining part of the third term as

$$\begin{aligned} &-\lim_{x\to\infty} \int_{-\epsilon}^{+\epsilon} d\rho \, \frac{g(\rho)}{\rho} \cos\rho x \\ &= -\lim_{x\to\infty} \left[g(0) \int_{-\epsilon}^{+\epsilon} d\rho \, \frac{\cos\rho x}{\rho} + g'(0) \int_{-\epsilon}^{+\epsilon} d\rho \cos\rho x + \frac{g''(0)}{2!} \int_{-\epsilon}^{+\epsilon} \rho \cos\rho x \, d\rho + \ldots \right]. \end{aligned} \tag{8.28}$$

Since $\frac{\cos\rho x}{\rho}$ is an odd function of ρ, the first integral on the right side of the above equation is

$$\int_{-\epsilon}^{+\epsilon} d\rho \, \frac{\cos\rho x}{\rho} = 0 \ . \tag{8.29}$$

The second, third, and subsequent integrals vanish according to the following integral from MacRobert (1947):

$$\lim_{m\to\infty} \int_a^b \phi(u) \cos mu \, du = 0 \tag{8.30}$$

if $\phi(u)$ satisfies Dirichlet's Conditions in the interval (a,b). Hence the third term of the right side of Eq. 8.21 makes zero contribution.

The fourth and last term of Eq. 8.21 is to be studied in the same manner as for the third term. One defines

$$\begin{aligned} \rho &\equiv \sigma' - \sigma \quad , \\ h(\rho) &\equiv \frac{1}{\pi} f(\sigma') \operatorname{Re}\left\{ A(\sigma, -\sigma', m) \right\} \end{aligned} \tag{8.31}$$

and writes the term as

$$\lim_{x\to\infty} \int_{-\sigma}^{\infty} d\rho \, \frac{h(\rho)}{\rho} \sin\rho x = \lim_{x\to\infty} \left(\int_{-\sigma}^{-\epsilon} + \int_{-\epsilon}^{+\epsilon} + \int_{+\epsilon}^{\infty} \right) d\rho \, \frac{h(\rho)}{\rho} \sin\rho x \; . \tag{8.32}$$

For $f(\sigma')$ in the interval $(0, \infty)$ and hence $\frac{h(\rho)}{\rho}$ in $(-\sigma, -\epsilon)$ and $(+\epsilon, \infty)$ obeying Dirichlet's Conditions, the integrals $\int_{-\sigma}^{-\epsilon}$ and $\int_{+\epsilon}^{\infty}$ equal zero. Assuming analyticity for $f(\sigma')$ in the interval $(\sigma - \epsilon, \sigma + \epsilon)$, one has $h(\rho)$ analytic in the interval $(-\epsilon, +\epsilon)$ and can make the Taylor expansion

$$h(\rho) = h(0) + \rho h'(0) + \frac{\rho^2}{2!} h''(0) + \cdots \quad . \tag{8.33}$$

Then

$$\begin{aligned} &\lim_{x\to\infty} \int_{-\epsilon}^{+\epsilon} d\rho \, \frac{h(\rho)}{\rho} \sin\rho x \\ &= \lim_{x\to\infty} \left[h(0) \int_{-\epsilon}^{+\epsilon} d\rho \, \frac{\sin\rho x}{\rho} + h'(0) \int_{-\epsilon}^{+\epsilon} \sin\rho x \, d\rho + \frac{h''(0)}{2!} \int_{-\epsilon}^{+\epsilon} \rho \sin\rho x \, d\rho + \cdots \right] . \end{aligned} \tag{8.34}$$

Only the first integral on the right side of the above equation is non-zero. It gives

$$\begin{aligned} \lim_{x\to\infty} h(0) \int_{-\epsilon}^{+\epsilon} d\rho \, \frac{\sin\rho x}{\rho} &= \pi h(0) = f(\sigma) \operatorname{Re}\left\{ A(\sigma, -\sigma, m) \right\} \\ &= \frac{\pi f(\sigma)}{\sigma \sinh(\pi\sigma) \left| \Gamma(\tfrac{1}{2} + i\sigma - m) \right|^2} \end{aligned} \tag{8.35}$$

by use of the definition of $A(\sigma, -\sigma, m)$ and of the following gamma function relationship from Erdélyi, et al. (1953):

$$\Gamma(z)\,\Gamma(-z) = -\pi z^{-1} \csc(\pi z) \quad . \tag{8.36}$$

The second, third, and subsequent integrals go to zero according to MacRobert's proof:

$$\lim_{m\to\infty} \int_a^b \phi(u) \sin mu \; du = 0 \tag{8.37}$$

for $\phi(u)$ satisfying Dirichlet's Conditions in (a,b).

Thus the only non-zero contribution to the integral of Eq. 8.21 is given by the expression in Eq. 8.35. One finally has

$$\lim_{N\to\infty} \int_0^\infty d\sigma' f(\sigma') \int_1^N \mathcal{P}^m_{-\frac{1}{2}-i\sigma}(z)\, \mathcal{P}^m_{-\frac{1}{2}-i\sigma'}(z)\, dz = \frac{\pi f(\sigma)}{\sigma \sinh(\pi\sigma) \left|\Gamma(\frac{1}{2}+i\sigma-m)\right|^2} \tag{8.38}$$

which is the desired orthogonality relationship. Letting $\sigma, \sigma' \to \frac{\nu}{2}, \frac{\nu'}{2}$ one obtains Eq. 8.9 as stated.

C. Expansions in Terms of the Associated Conical Functions

The orthogonality property of the associated conical functions $\mathcal{P}^m_{-\frac{1}{2}-\frac{i\nu}{2}}(\cosh u)$ suggests the possibility that arbitrary functions of the parameter u can be expanded in terms of them. This is in fact established here by use of the Titchmarsh theory (1962) of eigenfunction expansions.

There has been some previous work related to this subject of expansions by use of conical functions. Banerji (1938) gave the following expansion theorem for functions F(x) which can be written in the form $F(x) = \int_1^x \frac{u(\xi)\, d\xi}{(x-\xi)^{\frac{1}{2}}}$:

$$F(\cosh\psi) = \sum_{p=0}^{\infty} a_p\, \mathcal{P}_{-\frac{1}{2}+ip}(\cosh\psi)\ , \quad (0 \leq \psi \leq \pi)\ , \tag{8.39}$$

where

$$a_p = (2)^{\frac{1}{2}} \int_0^{\pi} u(\cosh\phi)\, \sinh\phi\, \cos p\phi\, d\phi \quad , \quad p \geq 1 \ ,$$

$$a_0 = (2)^{-\frac{1}{2}} \int_0^{\pi} u(\cosh\phi)\, \sinh\phi\, d\phi \quad . \tag{8.40}$$

The main drawback of the above series expansion is that the argument ψ is limited to the interval $0 \leq \psi \leq \pi$. Mehler (1881) has presented the inversion formulas:

$$f(x) = \int_0^{\infty} \mu\, d\mu\, \tanh(\mu\pi)\, C^{\mu}\, \mathcal{P}_{-\frac{1}{2}+i\mu}(x) \ ,$$

$$C^{\mu} = \int_1^{\infty} f(x)\, \mathcal{P}_{-\frac{1}{2}+i\mu}(x)\, dx \tag{8.41}$$

for functions $f(x)$ expressible as

$$f(x) = (x-y)^{-\delta} \quad , \quad \delta > \tfrac{1}{2} \quad , \quad 1 \leq x < \infty \ , \quad -\infty < y < 1 \quad . \tag{8.42}$$

Similar to Mehler's work is the following theorem given by Fock (1943):

If a function $\psi(x)$, where $1 \leq x < \infty$, is such that there exists an integral $\int_1^{\infty} \frac{|\psi(x)|\, dx}{(x+1)^{1/2}}$, then for every point x, in whose neighborhood $\psi(x)$ has a bounded variation, there holds

$$\frac{1}{2}\left[\psi(x+0) + \psi(x-0)\right] = \int_0^{\infty} \mathcal{P}_{i\mu-\frac{1}{2}}(x)\, f(\mu)\, d\mu \ , \tag{8.43}$$

where

$$f(\mu) = \mu \tanh(\mu\pi) \int_1^{\infty} \mathcal{P}_{i\mu-\frac{1}{2}}(x)\, \psi(x)\, dx \quad . \tag{8.44}$$

There are generalizations of the Mehler formulas whereby the associated conical function is the kernel of the integral inversion formulas:

$$\text{if} \qquad g(y) = \int_0^\infty f(x)\, \mathcal{P}^{\mu}_{-\frac{1}{2}+ix}(y)\, dx \qquad , \tag{8.45}$$

$$\text{then} \qquad f(x) = \pi^{-1} x \sinh(\pi x) \left|\Gamma(\tfrac{1}{2}-\mu+ix)\right|^2 \int_1^\infty g(y)\, \mathcal{P}^{\mu}_{-\frac{1}{2}+ix}(y)\, dy \tag{8.46}$$

or conversely. These formulas are listed in a book by Magnus, et al. (1966).

In none of the previous work listed above has there been an establishment of a general expansion theorem for the associated conical functions: that the functions $\mathcal{P}^{m}_{-\frac{1}{2}-\frac{i\nu}{2}}(\cosh u)$ form a complete set over the u-parameter space in the interval $(0, \infty)$ with weight function $\sinh u$. Titchmarsh's (1962) eigenfunction expansion theory can be used to show that any arbitrary function of parameter u which is square-integrable in the interval $(0, \infty)$ can be expanded in terms of the functions $\mathcal{P}^{m}_{-\frac{1}{2}-\frac{i\nu}{2}}(\cosh u)$. This is the work in the remainder of the section. The theory, results, and most of the notation is that of Titchmarsh (1962) for second-order ordinary differential operators. A paper by Sarginson (1946) is found to be a very helpful guide to the actual application of the Titchmarsh theory.

The Titchmarsh eigenfunction expansion theory is simply the application of the Cauchy contour integration method to the general singular case of the Sturm-Liouville expansion. A function $\Phi(x, \lambda)$ which satisfies the second-order differential equation

$$(L-\lambda)\,\Phi(x,\lambda) = -f(x) \tag{8.47}$$

is constructed in terms of two solutions ϕ and Ψ of the eigenvalue equation

$$(L-\lambda)W = 0 \quad . \tag{8.48}$$

ϕ is well-behaved at one endpoint of variable x and Ψ at the other. Integration of $\Phi(x, \lambda)$ around a large contour in the complex λ-plane gives f(x), the arbitrary function. The singularities of $\Phi(x, \lambda)$ on the real axis determine whether one gets a series expansion, an integral expansion, or both.

The details of the expansion theorem are now given. The associated conical functions satisfy the eigenvalue equation

$$(L-\lambda)g = 0 \quad , \tag{8.49}$$

where

$$L = -\frac{d^2}{du^2} - \coth u \frac{d}{du} + \frac{m^2}{\sinh^2 u} \quad ,$$
$$\lambda = \frac{\nu^2+1}{4} \quad . \tag{8.50}$$

This equation can be written in the standard Titchmarsh form of

$$\left[\frac{d^2}{du^2} + \left(\lambda' - q(u)\right)\right]h = 0 \quad , \tag{8.51}$$

where

$$g(u) = \frac{1}{(\sinh u)^{\frac{1}{2}}} h(u) \quad ,$$
$$q(u) = \frac{m^2 - \frac{1}{4}}{\sinh^2 u} \quad , \tag{8.52}$$
$$\lambda' = (\lambda - \tfrac{1}{4}) \quad .$$

Since $q(u) \longrightarrow 0$ as $u \longrightarrow \infty$, there is a continuous spectrum in the interval $(0, \infty)$ of λ-values with a point-spectrum (possibly null) in the interval $(0, -\infty)$ (Titchmarsh 1962). One makes the definitions

$$F_+(u,\lambda) \equiv (2\pi)^{-\frac{1}{2}} \int_0^\infty f(u,t)\, e^{i\lambda t}\, dt \quad (\mathrm{Im}\,\lambda > 0), \tag{8.53}$$

$$F_-(u,\lambda) \equiv (2\pi)^{-\frac{1}{2}} \int_{-\infty}^0 f(u,t)\, e^{i\lambda t}\, dt \quad (\mathrm{Im}\,\lambda < 0), \tag{8.54}$$

where λ is a complex variable and $f(u,t)$ is a solution of the partial differential equation

$$Lf = i\,\frac{\partial f}{\partial t} \quad . \tag{8.55}$$

The function $f(u,t)$ reduces to arbitrary function $f(u)$ when $t=0$. The functions $F_+(u, \lambda)$ and $F_-(u, \lambda)$ obey the equations

$$(L-\lambda)\, F_+(u,\lambda) = \frac{i}{(2\pi)^{1/2}} f(u) \quad , \tag{8.56}$$

$$(L-\lambda)\, F_-(u,\lambda) = -\frac{i}{(2\pi)^{1/2}} f(u) \tag{8.57}$$

and have the general form of

$$F_\pm = \mp \frac{i}{(2\pi)^{1/2}} \left[\Theta(u,\lambda) \int_0^u \frac{\Phi(y,\lambda)\, f(y)\, dy}{W_y(\Phi,\Theta)} + \Phi(u,\lambda) \int_u^\infty \frac{\Theta(y,\lambda)\, f(y)\, dy}{W_y(\Phi,\Theta)} \right], \tag{8.58}$$

where $\Phi(u, \lambda)$ and $\Theta(u, \lambda)$ are solutions of the eigenvalue equation

$$(L-\lambda)g = 0 \tag{8.59}$$

which are not divergent for $u=0$ and $u=\infty$, respectively, and where

W_y is the Wronskian evaluated at u=y:

$$W_y(\phi,\theta) = \left[\phi(u,\lambda)\frac{d}{du}\theta(u,\lambda) - \theta(u,\lambda)\frac{d}{du}\phi(u,\lambda)\right]_{u=y} . \tag{8.60}$$

The solutions $\phi(u, \lambda)$ and $\theta(u, \lambda)$ for operator L defined by Eq. 8.34 are

$$\phi(u,\lambda) = \mathcal{P}^{m}_{-\frac{1}{2}-i(\lambda')^{1/2}}(\cosh u) , \tag{8.61}$$

$$\theta(u,\lambda) = e^{im\pi}\,\mathcal{Q}^{-m}_{-\frac{1}{2}-i(\lambda')^{1/2}}(\cosh u) , \tag{8.62}$$

where $(\lambda')^{1/2} = (\lambda-\frac{1}{4})^{\frac{1}{2}} = \frac{\nu}{2}$ (with Im $\nu > 0$) . (8.63)

For the above equations m is integral and $\mathcal{P}$ and $\mathcal{Q}$ are the associated conical functions of the first and second kinds respectively. The Wronskian is

$$W_y(\phi, \theta) = -\frac{1}{\sinh y} . \tag{8.64}$$

The inverse formula to Eqs. 8.53 and 8.54 gives f(u,t):

$$f(u,t) = (2\pi)^{-\frac{1}{2}}\left(\int_{ic-\infty}^{ic+\infty} - \int_{ic'-\infty}^{ic'+\infty}\right) F_+(u,\lambda)\, e^{-i\lambda t}\, d\lambda , \tag{8.65}$$

where $c > 0$ and $c' < 0$ and where $F_+(u, \lambda) = -F_-(u, \lambda)$ has been used. By writing the preceding equation in terms of $\lambda' = \lambda - \frac{1}{4} = \frac{\nu^2}{4}$ and making the notational substitution $\Phi(u, \lambda')$ in place of $F_+(u, \lambda')$, one gets

$$f(u,t) = (2\pi)^{-\frac{1}{2}}\left(\int_{ic-\infty}^{ic+\infty} - \int_{ic'-\infty}^{ic'+\infty}\right)\Phi(u,\lambda')\, e^{-i(\lambda'+\frac{1}{4})t}\, d\lambda' . \tag{8.66}$$

The only singularity of $\Phi(u, \lambda')$ is a branch point at the origin $\lambda'=0$. There are no poles on the negative real axis and hence no point-spectrum. Hence Eq. 8.66 can be written as

$$f(u,t) = (2\pi)^{-\frac{1}{2}} \int_0^\infty \left[\Phi(u,\lambda' e^{i\pi}) - \Phi(u,\lambda' e^{-i\pi})\right] e^{-i(\lambda'+\frac{1}{4})t} d\lambda' . \qquad (8.67)$$

Since

$$\phi(u,\lambda' e^{-i\pi}) = \mathcal{P}^m_{-\frac{1}{2}+\frac{i\nu}{2}}(\cosh u) = \mathcal{P}^m_{-\frac{1}{2}-\frac{i\nu}{2}}(\cosh u) = \phi(u,\lambda' e^{+i\pi}) , \qquad (8.68)$$

$$\begin{aligned} \theta(u,\lambda' e^{-i\pi}) &= e^{im\pi} \mathcal{Q}^{-m}_{-\frac{1}{2}+\frac{i\nu}{2}} \\ &= e^{im\pi} \mathcal{Q}^{-m}_{-\frac{1}{2}-\frac{i\nu}{2}} + \cos\left(\tfrac{1}{2}+\tfrac{i\nu}{2}\right)\pi \left|\Gamma\left(\tfrac{1}{2}-m+\tfrac{i\nu}{2}\right)\right|^2 \mathcal{P}^m_{-\frac{1}{2}-\frac{i\nu}{2}} \qquad (8.69) \\ &= \theta(u,\lambda' e^{i\pi}) - i \sinh\left(\tfrac{\pi\nu}{2}\right) \left|\Gamma\left(\tfrac{1}{2}-m+\tfrac{i\nu}{2}\right)\right|^2 \phi(u,\lambda' e^{i\pi}) , \end{aligned}$$

one can use Eq. 8.58 to write:

$$\begin{aligned} &\Phi(u,\lambda' e^{i\pi}) - \Phi(u,\lambda' e^{-i\pi}) \\ &= \frac{\sinh\left(\frac{\pi\nu}{2}\right)}{(2\pi)^{1/2}} \left|\Gamma\left(\tfrac{1}{2}-m+\tfrac{i\nu}{2}\right)\right|^2 \mathcal{P}^m_{-\frac{1}{2}+\frac{i\nu}{2}}(\cosh u) \int_0^\infty \mathcal{P}^m_{-\frac{1}{2}+\frac{i\nu}{2}}(\cosh y) \sinh y \, f(y)\, dy . \qquad (8.70) \end{aligned}$$

By substituting the preceding equation into Eq. 8.67, setting $t=0$, and letting $\lambda' = \frac{\nu^2}{4}$, one obtains the final result

$$\begin{aligned} f(u) = \frac{1}{4\pi} \int_0^\infty \nu\, d\nu \sinh\left(\tfrac{\pi\nu}{2}\right) \left|\Gamma\left(\tfrac{1}{2}-m+\tfrac{i\nu}{2}\right)\right|^2 \mathcal{P}^m_{-\frac{1}{2}+\frac{i\nu}{2}}(\cosh u) \qquad (8.71) \\ \cdot \int_0^\infty \mathcal{P}^m_{-\frac{1}{2}+\frac{i\nu}{2}}(\cosh y) \sinh y\, f(y)\, dy \end{aligned}$$

for the expansion of an arbitrary function f(u) in terms of associated conical functions. The expansion theorem is valid provided that f(u) is square-integrable in the interval $(0, \infty)$. The form of the above expansion is exactly as suggested by the orthogonality property of the associated conical functions given by Eq. 8.10. Also the results of the expansion theorem agree with the generalized Mehler formulas expressed by Eqs. 8.45 and 8.46.

The expansion given by Eq. 8.71 can be used to make a further argument for the orthogonality of two associated conical functions corresponding to different ν, which was established in the previous section of this chapter. For a set f(u) equal to $\mathcal{P}^{m}_{-\frac{1}{2}+\frac{i\nu'}{2}}(\cosh u)$, one writes for Eq. 8.71 the following:

$$\mathcal{P}^{m}_{-\frac{1}{2}+\frac{i\nu'}{2}}(\cosh u) = \frac{1}{4\pi}\int_0^\infty \nu\, d\nu \sinh\left(\frac{\pi\nu}{2}\right)\left|\Gamma\left(\tfrac{1}{2}-m+\tfrac{i\nu}{2}\right)\right|^2 \mathcal{P}^{m}_{-\frac{1}{2}+\frac{i\nu}{2}}(\cosh u)$$
$$\cdot\int_0^\infty \mathcal{P}^{m}_{-\frac{1}{2}+\frac{i\nu}{2}}(\cosh y)\, \mathcal{P}^{m}_{-\frac{1}{2}+\frac{i\nu'}{2}}(\cosh y)\sinh y\, dy \quad . \tag{8.72}$$

One sees that $\mathcal{P}^{m}_{-\frac{1}{2}+\frac{i\nu'}{2}}(\cosh u)$ is expressed as a linear combination of functions $\mathcal{P}^{m}_{-\frac{1}{2}+\frac{i\nu}{2}}$ with coefficient proportional to the integral $\int_0^\infty \mathcal{P}^{m}_{-\frac{1}{2}+\frac{i\nu}{2}}\mathcal{P}^{m}_{-\frac{1}{2}+\frac{i\nu'}{2}}$ sinh y dy. Clearly the coefficient must be zero unless $\nu = \nu'$.

D. Expansions in Terms of the O(3,C) Harmonic Functions

The harmonic function solutions given by Eqs. 6.38 and 6.42 with ℓ set equal to zero are to be examined closely:

$$\mathcal{Z}^{o\nu}_{jm}(u,\psi,\theta,\phi) = \sum_{m''} f^{o\nu}_{jm''}(u)\, D^{j*}_{m''m}(\psi,\theta,\phi)$$

$$= \sum_{m',m''} i^{m'} \left[\frac{\Gamma(q-m'+1)}{\Gamma(q+m'+1)}\right]^{\frac{1}{2}} (p0qm'|pqjm')' \tag{8.73}$$

$$\cdot\, \mathcal{P}^{m'}_{q}(\cosh 2u)\; d^{j}_{m''m'}(-iu)\; D^{j*}_{m''m}(\psi,\theta,\phi),$$

$$\tilde{\mathcal{Z}}^{o\nu}_{jm}(u,\psi,\theta,\phi) = \sum_{m''} \tilde{f}^{o\nu}_{jm''}(u)\, D^{j*}_{m''m}(\psi,\theta,\phi)$$

$$= \sum_{m',m''} (-i)^{m'} \left[\frac{\Gamma(p-m'+1)}{\Gamma(p+m'+1)}\right]^{\frac{1}{2}} (pm'q0|pqjm')' \tag{8.74}$$

$$\cdot\, \mathcal{P}^{m'}_{p}(\cosh 2u)\; d^{j}_{m''m'}(+iu)\; D^{j*}_{m''m}(\psi,\theta,\phi),$$

where $p = \frac{-1 \pm i\nu}{2}$ and $q = \frac{-1 \pm i\nu}{2}$. The functions $\mathcal{Z}^{o\nu}_{jm}$ $(\tilde{\mathcal{Z}}^{o\nu}_{jm})$ correspond to a final orientation such that the rotated complex three-vector $\hat{z}'$ $(\hat{z}^{*\prime})$ lies along the 3-axis. If the complex numbers p and q are complex conjugates of each other, then the functions $f^{o\nu}_{jm''}$ are identical with $\tilde{f}^{o\nu\,*}_{jm''}$ except for a phase factor. If p and q are equal to each other, the relationship is slightly different: $f^{o\nu}_{jm''}$ and $\tilde{f}^{o-\nu\,*}_{jm''}$ are the same except for a phase factor. The exact relationships between the $f^{o\nu}_{jm''}$ and $\tilde{f}^{o\nu}_{jm''}$ cannot be found until the symmetry relations of the coupling coefficients have been derived.

Orthogonality of the $\mathcal{Z}^{o\nu}_{jm}$ and $\tilde{\mathcal{Z}}^{o\nu}_{jm}$ functions can be studied by use of the following orthogonality properties of the $D^{j*}_{m''m}$, $\mathfrak{d}^{j}_{m''m'}$, and $\mathcal{P}^{m'}_{-\frac{1}{2}-\frac{i\nu}{2}}$:

$$\int_0^{2\pi} d\psi \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \; D^{j_1 *}_{\mu_1 m_1}(\psi,\theta,\phi) \, D^{j_2}_{\mu_2 m_2}(\psi,\theta,\phi) = \frac{8\pi^2}{2j_1+1} \delta_{\mu_1\mu_2} \delta_{m_1 m_2} \delta_{j_1 j_2} \quad , \tag{8.75}$$

$$\sum_m \mathfrak{d}^{j}_{m'm}(-iu) \, \mathfrak{d}^{j}_{m m''}(+iu) = \delta_{m' m''} \quad , \tag{8.76}$$

$$\int_0^{\infty} \mathcal{P}^{m}_{-\frac{1}{2}-\frac{i\nu}{2}}(\cosh y) \, \mathcal{P}^{m}_{-\frac{1}{2}-\frac{i\nu'}{2}}(\cosh y) \sinh y \, dy = \frac{4\pi \, \delta(\nu-\nu')}{\nu \sinh(\frac{\pi\nu}{2}) \left| \Gamma(\frac{1}{2}-m+\frac{i\nu}{2}) \right|^2} \quad . \tag{8.77}$$

The first of the above equations is found in any standard angular momentum textbook (see for example, Rose 1957). The orthogonality property for the $\mathfrak{d}^{j}_{m''m'}$ follows readily from the corresponding property for the standard d-matrices, and the orthogonality of the associated conical functions was established in Section B.

It is easily seen that neither the $\mathcal{Z}^{o\nu}_{jm}$ functions nor the $\tilde{\mathcal{Z}}^{o\nu}_{jm}$ functions form an orthogonal set over the four-parameter space; that is, neither the integral $\int \mathcal{Z}^{o\nu}_{jm} \mathcal{Z}^{o\nu' *}_{j'm'} \, d\tau$ nor the integral $\int \tilde{\mathcal{Z}}^{o\nu}_{jm} \tilde{\mathcal{Z}}^{o\nu' *}_{j'm'} \, d\tau$ gives the delta functions $\delta_{jj'} \delta_{mm'} \delta(\nu-\nu')$ times constants. In both cases the sum over m-values does not remove the $\mathfrak{d}$-functions as needed to establish orthogonality. As expected, neither set of functions is square-integrable over the four-parameter space.

However, there does exist orthogonality of the form:

$$\int \mathcal{Z}^{o\nu}_{jm} \tilde{\mathcal{Z}}^{o\nu' *}_{j'm'} \, d\tau = \frac{4\pi^2 \coth(\frac{\pi\nu}{2})}{(2j+1)\nu} \alpha(\nu,j) \delta_{jj'} \delta_{mm'} \delta(\nu-\nu') \quad , \tag{8.78}$$

where $d\tau = \frac{\sinh 2u}{4} \sin\theta \, du \, d\Psi \, d\theta \, d\phi$,

$$\alpha(\nu, j) = \sum_{\mu} (p\,0\,q\,\mu | p\,q\,j\,\mu)'(q\,\mu\,p\,0 | q\,p\,j\,\mu)' \quad \text{if } p = q^{*}, \qquad (8.79)$$

$$\alpha(\nu, j) = \sum_{\mu} (p\,0\,q\,\mu | p\,q\,j\,\mu)'(p\,\mu\,q\,0 | p\,q\,j\,\mu)'^{*} \quad \text{if } p = q \;. \qquad (8.80)$$

The $\alpha(\nu, j)$ is proportional to either the sum $\sum_{\mu} (p\,0\,q\,\mu | p\,q\,j\,\mu)'^{2}$ or the sum $\sum_{\mu} |(p\,0\,q\,\mu | p\,q\,j\,\mu)'|^{2}$. As mentioned in Chapter 7, both of these sums should give finite constants. The appearance of delta-function normalization in the integral given by Eq. 8.78 reflects the non-compact nature of the complex rotation group.

The orthogonality property expressed by Eq. 8.78 suggests that only certain functions of the four-parameter space of the complex rotation group can be expanded in terms of either the $Z^{0\nu}_{jm}$ or the $\tilde{Z}^{0\nu}_{jm}$. For suitable functions $f(u, \Psi, \theta, \phi)$, one can write

$$f(u,\Psi,\theta,\phi) = \sum_{j,m} \int_{0}^{\infty} d\nu \; c_{jm}(\nu) \, Z^{0\nu}_{jm}(u,\Psi,\theta,\phi) \quad , \qquad (8.81)$$

where the expansion coefficients $C_{jm}(\nu)$ are given by

$$C_{jm}(\nu) = \frac{1}{A(\nu,j)} \int f(u',\Psi',\theta',\phi') \, \tilde{Z}^{0\nu\,*}_{jm}(u',\Psi',\theta',\phi') \, d\tau' \qquad (8.82)$$

with

$$A(\nu, j) \equiv \frac{4\pi^{2} \coth\left(\frac{\pi\nu}{2}\right)}{(2j+1)\,\nu} \, \alpha(\nu, j) \quad . \qquad (8.83)$$

Similarly in order to expand in terms of the $\tilde{Z}^{0\nu}_{jm}$ functions, one projects out with the $Z^{0\nu}_{jm}$ to find the expansion coefficients. This differs from the usual form of an expansion theorem but is a direct consequence of the orthogonality property given by Eq. 8.78. Although the orthogonality does not define a conventional state space, the $Z^{o\nu}_{jm}$ and $\tilde{Z}^{o\nu}_{jm}$ functions still do carry the $(0, \nu)$ irreducible representation of the complex rotation group.

A closer examination is now made of the class of functions which can legitimately be expanded in terms of $Z^{o\nu}_{jm}$. One can write the expandable functions as

$$f(u,\Psi,\theta,\Phi) \equiv \sum_{j,m,m'} g^{j}_{mm'}(u)\, D^{j*}_{mm'}(\Psi,\theta,\Phi) \tag{8.84}$$

or, alternatively

$$g^{j}_{mm'}(u) \equiv \frac{2j+1}{8\pi^2}\int d\Omega\; f(u,\Psi,\theta,\Phi)\, D^{j}_{mm'}(\Omega) \,, \tag{8.85}$$

where Ω is used for the Euler angles. Then if $f(u, \Psi, \theta, \Phi)$ has the expansion given by Eq. 8.81, it follows that

$$g^{j}_{mm'}(u) = \sum_{\mu}\int_0^\infty d\nu\; C_{jm'}(\nu)\, i^{\mu}\left[\frac{\Gamma(q-\mu+1)}{\Gamma(q+\mu+1)}\right]^{\frac{1}{2}} (p0q\mu|pqj\mu)' \cdot \mathcal{P}^{\mu}_{-\frac{1}{2}-\frac{i\nu}{2}}(\cosh 2u)\; d^{j}_{m\mu}(-iu) \,. \tag{8.86}$$

Hence

$$\sum_{m} g^{j}_{mm'}(u)\; d^{j}_{m''m}(+iu) \equiv h^{j}_{m''m'}(u) \tag{8.87}$$

must be square-integrable over the u-parameter space in order that the expansion expressed by Eq. 8.81 be valid. It is seen that the functions $f(u, \Psi, \theta, \Phi)$ which can be expanded in terms of the $Z^{o\nu}_{jm}$ are themselves not square-integrable. A similar analysis for the functions expandable in terms of the $\tilde{Z}^{o\nu}_{jm}$ can be done.

That the $Z^{o\nu}_{jm}$ and $\tilde{Z}^{o\nu}_{jm}$ do not form a complete set of functions over the u, Ψ, θ, ϕ -parameter space is quite obvious. The set that is complete over the four-parameter space can easily be found. It was proven in the last section that the associated conical functions $\mathcal{P}^{m}_{-\frac{1}{2}-\frac{i\nu}{2}}$ (cosh 2u) form a complete set over the u-parameter space. Also it is well established that the rotation matrices $D^{j*}_{mm'}(\Psi, \theta, \phi)$ form a complete set in the space of the Euler angles. Thus any arbitrary square-integrable function $W(u, \Psi, \theta, \phi)$ can be expanded as

$$W(u,\Psi,\theta,\phi) = \sum_{j,m,m'} \int_0^\infty d\nu \; b_{jmm'}(\nu) \, \mathcal{P}^{m}_{-\frac{1}{2}-\frac{i\nu}{2}}(\cosh 2u) \, D^{j*}_{mm'}(\Psi,\theta,\phi) , \tag{8.88}$$

where the expansion coefficients $b_{jmm'}(\nu)$ are given by

$$\begin{aligned} b_{jmm'}(\nu) = {} & \frac{(2j+1)}{16\pi^2} \nu \sinh\left(\frac{\pi\nu}{2}\right) \left| \Gamma\left(\tfrac{1}{2} - m + \tfrac{i\nu}{2}\right)\right|^2 \\ & \cdot \int W(u,\Psi,\theta,\phi) \, \mathcal{P}^{m}_{-\frac{1}{2}-\frac{i\nu}{2}}(\cosh 2u) \, D^{j}_{mm'}(\Psi,\theta,\phi) \, d\tau \end{aligned} \tag{8.89}$$

with

$$\int d\tau = \int_0^{2\pi} d\Psi \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \int_0^\infty \sinh 2u \; du \; . \tag{8.90}$$

Thus it is the functions $\left\{ \mathcal{P}^{m}_{-\frac{1}{2}-\frac{i\nu}{2}}(\cosh 2u) D^{j*}_{mm'}(\Psi,\theta,\phi) \right\}$ which form a complete set over the u, Ψ, θ, ϕ -parameter space.

Regarded as a mathematical complete set of functions suitable for use in the expansion of physical quantities such as scattering amplitudes, one would most naturally choose the expansion given by Eq. 8.88. On the other hand, if one wishes to follow the original motivation of exploiting symmetry properties, it is the expansion given by Eq. 8.81 which properly isolates the transformation properties under the complex rotation group.

IX. CONCLUSIONS

Three sets of functions of parameters u, Ψ, θ, ϕ which act as carriers of the unitary irreducible representations of the complex rotation group have been found. The sets $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ as given by Eqs. 8.73 and 8.74 respectively may be used in expansion theorems. However, the class of functions which may be expanded is somewhat unusual. Square-integrable functions may not be expanded, whereas there exists a class of functions, as discussed in Section D of Chapter 8, which may be expanded. The sets $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ are reciprocal to each other in a certain sense. The third set of functions as given by Eqs. 6.21 and 6.22 with $\ell = 0$ perhaps may be used for expanding a square-integrable function. However, no expansion theorem has been proved for these functions.

Thus the immediate mathematical problem dealing with the complex rotation group, as mentioned in Section A of Chapter 1, has received a partial solution. However it remains to be seen whether the sets of harmonic functions might be useful for physical problems. For example, does a realistic scattering amplitude have the appropriate properties for expansion in terms of the sets $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$? This would have to be tested using a simple model such as the Yukawa potential. If the rather stringent conditions necessary for an expansion in $\mathcal{Z}$ or $\tilde{\mathcal{Z}}$ are not met, this would raise the question of whether one should further study the third set of functions.

A further topic of study is the relationship between the three sets of harmonic functions. As mentioned in Chapter 5, the corresponding sets for the group R(4) are all equivalent. The group R(4) is however compact, whereas the complex rotation group O(3,C) and the Lorentz group L(4) are not. The fact that new features come in, particularly the apparent nonequivalence of the three sets of functions, comes as no great surprise.

Another possible topic is to consider alternative approaches to finding the harmonic functions associated with the complex rotation group. One possible attack immediately comes to mind. Consider the complex three-vector $\hat{z} = \vec{\lambda} + i\vec{\mu}$ in some standard form, i.e. with fixed length and direction. Then by studying the Lie algebra of the group one would hope to find a two-parameter subgroup of O(3,C) which leaves $\hat{z}$ invariant. Next the complex rotation group must be parametrized with these as two of the six parameters, say α_5 and α_6 where $\mathcal{D}^{\ell\nu}_{(jm)(j'm')}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ is the matrix representation of the complex rotation operator. The other four parameters can be interpreted similar to the u, ψ, θ, and ϕ used in this thesis. The harmonic functions would be a special case of $\mathcal{D}^{\ell\nu}_{(jm)(j'm')}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ such that the α_5 and α_6 dependence vanishes. This approach would be exactly analogous to the spherical harmonic function as a special case of the R(3) matrix representation: $Y^{m*}_j(\beta, \alpha) \propto D^j_{m0}(\alpha, \beta, \gamma)$.

A few possible non-relativistic and relativistic scattering applications for future investigation are now briefly discussed:

(1) Non-relativistic. One could consider elastic scattering in the center of mass system from initial momentum $\vec{k}$ to final momentum $\vec{k}'$. The idea would be to choose a complex three-vector $\vec{z}(\vec{k}, \vec{k}')$ such that $z^2=1$ and the complex orientation of $\vec{z}$ specifies both k^2 and $\cos\theta$, where θ is the scattering angle. Then an analysis of the

scattering amplitude would give an explicit display of the k^2 and $\cos\theta$ dependent terms. The first two cases to try would be to regard: (a) final momentum $\vec{k}'$ as a special case of the complex three-vector, and (b) $\vec{z} = (\vec{k} + \vec{k}') + i(\vec{k} - \vec{k}')$.

(2) Relativistic. The original motivation as stated in Chapter 1 was for the application of crossing symmetry in relativistic theory. This requires the complex Lorentz group L(4,C) which is identical to the group O(3,1,C). One should be able to develop a theory of O(3,1,C) by showing the close relationship of the Lie algebra of O(3,1,C) to that of R(4) x R(4), the direct product of two R(4) groups. Since the rotation group in four dimensions R(4) is well understood, one would then be able to understand the group O(3,1,C). However, before embarking on this program one should first clear up the remaining problems concerning the complex rotation group O(3,C).

X. LITERATURE CITED

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XII. APPENDIX

In this appendix an alternative method of finding the harmonic functions of the complex rotation group O(3,C) is presented. The following treatment is heuristic, with no pretense of being rigorous.

The operators $\vec{P}$ and $\vec{Q}$ are defined in terms of the generators $\vec{J}$ and $\vec{K}$ of the complex rotation group:

$$\vec{P} = \tfrac{1}{2}(\vec{J} + i\vec{K}) \quad , \tag{12.1}$$

$$\vec{Q} = \tfrac{1}{2}(\vec{J} - i\vec{K}) \quad . \tag{12.2}$$

They obey the commutation rules of two commuting angular momentum operators:

$$[P_i, Q_j] = 0 \quad , \tag{12.3}$$

$$[P_i, P_j] = i\,\epsilon_{ijk} P_k \quad , \tag{12.4}$$

$$[Q_i, Q_j] = i\,\epsilon_{ijk} Q_k \quad . \tag{12.5}$$

The Casimir operators of the complex rotation group are expressed as

$$F = \tfrac{1}{2}(J^2 - K^2) = P^2 + Q^2 \quad , \tag{12.6}$$

$$G = \vec{J}\cdot\vec{K} = -i(P^2 - Q^2) \, . \tag{12.7}$$

Each O(3,C) irreducible representation is labeled by the pair of indices (ℓ , ν), where $J^2 - K^2 = \ell^2 - \nu^2 - 1$ and $\vec{J}\cdot\vec{K} = \ell\nu$ for the (ℓ , ν) representation. The label ℓ is positive integral, half-integral, or zero whereas ν is any real number. Alternatively, an O(3,C) irreducible representation can be labeled by the pair (p,q), where $P^2 = p(p+1)$ and $Q^2 = q(q+1)$ for the (p,q) representation. The correspondence between the labels is as follows:

$$p = \frac{-1 \pm (\ell + i\nu)}{2} \quad , \tag{12.8}$$

$$q = \frac{-1 \pm (\ell - i\nu)}{2} \quad . \tag{12.9}$$

Instead of attempting to find direct solutions of the eigenvalue equations expressed explicitly in terms of four parameters one can obtain solutions by a close examination of the ($P^2 + Q^2$) and ($P^2 - Q^2$) operators. Relationships between the operators $\vec{P}$ and $\vec{Q}$ and the complex three-vectors $\hat{z}$ and $\hat{z}^*$ can be established:

$$\vec{P} = -i(\hat{z} \times \vec{\nabla}_{\hat{z}}) \quad , \tag{12.10}$$

$$\vec{Q} = -i(\hat{z}^* \times \vec{\nabla}_{\hat{z}^*}) \quad , \tag{12.11}$$

where the complex three-vector $\hat{z}$ is defined as before to be

$$\hat{z} = \vec{\lambda} + i\vec{\mu} \tag{12.12}$$

with

$$\lambda^2 - \mu^2 = 1 \quad , \tag{12.13}$$

$$\vec{\lambda} \cdot \vec{\mu} = 0 \quad . \tag{12.14}$$

$\vec{P}$ and $\vec{Q}$ can be interpreted as the "angular momentum" operators corresponding to the complex three-vectors $\hat{z}$ and $\hat{z}^*$ respectively. It follows that a set of eigenfunction solutions to the pair of operators ($P^2 + Q^2$) and ($P^2 - Q^2$) would consist of a product of generalized spherical harmonic functions:

$$\langle \theta_{\hat{z}}, \phi_{\hat{z}}, \theta_{\hat{z}^*}, \phi_{\hat{z}^*} | p m_p q m_q \rangle = Y_p^{m_p}(\theta_{\hat{z}}, \phi_{\hat{z}})\, Y_q^{m_q}(\theta_{\hat{z}^*}, \phi_{\hat{z}^*}) , \tag{12.15}$$

where the $\mathcal{Y}_p^{m_p}$ and $\mathcal{Y}_q^{m_q}$ are complex generalizations of the spherical harmonic functions corresponding to operators $\vec{P}$ and $\vec{Q}$ respectively. They have the form:

$$\mathcal{Y}_j^m(\theta,\phi) \sim (-i)^m \left[\frac{\Gamma(j-m+1)}{\Gamma(j+m+1)}\right]^{\frac{1}{2}} \mathcal{P}_j^m(\theta)\, e^{im\phi}, \qquad (12.16)$$

where $\mathcal{P}_j^m(\theta)$ is the associated Legendre function with j, m, and θ unrestricted. The angle variables $\theta_{\hat{z}}$, $\phi_{\hat{z}}$, $\theta_{\hat{z}^*}$, and $\phi_{\hat{z}^*}$ describing the positions of $\hat{z}$ and $\hat{z}^*$ are complex and are related by $\theta_{\hat{z}} = (\theta_{\hat{z}^*})^*$ and $\phi_{\hat{z}} = (\phi_{\hat{z}^*})^*$. Hereafter the subscripts on the angle variables will be dropped. The labels p, q, m_p, and m_q are all complex; p and q are given by Eqs. 12.8 and 12.9 and $m_p = m_q^*$ since $P_3 = Q_3^\dagger$. The m-values must be complex in order to insure the presence of a fourth angle variable. However, if active rotations of the complex three-vectors are made so that the azimuthal angles (ϕ 's) become real, then the m-values also become real. This is necessary in order for the rotated solutions to satisfy the eigenvalue equations.

The relationship between the generalized spherical harmonic functions corresponding to the original orientation of $\hat{z}$ and $\hat{z}^*$ and the functions corresponding to the new orientation of the vectors after an active rotation through (α , β , γ) is as follows:

$$\mathcal{Y}_p^{m_p}(\theta,\phi)\, \mathcal{Y}_q^{m_q}(\theta^*,\phi^*)$$
$$= \sum_{m_p', m_q'} \mathcal{Y}_p^{m_p'}(\theta',\phi')\, \mathcal{D}^{p*}_{m_p' m_p}(\alpha,\beta,\gamma)\, \mathcal{Y}_q^{m_q'}(\theta^{*\prime},\phi^{*\prime})\, \mathcal{D}^{q*}_{m_q' m_q}(\alpha,\beta,\gamma). \qquad (12.17)$$

This becomes

$$Y_p^{m_p}(\theta,\phi)\, Y_q^{m_q}(\theta^*,\phi^*)$$

$$= \sum_{m_p', m_q', j} Y_p^{m_p'}(\theta',\phi')\, Y_q^{m_q'}(\theta^{*\prime},\phi^{*\prime})\, (p m_p' q m_q' | p q j m')' \cdot (p m_p q m_q | p q j m)'\, \mathfrak{D}^{j*}_{m'm}(\alpha,\beta,\gamma) \qquad (12.18)$$

by use of the complex generalization of the Clebsch-Gordan series (Andrews and Gunson 1964). In the preceding equations, the θ', ϕ', $\theta^{*\prime}$, and $\phi^{*\prime}$ are the angles describing the rotated vectors; the labels m_p' and m_q' are both real and are either integral or half-integral; the $\mathfrak{D}^{j*}_{mm'}$ represent rotation matrices with complex angular momentum; and the $(j_1 m_2 j_2 m_2 | j_1 j_2 j m)'$ are complex generalizations of the R(3) Clebsch-Gordan coefficients. In coupled form (assuming that the complex coupling coefficients have a similar orthogonality property as do the ordinary coefficients), the harmonic function solutions can be written as

$$\langle \theta',\phi',\theta^{*\prime},\phi^{*\prime},\alpha,\beta,\gamma | \ell \nu j m \rangle = \sum_{m_p', m_q'} Y_p^{m_p'}(\theta',\phi')\, Y_q^{m_q'}(\theta^{*\prime},\phi^{*\prime}) \cdot (p m_p' q m_q' | p q j m')'\, \mathfrak{D}^{j*}_{m'm}(\alpha,\beta,\gamma) . \qquad (12.19)$$

Three different sets of solutions can be obtained by making three choices of final orientation for the complex three-vectors.

For a final orientation such that Re $\hat{z} \equiv \vec{\lambda}$ lies along the 3-axis and Im $\hat{z} \equiv \vec{\mu}$ lies along the 1-axis, the angle parameters are $\theta' = iu = \theta^{*\prime}$, $\phi' = 0$, and $\phi^{*\prime} = \pi$.

The corresponding O(3,C) harmonic functions are

$$\langle u,\alpha,\beta,\gamma \,|\, \ell\nu jm\rangle_1 = \sum_{m_p',m_q'} i^{m'}(-1)^{m_p'}\left[\frac{\Gamma(p-m_p'+1)\,\Gamma(q-m_q'+1)}{\Gamma(p+m_p'+1)\,\Gamma(q+m_q'+1)}\right]^{\frac{1}{2}} \tag{12.20}$$
$$\cdot\,(p m_p' q m_q' \,|\, pq j m')'\, \mathcal{P}_p^{m_p'}(\cosh u)\, \mathcal{P}_q^{m_q'}(\cosh u)\, \mathcal{D}^{j*}_{m'm}(\alpha,\beta,\gamma).$$

If the complex three-vectors $\hat{z}$ and $\hat{z}^*$, with orientation of $\vec{\lambda}$ along the 3-axis and $\vec{\mu}$ along the 1-axis, are given an additional pure imaginary rotation of $+iu$ about the 2-axis, they become

$$\begin{aligned} \hat{z} &\longrightarrow \hat{z}' = \hat{\iota}_3 \\ \hat{z}^* &\longrightarrow (\hat{z}^*)' = \hat{\iota}_1(-i\sinh 2u) + \hat{\iota}_3(\cosh 2u) \quad . \end{aligned} \tag{12.21}$$

Obviously $\hat{z}'$ lies along the 3-axis and $(\hat{z}^*)'$ lies in the 3-1 plane at an angle of 2iu with respect to the 3-axis. The angle variables in this case are $\theta' = 0$, $\theta^{*\prime} = 2iu$, ϕ', and $\phi^{*\prime} = \pi$. Then

$$\mathcal{Y}_p^{m_p'}(\theta',\phi')\,\mathcal{Y}_q^{m_q'}(\theta^{*\prime},\phi^{*\prime}) \propto (+i)^{m_q'}\,\mathcal{P}_q^{m_q'}(\cosh 2u)\,\delta_{m_p' 0} \quad . \tag{12.22}$$

The new Euler angles α', β', γ' are related to the old Euler angles α, β, γ by

$$\mathcal{D}^{j*}_{m'm}(\alpha',\beta',\gamma') = \sum_{m''} d^{j}_{m''m'}(-iu)\,\mathcal{D}^{j*}_{m''m}(\alpha,\beta,\gamma) \quad , \tag{12.23}$$

where $d^{j}_{m''m'}$ represent the matrix elements for complex rotations about the 2-axis. The harmonic function solutions are

$$\langle u,\alpha,\beta,\gamma \,|\, \ell\nu jm\rangle_2 = \sum_{m',m''} (+i)^{m'}\left[\frac{\Gamma(q-m'+1)}{\Gamma(q+m'+1)}\right]^{\frac{1}{2}} (p0qm' \,|\, pqjm')' \tag{12.24}$$
$$\cdot\,\mathcal{P}_q^{m'}(\cosh 2u)\, d^{j}_{m''m'}(-iu)\, \mathcal{D}^{j*}_{m''m}(\alpha,\beta,\gamma).$$

For the third set of solutions one makes an additional pure imaginary rotation of $-iu$ about the 2-axis for the complex three-vectors $\hat{z}$ and $\hat{z}^*$ from their orientation of $\vec{\lambda}$ along the 3-axis and $\vec{\mu}$ along the 1-axis to a new orientation:

$$\begin{aligned} \hat{z} &\longrightarrow \hat{z}' = \hat{\iota}_1 (i \sinh 2u) + \hat{\iota}_3 (\cosh 2u) \quad , \\ \hat{z}^* &\longrightarrow (\hat{z}^*)' = \hat{\iota}_3 \quad . \end{aligned} \tag{12.25}$$

Obviously $(\hat{z}^*)'$ lies along the 3-axis while $\hat{z}'$ lies in the 3-1 plane at an angle of $2iu$ with respect to the 3-axis. For this final orientation the angle variables are $\theta' = 2iu$, $\theta^{*\prime} = 0$, $\phi' = 0$, and $\phi^{*\prime}$. Then

$$\mathcal{Y}_p^{m_p'}(\theta',\phi')\, \mathcal{Y}_q^{m_q'}(\theta^{*\prime},\phi^{*\prime}) \propto (-i)^{m_p'}\, \mathcal{P}_p^{m_p'}(\cosh 2u)\, \delta_{m_q' 0} \ . \tag{12.26}$$

The new Euler angles α'', β'', γ'' can be related to the old Euler angles α, β, γ through the following:

$$\mathcal{D}^{j*}_{m'm}(\alpha'',\beta'',\gamma'') = \sum_{m''} d^{j}_{m''m'}(iu)\, \mathcal{D}^{j*}_{m''m}(\alpha,\beta,\gamma) \ . \tag{12.27}$$

The corresponding harmonic function solutions are

$$\langle u,\alpha,\beta,\gamma\,|\,\ell\nu jm\rangle_3 = \sum_{m',m''} (-i)^{m'} \left[\frac{\Gamma(p-m'+1)}{\Gamma(p+m'+1)}\right]^{\frac{1}{2}} (pm'q0\,|\,pqjm')' \cdot \mathcal{P}_p^{m'}(\cosh 2u)\, d^{j}_{m''m'}(iu)\, \mathcal{D}^{j*}_{m''m}(\alpha,\beta,\gamma) \ . \tag{12.28}$$