

MATRIX THEORY OF ELASTIC WAVE SCATTERING:  
APPLICATION TO SCATTERING OF TRANSVERSE WAVES

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ABSTRACT

New calculations of elastic wave scattering using Visscher's matrix theory have been made for transverse waves incident on axially symmetric defects. Longitudinal incident waves have also been considered with the results in agreement with those obtained previously by Visscher. A number of interesting features for incident transverse waves will be presented as well as some of the practical aspects of the calculation (e.g. convergence, accuracy, and computer time).

INTRODUCTION

The transition matrix (T-matrix) approach to solving elastic wave scattering problems has received considerable attention with essentially three methods having recently been developed.<sup>1-5</sup> Of these, the only method for which convergence has been demonstrated is the method of optimal truncation (MOOT) by Visscher, which he has proven converges uniformly to the exact solution. Some aspects of the rate of that convergence, however, have not yet been appreciated and will be discussed in the first part of this talk.

An attractive feature of any transition matrix theory is that the T-matrix is independent of the incident wave. Taking advantage of this, I have written a program based on MOOT which calculates scattering of both longitudinal and transverse waves incident on a defect from any number of prescribed angles. I should point out at this time that my results for the longitudinal case agree with Visscher's, suggesting that the method has been implemented correctly by both of us. Some examples for scattering from cylindrical voids will be shown in the latter portion of this presentation.

DISCUSSION

In any T-matrix theory, the incident and scattered waves are expanded in some finite subset of a complete set of functions, with the particular theory based on the method used to relate the unknown scattered wave coefficients to the known expansion coefficients of the incident wave. Using a finite number of terms in the expansion results in an error in satisfying the boundary conditions. MOOT optimizes the solution by a least squares minimization of that error which guarantees uniform convergence to the exact solution. A very interesting and significant point regarding the convergence of MOOT is perhaps best made with the following simple example.

Consider a system of two equations in two unknowns,

$$\begin{pmatrix} A_{\ell\ell} & A_{\ell\ell'} \\ A_{\ell'\ell} & A_{\ell'\ell'} \end{pmatrix} \begin{pmatrix} X_{\ell} \\ X_{\ell'} \end{pmatrix} = \begin{pmatrix} Y_{\ell} \\ Y_{\ell'} \end{pmatrix} \quad (1)$$

where A and Y depend on a parameter ka as  $A_{i,j} = a_{i,j}(i/ka)^{i+1}(j/ka)^{j+1}$ ,  $Y_i = y_i(i/ka)^{i+1}$ , with a and y independent of ka. This behavior is analogous to that encountered when implementing MOOT for terms in the spherical wave expansion corresponding to  $\ell \gg ka$ , where  $\ell$  is the order of the spherical Bessel, Neumann, or Hankel function, k the wave vector, and a the "size" of the defect (e.g., for a spherical defect, a equals the radius of the sphere). In this limit, the spherical Neumann function becomes very large, increasing approximately as  $(\ell/ka)^{\ell+1}$ . Inverting Eq.(1) one obtains for X,

$$\begin{pmatrix} X_{\ell} \\ X_{\ell'} \end{pmatrix} = \frac{1}{d} \begin{pmatrix} (a_{\ell'\ell}y_{\ell} - a_{\ell\ell'}y_{\ell'}) (ka/\ell)^{\ell+1} \\ (a_{\ell\ell}y_{\ell'} - a_{\ell'\ell}y_{\ell}) (ka/\ell')^{\ell'+1} \end{pmatrix} \quad (2)$$

where  $d = a_{\ell\ell}a_{\ell'\ell'} - a_{\ell\ell'}a_{\ell'\ell}$ .

The  $X_i$  in this example correspond to the scattered wave expansion coefficients and illustrate that once  $\ell$  exceeds ka by some significant amount, the series has converged. However, Eq.(2) also shows that if the off-diagonal elements of a are comparable to either of the diagonal elements of a, the full matrix needs to be inverted for an accurate determination of X. This would be the case even when  $X_{\ell}$  is found to be insignificant compared to  $X_{\ell'}$ . Another way of saying this in terms of the scattering problem is, that while it is wavelength relative to some characteristic size parameter that determines the number of terms required for the partial wave series to converge, the accurate solution for those expansion coefficients requires a matrix whose rank depends to a large extent on the shape of the scatterer (i.e., its degree of departure from sphericity).

RESULTS

There are several checks that one can make to test the convergence and accuracy of the calculations. In particular, the total cross-section, SIGMA, can be calculated at some order, N, (where N is the highest order spherical Bessel function in the expansion) and then at successively higher orders until no significant change in SIGMA is noted. The accuracy of the calculation is also reflected in the extent to

which the optical theorem is satisfied. This theorem states that the imaginary part of the forward scattering amplitude, IMF, is equal to the total cross-section and depends on having calculated the phase of the scattered wave correctly. A different kind of test, which to my knowledge hasn't been done before, is to calculate the scattering from a sphere displaced from the origin of the coordinate system in which the spherical wave functions are defined. In this displaced geometry, the wave functions are no longer orthogonal, consequently, a relatively larger matrix than in the undisplaced geometry will have to be inverted. Since the cross-sections are invariant, comparing these results with the known exact solution provides an additional test.

In Figs. 1-4, the total cross-section and optical theorem are shown to converge for longitudinal and shear waves incident on a spherical void which has been displaced by an amount equal to one-half its radius. The values for SIGMA at convergence are exactly those obtained in the undisplaced configuration. Although only terms of order up to and including  $N=4$  were needed in the resulting partial wave series, the calculation didn't converge until  $N=14$ , as indicated in the figures.

In these figures and in all remaining figures, THETA is the polar angle in the coordinate system whose polar axis is defined by the incident wave vector, and THETA0 is the angle of incidence measured from the symmetry axis of the scatterer.

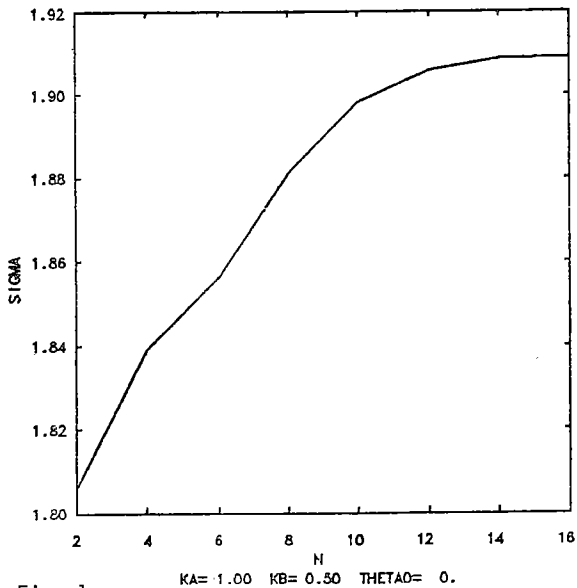


Fig. 1 Longitudinal waves incident on sphere.

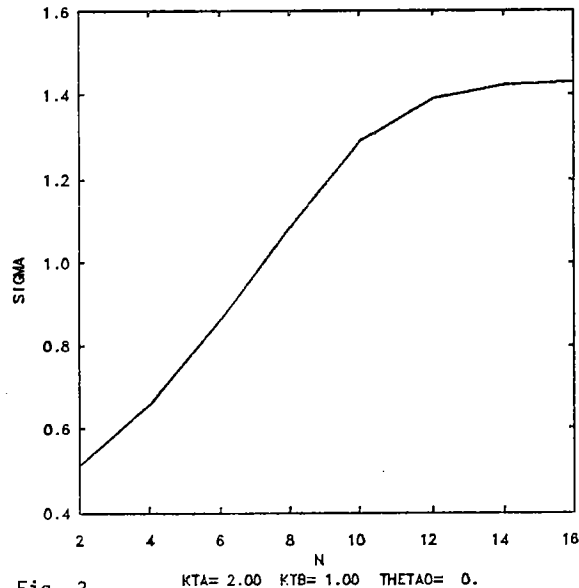


Fig. 3 Shear waves incident on sphere.

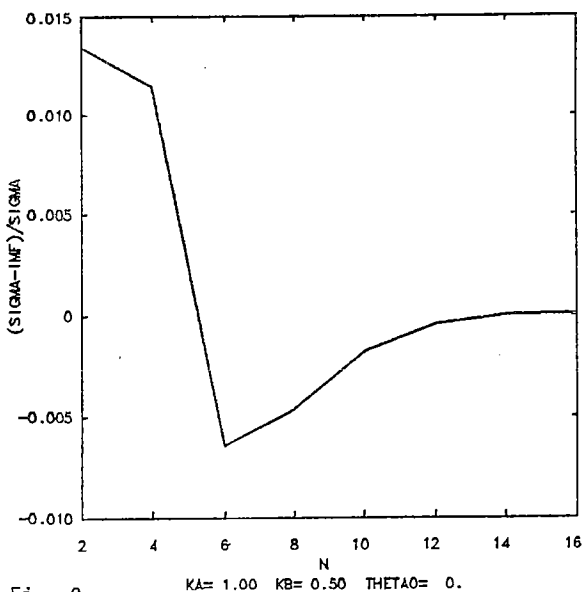


Fig. 2 Longitudinal waves incident on sphere.

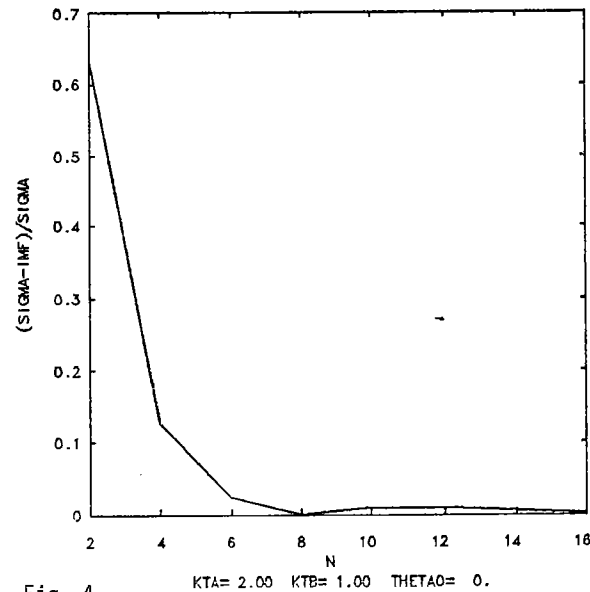


Fig. 4 Shear waves incident on sphere.

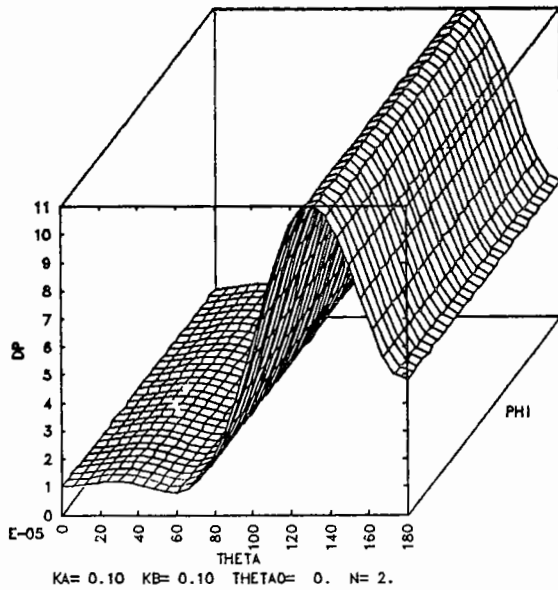


Fig. 5 Longitudinal waves incident on sphere.

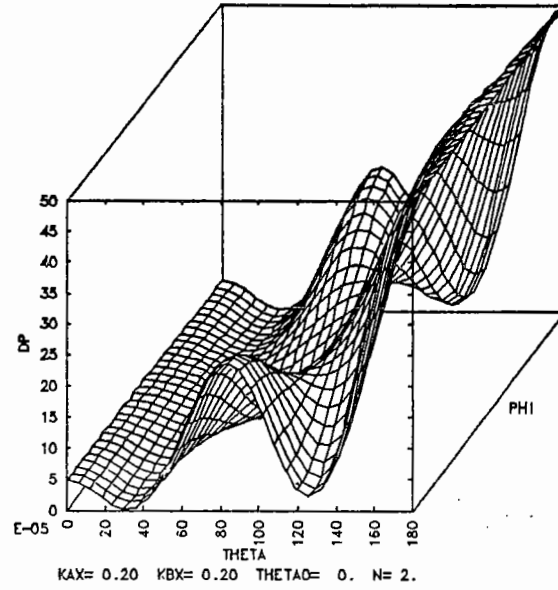


Fig. 7 Shear waves incident on sphere.

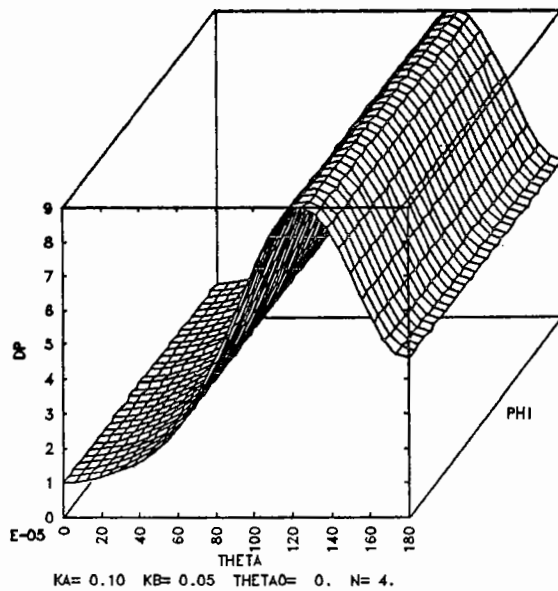


Fig. 6 Longitudinal waves incident on sphere(displaced)

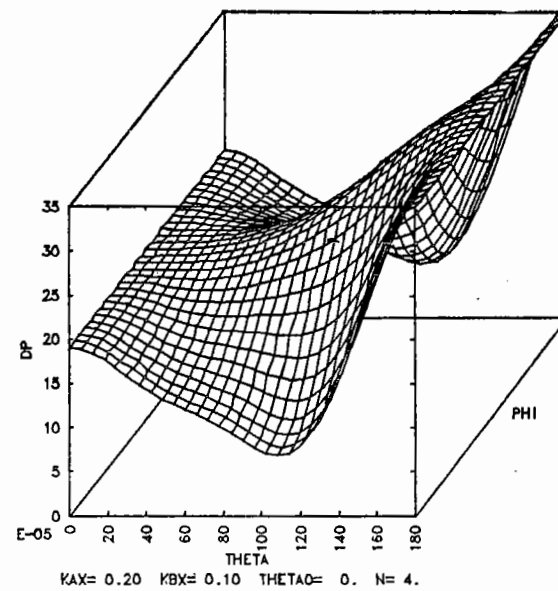
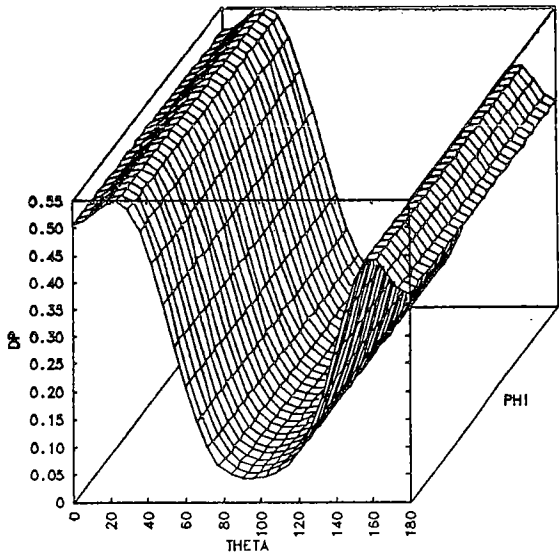


Fig. 8 Shear waves incident on sphere(displaced).

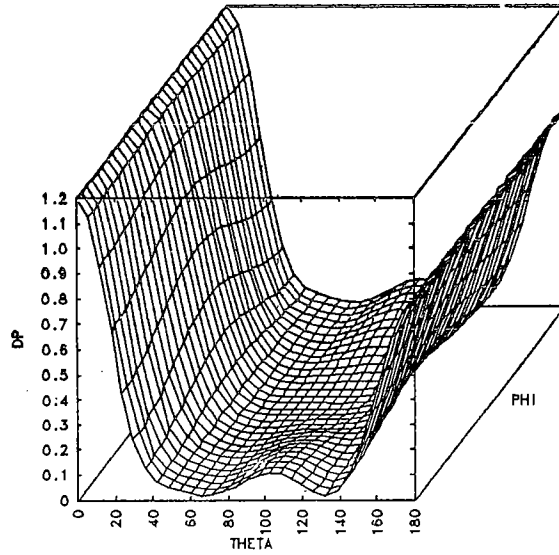
In Figs. 5-8 are shown differential cross-sections,  $DP$ , for scattering from a spherical void in the long-wavelength limit. For these calculations, and for all the calculations discussed in this talk, the cross-sections have been normalized to the geometrical cross-section (which for the sphere is  $\pi a^2$ ). The azimuthal angle  $\text{PHI}$ , ranges from 0 to 180 degrees and is measured from the x-axis which in the case of incident shear waves lies along the shear wave polarization vector. For convenience, the shear wave velocity has been set equal to one-half the longitudinal velocity in all calculations. In the long-wavelength limit,  $N=2$  is sufficient for convergence when the sphere is centered about the origin, the results for this case shown in Figs. 5 and 7. Figures 6 and 8 show calculations for the displaced sphere to order  $N=4$  which clearly have not converged. For this case,  $N=12$  is required for the

results to agree with those shown in Figs. 5 and 7.

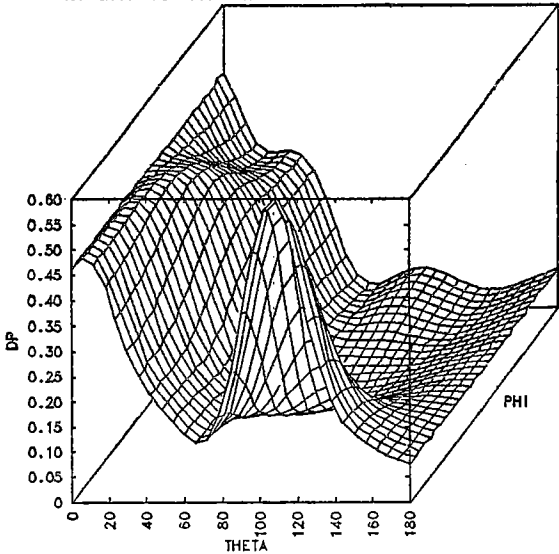
The results shown in Figs. 9-10 are for waves incident at  $0^\circ$ ,  $45^\circ$ , and  $90^\circ$  on a cylindrical void having aspect ratio of  $1/2$ . Evidence can be found in these figures that one is approaching the short wavelength limit, especially at the  $45^\circ$  incidence. For longitudinal incident waves, the peak centered about  $110^\circ$  is where one would expect a mode-converted specularly reflected shear wave. All of that peak is in fact due to the mode converted scattered wave. Strong backscattering at normal incidence and weak backscattering at  $90^\circ$  is also the expected behavior as one goes to short wavelengths. I should point out that for shear waves incident, mode conversion contributes about an order of magnitude less to the cross-section than in the case of incident longitudinal waves. All that one sees in Fig. 10 is the direct scattered wave.



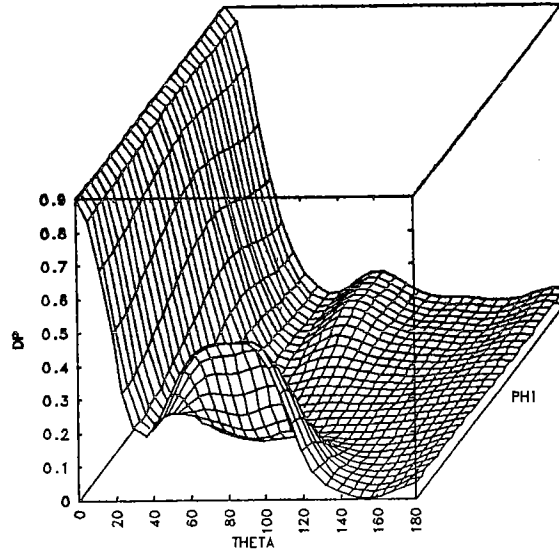
KA= 2.00 KB= 1.00 THETA0= 0. N=14.



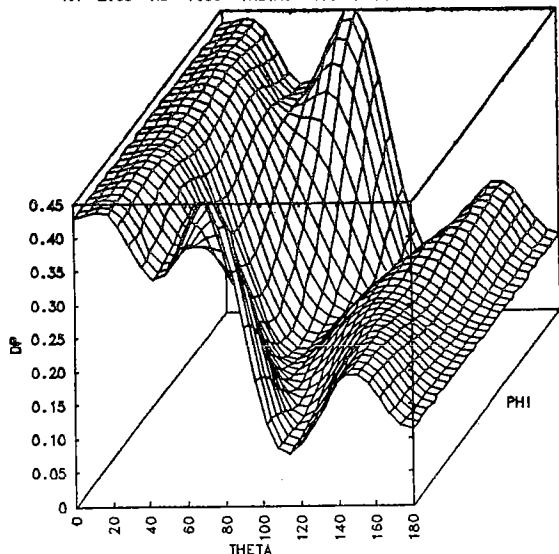
KAX= 4.00 KBX= 2.00 THETA0= 0. N=14.



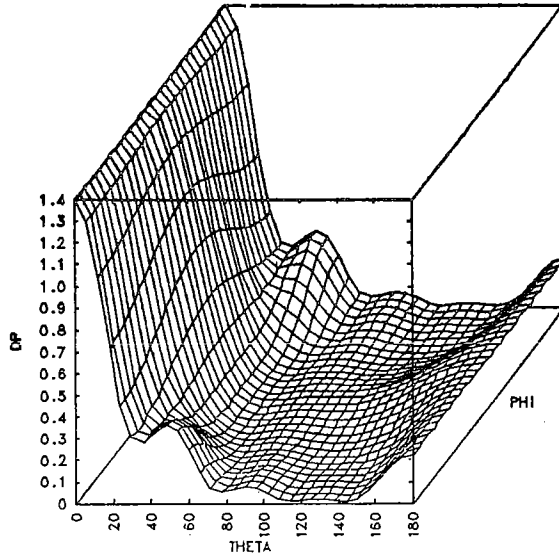
KA= 2.00 KB= 1.00 THETA0= 45. N=14.



KAX= 4.00 KBX= 2.00 THETA0= 45. N=14.



KA= 2.00 KB= 1.00 THETA0= 90. N=14.



KAX= 4.00 KBX= 2.00 THETA0= 90. N=14.

Fig. 9 Longitudinal waves incident on cylinder.

Fig. 10 Shear waves incident on cylinder.

In conclusion, I would like to express my opinion that the method of optimal truncation has advanced the theory of elastic wave scattering to the extent that it should be seriously considered as a standard with which to test the validity of other more approximate schemes. It is my intention to make such comparisons with other methods and place some quantitative limits on their range of usefulness.

#### ACKNOWLEDGMENT

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#### REFERENCES

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SUMMARY DISCUSSION  
(Jon Opsal)

Norman Bleistein (Denver Applied Analytics): I would like to address myself to this question of spherical harmonics. It would seem the difficulty is that you try to use spherical harmonics to do a problem where the coordinates are not the most desirable coordinates. Have you tried to use ellipsoidal harmonics?

Jon Opsal: In these kinds of problems it's only the spherical coordinates that you can actually separate the waves into longitudinal and transverse.

Norman Bleistein: Okay. So, the elasticity constrains against the mathematical.

Jon Opsal: You expect in the far fields spherical waves coming out, so it seems like a good choice. It's a matter of properly determining the coefficient.

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